Ordinal Pattern Features: Statistical Properties and Their Applications in Time Series Clustering

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Abstract

Time series analysis plays a vital role in understanding the underlying dynamics of complex systems across various domains such as engineering, economics, and the physical sciences. Traditional approaches, namely time-domain and frequency-domain methods often rely on strong assumptions such as stationarity, large sample sizes, or normality, which are frequently violated in real-world data. As a robust alternative, Bandt and Pompe [6] introduced a non-parametric method based on ordinal pattern symbolization and information theoretic descriptors. This approach transforms local segments of the time series into rank-based symbols (the ordinal patterns), constructs a histogram of ordinal patterns, and computes Shannon entropy, offering resistance to noise and model independence. Building on this, Lamberti et al [20] introduced the statistical complexity, allowing the joint representation of entropy and complexity in the entropy-complexity plane. This proposal investigates the use of various entropy measures, such as Shannon, Tsallis, and Rényi entropy, as well as the Fisher Information Measure, statistical complexity, and associated confidence intervals for time series analysis and clustering.

Contents

Chapter 1

Introduction

Time series contain valuable insights about the underlying system that generates the data. Their analysis is typically conducted using two primary approaches: time-domain and transformed-domain methods. In the context of time-domain analysis, Bandt and Pompe [6] introduced a novel methodology that is non-parametric and rooted in information theory descriptors: Ordinal Patterns symbolization.

Bandt and Pompe [6] proposed transforming small subsets of the time series observations into symbols that encode the sorting properties of the values in these subsets. Then, they computed a histogram of those symbols. The resulting distribution is less sensitive to outliers compared to the original data, and the histogram is independent of any specific model. The proposal proceeds by computing descriptors from this histogram, and extracting information about the system from these descriptors. As a result, this approach is versatile and applicable to a wide range of scenarios.

This study explores features derived from Bandt and Pompe symbolization, specifically Shannon entropy, complexity, and represents them graphically in the entropy-complexity plane, which is represented as a point in the $\mathbb{R}^2$ manifold. This proposal focuses on the independent analysis of permutation entropy and ordinal patterns in time series analysis. It involves calculating pattern histograms, entropy, complexity, and confidence intervals to better understand the statistical properties of these tools. Additionally, the role of confidence intervals in entropy and complexity, along with their applications in time-series clustering, will be explored. Future work will expand to include alternative measures, such as Rényi entropy, Tsallis entropy and Fisher information, with a focus on deriving confidence intervals for their entropy and complexity under the Multinomial model.

Time series analysis is widely applied across various fields, including engineering, economics, physical sciences, and more. A time series is defined as a collection of observations x_t, each representing a realized value of a particular random variable X_t, where time can be either discrete or continuous.

Examples of time series applications include finance (e.g., analyzing exchange rate movements or commodity prices), biology (e.g., modeling the growth and decline of bacterial populations), medicine (e.g., tracking the spread of diseases like COVID-19 or influenza), and geoscience (e.g., predicting wet or dry days based on past weather conditions).

The primary goal of time series analysis is to understand the nature of the phenomenon represented by the observed sequence. Time domain and frequency domain methods are the two primary approaches used in time series analysis. The temporal approach relies on concepts such as auto-correlation and regressions, where a time series' present value is analyzed in relation to its own past values or the past values of other series. This method represents time series directly as a function of time. On the other hand, the spectral approach represents time series through spectral expansions, such as wavelets or Fourier modes [55].

However, these methods often require assumptions such as large sample sizes or normally distributed observations that are rarely met in real-world empirical data. For many statistical techniques to be valid, these assumptions must hold, but in practice, they are frequently violated.

For example, traditional approaches to time series analysis, such as time domain and frequency domain methods, rely on assumptions that are not always valid in real-world data. The time domain approach, which uses techniques like auto-correlation and regression, assumes stationary and often struggles with nonlinear or non-stationary data. Similarly, the frequency domain approach, which represents time series through spectral expansions such as wavelets or Fourier modes, may require assumptions about periodicity and may not effectively capture short-term fluctuations.

Many statistical methods in these approaches depend on specific conditions, such as large sample sizes or normally distributed observations. However, these assumptions are often unrealistic, leading to inaccurate or biased results. When such conditions are not met, alternative methods must be considered.

As a result, alternative methods, often referred to as non-parametric techniques, must be considered. These methods rely on the rank R_t of the observations x_t rather than their actual values, making them robust and applicable to a wide range of data sets. Since non-parametric tests do not assume a normal distribution, they are highly reliable. For example, the Kruskal-Wallis H test and the Wilcoxon test are effective tools for comparing two or more population probability distributions from independent random samples. However, these techniques are not always suitable for time series data, which often require specialized methods tailored to their unique characteristics.

To address these challenges, ordinal pattern methods provide a robust alternative. Instead of analyzing the absolute values of a time series, these methods focus on the order relationships among consecutive data points.

This approach effectively captures the underlying dynamics of complex systems and offers several advantages.

The ordinal pattern-based method has become a widely used tool for characterizing complex time series. Since its introduction nearly twenty-three years ago by Bandt and Pompe in their foundational paper [6], it has been successfully applied across various scientific fields, including biomedical signal processing, optical chaos, hydrology, geophysics, econophysics, engineering, and biometrics. It has also been used in the characterization of pseudo-random number generators.

The Bandt and Pompe method successfully analyzes time series by transforming them into ordinal patterns, constructing a histogram, and computing Shannon entropy, making it robust against outliers and independent of predefined models.

