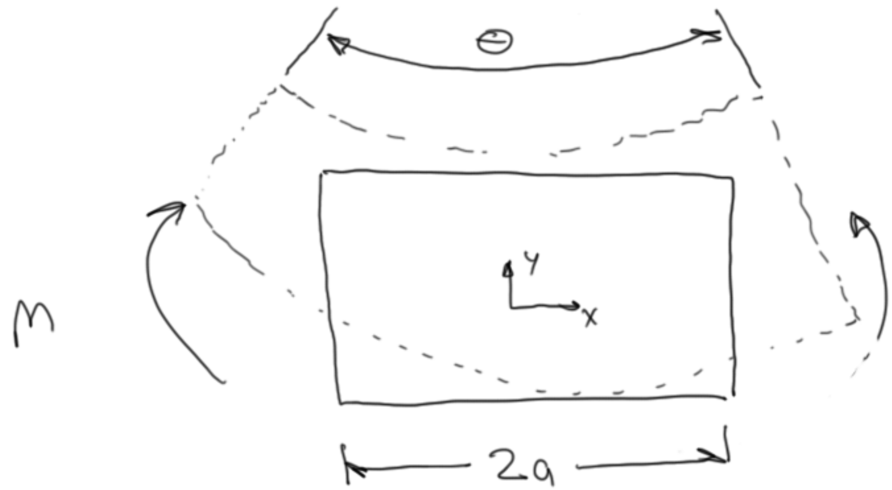
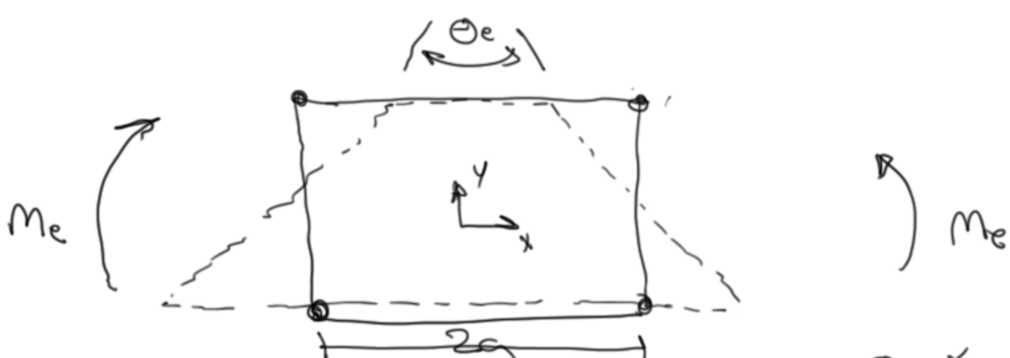
Euler beam



$$\varepsilon_{xx} = -\frac{\theta y}{2a}, \quad \varepsilon_{yy} = \nu\frac{\theta y}{2a}, \quad \varepsilon_{xy} = 0$$



$$\varepsilon_{xx} = -\frac{\theta_e y}{2a}, \quad \varepsilon_{yy} = 0, \quad \varepsilon_{xy} = -\frac{\theta_e x}{2a} \leftarrow \text{parasitic shear}$$

Elastic strain energy

$$U = \frac{1}{2}\int [\varepsilon]^T C \, [\varepsilon] \, dV \qquad \text{where} \qquad [\varepsilon] = [\, \varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{xy} \,]^T$$

$$\frac{M\theta}{2} = U \qquad\qquad \frac{M_e \theta_e}{2} = U_e$$

$M = M_e$ under plane stress $[C] \rightarrow$ plane stress

$$\frac{\theta_e}{\theta} = \frac{1-\nu^2}{1+\frac{1-\nu}{2}\left(\frac{a}{b}\right)^2}$$

the term $\left(\frac{a}{b}\right)^2$ is present due to parasitic shear

$$\lim_{\frac{a}{b}\to\infty} \frac{1-\nu^2}{1+\frac{1-\nu}{2}\left(\frac{a}{b}\right)^2} = 0 \qquad\qquad \frac{\theta_e}{\theta} \approx 0 \quad \text{as} \quad \frac{a}{b}\to\infty$$

the mesh "locks"

