Chaos Synchronization

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Abstract

This literature review begins with a brief introduction to chaos and then moves on to examples of chaotic systems such as the Lorenz and Rössler systems. Following that, a quick overview of synchronisation. The primary goal of this review is to investigate all types of chaos synchronisation (complete, phase, lag, and generalised). Finally, applications of chaos synchronisation are presented and discussed.

I. Introduction

Chaos and synchronization are quite counter-intuitive ideas on their own. Synchronization discusses the synchronized behaviour of coupled systems, in contrast to the chaos, which portrays total disorder and unpredictable behaviour of complex systems. However, the fact that two or more chaotic systems may be brought into synchronization is astounding. Also, coupled chaotic oscillations is a well-studied concept both mathematically and experimentally, which has provided a strong foundation and motivation for recent research work in secure communications, network dynamics, biological dynamics, etc.

1. Chaos

Chaos simply means the unpredictable nature of a deterministic system which has sensitive dependence on the initial conditions of the system. It implies if we have similar systems with slightly perturbed initial conditions, the trajectories of both systems will be uncorrelated and will diverge from each other exponentially [1, 2]. Chaotic systems are nonlinear dynamical systems or infinite-order linear systems which are bounded in phase space. Consequently, *Chaos Theory* emerged as a subfield of mathematics to analyse the behaviour of complex systems that show chaotic behaviour.

1.1. Attractors

An attractor is a set of states in phase space that explains the development of a dynamical system in time for variations of initial conditions of the system. Once the state of the system is in the attractor, it is bounded by it and all sufficiently nearby solutions converge to the attractor [1]. It attracts phase trajectories of the dynamical system and only one trajectory can evolve from a given point in a phase space [3].
An attractor for a chaotic system is known as *strange attractor* which exhibits sensitive dependence on initial conditions and has a fractal structure. Due to this sensitive dependence on initial conditions nearby states for the system on this attractor diverge exponentially from each other [1]. Despite the appearance of unpredictable movement of trajectories within them, chaotic attractors have notably defined geometric forms.

1.2. Lyapunov Exponents

The concept of a Lyapunov exponent is used to quantify the idea of exponential divergence discussed above. Lyapunov exponent is a quantity (characteristic of a dynamical system) which tells mathematically whether a system is chaotic or not. We have n-Lyapunov exponents (known as the spectrum of Lyapunov exponents) for a dynamical system in an n–dimensional phase space which explains the behaviour of state vector in the tangent space of the phase space thus, it is defined from the Jacobian matrix [1].
We have the same Lyapunov exponent for every point on a trajectory, and the trajectory is chaotic and initial-point dependent when a Lyapunov exponent is greater than zero. If the Lyapunov exponent for two originally identical trajectories is greater than zero then the difference between them will grow exponentially with time [4]. At least one Lyapunov exponent will be positive for a chaotic system as the chaotic system never converges to a stable trajectory in phase space.

1.3. Chaotic Systems

The following introduces some well-known chaotic systems that will be used later to examine chaos synchronisation.

Lorenz System:-

Lorenz system is a simplified model for general atmosphere circulation which was introduced by Edward Lorenz in 1963. This 3-dimensional system is explained using ordinary differential equations:

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= x(\rho - z) - y, \\ \dot{z} &= -\beta z + xy, \end{aligned} \tag{1}$$

where σ, ρ and β are parameters and x, y, z are coordinates representing the state of the system [5]. Lorenz chose σ=10, ρ=28 and β =8/3 for which the system showed chaotic behaviour. When the solution was visualized as a trajectory in phase space, we found a beautiful butterfly pattern that appears, known as *Lorenz attractor* (strange attractor for Lorenz system).