Later, Rosso [45] introduced an additional dimension to this analysis Statistical Complexity derived from the same histogram of causal patterns.

Introduction to Ordinal Pattern Analysis

Ordinal patterns are a non-parametric representation of real-valued time series and ordinal patterns are transformations that encode the sorting characteristics of values in $\mathbb{R}^D$ into $D!$ symbols, where D represents for the "Embedding Dimension" and usually ranges between three to six. One of the possible encoding is the set of indexes that sort the D values in non-decreasing order.

To illustrate this idea, let $\boldsymbol{x} = \{x_1, x_2, \ldots, x_{(n+D-1)}\}$ be a real valued time series of length $n + D - 1$ without ties. If the $\boldsymbol{x}$ takes infinitely many values, it is common to replace them with a symbol sequence $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ consisting of finitely many symbols and then compute the entropy from this sequence. The corresponding symbol sequence naturally emerges from the time series without requiring any model assumptions. We compute $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ symbols from sub-sequences of embedding dimension D. There are $D!$ possible symbols: $\pi_j \in \boldsymbol{\pi} = (\pi^1, \pi^2, \ldots, \pi^{D!})$. The histogram of proportions $h = (h_1, h_2, \ldots, h_{D!})$ in which the bin h_ℓ is the proportion of symbols of type π^ℓ of the total number of symbols. For convenience, we will model those symbols as a k dimensional random vector where $k = D!$.

Problem Statement

To illustrate this concept, imagine tracking the mean monthly humidity in Wellington. You want to analyze how humidity changes throughout the year. By examining this data, you can uncover interesting patterns that highlight the variations in humidity across different months.

Month	Mean of relative humidity
January	77.3
February	81.0
March	82.4
April	81.7
May	83.6
June	85.6
July	84.4
August	83.1
September	78.8
October	79.6
November	78.2
December	78.8

Table 1.1: Mean monthly humidity variations in Wellington throughout the year

Mean monthly humidity in Wellington is shown in Figure 1.1.

We can convert this actual data into ordinal patterns. To do this, for each month, we determine the order of the humidity values rather than their actual magnitudes. Each three-time-point sequence (which can be adjusted based on preference) is converted into an ordinal pattern. This "embedding dimension" usually varies between 3 and 6, but any dimension is possible. The conversion can be made in any way that uniquely maps the sorting properties of the sub sequence into a symbol. For example, consider the

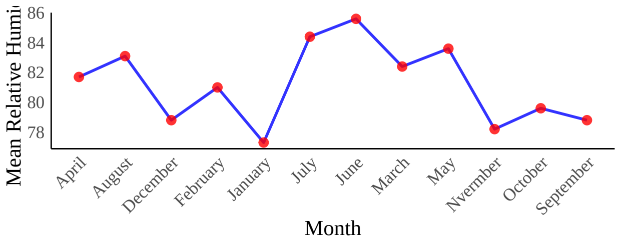

Figure 1.1: Mean Monthly humidity in Wellington

time series presented in Table 1.1. We can transform this series into ordinal patterns as follows. Assume we use patterns of length $D = 3$ with a time lag $\tau = 1$. The first overlapping window $(77.3, 81, 82.4)$ corresponds to the pattern $(1, 2, 3)$, where type is considering with the order of the real time series data. Here 77.3 is the smallest value and is assigned rank 1; 81 is the next highest and is assigned rank 2; and 82.4 is the largest, assigned rank 3. As another example, consider the overlapping window $(83.1, 78.8, 79.6)$. In this case, 78.8 is the smallest value and is assigned rank 1; 79.6 is the next highest, assigned rank 2; and 83.1 is the largest, assigned rank 3. Therefore, the pattern for this window is $(3, 1, 2)$. Table 1.2 has shown this scenario.

t	Mean Humidity sequence	Ordinal Pattern
1	(77.3,81,82.4)	(123) $= \pi^1$
2	(81,82.4,81.7)	(132) $= \pi^2$
3	(82.4,81.7,83.6)	(213) $= \pi^3$
4	(81.7,83.6,85.6)	(123) $= \pi^1$
5	(83.6,85.6,84.4)	(132) $= \pi^2$
6	(85.6,84.4,83.1)	(321) $= \pi^6$
7	(84.4,83.1,78.8)	(321) $= \pi^6$
8	(83.1,78.8,79.6)	(312) $= \pi^5$
9	(78.8,79.6,78.2)	(231) $= \pi^4$
10	(79.6,78.2,78.8)	(312) $= \pi^5$

Table 1.2: Ordinal Patterns

As shown in Table 1.2, we have six mutually exclusive events which we denote as $\{\pi^1, \pi^2, \ldots, \pi^6\} = \{(123), (132), (213), (231), (312), (321)\}$. The probability distribution of the mean humidity is calculated based on ordinal patterns as given below.

$$\hat{p}_i = \frac{\#\{\pi_j \in \boldsymbol{\pi} : \pi_j = \pi^i\}}{n}; 1 \leq i \leq 6, \tag{1.1}$$

where $\hat{\boldsymbol{p}} = (\hat{p}_1, \ldots, \hat{p}_6)$.

Notation	Probability
$p(\pi^1)$	$\frac{2}{10}$
$p(\pi^2)$	$\frac{2}{10}$
$p(\pi^3)$	$\frac{1}{10}$
$p(\pi^4)$	$\frac{1}{10}$
$p(\pi^5)$	$\frac{2}{10}$
$p(\pi^6)$	$\frac{2}{10}$

Table 1.3: Probability function

We construct the histogram of proportions $h = (h_1, h_2, h_3, h_4, h_5, h_6)$, where each bin h_ℓ represents the proportion of symbols of type π^ℓ out of the total six symbols. The histogram graph is shown Figure 1.2.

