Ordinal Pattern Features: Statistical Properties and Their Applications in Time Series Clustering

Rasika Dilhani

Supervisor: Alejandro C. Frery

Submitted in partial fullfilment of the requirements for
Doctor of Philosophy
in Data Science.

Victoria University of Wellington
2025

Abstract

Time series analysis plays a vital role in understanding the underlying dynamics of complex systems across various domains such as engineering, economics, and the physical sciences. Traditional approaches, namely time-domain and frequency-domain methods often rely on strong assumptions such as stationarity, large sample sizes, or normality, which are frequently violated in real-world data. As a robust alternative, Bandt and Pompe [15] introduced a non-parametric method based on ordinal pattern symbolization and information theoretic descriptors. This approach transforms local segments of the time series into rank-based symbols (the ordinal patterns), constructs a histogram of ordinal patterns, and computes Shannon entropy, offering resistance to noise and model independence. In addition to the foundational work by Bandt and Pompe, we refer to the contributions of López-Ruiz et al. [51], who introduced the concept of statistical complexity; Lamberti et al. [41], who implemented López-Ruiz's idea using the Euclidean distance; Rosso et al. [73], who proposed the joint representation of entropy and complexity in the entropy-complexity plane; and Martin et al. [53], who established the theoretical boundaries of the generalized statistical complexity measure. This proposal investigates the use of various entropy measures, such as Shannon, Tsallis, and Rényi entropy, as well as the Fisher Information Measure, statistical complexity, and associated confidence intervals for time series analysis clustering.

Contents

Chapter 1

Introduction

Time series contain valuable insights about the underlying system that generates the data. Their analysis is typically conducted using two primary approaches: time-domain and transformed-domain methods. In the context of time-domain analysis, Bandt and Pompe [15] introduced a novel methodology that is non-parametric and rooted in information theory descriptors: Ordinal Patterns symbolization.

Bandt and Pompe [15] proposed transforming small subsets of the time series observations into symbols that encode the sorting properties of the values in these subsets. Then, they computed a histogram of those symbols. The resulting distribution is less sensitive to outliers compared to the original data, and the histogram is independent of any specific model. The proposal proceeds by computing descriptors from this histogram, and extracting information about the system from these descriptors. As a result, this approach is versatile and applicable to a wide range of scenarios.

The proposal focuses on the statistical properties of features from ordinal patterns in time series clustering. It involves calculating pattern histograms, entropy, complexity, and confidence intervals to better understand the statistical properties of these tools. Additionally, the role of confidence intervals in entropy and complexity, along with their applications in time-series clustering, will be explored. Future work will expand to include alternative measures, such as Rényi entropy, Tsallis entropy and Fisher information, with a focus on deriving confidence intervals for their entropy and complexity under the Multinomial model.

Time series analysis is widely applied across various fields, including engineering, economics, physical sciences, and more. A time series is defined as a collection of observations x_t, each representing a realized value of a particular random variable X_t, where time can be either discrete or continuous.

Examples of time series applications include finance (e.g., analyzing exchange rate movements or commodity prices), biology (e.g., modeling the growth and decline of bacterial populations), medicine (e.g., tracking the spread of diseases like COVID-19 or influenza), and geoscience (e.g., predicting wet or dry days based on past weather conditions).

The primary goal of time series analysis is to understand the nature of the phenomenon represented by the observed sequence. Time domain and frequency domain methods are the two primary approaches used in time series analysis. The temporal approach relies on concepts such as auto-correlation and regressions, where a time series' present value is analyzed in relation to its own past values or the past values of other series. This method represents time series directly as a function of time. On the other hand, the spectral approach represents time series through spectral expansions, such as wavelets or Fourier modes [88].

However, these methods often require assumptions such as large sample sizes or normally distributed observations that are rarely met in real-world empirical data. For many statistical techniques to be valid, these assumptions must hold, but in practice, they are frequently violated.