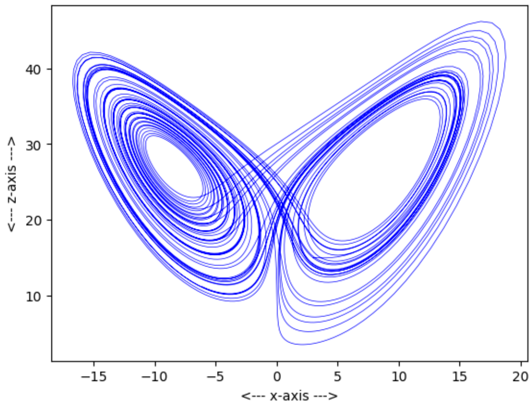


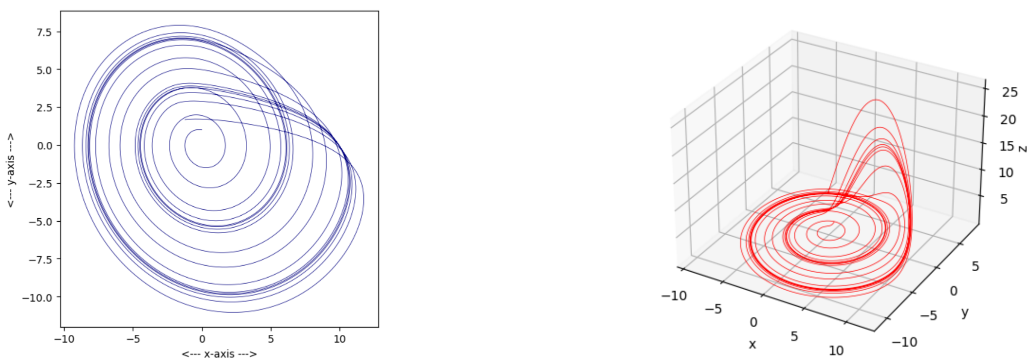
Figure 1: *(Original image) Projection of Lorenz attractor in x-z plane.*

Rössler System:-

Rössler system is an even simpler than the Lorenz system with only one nonlinear term. This system was introduced by Otto Rössler in 1976 and explained using the following ordinary differential equations:

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + ay, \\ \dot{z} &= b + z(x - c), \end{aligned} \tag{2}$$

where a, b, c are parameters and x, y, z are coordinates representing the state of the system [5].



(a) Projection of Rössler attractor in x-y plane

(b) Rössler attractor in 3-dimensions

Figure 2: *(Original image) Strange attractor for Rössler System.*

2. Synchronization

Synchronization is a behaviour which seems intrinsic to the universe, as we can observe synchronicity in various natural phenomena, from the flashing of thousands of fireflies in unison, the coherent firing of our hearts' hundreds of pacemaker cells, coordinated neutral activities, to the harmonized marching of the trillions of electrons in a superconductor allowing electricity to flow through it without any resistance. Various such examples show how deeply synchronization is embedded in the universe and how synchronicity arises spontaneously in complex systems [6].
In terms of classical physics, synchronisation is the coincidence in the rhythms of autonomous periodic systems caused by the weak interaction between them. It is coordination in the time of different processes. The moderation in the behaviour of the oscillating system to synchronise with another system can be explained in terms of phase locking and frequency locking [7].
There are various types of synchronization which are being observed and studied. When two or more identical coupled systems synchronize completely, it is referred to as *complete synchronization* or *Identical synchronization*. *Phase synchronization* occurs when the coupled systems maintain a constrained phase difference while keeping their amplitudes uncorrelated. Whenever there is a time lag in synchronised behaviour between coupled systems that is the case of *lag synchronization*. Finally, there is *generalized synchronization* in which we have synchronized behaviour between non-identical systems that are related by a function [8].
Many complex dynamical systems synchronize so why it's unlikely for chaotic systems to synchronise? This is to do with their positive Lyapunov exponents [Subsection 1.2.] because of which isolated chaotic systems resist synchronization with anything and the variations in system states increase exponentially with time. Therefore, the synchronization of chaotic systems is already quite intriguing.

II. Synchronization of Chaotic Systems

Synchronization of chaotic systems is one of the fundamental aspects of nonlinear dynamics, which is the convergence of the trajectories of the chaotic systems onto one. Chaotic systems that appear to oppose synchronization due to their sensitive dependence on the initial conditions explained by their positive Lyapunov exponents, can be synchronized with the help of proper coupling between them and the conditions for the synchronization can be explained mathematically. This coupling allows us to change the Lyapunov spectrum, either reducing or increasing the number of positive Lyapunov exponents [9].

