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PROJECT REPORT

SPACE-TIME NEAR BLACK HOLE

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ABSTRACT

In this theoretical project we firstly did foundation to General Theory of Relativity and got an insight of how important are Tensors which are used to describe various higher dimensional quantities and how gravity can be explained in terms of curvature of space time with help of metric tensor and Riemann curvature tensor. We then look into concepts of Minkowski spacetime. We also saw how we can define geodesic equation for any curve with help of parallel transport using covariant derivative. In covariant derivative we were introduced to connections particularly Christoffel symbols (or Christoffel connections) which is metric compatible and torsion free. After that we learnt about geodesic deviations then we move towards Einstein equation where we saw that curvature of spacetime is related to mass and energy.

Now having enough foundation of General relativity, we learnt about black hole particularly Schwarzschild black hole. We also checked Schwarzschild metric in different coordinate systems like Tortoise coordinate, Eddington-Finkelstein coordinate and Kruskal-Szekeres Coordinate. Then we moved towards Kruskal diagram and then we saw basic Penrose diagram.

At last we did some coding in python with help of EinsteinPy library of python which is developed to solve equations in general relativity. We did coding to calculate Christoffel symbols and Riemann tensor for given metric.

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INTRODUCTION

Earlier, In Newtonian Physics we thought space and time are totally independent of each other and there was the concept of absolute space and absolute time according to Newton. But, later Einstein's Theory of Relativity showed us that space and time are related to each other and they are not absolute.

What is General Relativity?

General Relativity is *the geometric theory of gravitation* published by Albert Einstein; it is also known as General Theory of Relativity. It tells us that the force of gravity *arises* from the curvature of space and time. It tells how matter determines the gravitational field and how gravitational field influences the behaviour of matter. The theory explains the behaviour of objects in space and time, and it can be used to understand the existence of black holes, bending of light due to gravity, gravitational lensing, etc.

How Special Theory of Relativity different from General Theory of Relativity?

The Special theory of relativity tells us about the relationship between space and time. It explains how space and time are linked for objects that are moving with respect to inertial frame of reference. It is based on two postulates:

- The laws of physics are invariant in all inertial frames of reference.
- The speed of light in a vacuum is the same for all observers, regardless of the motion of the light source or observer.

While General Relativity *generalizes the Special Theory of Relativity*, providing a unified description of gravity as a geometric property of space and time, or space-time.

Special relativity helped us to *explain events happening at speed of light in all inertial frames* but do not tell anything about gravity while General Relativity is totally based on theory of gravitation, associating force of gravity with changing geometry of space-time due to presence of mass and energy on it.

Special relativity is limited to objects that are moving with respect to inertial frames of reference while General theory of relativity deals with complete space-time continuum.

What is the Space-Time continuum?

Space-time is a mathematical model that has 3-dimensions of space and 1-dimension of time into a single 4-dimensional manifold. It is simply known as **space-time**.

Note:- *Manifold is a generalization of the notion of curved space. A manifold is a topological space that is modeled closely on Euclidean space (3-D flat space) locally, but may vary widely in global properties.*

We think space-time as fabric and according to general relativity, the presence of large amounts of mass or energy distorts the space-time, causing the fabric to "*warp*".

REVISITING GENERAL THEORY OF RELATIVITY

TENSORS

A tensor is an object which is *invariant* under a change in the coordinate system and has *components* that *change in a special and predictable* way under the coordinate transformation.
The main reason to understand tensor is geometry because it gives us insight into how geometry works. (Here in General Relativity, we need tensors to understand the geometry of space-time).
Rank of a Tensor = Total no. of indices required to identify each component uniquely
"A rank-n tensor in m-dimension is a mathematical object that has n indices and m^n components and obeys certain transformation rules." Example:

1. *Scalar: rank 0 tensor*
2. *Vector: rank 1 tensor*
3. *Matrices: rank 2 tensor*

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$ (vector: rank 1 tensor)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$ (matrix: rank 2 tensor)

NOTE: This array notation falls apart for higher rank tensors that's why <u>index notations</u> were introduced.

Index Notations:-

Rank 0 → τ
Rank 1 → τ_i
Rank 2 → τ_{ij} (For 2-d or 3-D) or $\tau_{\alpha\beta}$ (For 4-D)
Rank 3 → τ_{ijk}
Rank 4 → $\tau_{\alpha\beta\gamma\mu}$

<u>Einstein Summation Notation :-</u>

It is notational convention to simplify expression including summation of vectors, matrices and general tensors.

- a_i (i : subscript index)
- a^i (i : superscript index)

We will see lots of expression like bellow in tensor algebra:-

$$\sum_{i=1}^{3} a_i x_i = a_1 x_1 + a_2 x_2 + a_3 x_3$$

In Einstein summation notation we do not use summation symbol. This means

$$a_i x_i = a_1 x_1 + a_2 x_2 + a_3 x_3$$

Rules:-
1) An index is repeated twice in a single term is dummy index or we can say, summed over index.
Example:
i) $a_i\, x_i$ (***i*** is dummy index; where i = 1,2,----n)
ii) $a_{ij}\, b_j$ (***j*** is dummy index; where, i = 1,2,----n)
Dummy index is an index that is repeated twice in a term and we can replace it with any other dummy index as long as that index satisfies two conditions:

a) Index or letter can not already be in the term. Here in example (ii) dummy index j can not be replaced by ***i***.

b) The new index should be defined over same range as the dummy index.

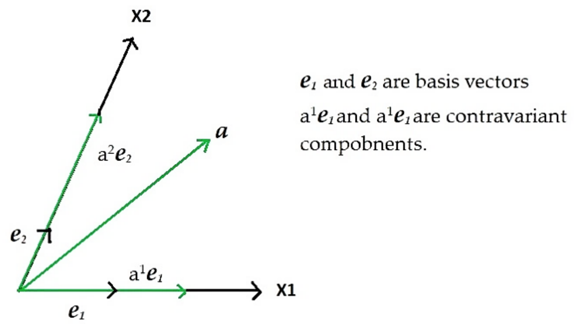
2) ***i*** is a free index(from example (ii)) that can take any value that the dummy index can take.
For example -If in the example (ii) above, j=1,2,3. So, ***i*** *can take either 1 or 2 or 3.*
I cannot always replace one free index with another free index. Free indices are not summed over. They occur only once in a term of the expression.

NOTE:- When we count no. of indices in a term, we are considering both superscript indices and subscript indices.

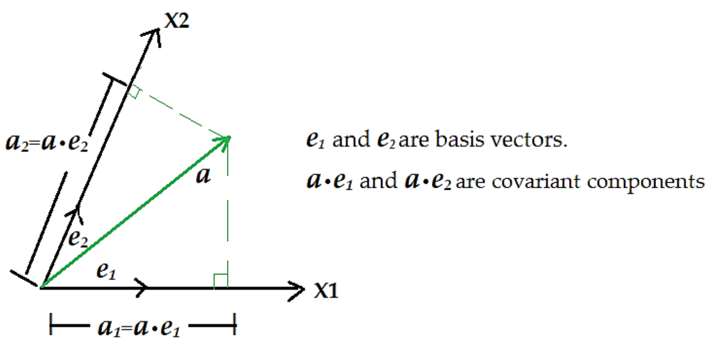
CONTRAVARIANT AND COVARIANT VECTORS

A. Theoretical understanding:-

a) We usually describe a vector **V** by *how many of each of the basis vectors we have to add together to produce it.* In such vectors, on changing coordinate system the vector components change in opposite manner to the basis vector. In this, vector components transform in contravariant fashion so they are called *contravariant components* and shown by superscript indices. The vector defined using these components known as *contravariant vectors*. ***Example:-*** Suppose I have 2D coordinate system where X1 and X2 are axes which are not perpendicular to each other(They are slanted at a non-right angle). Let $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ be ***basis vectors*** for X1 and X2 respectively. Let $\boldsymbol{a}$ be a vector in X1 and X2 coordinate system. $\boldsymbol{a}$ is equal to the ***sum*** *of no. of* $\boldsymbol{e}_1$ *need to specify* $\boldsymbol{a}$ (i.e., a^1) *and no. of* $\boldsymbol{e}_2$ *need to specify* $\boldsymbol{a}$ (i.e., a^2) in X1 and X2 axes respectively.



b) But, *there is also another way to describe a vector in terms of basis vector*. This is done by taking dot product of $\boldsymbol{a}$ with each of the ***basis vector*** i.e. $\boldsymbol{a}_1=\boldsymbol{a}\cdot\boldsymbol{e}_1$ and $\boldsymbol{a}_2=\boldsymbol{a}\cdot\boldsymbol{e}_2$**.** We can also say, these components are just the perpendicular projection of $\boldsymbol{a}$ onto $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. On changing the coordinate system, these vector components transform in same manner as the basis vector. Since, these vector components transform in covariant fashion thus, known as *covariant components*, shown with subscripts. The vectors defined using covariant components are known as *covariant vectors*.



B. Mathematical understanding:-

Assumption: *Suppose that V is a vector field defined on a subset of R^n. And let x^i and $x^{i\prime}$ are two coordinate systems related by coordinate transformation T.*
$x^{i\prime} = x^i(x^1, x^2, \ldots, x^n)$

CONTRAVARIANT VECTORS:-

The vector field A is said to be contravariant tensor of *rank 1* if its components A^μ in the x^μ coordinate system and A'^μ in the x'^μ coordinate system are related by following law of transformation:

$$A'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} A^{\nu}$$ ν is dummy index.

Example: *tangent vector is a contravariant tensor of rank 1.*

COVARIANT VECTORS:-

The vector field A is said to be covariant tensor of rank 1 if its components A_μ in the x^μ coordinate system and A'_μ in the x'^μ coordinate system are related by following law of transformation:

$$A'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} A_{\nu}$$ ν is dummy index.

Example: *Gradient vector of a differential scalar is a covariant vector of rank 1.*

MIXED TENSOR (rank=2):-

A matric field $\mathcal{Z}$ is a mixed tensor of *rank 2*, if it's components $\mathcal{Z}^i{}_j$ in the x^i coordinate system and $\mathcal{Z}^i{}_j{}'$ in the $x^{i\prime}$ coordinate system are related by following law of transformation:

$$\mathcal{Z}^i{}_j{}' = \mathcal{Z}^r{}_s \frac{\partial x^{i\prime}}{\partial x^r} \frac{\partial x^s}{\partial x^{j\prime}} \quad ; 1 \leq i \leq n \,,\, 1 \leq j \leq n$$

METRIC TENSOR

The metric tensor tells us how *to calculate distance between any two points in a given space*, means it help us to calculate space interval. It is a symmetric tensor. It is used to measure length of curve in any given manifold or space.