For example, traditional approaches to time series analysis, such as time domain and frequency domain methods, rely on assumptions that are not always valid in real-world data. The time domain approach, which uses techniques like auto-correlation and regression, assumes stationary and often struggles with nonlinear or non-stationary data. Similarly, the frequency domain approach, which represents time series through spectral expansions such as wavelets or Fourier modes, may require assumptions about periodicity and may not effectively capture short-term fluctuations.

Many statistical methods in these approaches depend on specific conditions, such as large sample sizes or normally distributed observations. However, these assumptions are often unrealistic, leading to inaccurate or biased results. When such conditions are not met, alternative methods must be considered.

As a result, alternative methods, commonly known as non-parametric techniques, are often considered. These approaches do not rely on the actual numerical values of the observations x_t, but rather on their ranks R_t, making them more robust and less sensitive to outliers and applicable to a wide range of data sets. Since non-parametric tests do not assume any specific distribution, such as normality, they are considered highly reliable for a range of data types.

However, while these techniques are powerful for general statistical analysis, they are not always well-suited for time series data.

To address these challenges, ordinal pattern methods provide a robust alternative. Instead of analyzing the absolute values of a time series, these methods focus on the order relationships among consecutive data points.

This approach effectively captures the underlying dynamics of complex systems and offers several advantages.

The ordinal pattern-based method has become a widely used tool for characterizing complex time series. Since its introduction nearly twenty-three years ago by Bandt and Pompe in their foundational paper [15], it has been successfully applied across various scientific fields, including biomedical signal processing, optical chaos, hydrology, geophysics, econophysics, engineering, and biometrics. It has also been used in the characterization of pseudo-random number generators.

The Bandt and Pompe method successfully analyzes time series by transforming them into ordinal patterns, constructing a histogram, and computing Shannon entropy, making it robust against outliers and independent of predefined models.

1.1 Introduction to Ordinal Pattern Analysis

Ordinal patterns are a non-parametric representation of real-valued time series by transforming small subsets of observations into symbols based on their relative order, rather than looking into actual values. This approach maps each segment of the time series in $\mathbb{R}^D$ into a finite set of $D!$ distinct symbols, where D represents for the "Embedding Dimension" and usually ranges between three to six. One of the possible encoding is the set of indexes that sort the D values in non-decreasing order.

To illustrate this idea, let $\boldsymbol{x} = \{x_1, x_2, \dots, x_{n+D-1}\}$ be a real valued time series of length $n + D - 1$ without ties. The corresponding symbol sequence naturally emerges from the time series without requiring any model assumptions. We compute $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_n)$ symbols from subsequences of embedding dimension D. There are $D!$ possible symbols: $\pi_j \in \{\pi^1, \pi^2, \dots, \pi^{D!}\}$, for each $1 \leq j \leq n$. The histogram of proportions $h = (h_1, h_2, \dots, h_{D!})$ in which the bin h_ℓ is the proportion of symbols of type π^ℓ of the total number of symbols. For convenience, we will model those symbols as a k dimensional random vector where $k = D!$.

1.2 Problem Statement

To illustrate this concept, imagine tracking the mean monthly humidity in Wellington. You want to analyze how humidity changes throughout the year. By examining this data, you can uncover interesting patterns that highlight the variations in humidity across different months.

The mean monthly humidity in Wellington is shown in Figure 1.1.

We can convert this actual data into ordinal patterns. To do this, for each month, we determine the order of the humidity values rather than

Month	Mean of relative humidity
January	77.3
February	81.0
March	82.4
April	81.7
May	83.6
June	85.6
July	84.4
August	83.1
September	78.8
October	79.6
November	78.2
December	78.8