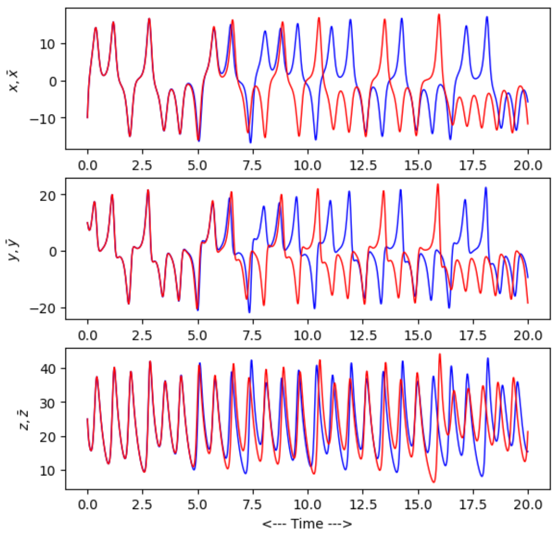


Figure 3: *(Original image) This image is an example of sensitive dependence of chaotic systems. This is a simulation of Lorenz system [Eq(1)] with parameter values σ=10, ρ=28, β=8/3, starting from two slightly different initial conditions (x,y,z) = (-10,10,25) and ($\bar{x}, \bar{y}, \bar{z}$) = (-10.01,10,25) yet exhibit no synchronized behaviour [10]. For the simulation, RK4 numerical methods technique is used from Ref.[11].*

The coupling can be done in two ways: *unidirectional coupling and bidirectional coupling*. A main system in the unidirectional situation is composed of two subsystems that operate in a drive-response configuration. This means that the evolution of one subsystem causes the evolution of the other and the response system is forced to mimic the dynamics of the drive system, which serves only as an external yet chaotic forcing on the response system. External synchronisation is produced in this instance and the best example of this is secure communication based on synchronization of chaotic systems. While in bidirectional coupling the two

subsystems are interconnected to one another and the coupling factor causes their rhythms to be adjusted to a shared synchronised behaviour. An example of this is the interaction between neurons [12].
For coupled chaotic oscillators, various types of synchronizations have been studied and explained mathematically and experimentally i.e., Complete Synchronization, Phase Synchronization, Lag Synchronization and Generalized Synchronization. And all of these synchronisations can be grouped together under a single term: *Time-Scale Synchronisation* [8].

1. Complete Synchronization

Complete synchronization is observed in identical chaotic systems. It occurs when the difference between state vectors or trajectories of coupled identical chaotic systems converges to zero as time (t) tends to infinity, for the large enough coupling strengths. If isolated identical chaotic systems are diffusively coupled, then the coupling disappears with time, and both systems oscillate in alignment at all coupling strengths and times [8, 10].
Let's consider two identical non-linear n-dimensional systems which are fully diffusively coupled [10]:-

$$\begin{aligned} \dot{x_1} &= f(x_1) + \alpha \boldsymbol{H}(x_2 - x_1) \\ \dot{x_2} &= f(x_2) + \alpha \boldsymbol{H}(x_1 - x_2) \end{aligned} \tag{3}$$

where $\boldsymbol{f}$ is general non-linear function, α is coupling strength parameter and $\boldsymbol{H}$ is a smooth coupling function. Here, we assume $\boldsymbol{H(0) = 0}$ to attain invariant synchronized subspace $x_1 = x_2$ for all coupling strength α. For sufficiently strong coupling, $x_1(t)$–$x_2(t) \to 0$ as t $\to \infty$. Let consider $z := x_1 - x_2$ such that synchronization is attained if

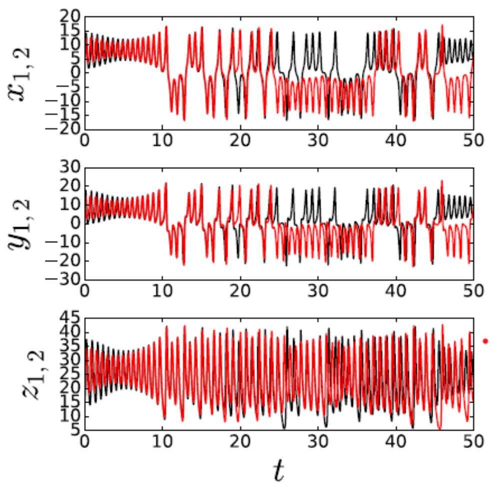
$$lim_{t\to\infty} z(t) = 0.$$

For the simplest case $\boldsymbol{H}$=$\boldsymbol{I}$ (Identity matrix) was considered. Following mathematical analysis, it was discovered that synchronisation occurs when $\alpha > \alpha_c$ [10], where α_c is critical coupling strength for synchronisation and $\alpha_c := \Lambda/2$, where Λ is the Lyapunov exponent for trajectory $x_1(t)$. This is the case because,