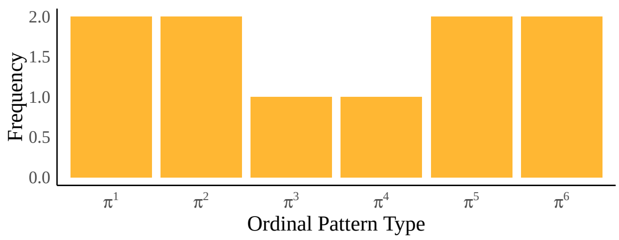

Figure 1.2: Histogram of proportions of the observed patterns according to Table 1.3.

This example explains how time series data can be converted into ordinal patterns and how the probability distribution function can be calculated from these patterns. Chapter Two will review the literature on ordinal pattern analysis, while Chapter Three will expand on this concept by exploring the characterization of time series. It will also cover the computation of two key descriptors entropy and complexity from the resulting histograms. Additionally, Chapter Three will outline the main ideas and objectives of this research project.

Chapter 2

Literature Review

The analysis of complex time series has long relied on both time-domain and frequency-domain techniques. However, traditional methods often fall short in capturing nonlinear dynamics or are limited by strict assumptions such as stationarity and Gaussianity. The ordinal symbolic approach introduced by Bandt and Pompe in 2002 marked a significant theoretical advance by enabling robust, model-free characterization of time series. Their approach, rooted in information theory, involves converting segments of time series data into symbols based on the ordinal (rank) relationships among the data points. These symbols are called ″ordinal patterns.″ After computing all the symbols, their relative frequencies are used to estimate the probability distribution of ordinal patterns. From this distribution, two key descriptors entropy and complexity are calculated to characterize the time series. The scaled Shannon entropy calculated from the distribution of these patterns, now widely known as permutation entropy, has become a central tool in nonlinear data analysis. To complement entropy which quantifies randomness, theoretical statistical complexity was introduced by López-Ruiz et al. [28] to capture the underlying structure of a system. By plotting permutation entropy against statistical complexity, Rosso et al. [45] use the entropy-complexity plane as a powerful diagnostic tool to distinguish between different dynamical regimes, such as chaos, noise, and periodicity.

This literature review discusses the theoretical foundations and applications of the Bandt-Pompe methodology, as well as recent developments and challenges outlined in contemporary works, including the reviews by Amigó et al. [1] and Leyva et al. [22].

2.1 Theoretical Foundations of the Bandt-Pompe Methodology

Bandt and Pompe [6] introduced in 2002 a novel approach to time series analysis by focusing on the ordinal relationships between data points rather than their actual values. This method involves mapping a time series into a sequence of symbols representing the relative ordering of values within embedding vectors. The authors computed the scaled Shannon entropy (the permutation entropy) of the frequency distribution of these patterns, and used it as a measure of the signal complexity. Since the symbols are invariant under monotonic transformations and robust to observational noise, so is any feature derived from them. This approach requires minimal assumptions about the data, it is computationally efficient, and effective for short time series, making it suitable for a wide range of applications [59].

A further theoretical development by López-Ruiz et al. [28] extended method by introducing the statistical complexity, which measures both the system's randomness and structure. Later, Rosso et al. [45] extended and applied statistical complexity in the entropy-complexity plane framework, Plotting the permutation entropy against the statistical complexity yields the entropy-complexity plane, a diagnostic tool for classifying different dynamical regimes.

2.2 Applications and Advances

The Bandt-Pompe framework has since found broad applications across domains, as reviewed by Amigó et al. [1]. This section is divided into three subsections that analyze the applications of the Bandt and Pompe framework across various domains.

On 21 February 2025, we collected all Scopus-indexed references that cited the seminal work by Bandt and Pompe. On that date, they were 3986. We marked them into categories based on application fields, such as biomedical signal processing, geophysics and hydrology, econophysics, optical systems and engineering.

2.2.1 Biomedical Signal Processing

In biomedical signal processing, it has been used to analyze EEG, ECG, and fMRI data, detecting anomalies such as epileptic seizures or sleep stage transitions.

- **EEG Analysis:** A study by Keller et al. [17] demonstrated the application of ordinal pattern based entropy measures, such as empirical permutation entropy (ePE), empirical conditional entropy (eCE), and robust empirical permutation entropy (rePE) to effectively analyze and classify EEG time series data, demonstrating their utility in detecting brain state transitions, segmenting non-stationary signals, and identifying change-points without relying on prior expert knowledge.

- **ECG Analysis:** Mansourian et al. [29] applied adaptive improved permutation entropy to extract fetal QRS complexes from single-channel abdominal ECG, enhancing the accuracy of fetal heart rate monitoring.

2.2.2 Geophysics and Hydrology

In the fields of geophysics and hydrology, PE has been applied to detect climatic variability and hydrological patterns.

- **Climate Variability:**
- **Hydrological Patterns:**

2.2.3 Econophysics, Optical Systems, and Engineering

Permutation entropy has found applications in econophysics for market behavior analysis, in optical systems for detecting chaos, and in engineering for fault detection and diagnostics.

- **Econophysics:**
- **Optical Systems:**
- **Engineering:**

2.3 Methodological Extensions

Recent developments have extended the methodology through multi-scale approaches, cross-entropy comparisons for multivariate signals, and weighted ordinal patterns.