In 4-D space time, space interval or we can say metric equation is given as: $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$

⇨ $ds^2 = g_{ab}dx^a\, dx^b$; where g_{ab} is metric tensor and $g_{ab} = (-c^2, 1, 1, 1)$

The components of a metric tensor in a coordinate basis take the form of a symmetric matrix whose entries transform covariantly under change of the coordinate system.

In basis, the component of a matrix are: $g_{ab} = g(e_a\, ,\, e_b\,)$

Example: In 2D flat space, $g_{ab} = \begin{bmatrix} \mathbf{e}_1 . \mathbf{e}_1 & \mathbf{e}_1 . \mathbf{e}_2 \\ \mathbf{e}_2 . \mathbf{e}_1 & \mathbf{e}_2 . \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$. Since, $\mathbf{e}_a . \mathbf{e}_b = \delta_{ab} = \begin{Bmatrix} 1, a = b \\ 0, a \neq b \end{Bmatrix}$, where δ_{ab} is **Kronecker delta.**

(i) In Euclidian space, $\mathbf{g}_{ij} = \boldsymbol{\delta}_{ij}$

(ii) The metric tensor is invariant under change in coordinate system but its components change under change in coordinate system. It is same metric tensor but represented by different matrices in different coordinate system for a given space.

MINKOWSKI METRIC:-

This is metric for Minkowski space where we take c=1 for $ds^2 = - c^2dt^2 + dx^2 + dy^2 + dz^2$; so metric equation for minkowski space is given as : $\mathbf{ds^2 = - dt^2 + dx^2 + dy^2 + dz^2}$

It is *flat space metric* denoted by η_{ij}

η_{ij} **= (-1 , 1, 1, 1)**

The metric signature of minkowski space is represented as (- +++) or (+ - - -) and it is always flat.

MINKOWSKI SPACE (or Minkowski Space-time)

Minkowski space is basically a combination of *3-dimensional Euclidean space* and *time* into 4-dimensional manifold, where space-time interval between any two points (events*) is independent of the inertial frame of reference in which they are recorded.*

The metric associated with it is the Minkowski metric η_{ij} **= (-1,1,1,1).**

NOTE:- *If we* ***use*** *units of feet for distance and nano-second for time,* **c** *is approximately* ***1****, so one foot per nanosecond, actually little less with an error of less than two percent.*

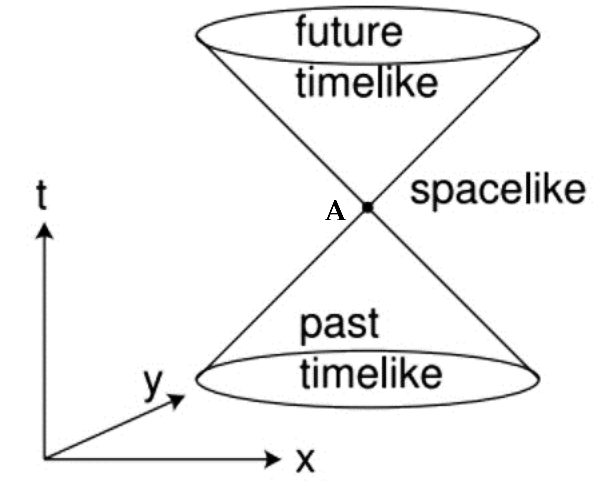
Henceforth, we take ***c =1.***

Let *S* be space interval between points ***X=(t,x,y,z)*** and ***X′ =(t′,x′,y′,z′)***

$S= -(t-t')^2 + (x-x')^2 + (y-y')^2 + (z-z')^2$; Here, *S* has units of distance squared; ***t*** is the time coordinate and ***x,y,z*** are space coordinates.

We say that points X and X′ are ***null*** or ***lightlike*** related if S=0. If X and X′ are null related, then X can communicate X′ by the signal traveling at the speed of light, if t<t′ and on the other hand, X can receive a signal from X′ traveling at speed of light if t>t′. Note that when S=0, we can only have t=t′ if X and X′ coincide. We can say that the points X and X′ are ***timelike*** related if S>0. If they are timelike related, then X can communicate X′ traveling at less speed of light, if t<t′. On the other hand, X can receive signal from X′ traveling at less than the speed of light, if t>t′. Note that when S>0, we cannot have t=t′. We say that X′ is in the ***causal future*** of X if t<t′ and S>=0. We say that X is the ***casual past*** of X if t>t′ and S>=0. We can say that X and X′ are acasual or ***spacelike*** related if and only if S<0. Such points cannot be connected by any unknown physical signal.

LIGHT CONE



The events that are ***lightlike*** separated from any particular event **A**, lie on a cone whose apex is A. This cone illustrated is called the ***light cone*** of A. All events within the light cone are timelike separated from A; all events outside it are spacelike separated. Therefore, all events inside the cone can be reached from A on a world line which everywhere moves in a timelike direction.

Light cone is defined in terms of all possible signals that could be sent.

Let have a point P. If we look at the case that point P′ differs infinitesimally from point P, we write the displacement dX from P to P′ as

$$dX = (dt,\, dx,\, dy,\, dz)$$

And the infinitesimal space-interval is given as

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

- A curve is a future pointing timelike if $ds^2 > 0$ and $dt > 0$.
- A curve is past pointing timelike if $ds^2 > 0$ and $dt < 0$.
- A curve is null or lightlike if $ds^2 = 0$.
- A curve is spacelike if $ds^2 < 0$.

For a timelike curve, the ***line integral*** *of* $ds = \sqrt{(-dt^2 + dx^2 + dy^2 + dz^2)}$ *along the curve is* ***the proper time*** *elapsed along the curve, which physically represents aging along the curve.*

THE SIGNATURE OF A METRIC:-

The sum of the diagonal elements in the metric is called the ***signature***.

If we have, $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

then, $g_{ij} = (-1, 1, 1, 1) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The signature is formed to be -1+1+1+1=2.

COVARIANT DERIVATIVE

Derivative of a ***tensor*** is not a tensor, instead it has ***one tensorial*** and ***one non-tensorial*** part. This is the reason that we come up through with the concept of ***Covariant Derivative***. This means *covariant derivative of tensor will results a tensor*. (But remember derivative of scalar is scalar, so there is no such problem with scalars.)

The covariant derivative is a way of introducing and working with a ***connection*** on a manifold by means of a ***differential operator***. *The covariant derivative of vector field* V^{ν}:

$$\nabla_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \Gamma_{\mu}{}^{\nu}{}_{\sigma} V^{\sigma}$$

where, ∇_{μ} is covariant derivative, ∂_{μ} is partial derivative and $\Gamma_{\mu}{}^{\nu}{}_{\sigma}$ is connection which will resolve our problem *(Here, Christoffel Symbol or we can say Christoffel connection).*

In geometry, ***the notion of connection makes precise the idea of transporting data along a curve in a parallel and consistent manner****. Connections are important because they allow a comparison between the local geometry at a point and local geometry at another point.* ***For example****, an affine connection the most elementary type of connection gives a mean for parallel transport of tangent vector on a manifold from one point to another along a curve. (We will see this later where it is used in Relativity)*

Few criteria for ∇_{μ} which specify us that the connection is Christoffel Symbol:-

a) ∇_{μ} should follow linearity and Leibnitz rule.

b) $\nabla_{\mu} V^{\nu}$ should be tensor.

c) ∇_{μ} should commutate with contraction.

d) ∇_μ should reduce to ∂_μ when acting on ***scalars.***

e) Connection is ***torsion free***

f) Metric Compatibility

These criteria specified us that the connection is Christoffel symbol.

- **NOTE:-** In mathematics and physics, the Christoffel symbols are an array of numbers describing a metric connection. [The metric connection is a specialization of the affine connection to surfaces or other manifolds endowed with a metric, allowing distances to be measured on that surface.]
- **NOTE:-** We can also think about other variants of gravity using different form of connections, for example if you want to look at how to incorporate spin into gravity that usually require torsion pre-condition so we cannot use Christoffel connection in this case as Christoffel connection is torsion free.

Generally, *connections can be represented as sum of two parts one anti-symmetric (tensorial part) and other symmetric (non-tensorial part)*. ***This symmetric connection is symmetric in its lower two indices*** and ***it is torsion free*** while ***anti-symmetric one is not torsion free***. Therefore, we can say *Christoffel connection is symmetric in its lower two indices.*

CHRISTOFFEL SYMBOL:-

As we know, for connections to be Christoffel Symbol the covariant derivative ∇_μ must be metric compatible. For being metric compatible $\nabla_\beta g_{\mu\nu} = 0$. Now, consider

(1) $\nabla_\beta g_{\mu\nu} = \partial_\beta g_{\mu\nu} - \Gamma_\beta{}^\lambda{}_\mu g_{\lambda\nu} - \Gamma_\beta{}^\lambda{}_\nu g_{\mu\lambda} = 0$

(2) $\nabla_\mu g_{\nu\beta} = \partial_\mu g_{\nu\beta} - \Gamma_\mu{}^\lambda{}_\nu g_{\lambda\beta} - \Gamma_\mu{}^\lambda{}_\beta g_{\nu\lambda} = 0$

(3) $\nabla_\nu g_{\beta\mu} = \partial_\nu g_{\beta\mu} - \Gamma_\nu{}^\lambda{}_\beta g_{\lambda\mu} - \Gamma_\nu{}^\lambda{}_\mu g_{\beta\lambda} = 0$

Now, calculate (1)-(2)-(3)=0 ; also using symmetric property of Christoffel symbol we get,

$$\partial_\beta g_{\mu\nu} - \partial_\mu g_{\nu\beta} - \partial_\nu g_{\beta\mu} + 2\Gamma_\mu{}^\lambda{}_\nu g_{\lambda\beta} = 0$$

⇨ $2\Gamma_\mu{}^\lambda{}_\nu g_{\lambda\beta} = \partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}$ *(multiply both side with $g^{\varrho\beta}$)*

⇨ $\Gamma_\mu{}^\lambda{}_\nu g_{\lambda\beta} g^{\varrho\beta} = \frac{1}{2} g^{\varrho\beta} [\, \partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}]$ *(since, $g_{\lambda\beta} g^{\varrho\beta} = \delta^\varrho{}_\lambda$ (Kronecker delta))*

⇨ $\Gamma_\mu{}^\varrho{}_\nu = \frac{1}{2} g^{\varrho\beta} [\, \partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}]$ → ***Equation(i)***

Formula for Christoffel connection: $\boxed{\Gamma_\mu{}^\varrho{}_\nu = \frac{1}{2} g^{\varrho\beta} [\, \partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}}$

PARALLEL TRANSPORT

Parallel transport *is a way of moving the vector on the surface keeping the vector as straight as possible*. In simple words, Parallel transport means *we are keeping the vector as constant as possible as we move along the path step by step.*

- *Parallel transport is the intrinsic method to define curvature.*
- *When parallel transporting a vector 'forward', the rate of change of vector is completely normal to the surface i.e., perpendicular to the tangent plane.*
- ***Covariant derivative** helps us find parallel transported vector field.*
- *If* $\nabla_{\vec{w}} \vec{V} = 0$ *then this means that the vector V is parallel transported in direction* $\vec{w}$

Directional Covariant Derivative : *It is parallel transport a vector but not by just some coordinate shift but also by a shift along some path which is parameterized by some quantity.*

- $\nabla_{\vec{x}^r}$ *is directional covariant derivative. It is covariant derivative along* $\mathbf{x}^r$.

$$\nabla_{\vec{x}^r} V^\nu = \frac{DV^\nu}{d\lambda} = \frac{dx^r}{d\lambda} \nabla_r V^\nu$$

Directional Covariant Derivative is parallel transporting vector V^ν along some path parameterized by λ.

