$$g(n) = \frac{n^2}{2} + \left(\frac{n^2}{2} - n\right) \geq \frac{n^2}{2} \quad \forall n \geq 2$$

$$g(n) \geq \frac{n^2}{2} = \frac{1}{2} f(n) \quad \forall n \geq 2$$

$$\Rightarrow f(n) \leq 2\, g(n) \quad \forall n \geq 2$$

$$P(n) = a_k n^k + a_{k-1} n^{k-1} + a_{k-2} n^{k-2} + \dots + a_1 n + a_0$$

Assumption : $a_k > 0$

$$P(n) \leq \sum_{i=0}^{k} |a_i| n^i \qquad [a_i \leq |a_i|]$$

$$\leq \sum_{i=0}^{k} |a_i| n^k \qquad (|a_k| + |a_{k-1}| + |a_{k-2}| + \dots |a_1| + |a_0|) n^k$$

$$P(n) \leq c_1 n^k \text{ ——— (1) } \forall n$$

$$\Rightarrow P(n) = O(n^k) \quad \forall n$$

$$P(n) = a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0$$

$\frac{1}{2} a_k n^k + \frac{1}{2} a_k n^k$    $k$ terms

$$= \frac{1}{2} a_k n^k + \left(\frac{a_k n^k}{2k} + a_{k-1} n^{k-1}\right) + \left(\frac{a_k n^k}{2k} + a_{k-2} n^{k-2}\right) + \dots$$

$$\dots + \left(\frac{a_k n^k}{2k} + a_1\right) + \left(\frac{a_k n^k}{2k} + a_0\right)$$

Since $a_k > 0$, we have for all large enough $n$ (say $\geq n_0$)

$$\boxed{\frac{a_k n^k}{2k}} + a_{k-1} n^{k-1} \geq 0 \quad n_1$$

$$\frac{a_k n^k}{2k} + a_{k-2} n^{k-2} \geq 0 \quad n_2$$

$$\vdots$$

$$\frac{a_k n^k}{2k} + a_0 \geq 0$$

$$\therefore P(n) \geq \underbrace{\frac{1}{2} a_k}_{c_2} n^k \quad \forall n \geq n_0$$

$$c_2 n^k \leq P(n) \leq c_1 n^k \quad \forall n \geq n_0$$

$$\Rightarrow P(n) = \Theta(n^k)$$

Alternative Proof :

$$0 < \left[\lim_{n \to \infty} \frac{f(n)}{g(n)}\right] < \infty$$

$$\Rightarrow f(n) = \Theta(g(n))$$

$$\&$$

$$g(n) = \Theta(f(n))$$

$$\lim_{n \to \infty} \frac{a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0}{n^k}$$

$$= \lim_{n \to \infty} \frac{a_k n^k}{n^k} + \underbrace{\lim_{n \to \infty} \frac{a_{k-1} n^{k-1}}{n^k} \dots + \lim_{n \to \infty} \frac{a_1 n}{n^k} + \lim_{n \to \infty} \frac{a_0}{n^k}}_{\text{all these terms evaluate to zero}}$$

$$= \sum_{i=0}^{k} \lim_{n \to \infty} \frac{a_i n^i}{n^k}$$

$$= \lim_{n \to \infty} \frac{a_k n^k}{n^k} = \lim_{n \to \infty} a_k = a_k \quad [> 0]$$

$$\Rightarrow P(n) = \Theta(n^k)$$

Given: $a_i > 0 \quad \forall i$

$$P(n) \geq c_2 n^k \quad \forall n \geq n_0$$

Given: ~~a~~ $a_i > 0 \quad \forall i$

$p(n) \geq c_2 n^k \quad \forall n \geq n_0$

$O(n^k)$ $\quad a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0$

$\leq \underbrace{(a_k + a_{k-1} + \dots + a_1 + a_0)}_{c_1} n^k$

$\Omega(n^k)$ $\quad a_k n^k + a_{k-1} n^{k-1} + \dots + a_0 \geq \underbrace{a_k}_{c_2} n^k$