Furthermore, the use of alternative entropy measures such as Tsallis and Renyi entropy has been explored to better capture non-extensive and multifractal behavior in complex systems.

Recent developments have extended the methodology through multi-scale approaches, cross-entropy comparisons for multivariate signals, and weighted ordinal patterns. These innovations aim to enhance the sensitivity of ordinal analysis across multiple temporal resolutions, account for signal interdependencies, and assign importance to certain ordinal patterns based on domain-specific knowledge.

Furthermore, the use of alternative entropy measures such as Tsallis and Rényi entropy has been explored to better capture non-extensive and multifractal behavior in complex systems, especially where the assumptions of Shannon entropy may not hold.

In addition to methodological enhancements, attention has also been given to the statistical properties of entropy and complexity estimates. One key focus is the estimation of confidence intervals for descriptors such as permutation entropy and statistical complexity. These intervals provide a quantifiable measure of uncertainty, which is crucial for distinguishing significant patterns from noise, validating comparisons between different systems, and assessing the stability of observed dynamics.

Incorporating statistical inference technique, such as asymptotic variance estimation, improves the robustness and interpretability of the results. These tools enable researchers to make reliable conclusions even in cases of short time series, thereby reinforcing the utility of ordinal methods in practical applications.

Statistical properties!

2.4 Challenges and Future Directions

Despite its success, several challenges remain in the application and development of the Bandt-Pompe approach. One major issue is the choice of embedding parameters (dimension and delay), which significantly affect the pattern distribution. Additionally, estimating entropy and complexity reliably in short time series remains difficult, prompting the need for improved statistical inference methods such as bootstrapping and confidence interval estimation.

Amigó et al. [1] emphasize the need for more robust inferential frameworks, better handling of multivariate and high-dimensional data, and integrating ordinal methods with machine learning for automated pattern recognition. As ordinal symbolic analysis continues to evolve, it holds promise for deeper insights into both theoretical dynamics and real-world systems.

As a final conclusion related to this literature review, the Bandt-Pompe method has become a foundational tool in time series analysis, offering a robust and intuitive approach to understanding complex dynamics. Through the use of ordinal patterns and entropy-based descriptors, researchers can extract meaningful information from noisy, nonlinear, and non-stationary data. While challenges remain, ongoing theoretical advances and applications across disciplines ensure the continued relevance and growth of this methodology.

Chapter 3

The Research Project

In this chapter, we outline the main ideas and objectives of this research project. Section 3.1 discusses entropy and complexity analysis in time series, highlighting its advantages an d limitations. Section 3.2 examines the advantages and limitations of the Bandt and Pompe method. Section 3.3 provides background knowledge on the entropy-complexity plane. Section 3.4 explores the entropy-complexity plane for a broad class of time series. Section 3.5 provides the asymptotic distribution of the entropy. Finally, the chapter concludes with the objectives of the research project and a case study related to our work.

3.1 Entropy and Complexity Analysis in Time Series: Advantages and Limitations

Entropy and complexity analysis in time series provides powerful tools for measuring the unpredictability and structural richness of dynamical systems, which means the systems that evolve in time. These methods help describe the behavior of a system using mathematical models. Entropy measures, such as Shannon entropy (which quantifies the uncertainty in a probability distribution) and permutation entropy (which measures the complexity of the order structure in time series using ordinal patterns), are used to quantify the degree of randomness or disorder in data. The major difference between Shannon entropy and permutation entropy is that permutation entropy is the Shannon entropy computed from the ordinal patterns (permutations) extracted from a time series. The complexity measures assess the balance between order and chaos. Entropy and complexity measures are powerful tools for identifying nonlinear patterns in time series data, and together, they are particularly valuable for distinguishing between deterministic and stochastic behavior by revealing hidden structures and irregular dynamics that traditional linear methods often overlook. While entropy and complexity analysis offers powerful insights into nonlinearity and chaos, capturing patterns that traditional linear methods often miss, it requires careful preprocessing, precise parameter tuning, and a solid understanding of the system's domain to avoid misleading conclusions. Non-linearity refers to relationships within the data where small changes in input can lead to disproportionately large or unpredictable changes in output, often seen in complex real-world systems such as biological signals or financial markets. Choosing the right parameters, such as embedding dimension and time delay, is crucial for accurately capturing the underlying dynamics, and without domain knowledge, interpreting the results and identifying meaningful patterns can be challenging. Despite these limitations, entropy and complexity remain essential in modern time series analysis for uncovering hidden dynamics beyond the reach of traditional linear methods. Linear methods, such as auto-correlation, linear regression, and Fourier analysis, assume that relationships within data are proportional and predictable, often focusing on averaged behavior, periodicity, or stationary patterns. However, many real-world systems (like the brain, heart, or climate) display nonlinear behavior, where the output does not change in a simple, direct way with the input. Entropy and complexity measures are specifically designed to capture these irregularities, revealing subtle structures, transient changes, or chaotic patterns that linear tools often overlook or misinterpret. This makes them invaluable for exploring complex, dynamic, and nonlinear systems where traditional approaches fall short.

3.2 The Bandt and Pompe Method: A Robust Approach

The concept of ordinal patterns in time series can be effectively studied through real world examples. Traditionally, numerous algorithms, techniques, and heuristics have been employed to estimate complexity measures from real world data.