GEODESIC

It is the *shortest possible path* between two points in a curved space. Geodesics are also defined as, *the curves which parallel transport their own tangent vectors*. It is the straightest possible path in a curved surface.

Tangent vectors of path parametrized by λ then the components of tangent vector given by $\frac{dx^\mu}{d\lambda}$

Then, Condition for $x^\mu(\lambda)$ to be geodesic is the tangent vector should be covariantly constant along the path. This means :

$$\frac{D}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\right) = \frac{dx^\nu}{d\lambda} \nabla_\nu \left(\frac{dx^\mu}{d\lambda}\right) = 0$$

After solving above equation we get,

$$\boxed{\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu s} \frac{dx^\nu}{d\lambda} \frac{dx^s}{d\lambda} = 0}$$

Geodesic Equation

Every curve must satisfy geodesic equation to be the geodesic of their surface or space or manifold.

Some Properties of Geodesics:-

- *The primary usefulness of geodesics in general relativity is that they are the paths followed by unaccelerated test particles* (A test particle is a body that does not itself influence the geometry through which it moves). This concept allows us to explore, for example, the properties of the gravitational field around the sun, without worrying about the field of the planet whose motion we are considering. *The geodesic equation can be thought as the generalization of Newtons law f = ma, for case f = 0, to be curved spacetime.*

- For *timelike* paths, we can write the ***geodesic equation in terms of the four-velocity*** $U = dx/d\tau$ as : $\mathbf{U^\lambda \nabla_\lambda U^\mu = 0}$
- Similarly, in terms of the four-momentum $\mathbf{p^\mu} = m\mathbf{U^\mu}$, the ***geodesic equation*** is simply $\mathbf{p^\lambda \nabla_\lambda p^\mu = 0}$

CURVATURE:- It is the amount by which a geometric object such as a surface deviates from being a flat plane, or a curve from being straight.

Riemann Curvature Tensor

- The simplest way to know about curvature on a space is to consider moving vector in closed path, by help of this we got to know that by considering commutator we can measure the curvature of the surface.
- The Commutator of two covariant derivatives on Vector field V^ρ :

$$[\nabla_\mu, \nabla_\nu]V^\rho = \nabla_\mu\nabla_\nu V^\rho - \nabla_\nu\nabla_\mu V^\rho \qquad \rightarrow \textit{Equation(i)}$$

$$= \partial_\mu(\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu\nu}\nabla_\lambda V^\rho + \Gamma^\rho_{\mu\sigma}\nabla_\nu V^\sigma - (\mu \leftrightarrow \nu)$$

$$= \partial_\mu\partial_\nu V^\rho + (\partial_\mu\Gamma^\rho_{\nu\sigma})V^\sigma + \Gamma^\rho_{\nu\sigma}\partial_\mu V^\sigma - \Gamma^\lambda_{\mu\nu}\partial_\lambda V^\rho - \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\sigma}V^\sigma$$

$$+ \Gamma^\rho_{\mu\sigma}\partial_\nu V^\sigma + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda}V^\lambda - (\mu \leftrightarrow \nu)$$

$$= (\partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma})V^\sigma - 2\Gamma^\lambda_{[\mu\nu]}\nabla_\lambda V^\rho \qquad \rightarrow \textit{Equation(ii)}$$

NOTE:- ($\mu \longleftrightarrow \nu$) means write same expression as before which we got after opening term 1 of eq(ii) with just interchanging position of μ and ν in the expression.

After solving we get equation(ii). The last term of eq(ii) is ***antisymmetric*** part of connection which is ***torsion tensor***. *Since, we are considering* ***torsion free*** *case* so therefore our final equation is :

$$[\nabla_\mu, \nabla_\nu]V^\rho = (\partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma})V^\sigma$$

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma - T^\lambda{}_{\mu\nu}\nabla_\lambda V^\rho$$

where $R^\rho{}_{\sigma\mu\nu}$ is ***Riemann Tensor*** and identified as:

$$\boxed{R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}} \qquad \rightarrow \textit{Equation(iii)}$$

We would like to have *coordinate invariant* way of saying space is flat or curved and Riemann tensor is that way with help of which we can say space is flat or curved.

- *If* ***Riemann tensor*** *is* ***non-zero*** *for some space then that* ***space is curved.***
- ***NOTE:-*** *If in some coordinate system metric tensor is constant then Riemann tensor vanishes. This means that space is not curved.*

Properties of Riemann Tensor :-

The Riemann tensor, with four indices, has n^4 independent components in an n-dimensional space. In fact the antisymmetry property means that there are only n(n-1)/2 independent values these last two indices can take on, leaving us with n^3 (n-1)/2 independent components. When we consider the Christoffel connection, however, a number of other symmetries reduce the independent components further. Let's consider these now. The simplest way to derive these additional symmetries is to examine the Riemann tensor with all lower indices,

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^{\lambda}{}_{\sigma\mu\nu}$$

Let us further consider the components of this tensor in locally inertial coordinates $x^{\hat{\mu}}$ established at a point p. Then the Christoffel symbols themselves will vanish, although their derivatives will not. We therefore have

$$\begin{aligned} R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}}(p) &= g_{\hat{\rho}\hat{\lambda}}(\partial_{\hat{\mu}}\Gamma^{\hat{\lambda}}_{\hat{\nu}\hat{\sigma}} - \partial_{\hat{\nu}}\Gamma^{\hat{\lambda}}_{\hat{\mu}\hat{\sigma}}) \\ &= \tfrac{1}{2} g_{\hat{\rho}\hat{\lambda}} g^{\hat{\lambda}\hat{\tau}}(\partial_{\hat{\mu}}\partial_{\hat{\nu}} g_{\hat{\sigma}\hat{\tau}} + \partial_{\hat{\mu}}\partial_{\hat{\sigma}} g_{\hat{\tau}\hat{\nu}} - \partial_{\hat{\mu}}\partial_{\hat{\tau}} g_{\hat{\nu}\hat{\sigma}} - \partial_{\hat{\nu}}\partial_{\hat{\mu}} g_{\hat{\sigma}\hat{\tau}} \\ &\qquad - \partial_{\hat{\nu}}\partial_{\hat{\sigma}} g_{\hat{\tau}\hat{\mu}} + \partial_{\hat{\nu}}\partial_{\hat{\tau}} g_{\hat{\mu}\hat{\sigma}}) \\ &= \tfrac{1}{2}(\partial_{\hat{\mu}}\partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\nu}} - \partial_{\hat{\mu}}\partial_{\hat{\rho}} g_{\hat{\nu}\hat{\sigma}} - \partial_{\hat{\nu}}\partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\mu}} + \partial_{\hat{\nu}}\partial_{\hat{\rho}} g_{\hat{\mu}\hat{\sigma}}) \end{aligned}$$ → ***Equation(iv)***

NOTE:- Remember that *for any point we can have* ***Local Inertial Coordinates*** and for local inertial coordinates metric tensor $g_{\mu\nu}$ is same as metric for flat space. Therefore, **Γ= 0** *for local inertial coordinates* even if the manifold or space is curved.

Some important properties for Reimann tensor are:-

1) $R_{\rho\sigma\mu\nu} = - R_{\rho\sigma\nu\mu}$ (antisymmetry in last two indices)
2) $R_{\rho\sigma\mu\nu} = - R_{\sigma\rho\mu\nu}$ (antisymmetry in first two indices)
3) $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$ (symmetry in pair of indices)
4) $R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$ or $R_{\rho[\sigma\mu\nu]} = 0$

Note:- If I take symmetric tensor $T^{(\mu\nu)}$ and contract it with antisymmetric tensor $W_{[\mu\nu]}$ then, $T^{(\mu\nu)} W_{[\mu\nu]} = 0$

All Contractions of $R^{\lambda}{}_{\sigma\mu\nu}$:-

1) $R^{\lambda}{}_{\lambda\mu\nu} = g^{\lambda\alpha} R_{\alpha\lambda\mu\nu} = g^{(\lambda\alpha)} R_{[\alpha\lambda]\mu\nu} = 0$
2) $R^{\lambda}{}_{\mu\lambda\nu} = g^{\lambda\alpha} R_{\alpha\mu\lambda\nu} = R_{\mu\nu}$ ***(Ricci Tensor)***
3) $R = R^{\nu}{}_{\nu} = g^{\nu\mu} R_{\mu\nu}$ ***(Ricci Scalar)***

- $R^{\lambda}{}_{\alpha\mu\nu}$ – (all contractions)= $C_{\alpha\lambda\mu\nu}$ ***(Weyl Tensor)***

- For the curvature tensor formed from arbitrary connection, there are no. of independent contractions to take. Our primary concern is with Christoffel connection, for which $\mathbf{R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}}$ is the ***only independent contraction***.

- The Ricci tensor associated with Christoffel connection is automatically symmetric, as the consequence of the symmetries of the Riemann tensor.

$$\mathbf{R_{\mu\nu} = R_{\nu\mu}}$$

- The trace of the Ricci tensor is ***Ricci Scalar*** (The **trace** of a rank 2 tensor is equal to sum of the diagonal elements of its matrix representation). The trace of a matrix is invariant with respect to change of basis.