Table 1.1: Mean monthly humidity variations in Wellington throughout the year

their actual magnitudes. Each three-time-point sequence (which can be adjusted based on preference) is converted into an ordinal pattern. This "embedding dimension" usually varies between 3 and 6, but any dimension is possible. The conversion can be made in any way that uniquely maps the sorting properties of the sub sequence into a symbol. For example, consider the time series presented in Table 1.1. We can transform this series into ordinal patterns as follows. Assume we use patterns of length $D = 3$. The first overlapping window $(77.3, 81, 82.4)$ corresponds to the pattern $(1, 2, 3)$, where type is considering with the order of the real time series data. Here 77.3 is the smallest value and is assigned rank 1; 81 is the next highest and is assigned rank 2; and 82.4 is the largest, assigned rank 3. As another example, consider the overlapping window $(83.1, 78.8, 79.6)$. In this case, 78.8 is the smallest value and is assigned rank 1; 79.6 is the next highest, assigned rank 2; and 83.1 is the largest, assigned rank 3. Therefore, the

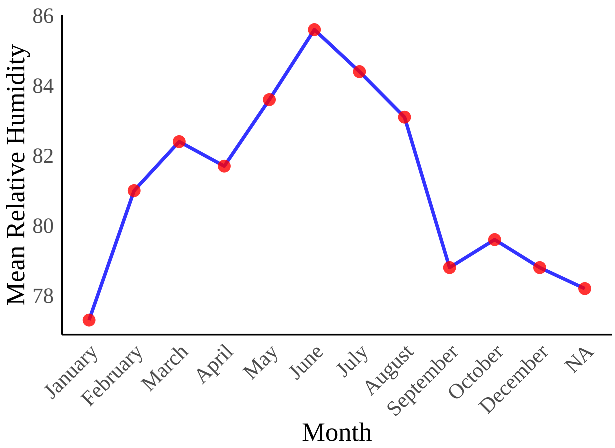

Figure 1.1: Mean Monthly humidity in Wellington

pattern for this window is $(3, 1, 2)$. Table 1.2 has shown this scenario.

t	Mean Humidity sequence	Ordinal Pattern
1	(77.3,81,82.4)	(123) $= \pi^1$
2	(81,82.4,81.7)	(132) $= \pi^2$
3	(82.4,81.7,83.6)	(213) $= \pi^3$
4	(81.7,83.6,85.6)	(123) $= \pi^1$
5	(83.6,85.6,84.4)	(132) $= \pi^2$
6	(85.6,84.4,83.1)	(321) $= \pi^6$
7	(84.4,83.1,78.8)	(321) $= \pi^6$
8	(83.1,78.8,79.6)	(312) $= \pi^5$
9	(78.8,79.6,78.2)	(231) $= \pi^4$
10	(79.6,78.2,78.8)	(312) $= \pi^5$

Table 1.2: Ordinal Patterns

As shown in Table 1.2, we have six mutually exclusive events which we denote as $\{\pi^1, \pi^2, \ldots, \pi^6\} = \{(123), (132), (213), (231), (312), (321)\}$. The probability distribution of the mean humidity is calculated based on ordinal patterns as given below.

$$\hat{p}_i = \frac{\#\{\pi_j \in \boldsymbol{\pi} : \pi_j = \pi^i\}}{n}; 1 \leq i \leq 6, \tag{1.1}$$

where $\hat{\boldsymbol{p}} = (\hat{p}_1, \ldots, \hat{p}_6)$.

Notation	Probability
$p(\pi^1)$	$\frac{2}{10}$
$p(\pi^2)$	$\frac{2}{10}$
$p(\pi^3)$	$\frac{1}{10}$
$p(\pi^4)$	$\frac{1}{10}$
$p(\pi^5)$	$\frac{2}{10}$
$p(\pi^6)$	$\frac{2}{10}$

Table 1.3: Probability function

We construct the histogram of proportions $h = (h_1, h_2, h_3, h_4, h_5, h_6)$, where each bin h_ℓ represents the proportion of symbols of type π^ℓ out of the total six symbols. The histogram graph is shown Figure 1.2.