However, these methods often perform well only for low-dimensional dynamical systems and struggle when noise is introduced. Low-dimensional dynamical systems are systems whose behavior can be described using a small number of variables or equations, typically two or three, such as the logistic map, or pendulum. These systems exhibit rich and often chaotic dynamics but remain mathematically tractable and easier to analyze using entropy and complexity measures. Because of their limited dimensionality, the patterns within the data are more distinct, making it easier to extract meaningful information.

The Bandt and Pompe method overcomes this limitation by providing a robust approach that remains reliable even in noisy environments. In time series analysis, key complexity parameters such as entropy, fractal dimension, and Lyapunov exponents play a crucial role in comparing neighboring values and uncovering the underlying structure and dynamics of the data. A Lyapunov exponent measures the average rate at which nearby trajectories in a dynamical system diverge or converge. It provides deeper understanding of system's behavior.

The advantages of Bandt & Pompe methods:

- Simplicity

- Extremely fast calculation
- Robustness
- Invariance to nonlinear monotonous transformations

This method exhibits low sensitivity to noise and naturally accounts for the causal order of elements in a time series. As a result, it can be applied to various real-world problems, particularly in differentiating between chaotic and stochastic signals.

Despite its limitations, researchers have developed extensions to the original method to address its shortcomings and enhance its applicability to a broader range of complex systems.

3.3 Statistical Complexity measures

Bandt and Pompe introduced a highly effective method for analyzing time series within this framework. They calculated Shannon entropy based on the histogram of causal patterns and successfully identified chaotic components in sequences of words, among other applications.

Later, Rosso et al. [45] expanded this analysis by introducing an additional dimension: the statistical complexity derived from the same histogram of causal patterns. The authors have contributed to a wide range of applications. This approach, which utilizes the entropy-complexity plane, has been successfully applied to the visualization and characterization of different dynamical regimes as system parameters change [5, 10, 13, 18, 19, 46, 62, 63], as well as to optical chaos [27, 52, 54, 58, 61], hydrology [21, 50, 53], geophysics [12, 48, 51], engineering [3, 4, 39, 57], biometrics [47], characterization of pseudo-random number generators [14, 15], biomedical signal analysis [23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 37, 38, 59],and econophysics [7, 8, 9, 59, 60, 64, 65], to name a few.

After computing all symbols as described in Chapter 1, the histogram proportions are used to estimate the probability distribution of ordinal patterns. From this distribution, two key descriptors are calculated to characterize the time series:

1. Entropy
2. Statistical complexity

The most common metric for the first descriptor is the normalized Shannon entropy, defined as:

$$H(\mathbf{p}) = -\frac{1}{\log k} \sum_{\ell=1}^{k} p_\ell \ln p_\ell. \tag{3.1}$$

Here, $k = D!$ represents the total number of possible permutation patterns. This entropy is bounded within the unit interval:

- It reaches its minimum value $(H = 0)$ when a single pattern dominates for some $p_\ell = 1$ for some ℓ
- It achieves its maximum $(H = 1)$ under uniform probability $p_\ell = 1/k$ for all ℓ.

This normalized entropy is often termed permutation entropy in time series analysis.

While normalized Shannon entropy is a powerful tool for quantifying disorder, it fails to fully characterize complex dynamics. To address this limitation, López-Ruiz et al. [28] introduced the disequilibrium Q concept, which quantifies the deviation of a probability distribution $\mathbf{p}$ from a uniform (non-informative) equilibrium state. López-Ruiz and the team employed the Euclidean distance between $\mathbf{p}$ and the uniform distribution, providing a complementary metric to Shannon entropy for assessing structural complexity in systems.

The Jensen-Shannon distance between histogram of proportion $\mathbf{p}$ and the uniform probability function $\mathbf{u} = (1/k, 1/k, \ldots, 1/k)$, where $k = D!$ corresponds to the number of possible permutation patterns provides a robust metric for quantifying deviations from uniformity. This distance measure, derived from the symmetric Jensen-Shannon divergence, is particularly suited for analyzing ordinal pattern distributions due to its ability to capture both structural differences and statistical equilibrium in time series data. It is defined as:

$$Q'(\mathbf{p}, \mathbf{u}) = \sum_{\ell=1}^{k} p_\ell \log \frac{p_\ell}{u_\ell} + u_\ell \log \frac{u_\ell}{p_\ell}. \tag{3.2}$$

Lamberti et al. [20] proposed Jensen-Shannon distance as a symmetric metric rooted in the Jensen-Shannon divergence. As the reference model, most works consider the uniform distribution $\mathbf{u} = (1/k, 1/k, \dots, 1/k)$. The normalized disequilibrium is defined as follows

$$Q = \frac{Q'}{\max(Q')}, \tag{3.3}$$

where $\max(Q')$ is defined as follows

$$\max(Q') = -2\left[\frac{k+1}{k}\log(k+1) - 2\log(2k) + \log k\right]. \tag{3.4}$$

With this, Lamberti et al. [20] proposed complexity as a measure of the statistical complexity of the underlying dynamics, which is defined as

$$C = HQ, \tag{3.5}$$

where both H and Q are normalized quantities, therefore C is also normalized.

3.4 The Entropy Complexity Plane

The entropy-complexity plane is a two-dimensional representation where time series are mapped based on their entropy and statistical complexity. These metrics are derived from ordinal pattern distributions obtained through embedding dimension D that are mapped on histograms of $D!$ bins.

3.4.1 Key Dynamics in the plane

1. Highly Ordered Systems, where the behavior is very predictable, structured, and often repeats in a regular pattern over time.

   Example: Strictly monotonic time series.