- ***Ricci Tensor*** tracks ***volume change*** along geodesics. In other words, Ricci tensor can be characterized by measurement of how a shape is deformed as it moves along geodesics in space.

Consider the covariant derivative of Reimann Tensor, evaluated in local inertial coordinates:

$$\begin{aligned}\nabla_{\hat{\lambda}} R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} &= \partial_{\hat{\lambda}} R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} \\ &= \tfrac{1}{2}\partial_{\hat{\lambda}}(\partial_{\hat{\mu}}\partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\nu}} - \partial_{\hat{\mu}}\partial_{\hat{\rho}} g_{\hat{\nu}\hat{\sigma}} - \partial_{\hat{\nu}}\partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\mu}} + \partial_{\hat{\nu}}\partial_{\hat{\rho}} g_{\hat{\mu}\hat{\sigma}})\end{aligned}$$

Now, consider cyclic permutation of first three indices:

$$\begin{aligned}&\nabla_{\hat{\lambda}} R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} + \nabla_{\hat{\rho}} R_{\hat{\sigma}\hat{\lambda}\hat{\mu}\hat{\nu}} + \nabla_{\hat{\sigma}} R_{\hat{\lambda}\hat{\rho}\hat{\mu}\hat{\nu}} \\ &\quad= \tfrac{1}{2}(\partial_{\hat{\lambda}}\partial_{\hat{\mu}}\partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\nu}} - \partial_{\hat{\lambda}}\partial_{\hat{\mu}}\partial_{\hat{\rho}} g_{\hat{\nu}\hat{\sigma}} - \partial_{\hat{\lambda}}\partial_{\hat{\nu}}\partial_{\sigma} g_{\hat{\rho}\hat{\mu}} + \partial_{\hat{\lambda}}\partial_{\hat{\nu}}\partial_{\hat{\rho}} g_{\hat{\mu}\hat{\sigma}} \\ &\qquad + \partial_{\hat{\rho}}\partial_{\hat{\mu}}\partial_{\hat{\lambda}} g_{\hat{\sigma}\hat{\nu}} - \partial_{\hat{\rho}}\partial_{\hat{\mu}}\partial_{\hat{\sigma}} g_{\hat{\nu}\hat{\lambda}} - \partial_{\hat{\rho}}\partial_{\hat{\nu}}\partial_{\hat{\lambda}} g_{\hat{\sigma}\hat{\mu}} + \partial_{\hat{\rho}}\partial_{\hat{\nu}}\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\lambda}} \\ &\qquad + \partial_{\hat{\sigma}}\partial_{\hat{\mu}}\partial_{\hat{\rho}} g_{\hat{\lambda}\hat{\nu}} - \partial_{\hat{\sigma}}\partial_{\hat{\mu}}\partial_{\hat{\lambda}} g_{\hat{\nu}\hat{\rho}} - \partial_{\hat{\sigma}}\partial_{\hat{\nu}}\partial_{\hat{\rho}} g_{\hat{\lambda}\hat{\mu}} + \partial_{\hat{\sigma}}\partial_{\hat{\nu}}\partial_{\hat{\lambda}} g_{\hat{\mu}\hat{\rho}}) \\ &\quad= 0\end{aligned}$$

→ *Equation(v)*

Since, above equation(v) is equation between tensors so it is true in any coordinate system, even though we derived it in particular one. We recognize by now that the antisymmetry $R_{\rho\sigma\mu\nu} = - R_{\sigma\rho\mu\nu}$ allow us to write this result as

$$\boldsymbol{\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0}$$

This is known as ***Bianchi Identity***.

An especially useful form of the Bianchi identity comes from contracting twice on equation(v)

$$\begin{aligned}0 &= g^{\nu\sigma} g^{\mu\lambda}(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu}) \\ &= \nabla^\mu R_{\rho\mu} - \nabla_\rho R + \nabla^\nu R_{\rho\nu},\end{aligned}$$

In this expression dummy index μ and ν can be replace by single dummy index since both of them have same range. So, at last we get equation(vi)

or

$$\nabla^\mu R_{\rho\mu} = \tfrac{1}{2}\nabla_\rho R$$

→ *Equation(vi)*

We define **Einstein Tensor** as,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \qquad \rightarrow \textit{Equation(vii)}$$

The Einstein tensor is symmetric due to symmetry of Ricci tensor and metric.
In four dimension the Einstein tensor can be thought of as a traced-reversed-version of Ricci Tensor. We then see that ***the twice contracted Bianchi identity is equivalent*** to

$$\nabla^{\mu} G_{\mu\nu} = 0 \qquad \rightarrow \textit{Equation(viii)}$$

GEODESIC DEVIATION:-

Geodesic deviation describes the tendency of objects to approach or move away from one another while moving under the influence of a spatially varying gravitational field.
When we have family of geodesic curves with a tangent velocity vector field $\mathbf{T}^{\mu}$, we can also look at the separation vectors along these curves which are given by vector field $\mathbf{S}^{\mu}$ (known as deviation vector field).
Consider one parameterized family of geodesic $\gamma_s(t)$. The collection of these curves defines a smooth 2-D surface. The tangent vectors to the geodesics is $\mathbf{T}^{\mu} = \partial x^{\mu}/\partial t$ and deviation vectors $\mathbf{S}^{\mu} = \partial x^{\mu}/\partial s$

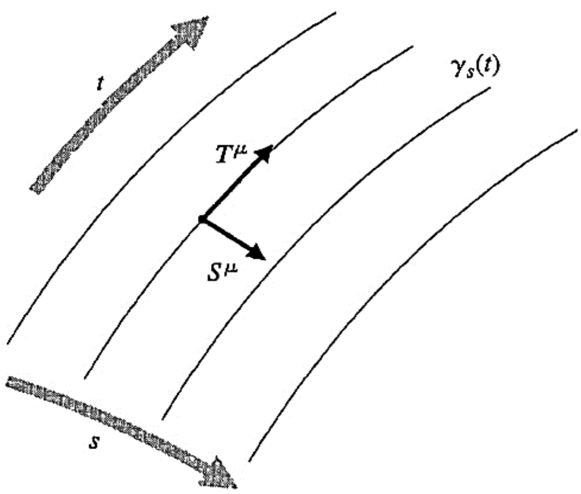


FIGURE (a). A set of geodesics $\gamma_s(t)$, with tangent vectors T^{μ}. The vector field S^{μ} measures the deviation between nearby geodesics.

The idea that S^{μ} points form one geodesic to next inspires us to define "relative velocity of geodesics": $V^{\mu} = (\nabla_T S)^{\mu} = T^{\rho} \nabla_{\rho} S^{\mu}$
And the relative acceleration of geodesics: $A^{\mu} = (\nabla_T V)^{\mu} = T^{\rho} \nabla_{\rho} V^{\mu}$
Since **S** and **T** are *the basis vectors adapted to a coordinate system,* ***their commutator vanishes***:

$$[S,T]=0$$

Note: formula for commutator is $[X, Y]^{\mu} = X^{\lambda}\partial_{\lambda}Y^{\mu} - Y^{\lambda}\partial_{\lambda}X^{\mu} = X^{\lambda}\nabla_{\lambda}Y^{\mu} - Y^{\lambda}\nabla_{\lambda}X^{\mu}$

So then we have,

$$\boxed{S^{\rho} \nabla_{\rho} T^{\mu} = T^{\rho} \nabla_{\rho} S^{\mu}} \qquad \rightarrow \textit{Equation(ix)}$$

Now, let compute acceleration by considering equation(ix) in the equation of acceleration we wrote before:

$$\begin{aligned} A^{\mu} &= T^{\rho}\nabla_{\rho}(T^{\sigma}\nabla_{\sigma}S^{\mu}) \\ &= T^{\rho}\nabla_{\rho}(S^{\sigma}\nabla_{\sigma}T^{\mu}) \\ &= (T^{\rho}\nabla_{\rho}S^{\sigma})(\nabla_{\sigma}T^{\mu}) + T^{\rho}S^{\sigma}\nabla_{\rho}\nabla_{\sigma}T^{\mu} \\ &= (S^{\rho}\nabla_{\rho}T^{\sigma})(\nabla_{\sigma}T^{\mu}) + T^{\rho}S^{\sigma}(\nabla_{\sigma}\nabla_{\rho}T^{\mu} + R^{\mu}{}_{\nu\rho\sigma}T^{\nu}) \\ &= (S^{\rho}\nabla_{\rho}T^{\sigma})(\nabla_{\sigma}T^{\mu}) + S^{\sigma}\nabla_{\sigma}(T^{\rho}\nabla_{\rho}T^{\mu}) - (S^{\sigma}\nabla_{\sigma}T^{\rho})\nabla_{\rho}T^{\mu} \\ &\quad + R^{\mu}{}_{\nu\rho\sigma}T^{\nu}T^{\rho}S^{\sigma} \\ &= R^{\mu}{}_{\nu\rho\sigma}T^{\nu}T^{\rho}S^{\sigma} \qquad \rightarrow \textit{Equation(x)} \end{aligned}$$

The first line is just the definition of A^{μ} (relative acceleration of geodesics) and the second line is given directly by equation(ix). The third line is simply the Leibniz rule. The fourth line is replaces the double covariant derivative by the derivatives in the opposite order plus Riemann tensor. Fifth line uses Leibniz Equation again (in opposite order). And then we cancel the identical terms and notice that the term involving vanishes $T^{\rho}\nabla_{\rho}T^{\mu}$ because T^{μ} is the tangent vector to a geodesic. The result obtained,

$$\boxed{A^{\mu} = \frac{D^2}{dt^2}S^{\mu} = R^{\mu}{}_{\nu\rho\sigma}T^{\nu}T^{\rho}S^{\sigma}} \qquad \rightarrow \textit{Equation(xi)}$$

is the ***Geodesic Deviation Equation.*** It expresses something that we might have expected: *the relative acceleration between two neighbouring geodesics is proportional to the curvature.*

EINSTEIN EQUATION

"In general theory of relativity the Einstein equation relate the geometry of spacetime with the distribution of matter within it. Einstein equations directly relate the *curvature of spacetime* to the *energy* and *momentum* of whatever matters and radiations are present."