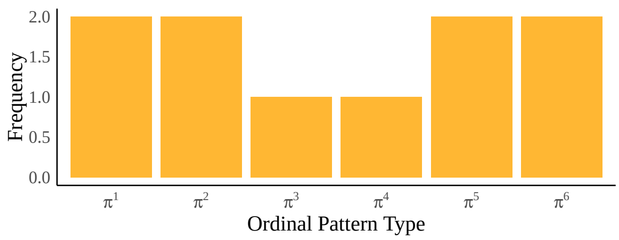

Figure 1.2: Histogram of proportions of the observed patterns according to Table 1.3.

This example explains how time series data can be converted into ordinal patterns and how the probability distribution function can be calculated from these patterns. Chapter 2 will expand on this concept by exploring the characterization of time series and cover the computation of two key descriptors entropy and complexity from the resulting histograms. Chapter 3 will review the literature on ordinal pattern analysis. Additionally, Chapter 4 will outline the main ideas and objectives of this research project.

Chapter 2

The Research Project

In this chapter, we outline the main ideas and objectives of this research project. Section 2.1 discusses entropy and complexity analysis in time series, highlighting its advantages and limitations. Section 2.2 examines the advantages and limitations of the Bandt and Pompe method. Section 2.3 provides background knowledge on the entropy-complexity plane. Section 2.4 explores the entropy-complexity plane for a broad class of time series. Section 2.5 provides the asymptotic distribution of the entropy. Finally, the chapter concludes with the objectives of the research project and a case study related to our work.

2.1 Entropy and Complexity Analysis in Time Series: Advantages and Limitations

Entropy and complexity analysis in time series provides powerful tools for measuring the unpredictability and structural richness of dynamical systems, which means the systems that evolve in time. These methods help describe the behavior of a system using mathematical models. Entropy measures, such as Shannon entropy (which quantifies the uncertainty in a probability distribution) and permutation entropy (which measures the complexity of the order structure in time series using ordinal patterns), are used to quantify the degree of randomness or disorder in data. The major difference between Shannon entropy and permutation entropy is that permutation entropy is the Shannon entropy computed from the ordinal patterns (permutations) extracted from a time series. The complexity measures assess the balance between order and chaos. Entropy and complexity measures are powerful tools for identifying nonlinear patterns in time series data, and together, they are particularly valuable for distinguishing between deterministic and stochastic behavior by revealing hidden structures and irregular dynamics that traditional linear methods often overlook. While entropy and complexity analysis offers powerful insights into nonlinearity and chaos, capturing patterns that traditional linear methods often miss, it requires careful preprocessing, precise parameter tuning, and a solid understanding of the system's domain to avoid misleading conclusions. Non-linearity refers to relationships within the data where small changes in input can lead to disproportionately large or unpredictable changes in output, often seen in complex real-world systems such as biological signals or financial markets. Choosing the right parameters, such as embedding dimension and time delay, is crucial for accurately capturing the underlying dynamics, and without domain knowledge, interpreting the results and identifying meaningful patterns can be challenging. Despite these limitations, entropy and complexity remain essential in modern time series analysis for uncovering hidden dynamics beyond the reach of traditional linear methods. Linear methods, such as auto-correlation, linear regression, and Fourier analysis, assume that relationships within data are proportional and predictable, often focusing on averaged behavior, periodicity, or stationary patterns. However, many real-world systems (like the brain, heart, or climate) display nonlinear behavior, where the output does not change in a simple, direct way with the input. Entropy and complexity measures are specifically designed to capture these irregularities, revealing subtle structures, transient changes, or chaotic patterns that linear tools often overlook or misinterpret. This makes them invaluable for exploring complex, dynamic, and nonlinear systems where traditional approaches fall short.

2.2 The Bandt and Pompe Method: A Robust Approach

The concept of ordinal patterns in time series can be effectively studied through real world examples. Traditionally, numerous algorithms, techniques, and heuristics have been employed to estimate complexity measures from real world data.