   - Produces a single ordinal pattern ($H = 0$).
   - Maximal disequilibrium (distance from uniform distribution).
   - Maps to $(0, 0)$, indicating minimal complexity.

2. Perfectly Random Systems

   Example: White noise

   - Uniform ordinal pattern distribution ($H = 1$).
   - Disequilibrium vanishes (distance $= 0$).
   - Maps to $(1, 0)$, reflecting maximal entropy without structural complexity.

The two extreme values are proved by Anteneodo & Plastino [2]. Expressions for the boundaries, derived using geometrical arguments within space configurations, were proposed by Martin et al. [30]. These formulations provide a structured approach to understanding and analyzing the spatial behavior of specific systems or models. The lower boundary is characterized by a smooth curve, whereas the upper boundary consists of $D! - 1$ distinct segments. As the embedding dimension D approaches infinity, the upper boundary gradually converges into a smooth curve. Example for the entropy complexity plane is shown in Figure 3.1

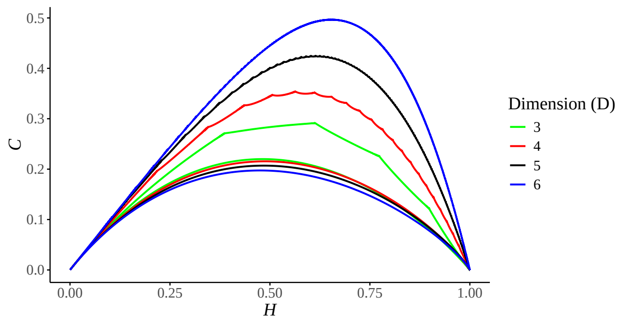

Figure 3.1: Entropy Complexity Plane for Embedding dimension 3, 4,5, and 6

3.5 Asymptotic Distribution of the Shannon Entropy under the Multinomial Model

The Multinomial distribution describes how observations fall into categories when an adequate model is available. It is similar to the Multivariate normal distribution, which is one of the continuous Multivariate distributions. Furthermore, it has received considerable attention from researchers, both in theoretical studies and in applications related to discrete Multivariate distributions. The normalized Shannon entropy, often employed in applications like permutation entropy, can be rigorously connected to its asymptotic distribution through the lens of statistical estimation theory. When estimating entropy from finite data, the plug-in estimator (computed directly from observed frequencies) converges to a normal distribution as the sample size $N \longrightarrow \infty$ even for dependent processes such as Markov chains. The Statistical properties of entropy measures under Multinomial distributions are crucial for analyzing complex systems where entropy serves as a key descriptors. Rey at al [43] investigate the asymptotic distributions of various entropy measures specifically, the Rényi and Tsallis entropies of order q, as well as Fisher information when these are computed using maximum likelihood estimators of probabilities from Multinomial random samples. The authors demonstrate that the Tsallis entropy and Fisher information asymptotically follow a normal distribution, whereas the Rényi entropy does not exhibit asymptotic normality. Through simulation studies, the paper validates that these asymptotic models effectively describe a variety of data scenarios. Additionally, the study introduces test statistics for comparing different types of entropies derived from two samples, even when the samples have differing numbers of categories. An application of these tests to social survey data indicates that the results are consistent and offer a more general approach compared to traditional chi-squared tests.

In a subsequent study, Rey et al [41] focus on the statistical complexity measure defined as the product of normalized Shannon entropy and the Normalized Jensen-Shannon divergence between a given probability distribution and the uniform distribution. They derive the asymptotic distribution of this complexity measure under the assumption that the observed data follow a Multinomial distribution. Further the study demonstrates that, as the sample size increases, the distribution of the statistical complexity converges to a normal distribution, with its variance and bias determined by the dynamics of the underlying system. This result provides a theoretical foundation for using statistical complexity as a tool for analyzing systems where the probability distributions are estimated from finite samples. The results are validated with theoretical findings through numerical experiments, showing that the asymptotic normality holds even in scenarios where the Multinomial model is not strictly applicable, such as in applications involving Bandt and Pompe ordinal patterns.

Crucially, the convergence rate and limiting distribution depend on the system's correlation structure, which deviates from the standard Multinomial case. However, they remain tractable through spectral analysis, which involves examining the eigenvalues and eigenvectors of the transition matrix or the distributions of ordinal patterns [11, 44]. This connection underscores the reliability of normalized entropy measures in large data regimes while highlighting the need to account for dependence structures in finite sample applications.

Imagine a sequence of n independent trials, each resulting in precisely one outcome from a set of k distinct possibilities labeled $\pi^1, \pi^2, \ldots, \pi^k$ and so on. These outcomes are mutually exclusive, meaning only one can occur per trial, with respective probabilities $\boldsymbol{p} = \{p_1, p_2, \ldots, p_k\}$, such that $p_\ell \geq 0$ and $\sum_{\ell=1}^{k} p_\ell = 1$. The random vector $\boldsymbol{N} = (N_1, N_2, \ldots, N_k)$ counts the number of occurrences of the events $\pi^1, \pi^2, \ldots, \pi^k$ in the n trials, with $N_\ell \geq 0$ and $\sum_{\ell=1}^{k} N_\ell = n$. A $\boldsymbol{n}$ is a sample from $\boldsymbol{N}$ and it has a $k-$variate vector of integer values $\boldsymbol{n} = (n_1, n_2, \ldots, n_k)$. Then the joint distribution of $\boldsymbol{N}$ is

$$Pr(\boldsymbol{N} = \boldsymbol{n}) = Pr(N_1 = n_1, N_2 = n_2, \ldots, N_k = n_k) = n! \prod_{\ell=1}^{k} \frac{{p_\ell}^{n_\ell}}{n_\ell!}. \tag{3.6}$$