PHYSICS IN CURVED SPACE:-

General relativity explains *how the gravitational field influences the behavior of matter, and how matter determines the gravitational field.* In *Newtonian gravity*, these two elements consist of the expression for the acceleration of a body in a gravitational potential Φ,

$$a = -\nabla\Phi$$

and ***Poisson's differential equation*** for the potential in terms of the matter density ρ and Newton's gravitational constant **G**:

$$\nabla^2\Phi = 4\pi G\rho$$

So, now in general relativity, the analogous statements will describe how the curvature of spacetime acts on matter to manifest itself as gravity, and how energy and momentum influence spacetime to create curvature.

According to ***Einstein Equivalence Principle (EEP)***, "*In small enough regions of spacetime, the laws of physics reduce to those of special relativity, it is impossible to detect the existence of a gravitational field by means of local experiments.*" *The EEP arises from the idea that gravity is universal*; it affects all particles (and indeed all forms of energy-momentum) in the same way. ***This feature of universality led Einstein to propose that what we experience as gravity is a manifestation of the curvature of spacetime.***

The law of energy momentum conservation in flat spacetime is given as: $\boldsymbol{\partial_\mu T^{\mu\nu} = 0}$

Approx. generalization of law of energy momentum conservation in curved spacetime: $\boldsymbol{\nabla_\mu T^{\mu\nu} = 0}$

It is one thing to generalize an equation from flat to curved spacetime; it is something altogether different to argue that the result descries gravity. To do so, we can show how the usual results of Newtonian Gravity fit into the picture.

We define Newtonian Limits by three requirements:

- the particle are moving slowly (with respect to speed of light)
- the Gravitational field is weak
- the field is also static

Einstein's field equation governs how the metric responds to energy and momentum.

The informal argument begins with the realization that we would like to find an equation that supersedes the Poisson equation for the Newtonian potential: $\boldsymbol{\nabla^2 \Phi = 4\pi G\rho}$

where $\boldsymbol{\nabla^2}$ is Laplacian operator (Laplacian in space) and $\boldsymbol{\rho}$ is mass density. So, ***what characteristics should above equation possess?*** On the left-hand side of above we have a second-order differential operator acting on the gravitational potential, and on the right-hand side a measure of the mass distribution. A relativistic generalization should take the form of an equation between tensors. We know what the tensor generalization of the mass density is; it's the energy-momentum tensor $\mathbf{T}_{\mu\nu}$. The gravitational potential, meanwhile, should get replaced by the metric tensor.

We might therefore guess that our new equation will have $\mathbf{T}_{\mu\nu}$ set proportional to some tensor, which is second-order in derivatives of the metric; something like:

$$[\boldsymbol{\nabla}^2 \mathbf{g}]_{\mu\nu} \propto \mathbf{T}_{\mu\nu}$$

The left-hand side of above equation is not a sensible tensor: it's just a suggestive notation to indicate that we would like a symmetric (0, 2) tensor that is second-order in derivatives of the metric. After considering different tensor we got to know ***Ricci Tensor*** fits best to all conditions.

So, now gravitational field equations are: $\mathbf{R}_{\mu\nu} = \kappa \mathbf{T}_{\mu\nu}$, for some constant k.

But there is one problem with this equation that with conservation of energy $\boldsymbol{\nabla^\mu T_{\mu\nu} = 0}$ **;** this means also that $\boldsymbol{\nabla^\mu R_{\mu\nu} = 0}$

This is certainly not true in arbitrary geometry; we have seen from the ***Bianchi identity*** that

$$\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R$$

But our proposal field equation implies that $\mathbf{R} = \kappa\, \mathbf{g}^{\mu\nu}\, \mathbf{T}_{\mu\nu}$,so taking these together we have $\boldsymbol{\nabla_\mu T = 0}$.

The covariant derivate of scalar is just the partial derivative this means T is constant through spacetime. T=0 in vacuum while T≠0 in matter. We know one symmetric (0,2) tensor constructed from Ricci tensor, which is automatically conserved i.e., **Einstein Tensor :**

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}.$$

which always obey $\nabla^{\mu}G_{\mu\nu} = 0$

We therefore are led to propose, $\mathbf{G_{\mu\nu} = \kappa T_{\mu\nu}}$ as the field equation for the metric.
When we try to check does this equation predict Poisson equation for the gravitational potential in Newtonian Limits which works out correctly and we got to know $\kappa=8\pi G$.
So now, **Einstein Equation** for general relativity:

$$\boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}} \qquad \rightarrow \textit{Equation(xii)}$$

This tells us how the curvature of spacetime react to the presence of energy-momentum.
Here, ***G*** is Newtonian Gravitational Constant.

<u>Einstein-Hilbert Action (Action for gravity)</u>:-

Einstein-Hilbert action or simply Hilbert action in general relativity is the ***action*** that yields the Einstein Field Equations through the ***principle of least action***.
With the (- +++) metric signature, the action for gravity is given as: $S_H = \int \sqrt{-g} R \, d^n x$

; where $\mathbf{g = det(g_{\mu\nu})}$ and **R** is Ricci scalar
<u>NOTE</u>:- *Action* is a mathematical function which takes trajectory of the system as its argument and from which the equations of motion of the system can be derived through principle of least action. *Principle of least action* or principle of stationary action states, "The path taken by system between times t_1 and t_2 and configurations q_1 and q_2 is the one for which the *action* is *stationary* (no change) to first order."

<u>BLACK HOLE</u>

- One of the intriguing predictions of Einstein's Theory of Relativity is the existence of black holes.
- *A black hole is the region in space-time where gravitational field is so strong that even nothing can escape from it not even electromagnetics radiation such as light.*
- A black hole is formed when the size of gravitating object of mass M becomes smaller than its *gravitational radius*: $\mathbf{r_s = 2GM/c^2}$; where G is Newton's Gravitational constant and c is speed of light.
- The boundary of the region from which no escape is possible is called the *event horizon*. We can also say event horizon is a boundary beyond which events cannot affect an observer.
- The velocity required to leave the boundary of black hole and move away to infinity is equal to speed of light. This means no signal or particle can escape from region inside black hole since speed of light is limiting propagation velocity for physical signals.

SCHWARZSCHILD BLACK HOLE

A ***Schwarzschild black hole*** is a black hole that has mass but neither have electric charge nor angular momentum. It is also known as ***static black hole***.
A Schwarzschild black hole is described by the Schwarzschild metric, and can be distinguished from any other Schwarzschild black hole by its mass only.

Schwarzschild Metric:-

Consider a spherical gravitational field in vacuum to seek spherically symmetric solutions to Einstein equation because they are easier and relevant for astrophysical objects.
One such solution to Einstein's Equations was given by Schwarzschild. In *Schwarzschild coordinates* i.e.; ***(t, r, θ, Φ)*** , Schwarzschild geometry is *symmetric* and *static* and metric tensor is given as :

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) = g_{ab} dx^a\, dx^b \quad \rightarrow \textit{Equation(xiii)}$$

where, **G** is Newton's Gravitational constants, **r** is radial distance, **t** is Schwarzschild time and it is time observed by an observer at infinity. *An important property of this solution is it is independent of temporal coordinate **t*** and *only depend on one factor that is Mass of gravitational source* (**M**). ***The quantity*** $r = r_s = 2GM/c^2$ ***is called Schwarzschild radius.***

According to **Birkhoff's theorem,** the Schwarzschild solution is also a *unique spherically symmetric solution of the vacuum Einstein equation.*

We can also represent Schwarzschild metric **(c=1)** as:

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2} \quad \rightarrow \textit{Equation(xiv)}$$

where $d\Omega^2$ is the metric on a unit two-sphere, $r_s = 2GM$ **is Schwarzschild radius**

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2.$$

For ordinary measurement of length in a small neighbourhood of each spatial point, we can use a local Cartesian coordinate system (x, y, z). Let (e_1 , e_2 , e_3) be three spatial orthonormal vectors at a chosen point **p.** Then the displacement vector **δr** for the position of a point in a small neighbourhood of **p** can be characterized by three numbers **(δx , δy , δz) :** $\delta \mathbf{r} = \delta x \mathbf{e}_1 + \delta y \mathbf{e}_2 + \delta z \mathbf{e}_3$. If $\mathbf{e}_1$, $\mathbf{e}_2$ **and** $\mathbf{e}_3$ are directed along ***r, θ*** and ***Φ*** , respectively then from ***equation(xiii),***

$$\delta x = \sqrt{g_{11}}\, dr = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} dr$$

$$\delta y = \sqrt{g_{22}}\, d\theta = r\, d\theta\,,$$

$$\delta z = \sqrt{g_{33}}\, d\phi = r\, \sin\theta\, d\phi\,.$$

The factor $(1 - 2GM/c^2 r)^{-1/2}$ *reflect curvature of 3-D space.*

The *physical time* **τ** at a point **r** is given by expression:

$$d\tau = c^{-1}\sqrt{(-g_{00}\,dx^0)} = \sqrt{(-g_{00}\,dt)} = (1 - 2GM/c^2 r)^{-1/2} dt$$

,where x^0=ct. If $\mathbf{e}_0$ is time directed vector orthogonal to $\mathbf{e}_m$ (m = 1,2,3) then the position of any event with respect to local frame can be presented as: $c\,\delta\tau\,\mathbf{e}_0 + \mathbf{dr}$. *Far from the gravitational center (r → ∞), we have* $d\tau = dt$ *; that is,* t *is physical time of the observer at infinity. At smaller* r*, time* τ *runs progressively slower in comparison with time* t *at infinity.* As $r \to 2GM/c^2$,we find $d\tau \to 0$.

If r is Schwarzschild radius i.e.; $r = r_s = 2GM$ (equation(xii)), *Schwarzschild reference frame (reference frame formed using Schwarzschild coordinates) has a* ***Physical Singularity***. The sphere of radius r_s is called Schwarzschild sphere.

Singularity is a location in spacetime where the gravitational field of a celestial body is predicted to become infinite by general relativity in a way that does not depend on the coordinate system.

Over all, The Schwarzschild solution in Schwarzschild Coordinates appears to have singularities at $r = 0$ and $r = r_s$.
The *exterior Schwarzschild solution* with $r > r_s$ is the one that is related to the gravitational fields of stars and planets. The *interior Schwarzschild solution* with $0 \le r < r_s$, which contains the singularity at $r = 0$, is completely separated from the outer patch by the singularity at $r = r_s$.
The Schwarzschild reference frame is static. The three-dimensional geometry of the space g_{ij} and the gravitational potential g_{00} in it do not depend on time **t.** ***The generator of this time-symmetry transformation*** **is killing vector** $\varepsilon^{\mu}{}_{(t)} : \delta^{\mu}{}_{0}$.