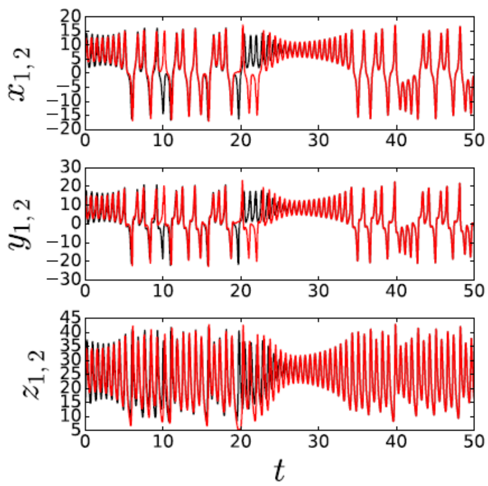
$$||z(t)|| \leq Ce^{(\Lambda - 2\alpha)t} \tag{4}$$

where C is constant and $C > 0$. Therefore, values of coupling strength α that are above α_c would result in synchronization of the coupled identical chaotic systems [10].

Now, let's understand the above concept with the help of an example of coupled Lorenz Systems.
Lorenz System:- Consider two coupled Lorenz systems [Eq(1)], as in Equation(3). Using numerical analysis it was found that $\alpha_c \approx 0.453$ for two coupled Lorenz systems. In the Figure (4) below it is shown that indeed, for $\alpha < \alpha_c$ we see no synchronization and for $\alpha > \alpha_c$ there is synchronization in the trajectories of two coupled Lorenz system.



(a) No synchronization



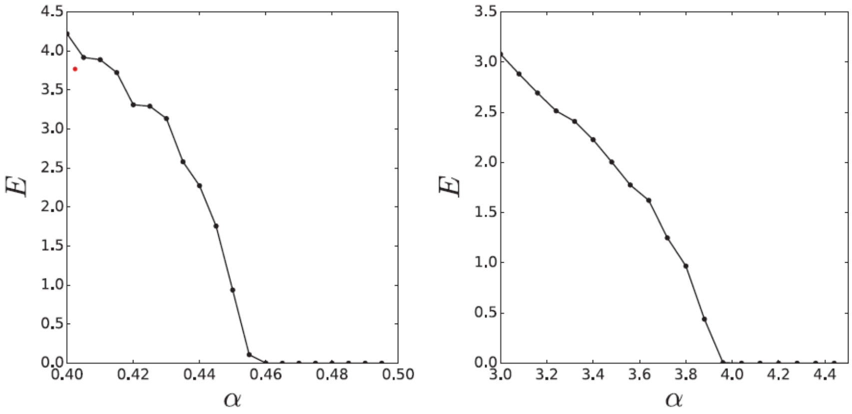
(b) Synchronization

Figure 4: Two coupled Lorenz systems with initial conditions selected as (x,y,z) = (3,10,15) and $(\tilde{x}, \tilde{y}, \tilde{z})$ = (10,15,25), with parameter values σ=10, ρ=28, β=8/3. (a) When α=0.4 $< \alpha_c$, there is no synchronization seen between coupled Lorenz systems. (b) For α=0.5 $> \alpha_c$, there is synchronization of trajectories (Ref[10]).

For a time-interval of length T, let consider the average deviation from synchronization (E) as [10]:

$$E = \frac{1}{T}\int_{t=0}^{T} ||x_1(t) - x_2(t)||dt \tag{5}$$

In Figure (5) the relation between average deviation from synchronisation (E) and coupling strength α is analysed. We find a fair correlation with the derived value of α_c for two coupled Lorenz systems with coupling matrix **H=I**. Also, some other coupling matrix can be used other than Identity **I**, for example x-coupling for which H=$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. For this, the critical coupling strength α_c is observed around ~ 3.75 as shown in Figure (5).