Consider volumetric strain

$$\frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy}$$

Under plain strain conditions, the pressure

$$p = \frac{E_y \theta (y - \nu)}{2a(1+\nu)(2\nu - 1)}$$

$\nu \to \frac{1}{2}$ $\quad p \to \infty$ $\quad$ known as volumetric locking

$$p_e = \frac{E_y\, y\, \theta_e}{2a(-1 + \nu + 2\nu^2)}$$

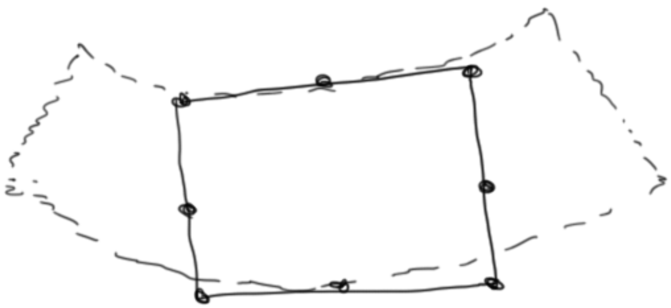
$p \Rightarrow \infty$, as $\nu \Rightarrow \frac{1}{2}$

3 node tri (constant strain tri) also similar locking

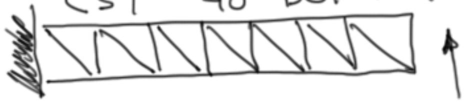
"locking" $\neq$ immovability

$=$ excessive stiffness

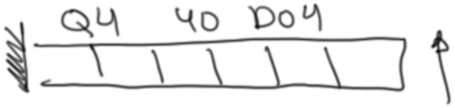
For example (Q8)



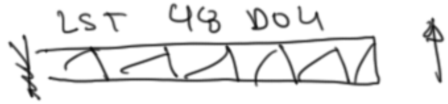
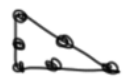
CST 40 DOF (not to scale)

 0.25 

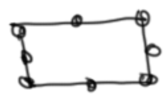
Q4 40 DOF

 0.67 

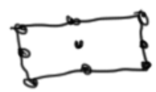
LST 48 DOF

 0.99 

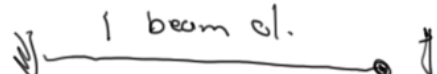
Q8 20 DOF

 0.93 

Q9 24 DOF

 0.99 

1 beam el.

 1.00 

$$0 = \left[\int_{\Omega} h_e B^T C B \, d\vec{x} \right] \vec{u} + \left[\int_{\Omega} h_e \rho [N]^T [N] \, d\vec{x} \right] \ddot{\vec{u}} - \int h_e \rho [N]^T \vec{b} \, d\vec{x}$$

$$- \oint_{\Gamma} h_e [N]^T \vec{t} \, dS$$

$$\sigma = C B \vec{u}$$

$$\vec{\sigma} = \vec{\sigma}'' - \alpha \vec{m} p \qquad \text{where } \vec{m} = [1\ 1\ 1\ 0\ 0\ 0]^T \quad \text{in 3D}$$

$$\vec{m} = [1\ 1\ 0]^T$$

$$\vec{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} \qquad p \approx p^h = N_j^p p_j$$

?

$$0 = \underbrace{\left[\int_\Omega h_e B^T C B \, d\vec{x}\right]}_{K} \vec{u} - \underbrace{\left[\int_\Omega B^T \alpha \vec{m} N^p \, d\vec{x}\right]}_{Q} \vec{p} + \underbrace{\left[\int_\Omega \rho [N^u]^T [N^u] \, d\vec{x}\right]}_{M} \ddot{\vec{u}}$$

$$\underbrace{- \int h_e \rho [N^u]^T \vec{b} \, d\vec{x} - \oint_\Gamma h_e [N^u]^T \vec{t} \, dS}_{F^{(1)}}$$

u-p form ignore solid inertia $\ddot{u} \to 0$

$$0 = \underbrace{\left[\int_\Omega h_e B^T \alpha \vec{m} N^p \, d\vec{x}\right]}_{Q} \dot{\vec{u}} + \underbrace{\left[\int_\Omega h_e (\nabla N^p)^T \underline{\underline{K}} \nabla N^p \, d\vec{x}\right]}_{H} \vec{p} + \underbrace{\left[\int_\Omega h_e N^p \frac{1}{Q} N^p \, d\vec{x}\right]}_{S} \dot{\vec{p}}$$

$$\underbrace{+ \int h_e (\nabla N^p)^T \nabla^T (\underline{K} \rho \vec{b}) \, d\vec{x} - \oint (N^p)^T \vec{q} \, dS}_{F^{(2)}}$$

where $\frac{1}{Q} = \frac{n}{K_f} + \frac{\alpha - n}{K_s}$

$$\begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\vec{u}} \\ \ddot{\vec{p}} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ Q^T & S \end{bmatrix} \begin{Bmatrix} \dot{\vec{u}} \\ \dot{\vec{p}} \end{Bmatrix} + \begin{bmatrix} K & -Q \\ 0 & H \end{bmatrix} \begin{Bmatrix} \vec{u} \\ \vec{p} \end{Bmatrix} - \begin{Bmatrix} F^{(1)} \\ F^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

drained behavior "sands + gravel" high-perm.