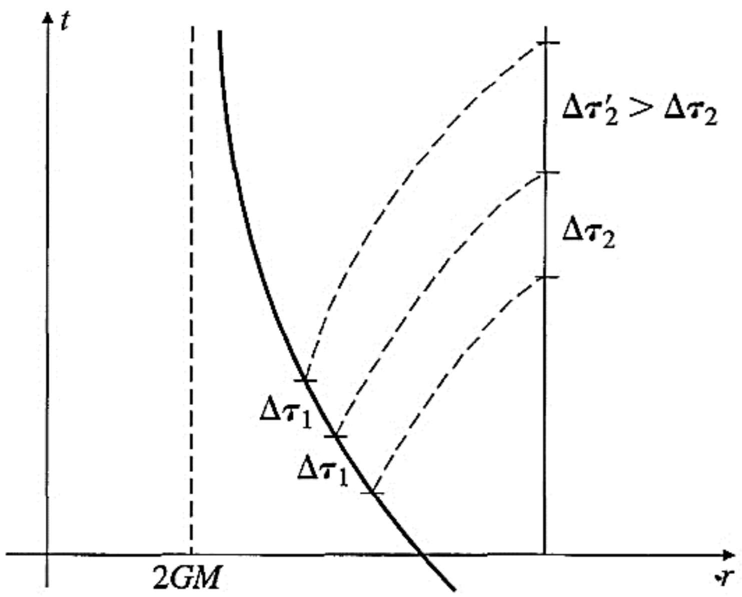


FIGURE (b) A **object** falling freely into a black hole emits signals at intervals of constant proper time $\Delta\tau_1$. An observer at fixed r receives the signals at successively longer time intervals $\Delta\tau_2$.

NOTE:- As we know the ***metric coffecients are coordinate-dependent quantities***, so values we obtained using them may change in different coordinate system. Then it is also certainly possible to have the coordinate singulaity at a value in only one coordiante system but not all.
So we need to think of such ***coordinate-independent*** quantity which tells us about geometry of space-time. And the best suitable simplest answer is ***Ricci Scalar*** since scalars are coordinate-independent and it is meaningful to say that it become infinite for some geometry. So, therefore *for a point to be* ***true singularity*** *then metric and Ricci Scalar must be infinite for that given point*.

Tortoise Coordinate And Eddington-Finkelstein Coordinate

We know behavior of geodesics outside Schwarzschild Radius. Now we will try to know and describe objects even at radii less than Schwarzschild radius.
One way to understand geometry of spacetime is to understand casual structure, as defined by light cones. We therefore consider radial null curves for which ***θ*** and ***Φ*** are constant and $\mathbf{ds^2 = 0}$.

$$ds^2 = 0 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 \qquad \rightarrow \textit{Equation(a)}$$

$$ds^2 = 0 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 \qquad \rightarrow \textit{Equation(b)}$$

from which we see that

$$\frac{dt}{dr} = \pm\left(1 - \frac{2GM}{r}\right)^{-1}$$

This is the slope of the light cones on the space time diagram of t-r.

For larger r the slope is $\pm$ **1** as same for flat space, while we approach $r = 2GM$ we get dt/dr → $\pm\infty$ there is no light cone. Thus, an observer outside can never see light ray or object that approaches r = 2GM to cross it.
The problem with our coordinate system is that dt/dr → ∞ along radial null geodesic which approaches r=2GM; progress in r direction become smaller and smaller with respect to coordinate time *t*.
So, now we try to solve this problem by replacing *t* with a coordinate that move more slowly along null geodesics. Let it be

$$\mathbf{t = \pm r^* + constant}$$

where r^* is ***tortoise coordinate.***

Tortoise coordinate r^* is defined by:

$$r^* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$$

Now put t = r^* + constant in ***equation(b):***

$$\frac{dt}{dr} = \frac{d}{dr}(r^* + \text{constant}) = \frac{d}{dr}(r^*)$$

$$= \frac{d}{dr}\left(r + 2GM\ \ln\left(\frac{r}{2GM} - 1\right)\right)$$

$$= 1 + 2GM\left(\frac{1}{r/2GM - 1}\right)\frac{d}{dr}\left(\frac{r}{2GM} - 1\right)$$

$$= 1 + 2GM\left(\frac{1}{r/2GM - 1}\right)\frac{1}{2GM}$$

$$= 1 + \left(\frac{2GM}{r - 2GM}\right) = \left(\frac{r}{r - 2GM}\right) = \left(\frac{1}{1 - 2GM/r}\right)$$

$$\mathbf{\frac{d}{dr}(r^*) = \left(1 - \frac{2GM}{r}\right)^{-1}} \quad \text{or}$$

$$dr = dr^*\left(1 - \frac{2GM}{r}\right)$$

Put value of **dr** from above in general Schwarzschild metric (equation(xiv)):

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(1 - \frac{2GM}{r}\right)^2 (dr^*)^2 + r^2\, d\Omega^2$$

$$= -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right) dr^{*2} + r^2\, d\Omega^2$$

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right)\left(dt^2 + dr^{*2}\right) + r^2\, d\Omega^2} \quad \rightarrow \textit{Equation(c)}$$

This equation(c) is Schwarzschild metric in tortoise coordinate where r is thought of as function of r^*.

Now, next move is to define coordinates that are naturally adapted to null geodesics. So, firstly for *future directed curves* replace ***t*** with a null coordinate $\mathbf{v = t + r^*}$

Differentiating both side we get $\mathbf{dv = dt + dr^*}$

In terms of radial coordinate **r,**

$$dv = dt + \left(1 - \frac{2GM}{r}\right)^{-1} dr$$

$$dt = dv - \left(1 - \frac{2GM}{r}\right)^{-1} dr$$

Put above value in equation(xii) i.e., equation of metric in radial coordinates:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)\left(dv - \left(1 - \frac{2GM}{r}\right)^{-1} dr\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}(dr)^2 + r^2\, d\Omega^2$$

$$= -\left(1 - \frac{2GM}{r}\right)dv^2 - \left(1 - \frac{2GM}{r}\right)dr^2 + 2dv\, dr\left(1 - \frac{2GM}{r}\right)\left(1 - \frac{2GM}{r}\right)^{-1}$$
$$+ \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2$$

$$\mathbf{ds^2 = -\left(1 - \frac{2GM}{r}\right)dv^2 + 2dv\, dr + r^2\, d\Omega^2}$$

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right)dv^2 + (dv\, dr + dr\, dv) + r^2\, d\Omega^2}$$

These new coordinates are known as ***Eddington-Finkelstein Coordinates*** in which condition for radial null curves is solved by $\frac{dv}{dr} = 0$ and $\frac{dv}{dr} = 2(1 - \frac{2GM}{r})^{-1}$ which describe *in-falling* and *outgoing* future directed null trajectories respectively.
We therefore can see that in this coordinate system the light cones remain well-behaved at r=2GM, and this surface is at finite coordinate values.

Alternatively, we could also choose *past directed curves* defined by $\mathbf{u = t - r^*}$ now, replace ***t*** with a null coordinate **u.**
Differentiating both side we get $\mathbf{du = dt - dr^*}$
In terms of radial coordinate **r,**

$$du = dt - (1 - \frac{2GM}{r})^{-1} dr$$

$$dt = du + (1 - \frac{2GM}{r})^{-1} dr$$

Put above value in equation(xiv) i.e., equation of Schwarzschild metric in radial coordinates:

$$ds^2 = -(1 - \frac{2GM}{r})(du + (1 - \frac{2GM}{r})^{-1} dr)^2 + (1 - \frac{2GM}{r})^{-1} (dr)^2 + r^2 d\Omega^2$$

$$= -(1 - \frac{2GM}{r})du^2 - (1 - \frac{2GM}{r})dr^2 - 2du\, dr(1 - \frac{2GM}{r})(1 - \frac{2GM}{r})^{-1}$$

$$+ (1 - \frac{2GM}{r})^{-1} dr^2 + r^2 d\Omega^2$$

$$\mathbf{ds^2 = -(1 - \frac{2GM}{r})du^2 - 2du\, dr + r^2 d\Omega^2}$$

$$\boxed{\mathbf{ds^2 = -(1 - \frac{2GM}{r})du^2 - (du\, dr + dr\, du) + r^2 d\Omega^2}}$$

These coordinates in which condition for radial null curves is solved by $\frac{du}{dr} = 0$ and $\frac{du}{dr} = -2(1 - \frac{2GM}{r})^{-1}$ which describe *in-falling* and *outgoing* past directed null trajectories respectively.
Note:- We can consistently follow either future-directed or past-directed curves through r=2GM, but we arrives in different places.

Now, we follow to spacelike geodesics to see if we would uncover still more regions.
Let firstly use both **u** and **v** at once (in place of t and r).

$$\mathbf{v = t + r^*},$$

$$\mathbf{u = t - r^*}$$

differentiating above both equations both side,

$$\mathbf{dv = dt + dr^*},$$

$$\mathbf{du = dt - dr^*}$$

Now add above equations

$$\mathbf{dt = \frac{1}{2}(dv + du)}$$

now subtract $dv = dt + dr^*$ *and* $du = dt - dr^*$,

$$\mathbf{dr} = \frac{1}{2}\left(1 - \frac{2GM}{r}\right)(\mathbf{dv} - \mathbf{du})$$

Put value of **dt** and **dr** in equation(xiv) i.e., equation of Schwarzschild metric in radial coordinates:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)\left(\frac{1}{2}\,(dv + du)\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{1}{2}\left(1 - \frac{2GM}{r}\right)(dv - du)\right)^2 + r^2\,d\Omega^2$$

$$= -\frac{1}{4}\left(1 - \frac{2GM}{r}\right)(dv^2 + du^2 + 2dvdu) + \frac{1}{4}\left(1 - \frac{2GM}{r}\right)(dv^2 + du^2 - 2dvdu) + r^2\,d\Omega^2$$

$$\boxed{ds^2 = -\frac{1}{2}\left(1 - \frac{2GM}{r}\right)(dv\,du + du\,dv) + r^2\,d\Omega^2} \quad \rightarrow \textit{Equation(d)}$$

Kruskal-Szekeres Coordinates

We again reintroduced degeneracy as we can see in equation (d) with which we started out in these coordinates r=2GM is "infinitely far away" (at either v= -∞ or u= ∞). The thing to do is to change coordinates that pull these points into finite coordinate values. Therefore, let

$$\mathbf{v'} = e^{v/4GM}$$

$$\mathbf{u'} = -\, e^{-u/4GM}$$

which in terms of original (t, r) system is

$$\mathbf{v'} = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{(r+t)/4GM}$$

$$\mathbf{u'} = -\left(\frac{r}{2GM} - 1\right)^{1/2} e^{(r-t)/4GM}$$

In the (v′, u′, $\theta, \emptyset$) system the Schwarzschild metric is

$$ds^2 = -\frac{16G^3M^3}{r} e^{-r/2GM}(dv'du' + du'dv') + r^2\,d\Omega^2$$

Finally, the non-singular nature of r=2GM can be seen true through equation of metric. Both v′ and u′ are null coordinates.
We therefore define,

$$T = \frac{1}{2}(v' + u') = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right)$$

and

$$R = \frac{1}{2}(v' - u') = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right)$$

The coordinates **(T, R , $\boldsymbol{\theta}, \boldsymbol{\emptyset}$)** are known as ***Kruskal-Szekeres Coordinates*** and the Schwarzschild metric in the (T, R , $\theta, \emptyset$) coordinates is

$$\boxed{ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}(-dT^2 + dR^2) + r^2\,d\Omega^2.}$$

where r is defines as

$$\boxed{T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM}} \quad \rightarrow \textit{Equation(e)}$$

For Kruskal coordinates radial null curves look like they look for flat space (i.e; (t , r*)):

$$\mathbf{T = \pm R + constant}$$

Unlike the (t , r) the event horizon r =2GM is not infinitely far away* and is defined as **T= ± R ,** consistent with it being a null surface.