However, these methods often perform well only for low-dimensional dynamical systems and struggle when noise is introduced. Low-dimensional dynamical systems are systems whose behavior can be described using a small number of variables or equations, typically two or three, such as the logistic map, or pendulum. These systems exhibit rich and often chaotic dynamics but remain mathematically tractable and easier to analyze using entropy and complexity measures. Because of their limited dimensionality, the patterns within the data are more distinct, making it easier to extract meaningful information.

The Bandt and Pompe method overcomes this limitation by providing a robust approach that remains reliable even in noisy environments. In time series analysis, key complexity parameters such as entropy, fractal dimension, and Lyapunov exponents play a crucial role in comparing neighboring values and uncovering the underlying structure and dynamics of the data. A Lyapunov exponent measures the average rate at which nearby trajectories in a dynamical system diverge or converge. It provides deeper understanding of system's behavior.

The advantages of Bandt & Pompe methods:

- Simplicity

- Extremely fast calculation
- Robustness
- Invariance to nonlinear monotonous transformations

This method exhibits low sensitivity to noise and naturally accounts for the causal order of elements in a time series. As a result, it can be applied to various real-world problems, particularly in differentiating between chaotic and stochastic signals.

Despite its limitations, researchers have developed extensions to the original method to address its shortcomings and enhance its applicability to a broader range of complex systems.

2.3 Statistical Complexity measures

Bandt and Pompe introduced a highly effective method for analyzing time series within this framework. They calculated Shannon entropy based on the histogram of causal patterns and successfully identified chaotic components in sequences of words, among other applications.

Later, Rosso et al. [73] expanded this analysis by introducing an additional dimension: the statistical complexity derived from the same histogram of causal patterns. The authors have contributed to a wide range of applications. This approach, which utilizes the entropy-complexity plane, has been successfully applied to the visualization and characterization of different dynamical regimes as system parameters change [14, 22, 28, 38, 39, 72, 102, 103], as well as to optical chaos [49, 85, 87, 95, 101], hydrology [42, 80, 86], geophysics [27, 77, 84], engineering [9, 10, 65, 93], biometrics [74], characterization of pseudo-random number generators [29, 30], biomedical signal analysis [44, 46, 47, 48, 57, 58, 59, 60, 61, 62, 63, 64, 97],and econophysics [17, 18, 19, 97, 100, 104, 105], to name a few.

After computing all symbols as described in Chapter 1, the histogram proportions are used to estimate the probability distribution of ordinal patterns. From this distribution, two key descriptors are calculated to characterize the time series:

1. Entropy
2. Statistical complexity

The most common metric for the first descriptor is the normalized Shannon entropy, defined as:

$$H(\mathbf{p}) = -\frac{1}{\log k}\sum_{\ell=1}^{k} p_\ell \ln p_\ell. \tag{2.1}$$

Here, $k = D!$ represents the total number of possible permutation patterns. This entropy is bounded within the unit interval:

- It reaches its minimum value $(H = 0)$ when a single pattern dominates for some $p_\ell = 1$ for some ℓ
- It achieves its maximum $(H = 1)$ under uniform probability $p_\ell = 1/k$ for all ℓ.

This normalized entropy is often termed permutation entropy in time series analysis.

While normalized Shannon entropy is a powerful tool for quantifying disorder, it fails to fully characterize complex dynamics. To address this limitation, López-Ruiz et al. [51] introduced the disequilibrium Q concept, which quantifies the deviation of a probability distribution $\mathbf{p}$ from a uniform (non-informative) equilibrium state. López-Ruiz and the team employed the Euclidean distance between $\mathbf{p}$ and the uniform distribution, providing a complementary metric to Shannon entropy for assessing structural complexity in systems.