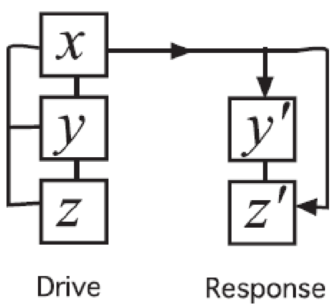


(a) Identity coupling

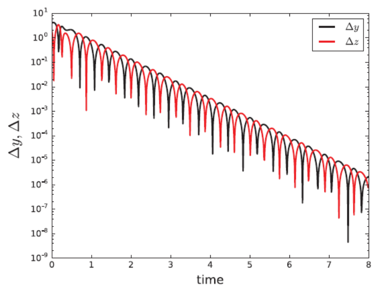
(b) x-coupling

Figure 5: *Average deviation from synchronisation (E) versus coupling strength α. (a) For **H=I**, α_c correlates the derived value i.e., $\alpha_c \sim 0.453$. (b) For **H** as x-coupling matrix, α_c corresponds to $\alpha_c \sim 3.75$ (Ref[10]).*

Complete Synchronization in drive-response (or master-slave) configuration:
In this configuration, the drive (transmitter) system sends a signal from one of its component to the response (or receiver) system where the receiver was missing the part of the system which was compensated for by using the received signal from the drive system. In this situation, complete synchronisation is possible only if all of the Lyapunov exponents of the response system under the action of the drive system (the conditional Lyapunov exponents) are negative [2, 12]. It is important to note that not all feasible driving signal selections result in a synchronised state. An example of this is shown Figure 6.



(a) Schematic diagram (Ref[9]).



(b) Simulation of drive-response coupling for two Lorenz systems for which x is the drive system. (Ref[10]).

Figure 6: *Drive-response configuration for coupled chaotic systems*

In Figure 6(b), the drive system sends the x signal to the response system, which doesn't have the x component but is otherwise similar to the drive, i.e., the y' and z' subsystems are identical to the drive's y and z subsystem. And for the systems to synchronize the conditional Lyapunov exponents must be negative [9, 10, 12]. These Lyapunov exponents are called conditional because of their dependence on the driving signal, here the x signal in Figure 6(b).

2. Phase Synchronization

Phase synchronisation is the locking of coupled chaotic systems phases which lead to the frequency entrainment. Whenever for two coupled chaotic dynamical systems $x_{1,2}$ the phase difference $\Delta\phi$ for instantaneous phase $\phi_{1,2}$ is bounded by some constant, we get phase synchronization between the systems i.e., $|\phi_1(t) - \phi_2(t)| <$ constant. Because of the chaotic nature of the system, the phase difference will not be exactly zero. In phase synchronization of coupled chaotic dynamical systems, only phases of the subsystems are locked, while the dynamics remain hyper-chaotic. Note that phase synchronization does not imply complete synchronization because amplitudes of the coupled chaotic systems remain uncorrelated [8, 10, 12].
Using the phase angle $\phi(t)$ it is possible to define a mean rotation frequency,

$$\Omega = lim_{t\to\infty} \frac{\phi(t)}{t} \tag{6}$$

and the *frequency mismatch* is defined as $\Delta\Omega = \Omega_1 - \Omega_2$. Thus, for phase synchronization to occur the mean rotation frequency must be the same for the drive and the response system i.e., $\Delta\Omega = 0$. This leads to the frequency entrainment known from coupled periodic oscillations [13]. An example of this is shown below with the help of Figure (7) using two unidirectionally coupled Rössler systems. A detailed analysis of this example is in Refs.[10, 12, 13].

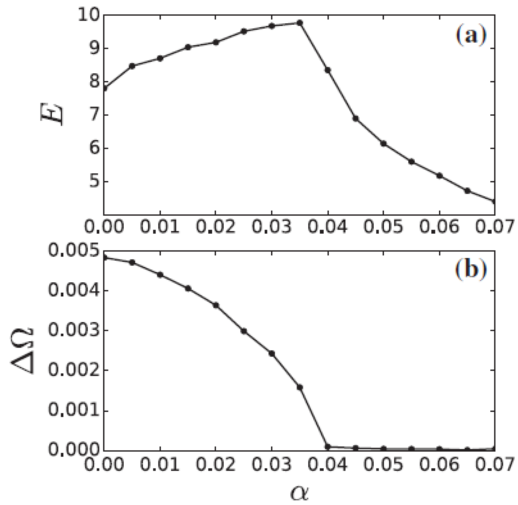


Figure 7: *The synchronisation error E and frequency mismatch $\Delta\Omega$ relation with coupling strength α. Phase synchronization occurs for $\alpha > \alpha_c (\sim 0.04)$ (Ref.[10]).*