More in general we can consider the surfaces r = constant and therefore form equation(e) we have,

$$\mathbf{T^2 - R^2 = constant}$$

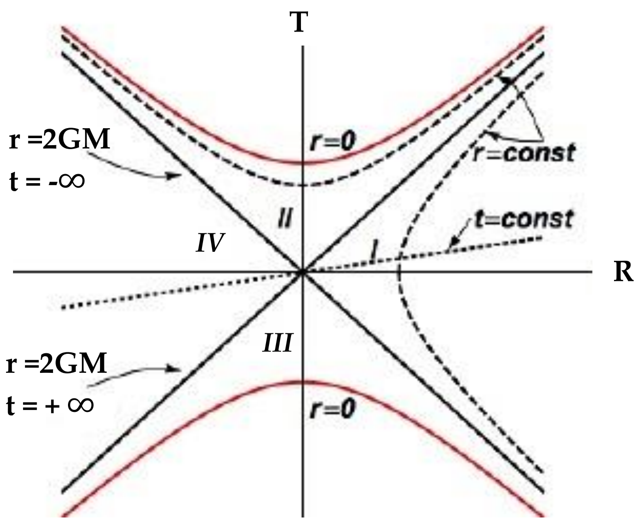
Thus, they appear as hyperbola in R-T plane. Furthermore, the surfaces of constant *t* is given by

$$\frac{T}{R} = \tanh\left(\frac{t}{4GM}\right)$$

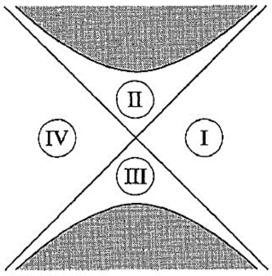
which defines straight lines through the origin with slope tanh(t/4GM). If **t → ±** ∞ then above equation become same as **T= ± R,** therefore **t = ±** ∞ represent same surfaces as r = 2GM.

Kruskal-Szekeres diagram:-

In (t , r) coordinate system **T** and **R** become imaginary for r<2GM which is of no good**.** So, we choose to draw spacetime in **T-R** plane (with θ *and* $\emptyset$ suppressed) known as *Kruskal-Szekeres diagram.* Each point in the diagram is two-sphere. This diagram represent maximal extension of Schwarzschild geometry. This diagram is divided into four regions as shown bellow



The maximal extension of the Schwarzschild spacetime. It is represented in the diagram as hyperbolae with the R-axis as symmetry axis. Straight lines correspond to a constant time ***t*** *. However, at the two* ***45°*** *diagonal lines r = r_s which represents a limiting case where a timelike line goes over in a spacelike line.*



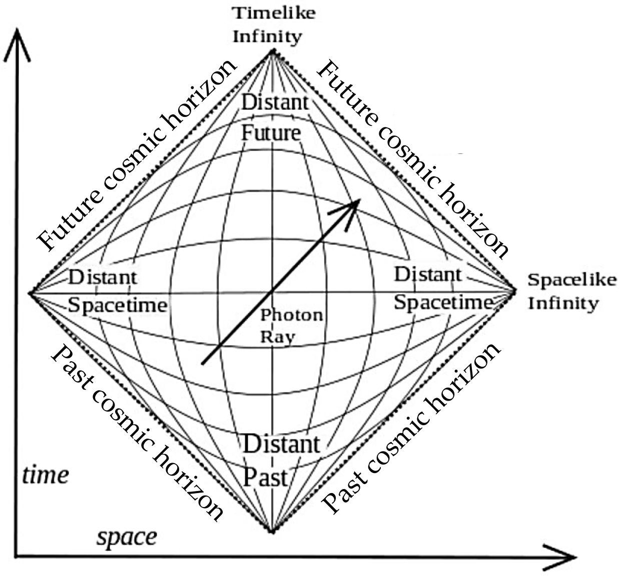
Regions in Kruskal Diagram

1) Region (I) is the original spacetime which is observable by physical instruments. It is our world. It corresponds to r > 2GM in which our original coordinates are well defined.
2) By following future-directed null rays we reach region (II) and Infalling matter will fall into the singularity at r = 0. Any light signal from region II will remain there and also fall in the singularity. Region II describes the **Black Hole**. Once anything travelled from (I) to (II) it never returns. Every future directed path in (II) ends up hitting singularity at r=0. For r < 2GM , ***t*** become spacelike and ***r*** become timelike.
3) By following pas-directed null rays we reach region (III). Region (III) is just time-reverse of region(II) from which things can escape to us. An observer present in III must have been originated in the singularity and must leave region III again to region I. Therefore III is called a **White Hole.** There is singularity in past from which universe appears. The boundary of region(III) is past event horizon.
4) Region(IV) cannot be reached from our region(I) or no one from there can reach us; neither by forward in time nor by backward in time.

PENROSE DIAGRAM

- Penrose diagram is a special type of space time diagram *defined to clarify the nature of event horizons.*
- Penrose diagram is a Space-time diagram capturing casual relationship between different point in space
- *It is extended version of Minkowski diagram where vertical dimension represent time, horizontal dimension represents space and slanted lines at angle of 45° corresponds to light rays.*
- Locally the metric in Penrose diagram is equivalent to metric of Minkowski space time.
- Penrose diagram transform the regular space-time diagram to give it two powerful features: **(i)** *It compactifies the grid lines to fit infinite space-time on one graph which is very useful for blackholes.* **(ii)** *It also curves the lines of constant time and constant space in what call a conformal transformation i.e., light always follows*

a 45° path. ***That means light cone always have same orientation everywhere in Penrose diagram*** *(It preserves light cone).*

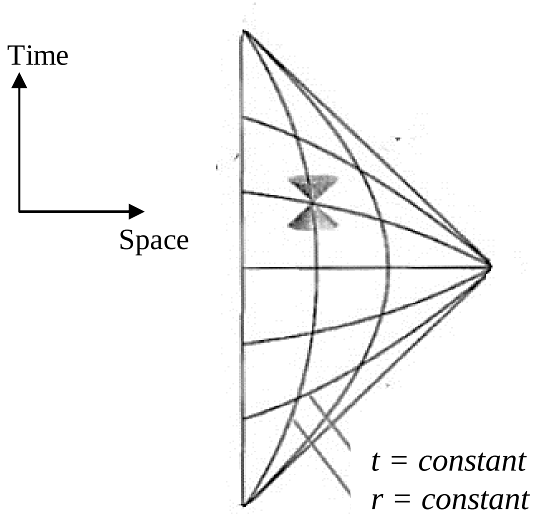


Figure(c). Penrose Diagram for flat Space-time with no Black hole

Lines in *horizontal dimension* represents fixed locations in space and lines in *vertical dimension* are fixed moments in time

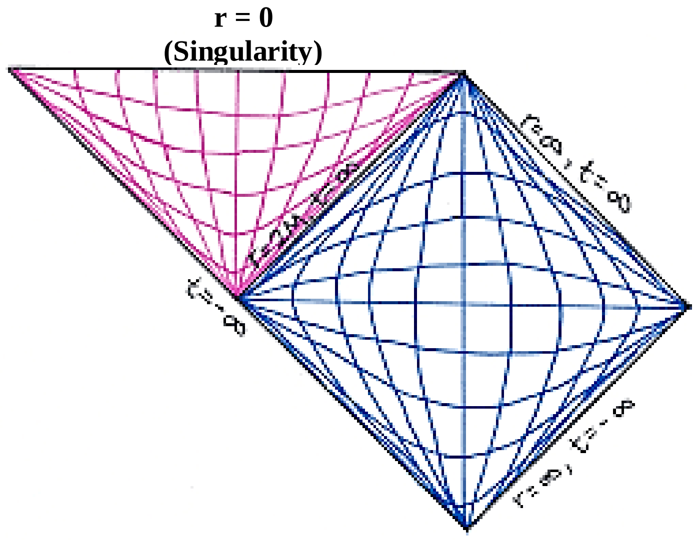
- *Those lines in one dimension of space and one dimension of time get closer and closer together towards the edge of the plot to encompass more and more space-time. They are extremely finely separated at the edges, so that any tiny stretch on the graph represents the vast distances and/or time.*
- Those lines also converge together towards the corners so that light travels at a 45° everywhere on diagram.

Figure(d). The conformal diagram of Minkowski space. Light cones are 45° throughout the diagram.



Now, let see what will happen to Penrose diagram when we put one Schwarzschild Black hole to the left in the Space-time as shown by Figure(c).
Since, *we have only one-dimension of space so any motion to the left brings us closer to the Black hole*. Its event horizon becomes the end of the line in that direction. The future cosmic horizon on the Penrose diagram is replaced with a plunge into a black-hole.
The compactified grid lines there now represent the stretched space-time near the event horizon. And an entirely ***new Penrose region*** *represents the* ***interior of the Black Hole****.*

Figure(e). Penrose Diagram with one **Schwarzschild Black Hole**



The lines that represents fixed locations in space and fixed moments in time, ***switch*** *in this* ***new region*** *of Penrose diagram. Space flows at* ***greater than the speed of light*** *inwards towards the central* ***singularity****; space becomes unidirectional flowing downwards into singularity and all paths lead to the inevitable singularity.*
Once you are beneath horizon the future light cone still represents all possible paths that you could take but all of them end up at the singularity.