$$\begin{bmatrix} K & -Q \\ 0 & H \end{bmatrix} \begin{Bmatrix} \vec{u} \\ \vec{p} \end{Bmatrix} = \begin{Bmatrix} F^{(1)} \\ F^{(2)} \end{Bmatrix}$$

undrained behavior "silts + clays" low-perm

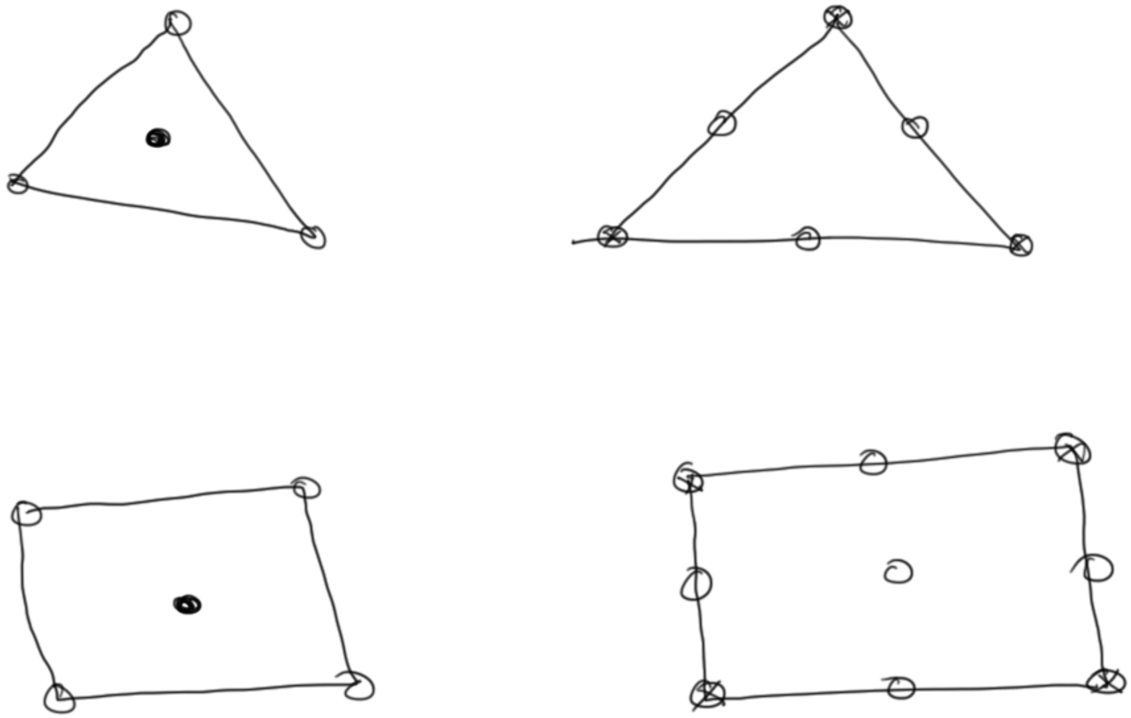
$$H = 0 \qquad F^{(2)} = 0$$

$$Q^T \dot{\vec{u}} + S \dot{\vec{p}} = 0 \quad \Rightarrow \quad \vec{u}(t=0) = \vec{p}(t=0) = 0$$

$$Q^T \vec{u} + S \vec{p} = 0$$

N^p must be lower order that N^u

$$\begin{bmatrix} K & -Q \\ Q^T & 0 \end{bmatrix} \begin{Bmatrix} \vec{u} \\ \vec{p} \end{Bmatrix} = \begin{Bmatrix} \vec{F}^{(1)} \\ 0 \end{Bmatrix}$$



○ nodes w/ $\vec{u}$ only

⊗ nodes w/ $\vec{u}$ + p

● nodes w/ p only