3. Lag Synchronization

In lag synchronization, the states of the two chaotic dynamical systems become identical with a proper time shifting i.e., $x_1(t + \tau) = x_2$(t). The time lag t reduces as the coupling between the oscillators increases, and the lag synchronisation regime tends to be complete synchronisation. Lag synchronization is mostly studied between symmetrically coupled non-identical dynamical systems and in time-delayed systems [14]. For identical coupled systems, complete synchronization is a special case of lag synchronization when τ=0. The similarity function is used to determine the lag synchronisation [8, 12].
Also, recently it has been revealed that dissipative chaotic systems with time-delayed feedback can drive identical systems in such a way that the driven systems anticipate the drivers by synchronising with their future states, y(t) $\approx x(t + \tau)$. Detailed analysis of lag synchronization can be found in Refs. [12, 14].

4. Generalized Synchronization

Generalized synchronization is the synchronization between two non-identical chaotic dynamical systems. It is introduced for drive-response configuration. The chaotic dynamics of the driving system are independent of the response system parameters. For generalized synchronization to occur coupled chaotic systems must have a functional relationship, i.e., $x_2(t) = F[x_1(t)]$. The case of generalised synchronisation is more challenging since the functional relation F[.] is involved. Yet, there are numerous approaches to discover synchronised behaviour of coupled chaotic systems, such as *the auxiliary system approach* which links the generalized synchronization problem to the complete synchronisation problem and *the method of the nearest neighbourhood (mutual false nearest neighbours)* which determine when closeness in response space implies closeness in driving space. The synchronisation condition in generalized synchronization means that the drive alone determines the response dynamics [8, 10, 12, 15]. For unidirectionally coupled dynamical systems, the drive x and the response y systems are coupled as

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{y} &= g(y, h(x)) \end{aligned} \tag{7}$$

where x $\in R^n$, y $\in R^m$ and h(x) is the coupling, and for certain coupling strengths, the dynamics of system y are completely determined by the dynamics of system x. That is, x solutions can be mapped into y solutions.

$$y = \Psi(x) \tag{8}$$

where Ψ is the functional relationship between coupled chaotic systems This results in generalized synchronization between these two systems. When Ψ is the identity, *complete synchronization* is a special case of generalized synchronization [10].
Now, let's review the methodologies stated above for discovering the synchronised behaviour of connected chaotic dynamical systems.
Auxiliary System Approach:-
In this, we have an auxiliary system which is a copy of the response (or slave) system and the drive (or master) system drives the slave system and an auxiliary system. If the auxiliary and response systems exhibit complete synchronization then the drive and response systems are in generalized synchronization.

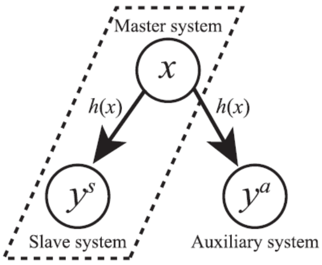
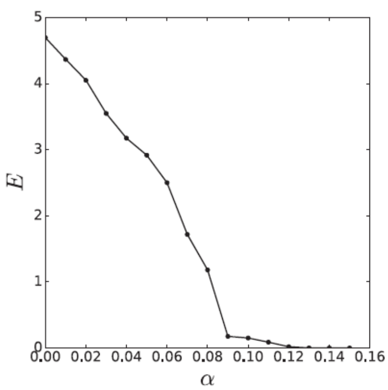
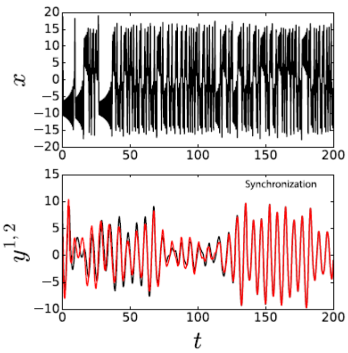