The Jensen-Shannon distance between histogram of proportion $\mathbf{p}$ and the uniform probability function $\mathbf{u} = (1/k, 1/k, \ldots, 1/k)$, where $k = D!$ corresponds to the number of possible permutation patterns provides a robust metric for quantifying deviations from uniformity. This distance measure, derived from the symmetric Jensen-Shannon divergence, is particularly suited for analyzing ordinal pattern distributions due to its ability to capture both structural differences and statistical disequilibrium in time series data. It is defined as:

$$Q'(\mathbf{p}, \mathbf{u}) = \sum_{\ell=1}^{k} p_\ell \log \frac{p_\ell}{u_\ell} + u_\ell \log \frac{u_\ell}{p_\ell}. \tag{2.2}$$

Lamberti et al. [41] proposed Jensen-Shannon distance as a symmetric metric rooted in the Jensen-Shannon divergence. As the reference model, most works consider the uniform distribution $\mathbf{u} = (1/k, 1/k, \dots, 1/k)$. The normalized disequilibrium is defined as follows

$$Q = \frac{Q'}{\max(Q')}, \tag{2.3}$$

where $\max(Q')$ is defined as follows

$$\max(Q') = -2\left[\frac{k+1}{k}\log(k+1) - 2\log(2k) + \log k\right]. \tag{2.4}$$

With this, Lamberti et al. [41] proposed complexity as a measure of the statistical complexity of the underlying dynamics, which is defined as

$$C = HQ, \tag{2.5}$$

where both H and Q are normalized quantities, therefore C is also normalized.

2.4 The Entropy Complexity Plane

The entropy-complexity plane is a two-dimensional representation where time series are mapped based on their entropy and statistical complexity. These metrics are derived from ordinal pattern distributions obtained through embedding dimension D that are mapped on histograms of $D!$ bins.

2.4.1 Key Dynamics in the plane

1. Highly Ordered Systems, where the behavior is very predictable, structured, and often repeats in a regular pattern over time.

   Example: Strictly monotonic time series.

   - Produces a single ordinal pattern ($H = 0$).
   - Maximal disequilibrium (distance from uniform distribution).
   - Maps to $(0, 0)$, indicating minimal complexity.

2. Perfectly Random Systems

   Example: White noise

   - Uniform ordinal pattern distribution ($H = 1$).
   - Disequilibrium vanishes (distance $= 0$).
   - Maps to $(1, 0)$, reflecting maximal entropy without structural complexity.

The two extreme values are proved by Anteneodo & Plastino [8]. Expressions for the boundaries, derived using geometrical arguments within space configurations, were proposed by Martin et al. [53]. These formulations provide a structured approach to understanding and analyzing the spatial behavior of specific systems or models. The lower boundary is characterized by a smooth curve, whereas the upper boundary consists of $D! - 1$ distinct segments. As the embedding dimension D approaches infinity, the upper boundary gradually converges into a smooth curve. Example for the entropy complexity plane is shown in Figure 2.1

