Time dependendent problems

(a) coupled

$$u(\vec{x},t) \approx u^h(\vec{x},t) = u_j\, N_j(\vec{x},t)$$

(b) decoupled, i.e. seperable

$$u(x,t) \approx u_h(\vec{x},t) = \Sigma\, u_j(t)\, N_j(\hat{x})$$

$$\underline{u(x,t) = T(t)\, X(x)}$$

1. Spatial approximat w/ FEM

2. Temporal approx. use finite diff.

$$\frac{du}{dt} = \frac{\Delta u}{\Delta t}$$

$$\frac{\partial x}{\partial t} = \dot{x} = f(x,t)$$

<u>Explicit</u>

$$t = t_0 \, , \, t_0 + \Delta t \, , \, t_0 + 2\Delta t$$

$$\dot{x}_n = f(x_n, t_n)$$

$$\dot{x}_n = \frac{(x_{n+1} - x_n)}{\Delta t} = \frac{\Delta x}{\Delta t}$$

$$\frac{(x_{n+1} - x_n)}{\Delta t} = f(t_n, x_n)$$

$$x_{n+1} = x_n + \Delta t \, f(x_n, x_n)$$

$$x_0 = c$$

$$x_1 = \Delta t \, f(t_0, c) + c$$

$$x_2 = \Delta t \, f(t_1, x_1) + x_1$$

$$\vdots$$

$$\dot{x} = \lambda x(t) \qquad x(0) = c \qquad x(t) = c\, e^{\lambda t}$$

$$R(\lambda) \leq 0$$

$$\begin{aligned} x_{n+1} &= \lambda \Delta t\, x_n + x_n \\ &= (1 + \lambda \Delta t) x_n \\ &= (1 + \lambda \Delta t)^2 x_{n-1} \\ &\ \vdots \\ &= \underline{(1 + \lambda \Delta t)}^{n+1} x_0 \end{aligned}$$

$$|1 + \lambda \Delta t| \leq 1$$

$$\boxed{\Delta t \leq \frac{2}{|\lambda|}} \qquad \sim \frac{l_e}{\underline{c}} \qquad c = \sqrt{\frac{E}{\rho}}$$

Implicit

$$\dot{x}_n = \frac{(x_n - x_{n-1})}{\Delta t}$$

$$\frac{(x_n - x_{n-1})}{\Delta t} = f(x_n, t_n)$$

$$x_n = \Delta t \, f(x_n, t_n) + x_{n-1}$$

$\dot{x} = \lambda x(t)$

$$x_{n+1} = \Delta t \, f(x_{n+1}, t_{n+1}) + x_n$$

$$x_{n+1} = \lambda \Delta t \, x_{n+1} + x$$

$$x_{n+1} - \lambda \Delta t \, x_{n+1} = x_n$$

$$(1 - \lambda \Delta t) x_{n+1} = x_n$$

$$x_{n+1} = \frac{x_n}{(1 - \lambda \Delta t)^2} \qquad |1 - \lambda \Delta t| \geq 1$$

$$\vdots$$

$$= \frac{x_0}{(1 - \lambda \Delta t)^{n+1}}$$

Explicit Analysis

$$\cancel{[M]\ddot{u}} + [K]\vec{u} = \vec{F} \Rightarrow M\ddot{\vec{u}} = f^{ext} - f^{int}$$

where $[M] = \int \rho N_i N_j \, d\Omega$

$$[K] = \int B^T C B \, d\Omega$$

$$\vec{F} = \vec{f} + \vec{Q} = \underbrace{\int N_i f_i \, d\Omega - \oint_\Gamma N_i t_i \, dS}$$