This situation is denoted as $\boldsymbol{N} \sim \text{Mult}(n, \boldsymbol{p})$. [43]

In practical applications, the true probability distribution $\boldsymbol{p}$ governing a Multinomial system is typically unknown. Instead, estimators $\widehat{p}_l$, are derived empirically by calculating the observed frequency of each event π^l within the set of k possible outcomes $\boldsymbol{\pi} = \pi^1, \pi^2, \ldots, \pi^k$ across n independent trials. These frequencies approximate the underlying probabilities, enabling inference about the system's behavior. This maximum likelihood estimator (MLE) aligns with the empirical estimator derived from first-moment matching of the distribution. Due to its consistency, asymptotic normality, and computational tractability under regularity conditions, it remains the predominant choice in applied statistical modeling.

Shannon entropy quantifies the level of disorder within a system. When the system's behavior is entirely predictable, the Shannon entropy reaches its minimum, indicating complete knowledge of future observations. Conversely, when the system follows a uniform distribution where all possible outcomes have equal probability—the entropy is maximized, reflecting minimal knowledge about the system's behavior. Chagas et al. [11] have analyzed the asymptotic distribution of Shannon entropy in their study.

Moreover, other types of descriptors, such as Rényi entropy[40], Tsallis entropy[56], and Fisher information [16], have been proposed to extract additional information that is not captured by Shannon entropy. From these entropy measures, Fisher information has garnered more attention due to its unique properties. Fisher information is defined as the average logarithmic derivative of a continuous probability density function.

For discrete probability distributions, Fisher information can be approximated by calculating the differences between probabilities of consecutive distribution elements. A key distinction between Shannon entropy and Fisher information lies in their focus: Shannon entropy quantifies the overall unpredictability of a system, while Fisher information measures the rate of change between consecutive observations, making it more sensitive to small changes and perturbations.

The following equations define Tsallis entropy $(H_T^q(\widehat{\boldsymbol{p}}))$, Rényi entropy $(H_R^q(\widehat{\boldsymbol{p}}))$, and Fisher information measures $(H_F(\widehat{\boldsymbol{p}}))$ [49] :

$$H_T^q(\widehat{\boldsymbol{p}}) = \sum_{\ell=1}^{k} \frac{\widehat{p_\ell} - \widehat{p_\ell}^q}{q-1}, \tag{3.7}$$

where the index $q \in \mathbb{R}\backslash\{1\}$

$$H_R^q(\widehat{\boldsymbol{p}}) = \frac{1}{1-q} \log \sum_{\ell=1}^{k} \widehat{p_\ell}^q, \tag{3.8}$$

where the index $q \in \mathbb{R}^+\backslash\{1\}$

$$H_F(\widehat{\boldsymbol{p}}) = F_0 \sum_{\ell=1}^{k-1} (\sqrt{\widehat{p}_{\ell+1}} - \sqrt{\widehat{p_\ell}})^2, \tag{3.9}$$

where the re-normalization coefficient is $F_0 = 4$ [49]

Rey et al. [43] investigated the asymptotic distribution of several entropy measures, including Shannon, Tsallis, Rényi entropy, and Fisher information, and provided the following formulation:

Let $\boldsymbol{Z} \sim N(\boldsymbol{\mu}, \sum)$, be a $k-$dimensional multivariate normal distribution with mean vector $\mu \in \mathbb{R}^k$ and covariance matrix $\sum = (\sigma_{\ell j})$. Then, for any

$\boldsymbol{a} \in \mathbb{R}^k$, the linear combination $W = \boldsymbol{a}^T \boldsymbol{Z}$, is normally distributed as:

$$W \sim N(\boldsymbol{a}^T \boldsymbol{\mu}, \sum_{\ell=1}^{k} a_\ell^2 \sigma_{\ell\ell} + 2 \sum_{j=1}^{k-1} \sum_{\ell=j+1}^{k} a_\ell a_j \sigma_{\ell j}). \tag{3.10}$$

The estimated Shannon entropy is defined as:

$$H_s(\widehat{\boldsymbol{p}}) = -\sum_{\ell=1}^{k} \widehat{p_\ell} \log \widehat{p_\ell}. \tag{3.11}$$

The asymptotic variance $\widehat{\sigma}^2$ of the entropy estimator is given by:

$$\widehat{\sigma}^2 = \frac{1}{n} \sum_{\ell=1}^{k} p_\ell (1 - p_\ell)(\log p_\ell + 1)^2 - \frac{2}{n} \sum_{j=1}^{k-1} \sum_{\ell=j+1}^{k} p_\ell p_j (\log p_\ell + 1)(\log p_j + 1). \tag{3.12}$$

These formulas serve as the basis for the subsequent case study in our research.

3.6 Case Study of Asymptotic Distribution of the Complexity

Statistical complexity is defined as the product of two normalized quantities:

- The Shannon entropy
- The Jensen-Shannon distance between the observed probability distribution and the uniform distribution

In this section we discuss three key aspects with real world scenario:

1. **Significance of Asymptotic Complexity Distributions**: Why understanding large-sample behavior matters for statistical inference
2. **Multinomial Model Framework**: Derivation of the asymptotic distribution for statistical complexity under Multinomial assumptions

3. **Practical Formula**: A working equation for calculating the asymptotic distribution of complexity

As a case study for our work, we consider data from the Bearing Data Center and the seeded fault test data from Case Western Reserve University, School of Engineering. The datasets includes ball bearing test data for normal bearings as well as single-point defects on the fan end and drive end. Data were collected at a rate of $48,000(48k$ drive-end) data points per second during bearing tests. Each file contains motor loads $(0, 1, 2$, and $3)$, drive-end vibration data, and fan-end vibration data. The approximate motor speeds in RPM during testing: $1797, 1772, 1750$, and 1730. For our case study, we consider two time series (Normal Baseline and $48k$ Drive-End) with a motor load of 0 and an RPM of 1797.