Figure 8: *Schematic diagram for auxiliary system approach for generalized synchronization. If there is complete synchronization between y^s and y^a, then the generalized synchronization occurs between x and $y^{a,s}$ (Ref[10]).*

Let's see an example of it. Consider two identical Rössler systems [Eq(2)] i.e., one response and other auxiliary with the parameters (a = 0.2, b = 0.2 and c = 5.7) are driven by a Lorenz system [Eq(1)] with the parameters (ρ = 28, σ = 10 and $\beta = 8/3$) through x-components. Here, by using the auxiliary system approach we get critical coupling for the generalized synchronization as $\alpha_c \sim 0.12$ as shown in Figure 9. The behaviour of the above system and this approach has been thoroughly studied in Ref.[10, 12]



(a) Critical coupling strength $\alpha_c \sim 0.12$



(b) The drive system x is Lorenz system and the response and auxiliary systems are two identical Rössler systems. For coupling strength α = 0.2 which is greater than α_c ($\sim$ 0.12), we get the synchronized behaviour.

Figure 9: *Auxiliary system approach for generalized synchronization (Ref[10]).*

Mutual False Nearest Neighbours Method:-
The key idea is to look at how close points are mapped through the dynamics. The existence of the mapping Ψ can be inferred by evaluating the attributes of surrounding points [10]. Let D be the phase space of the driving system x and R be the phase space for y. If there exists a transformation Ψ from the trajectories of the attractor in D space to the trajectories in R space then we have synchronization between the x and y systems. Mutual false nearest neighbours numerical approach detects the presence of the continuous transformation Ψ and hence distinguishes between synchronised and unsynchronized behaviour in coupled chaotic dynamical systems. In simple words, in this approach, we are trying to find some geometric relation between the driving and response systems which do not alter the identity of neighbourhoods in state space. It is kind of finding the correlation between driving and response system. This approach has been thoroughly studied in Ref.[15].

III. Applications of Synchronization

Secure communication: Chaos synchronisation is an exciting area of research in cryptography. Since, chaotic signals are noise-like and broadband, it is difficult to read the message and thus, masking information in chaotic signals could provide a way for secure communication [10]. In Ref.[16] it has been shown that the isochronal synchronisation of two chaotic units can be utilised to transmit messages with a low bit error rate using the chaos pass filter configuration, or even error-free utilising chaos modulation. In both cases, bidirectional coupling has the potential to be employed in secure communication, whereas unidirectional coupling has been found to be insecure.

Parameter estimation and prediction: Predicting future behaviour (or data) for the study of a system is a critical part of any data analysis that has its own importance in various research areas. But the issue lies in figuring out the underlying parameters required for the prediction model. Now, chaos synchronization techniques are also being utilized for this purpose. Here, we blend the data with equations and the data are then used to drive the equations [10].

Natural Systems: Because natural systems are generically heterogeneous and coupling is mostly complicated and weak, complete synchronisation may be exceptional but still many natural systems can be explained with the help of other types of synchronization. Stable biological systems like the dynamics of the pacemaker cells where millions of the heart's pacemaker cells fire in unison can be modelled by phase synchronization. Phase synchronization methods have also been used to study spatial synchronization of oscillations in blood distribution systems and synchronization of biological neuron activities. For solar activities also phase analysis methods have been used like phase synchronization of the sunspot cycle and a fast component of the solar inertial motion with statistical significance in different epochs [10, 12].

IV. Discussion and Conclusion

Chaotic synchronisation is extremely sensitive to noise introduced into the coupling signal; it combines noise and signal in a nonlinear manner, making it impossible to separate the two using standard methods such as filtering or correlation [9]. This is an active research problem to reduce the noise influence on chaotic systems. There are also lots of issues in secure communication using chaos synchronization like precision issues. In addition, various methods are provided in Ref.[17] to breach secure communication generated using chaotic synchronisation, which was previously assumed to be unbreakable. Thus, active research on secure communication via chaos synchronization is ongoing by utilizing chaos synchronization for complex systems also.
Recently, the concept of synchronization is also changed to a more geometric view using synchronization manifolds, which itself is a whole new concept [9].

V. Acknowledgements

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