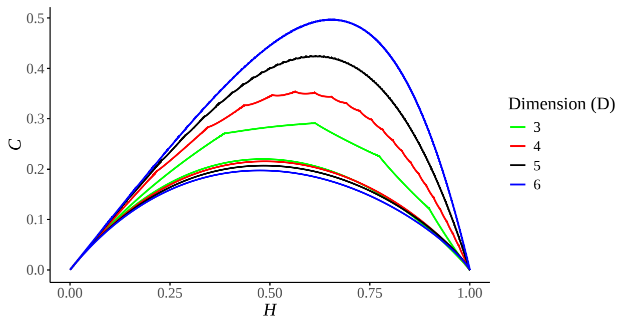

Figure 2.1: Entropy Complexity Plane for Embedding dimension 3, 4,5, and 6

2.5 Asymptotic Distribution of the Shannon Entropy under the Multinomial Model

The Multinomial distribution describes how observations fall into categories when an adequate model is available. It is similar to the Multivariate normal distribution, which is one of the continuous Multivariate distributions. Furthermore, it has received considerable attention from researchers, both in theoretical studies and in applications related to discrete Multivariate distributions. The normalized Shannon entropy, often employed in applications like permutation entropy, can be rigorously connected to its asymptotic distribution through the lens of statistical estimation theory. When estimating entropy from finite data, the plug-in estimator (computed directly from observed frequencies) converges to a normal distribution as the sample size $N \longrightarrow \infty$ even for dependent processes such as Markov chains. The Statistical properties of entropy measures under Multinomial distributions are crucial for analyzing complex systems where entropy serves as a key descriptors. Rey at al [70] investigate the asymptotic distributions of various entropy measures specifically, the Rényi and Tsallis entropies of order q, as well as Fisher information when these are computed using maximum likelihood estimators of probabilities from Multinomial random samples. The authors demonstrate that the Tsallis entropy and Fisher information asymptotically follow a normal distribution, whereas the Rényi entropy does not exhibit asymptotic normality. Through simulation studies, the paper validates that these asymptotic models effectively describe a variety of data scenarios. Additionally, the study introduces test statistics for comparing different types of entropies derived from two samples, even when the samples have differing numbers of categories. An application of these tests to social survey data indicates that the results are consistent and offer a more general approach compared to traditional chi-squared tests.

In a subsequent study, Rey et al [67] focus on the statistical complexity measure defined as the product of normalized Shannon entropy and the Normalized Jensen-Shannon divergence between a given probability distribution and the uniform distribution. They derive the asymptotic distribution of this complexity measure under the assumption that the observed data follow a Multinomial distribution. Further the study demonstrates that, as the sample size increases, the distribution of the statistical complexity converges to a normal distribution, with its variance and bias determined by the dynamics of the underlying system. This result provides a theoretical foundation for using statistical complexity as a tool for analyzing systems where the probability distributions are estimated from finite samples. The results are validated with theoretical findings through numerical experiments, showing that the asymptotic normality holds even in scenarios where the Multinomial model is not strictly applicable, such as in applications involving Bandt and Pompe ordinal patterns.

Crucially, the convergence rate and limiting distribution depend on the system's correlation structure, which deviates from the standard Multinomial case. However, they remain tractable through spectral analysis, which involves examining the eigenvalues and eigenvectors of the transition matrix or the distributions of ordinal patterns [23, 71]. This connection underscores the reliability of normalized entropy measures in large data regimes while highlighting the need to account for dependence structures in finite sample applications.

Imagine a sequence of n independent trials, each resulting in precisely one outcome from a set of k distinct possibilities labeled $\pi^1, \pi^2, \ldots, \pi^k$ and so on. These outcomes are mutually exclusive, meaning only one can occur per trial, with respective probabilities $\mathbf{p} = \{p_1, p_2, \ldots, p_k\}$, such that $p_\ell \geq 0$ and $\sum_{\ell=1}^{k} p_\ell = 1$. The random vector $\mathbf{N} = (N_1, N_2, \ldots, N_k)$ counts the number of occurrences of the events $\pi^1, \pi^2, \ldots, \pi^k$ in the n trials, with $N_\ell \geq 0$ and $\sum_{\ell=1}^{k} N_\ell = n$. A $\mathbf{n}$ is a sample from $\mathbf{N}$ and it has a $k-$variate vector of integer values $\mathbf{n} = (n_1, n_2, \ldots, n_k)$. Then the joint distribution of $\mathbf{N}$ is

$$Pr(\mathbf{N} = \mathbf{n}) = Pr(N_1 = n_1, N_2 = n_2, \ldots, N_k = n_k) = n! \prod_{\ell=1}^{k} \frac{{p_\ell}^{n_\ell}}{n_\ell!}. \quad (2.6)$$

This situation is denoted as $\mathbf{N} \sim \text{Mult}(n, \mathbf{p})$. [70]

In practical applications, the true probability distribution $\mathbf{p}$ governing a Multinomial system is typically unknown. Instead, estimators $\widehat{p}_\ell$, are derived empirically by calculating the observed frequency of each event π^l within the set of k possible outcomes $\pi = \pi^1, \pi^2, \ldots, \pi^k$ across n independent trials. These frequencies approximate the underlying probabilities, enabling inference about the system's behavior. This maximum likelihood estimator (MLE) aligns with the empirical estimator derived from first-moment matching of the distribution. Due to its consistency, asymptotic normality, and computational tractability under regularity conditions, it remains the predominant choice in applied statistical modeling.