The primary objective of this study is to detect malfunctioning machinery by analyzing two time series using ordinal patterns. We introduce a distance metric based on the ordinal structure of the segments to quantify similarity. This metric facilitates the identification of faulty machines across various embedding dimensions, ranging from 3 to 6. For this case study, we employ an embedding dimension of 3 for convenience; subsequent analyses will extend to the remaining dimensions to compare results. Permutation entropy under asymptotic conditions is computed by considering the probability distribution of ordinal patterns. The results are further analyzed using the complexity–entropy plane, providing insights into the system's dynamics.

Initially, we analyzed the complete datasets from two time series such as one comprising $250,000$ data points representing the normal baseline at motor load 0, and another containing $2,540,000$ data points from the $48k$ drive end under the same motor load. We computed entropy and complexity measures fro these entire datasets, with the results presented in the Table 3.1

Entropy	Complexity	Std Deviation	Semi Length
0.665235	0.226447	0.358893	0.000441
0.772973	0.170954	0.324376	0.001287

Table 3.1: Entropy Complexity Results

Subsequently, we segmented the data into batches of $10,000$ points, categorizing them as either 'Normal' or '48k Drive End'. We then performed a batch wise comparison of entropy and complexity metrics to identify fault data segments. The normal dataset comprises 25 batches, all corresponding to motor load 0, while the $48k$ drive end dataset includes 254 batches. Due to the extensive volume of entropy and complexity data generated, the complete results table is not included in this report. However, the entropy–complexity plane effectively illustrates both batch-wise and full-data analyses. As depicted in Figure 3.2 below, faulty machines form a distinct cluster in the entropy–complexity plane, highlighting their deviation from normal operational patterns. It is clear from the graph that there are both overlapping and non-overlapping confidence intervals. This indicates that some machines differ significantly, while others do not. The main purpose of our experiment is to identify faulty machines. Therefore, we highly recommend extending these results by increasing the embedding dimension to better understand the final outcomes. The general framework of this experiment is also provided in this chapter to clarify the main objective of the research.

The general framework for analyzing entropy-complexity planes with confidence intervals are given as follows.

1. **Calculate Entropy (H) and Complexity (C):** appropriate estimator are Shannon entropy, statistical complexity measures

2. **Compute Confidence Intervals:** Generate multiple resampled datasets to estimate the variance of H and C.

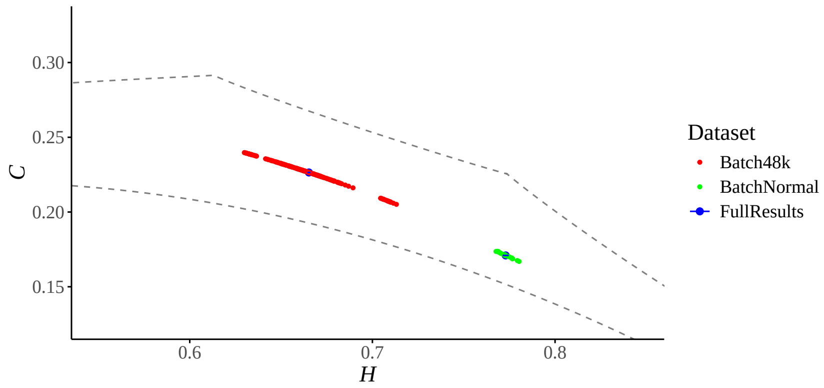

Figure 3.2: Entropy Complexity Plane

3. **Plot on Entropy-Complexity Plane:**

    - Axes:x-axis: Entropy (H), y-axis: Statistical complexity (C)
    - Data Points: Plot individual or aggregated results.
    - Confidence Regions: Represent uncertainty

4. **Interpretation**

Region of Plane	Interpretation
High H and High C	Complex, structured systems
Low H and Low C	Simple, predictable systems
High H and Low C	Random/noisy systems
Low H and High C	Non-random systems

5. **Statistical testing:**

    - Compare confidence intervals between groups to assess significant differences.
    - Overlapping intervals $\rightarrow$ No significant difference.
    - Non-overlapping intervals $\rightarrow$ Potential significance.

Chapter 4

Future Works

As a short summary, we have completed the necessary preliminaries studies on various topics such as:

- Entropy
- Complexity
- Entropy Complexity Plane
- Confidence interval
- Multinomial distribution.

Also we have examined the relevant research articles by Bandt & Pompe [6], and statistical properties of the entropy from ordinal patterns, asymptotic distribution of certain types of entropy under the Multinomial law, and the asymptotic distribution of the permutation entropy studied by Rey et.al [41, 42, 43].

This proposal has three objectives in order to continue this research work.

- Define a data base of time series for clustering, i.e., finding similar time series.

- Extract all the features we know from their Bandt & Pompe symbolization (Shannon, Tsallis and Renyi entropies, Fisher information measure, complexities, and the available confidence intervals)
- Use those features for time series clustering

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