Coding Using EinstienPy

EinsteinPy is a python library for purpose to solve equations and problems in General Relativity and for visualization of various results. It supports numerical relativity and relativistic astrophysics research.
NOTE:- Before writing any bellow code we need to import required Python libraries. These libraries have direct function for the work we need to do or various operation we want to perform and give required output.

```
import sympy
import numpy as np
from einsteinpy.symbolic import MetricTensor, ChristoffelSymbols, RiemannCurvatureTensor
```

A. Calculating the Christoffel Symbols for given metric:-

```
syms = sympy.symbols('r theta phi')
#define the metric for 3-D spherical coordinates
metric =  np.zeros([3, 3], dtype = list)
metric[0,0] = 1
metric[1,1] = syms[0]**2
metric[2,2] = (syms[0]**2)*(sympy.sin(syms[1])**2)

#creating metric object
m_obj = MetricTensor(metric, syms)
print("Metric tensor for 3-d spherical coordinates:\n",m_obj.tensor())

#Calculating the christoffel symbols
ch = ChristoffelSymbols.from_metric(m_obj)
print("\nChristoffel symbols:")
print(ch.tensor())
#We are using indices of array to get value of single element at given position. Example is bellow code line.
print("Element of christoffel symbols at position [0,1,1]:", ch.tensor()[0,1,1])
```

We have given metric for 3-D flat space in spherical coordinates. And with help of ChristoffelSymbols.form_metric() function we calculated Christoffel symbols for given metric.

OUTPUT:

```
Metric tensor for 3-d spherical coordinates:
 [[1, 0, 0], [0, r**2, 0], [0, 0, r**2*sin(theta)**2]]

Christoffel symbols:
[[[0, 0, 0], [0, -r, 0], [0, 0, -r*sin(theta)**2]], [[0, 1/r, 0], [1/r, 0, 0],
[0, 0, -sin(theta)*cos(theta)]], [[0, 0, 1/r], [0, 0, cos(theta)/sin(theta)],
[1/r, cos(theta)/sin(theta), 0]]]

Element of christoffel symbols at position [0,1,1]: -r
```

Simply Above Output Will Look Like:

Metric tensor for 3-d spherical coordinates:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(\theta) \end{bmatrix}$$

Christoffel symbols:

$$\begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r\sin^2(\theta) \end{bmatrix} & \begin{bmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & -\sin(\theta)\cos(\theta) \end{bmatrix} & \begin{bmatrix} 0 & 0 & \frac{1}{r} \\ 0 & 0 & \frac{\cos(\theta)}{\sin(\theta)} \\ \frac{1}{r} & \frac{\cos(\theta)}{\sin(\theta)} & 0 \end{bmatrix} \end{bmatrix}$$

Element of Christoffel symbols at position [0,1,1] : -r

B. Calculating Riemann Tensor from Christoffel Symbols:-

```
#Calculating Riemann Tensor from Christoffel Symbols
rm1 = RiemannCurvatureTensor.from_christoffels(ch)
print("\nCalculating Riemann Tensor from Christoffel Symbols:")
print(rm1.tensor())
```

After getting Christoffel symbols from code(A) we calculated Riemann tensor from Christoffel symbols using function of RiemannCurvatureTensor.from_Christoffels() function of EinsteinPy library.

OUTPUT:

```
Calculating Riemann Tensor from Christoffel Symbols:
[[[[0, 0, 0], [0, 0, 0], [0, 0, 0]], [[0, 0, 0], [0, 0, 0], [0, 0, 0]], [[0, 0,
0], [0, 0, 0], [0, 0, 0]]], [[[0, 0, 0], [0, 0, 0], [0, 0, 0]], [[0, 0, 0], [0,
0, 0], [0, 0, 0]], [[0, 0, 0], [0, 0, 0], [0, 0, 0]]], [[[0, 0, 0], [0, 0, 0],
[0, 0, 0]], [[0, 0, 0], [0, 0, 0], [0, 0, 0]], [[0, 0, 0], [0, 0, 0], [0, 0, 0]
]]]
```

Simply Above Output Will Look Like:

Calculating Riemann Tensor from Christoffel Symbols:

$$\begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

C. Calculating Riemann Tensor from Metric Tensor:-

```
#Calculating Riemann Tensor from Metric Tensor
rm2 = RiemannCurvatureTensor.from_metric(m_obj)
print("\nCalculating Riemann Tensor from Metric Tensor:")
print(rm2.tensor())
```

Now calculating Riemann tensor from metric tensor using function of RiemannCurvatureTensor.from_metric() function of EinsteinPy library.

OUTPUT:

```
Calculating Riemann Tensor from Metric Tensor:
[[[[0, 0, 0], [0, 0, 0], [0, 0, 0]], [[0, 0, 0], [0, 0, 0], [0, 0, 0]], [[0, 0,
0], [0, 0, 0], [0, 0, 0]]], [[[0, 0, 0], [0, 0, 0], [0, 0, 0]], [[0, 0, 0], [0,
0, 0], [0, 0, 0]], [[0, 0, 0], [0, 0, 0], [0, 0, 0]]], [[[0, 0, 0], [0, 0, 0],
[0, 0, 0]], [[0, 0, 0], [0, 0, 0], [0, 0, 0]], [[0, 0, 0], [0, 0, 0], [0, 0, 0]
]]]
```

Simply Above Output Will Look Like:

Calculating Riemann Tensor from Metric Tensor:

$$\begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

D. Calculating the Christoffel symbols for Schwarzschild Metric:-

```
"""Calculating the christoffel symbols for Schwarzschild Spacetime Metric"""
syms = sympy.symbols("t r theta phi")
G, M, c, a = sympy.symbols("G M c a")
#using metric values of schwarzschild space-time
#"a" is schwarzschild radius

list2d =  np.zeros([4, 4], dtype = list)
list2d[0][0] = 1 - (a / syms[1])
list2d[1][1] = -1 / ((1 - (a / syms[1])) * (c ** 2))
list2d[2][2] = -1 * (syms[1] ** 2) / (c ** 2)
list2d[3][3] = -1 * (syms[1] ** 2) * (sympy.sin(syms[2]) ** 2) / (c ** 2)
sch = MetricTensor(list2d, syms)
print("\nSchwarzschild Spacetime Metric:\n", sch.tensor())

# single substitution
subs1 = sch.subs(a,0)
print("\nFor a=0\n",subs1.tensor())

# multiple substitution
subs2 = sch.subs([(a,0), (c,1)])
print("\nFor a=0,c=1\n",subs2.tensor())

sch_ch = ChristoffelSymbols.from_metric(sch)
print("\nChristoffel symbols for Schwarzschild Spacetime Metric:")
print(sch_ch.tensor())
```

Firstly, we defined Schwarzschild metric and printed as Schwarzschild Spacetime Metric. After that in single substitution we gave value to a= 0 as (a,0) with help of subs( ) method. Similarly, we gave a=0 and c=1 in multiple substitution code block.

After that with help of ChristoffelSymbols.from_metric() we calculated Christoffel symbols for Schwarzschild metric.

OUPUT:

```
Christoffel symbols for Schwarzschild Spacetime Metric:

[[[0, a/(2*r**2*(-a/r + 1)), 0, 0], [a/(2*r**2*(-a/r + 1)), 0, 0, 0], [0, 0, 0, 0], [
0, 0, 0, 0]], [[-a*(a*c**2/(2*r) - c**2/2)/r**2, 0, 0, 0], [0, a*(a*c**2/(2*r) - c**2
/2)/(c**2*r**2*(-a/r + 1)**2), 0, 0], [0, 0, 2*r*(a*c**2/(2*r) - c**2/2)/c**2, 0], [0
, 0, 0, 2*r*(a*c**2/(2*r) - c**2/2)*sin(theta)**2/c**2]], [[0, 0, 0, 0], [0, 0, 1/r,
0], [0, 1/r, 0, 0], [0, 0, 0, -sin(theta)*cos(theta)]], [[0, 0, 0, 0], [0, 0, 0, 1/r]
, [0, 0, 0, cos(theta)/sin(theta)], [0, 1/r, cos(theta)/sin(theta), 0]]]
```

Simply Above Output Will Look Like:

$$\left[\left[\begin{matrix} 0 & \frac{a}{2r^2\left(-\frac{a}{r}+1\right)} & 0 & 0 \\ \frac{a}{2r^2\left(-\frac{a}{r}+1\right)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}\right] \left[\begin{matrix} \frac{ac^2\left(-\frac{a}{r}+1\right)}{2r^2} & 0 & 0 & 0 \\ 0 & -\frac{a}{2r^2\left(-\frac{a}{r}+1\right)} & 0 & 0 \\ 0 & 0 & -r\left(-\frac{a}{r}+1\right) & 0 \\ 0 & 0 & 0 & -r\left(-\frac{a}{r}+1\right)\sin^2(\theta) \end{matrix}\right] \left[\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{matrix}\right] \left[\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos(\theta)}{\sin(\theta)} \\ 0 & \frac{1}{r} & \frac{\cos(\theta)}{\sin(\theta)} & 0 \end{matrix}\right]\right]$$

CONCLUSION

In this project we learnt how gravity can be visualized as an effect of change in curvature of spacetime which occur due to presence of mass or energy in it. For curved spacetime we used concept of parllel transport to know about intrinsic geometry and by using this concept we calculated geodesic equtaion for curve. Every curve must satisfy geodesic equtaion to be geodesic of its curved manifold. In Future we can also study how light follow geodesic path near heavy mass or energy in universe. Schwarzschild solution to Einstein equations showed us singularity at r = 0 and r = 2GM in spacetime which after analysing in different other coordinate sysytems we concluded that r = 0 is true Physical singularity and r=2GM is Schwarzschild radius. After that we saw Penrose diagram for Schwarzschild Black Hole. We got to know that an entirely new Penrose region represents the interior of the Black Hole and in this new region of Penrose diagram the lines that represents fixed position in space and the lines that represents fixed moments in time got switched. There space flows at greater than the speed of light inwards towards the central singularity of Black hole.

In this we considered every thing static and general from Christoffel connection (torsion free) to Schawrzschild Black hole. For Future work we can work on rotation conditions also by considering connection for covariant derivative having tortion part and also after finding Einstein equation we can see various other type of Black hole like some having electric charge, spin, etc.

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