Assume $\Delta t^{n+1/2} = t^{n+1} - t^n$

Central Diff

$$\Delta t^{n+1/2} = t^{n+1} - t^n$$

$$\dot{u}^{n+1/2} = v^{n+1/2} = \frac{u^{n+1} - u^n}{\Delta t^{n+1/2}}$$

$$t^{n+\frac{1}{2}} = \frac{1}{2}(t^{n+1} + t^n)$$

$$\Delta t^n = t^{n+1/2} - t^{n-1/2}$$

$$\ddot{u}^n = a^n = \left(\frac{v^{n+1/2} - v^{n-1/2}}{t^{n+1/2} - t^{n-1/2}}\right) = \frac{v^{n+1/2} - v^{n-1/2}}{\Delta t^n}$$

$$\ddot{u}^n = \frac{\Delta t^{n-1/2}(u^{n+1} - u^n) - \Delta t^{n+1/2}(u^n - u^{n-1})}{\Delta t^{n+1/2}\,\Delta t^n\,\Delta t^{n-1/2}} =$$

for Δt same $\dfrac{(u^{n+1} - 2u^n + u^{n-1})}{\Delta t^2}$

Flow chart for explicit

1. Initialize v^0, σ^0, u^0, $n=0$, $t=0$ $\rightarrow$ Compute $[M]$
2. Compute $f_0 = f^{ext} - f^{int}$
3. Compute $a^n = M^{-1} f^0$
4. Update $t^{n+1} = t^n + \Delta t^{n+1/2}$, $t^{n+1/2} = \frac{1}{2}\left(t^n + t^{n-1}\right)$
5. Update vel $v^{n+1/2} = v^n + \left(t^{n+1/2} - t^n\right) a^n$
6. Enforce B.C's on $v^{n+1/2}$

8. Compute f^{n+1}
9. Compute a^{n+1}
10. Update $v^{n+1} = v^{n+1/2} + \left(t^{n+1} - t^{n+1/2}\right) a^{n+1}$
11. Check Energy Balance K.E. $\frac{1}{2}[m]\vec{v}^{n+1}\vec{v}^{n+1}$

Implicit

Newmark $-\beta$

$$u^{n+1} = \tilde{u}^{n+1} + \beta \Delta t^2 a^{n+1}$$

$$\tilde{u}^{n+1} = u^n + \Delta t v^n + \frac{\Delta t^2}{2}(1-2\beta) a^n$$

$$v^{n+1} = \tilde{v}^{n+1} + \gamma \Delta t a^{n+1}$$

$$\tilde{v}^{n+1} = v^n + (1-\gamma)\Delta t a^n$$

β

$\gamma \Rightarrow$ damping parameter

Solve $a^{n+1} = \frac{1}{\beta \Delta t^2}\left(u^{n+1} - \tilde{u}^{n+1}\right)$ for $\beta > 0$

$\beta = 0$, $\gamma = \frac{1}{2}$ explicit central difference

$\beta = \frac{1}{2}$, $\gamma = \frac{1}{2}$ undamped trapezoid rule

$\gamma > \frac{1}{2}$ damped response $\propto \left(\gamma - \frac{1}{2}\right)$

Unconditionally stable

$$\beta \geq \frac{\gamma}{2} \geq \frac{1}{4}$$

$$0 = R = \frac{1}{\beta \Delta t^2} M(u^{n+1} - \tilde{u}^{n+1}) - f^{ext} - f^{int}$$

$$K^T = \frac{\partial r}{\partial \vec{u}} = \frac{1}{\beta \Delta t^2} M + \frac{\partial f^{ext}}{\partial \hat{u}} - \frac{\partial f^{int}}{\partial \hat{u}}$$

Flow chart for implicit

1. Initialize $v^0, u^0, \sigma^0, n=0, t=0$
2. Compute f^0
3. Compute $a^n = M^{-1} f^n$
4. Estimate $u_{new} = u^n$ or $u_{new} = \tilde{u}^{n+1}$
5. Newton Iteration
   a) Compute $f(u_{new})$
   b) $a^{n+1} = \frac{1}{\beta \Delta t^2}(u_{new} - \tilde{u}^{n+1})$
   c) $r = m a^{n+1} - f$
   d) $K^T = \frac{\partial r}{\partial u}$
   e) Modify K^T for B.C.'s
   f) Solve $\Delta u = -(K^T)^{-1} r$
   e) check convergence

6.) Update disp $u^{n+1} = u_{new}$
7.) Check Energy Balance