Shannon entropy quantifies the level of disorder within a system. When the system's behavior is entirely predictable, the Shannon entropy reaches its minimum, indicating complete knowledge of future observations. Conversely, when the system follows a uniform distribution where all possible outcomes have equal probability, the entropy is maximized, reflecting minimal knowledge about the system's behavior. Chagas et al. [23] have analyzed the asymptotic distribution of Shannon entropy in their study.

Moreover, other types of descriptors, such as Rényi entropy[66], Tsallis entropy[89], and Fisher information [33], have been proposed to extract additional information that is not captured by Shannon entropy. From these entropy measures, Fisher information has garnered more attention due to its unique properties. Fisher information is defined as the average logarithmic derivative of a continuous probability density function.

For discrete probability distributions, Fisher information can be approximated by calculating the differences between probabilities of consecutive distribution elements. A key distinction between Shannon entropy and Fisher information lies in their focus: Shannon entropy quantifies the overall unpredictability of a system, while Fisher information measures the rate of change between consecutive observations, making it more sensitive to small changes and perturbations.

The following equations define Tsallis entropy ($H_T^q(\widehat{\mathbf{p}})$), Rényi entropy ($H_R^q(\widehat{\mathbf{p}})$), and Fisher information measures ($H_F(\widehat{\mathbf{p}})$) [79] :

$$H_T^q(\widehat{\mathbf{p}}) = \sum_{\ell=1}^{k} \frac{\widehat{p}_\ell - \widehat{p}_\ell{}^q}{q-1}, \tag{2.7}$$

where the index $q \in \mathbb{R}\backslash\{1\}$

$$H_R^q(\widehat{\mathbf{p}}) = \frac{1}{1-q} \log \sum_{\ell=1}^{k} \widehat{p}_\ell{}^q, \tag{2.8}$$

where the index $q \in \mathbb{R}^+\backslash\{1\}$

$$H_F(\widehat{\mathbf{p}}) = F_0 \sum_{\ell=1}^{k-1} (\sqrt{\widehat{p}_{\ell+1}} - \sqrt{\widehat{p}_\ell})^2, \tag{2.9}$$

where the re-normalization coefficient is $F_0 = 4$ [79]

Rey et al. [70] investigated the asymptotic distribution of several entropy measures, including Shannon, Tsallis, Rényi entropy, and Fisher information, and provided the following formulation:

Let $\mathbf{Z} \sim N(\mu, \sum)$, be a $k-$dimensional multivariate normal distribution with mean vector $\mu \in \mathbb{R}^k$ and covariance matrix $\sum = (\sigma_{\ell j})$. Then, for any

$\mathbf{a} \in \mathbb{R}^k$, the linear combination $W = \mathbf{a}^T\mathbf{Z}$, is normally distributed as:

$$W \sim N(\mathbf{a}^T\boldsymbol{\mu}, \sum_{\ell=1}^{k} a_\ell^2\sigma_{\ell\ell} + 2\sum_{j=1}^{k-1}\sum_{\ell=j+1}^{k} a_\ell a_j \sigma_{\ell j}). \tag{2.10}$$

The estimated Shannon entropy is defined as:

$$H_s(\widehat{\mathbf{p}}) = -\sum_{\ell=1}^{k} \widehat{p_\ell} \log \widehat{p_\ell}. \tag{2.11}$$

The asymptotic variance $\widehat{\sigma}^2$ of the entropy estimator is given by:

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{\ell=1}^{k} p_\ell(1-p_\ell)(\log p_\ell + 1)^2 - \frac{2}{n}\sum_{j=1}^{k-1}\sum_{\ell=j+1}^{k} p_\ell p_j(\log p_\ell + 1)(\log p_j + 1). \tag{2.12}$$

These formulas serve as the basis for the subsequent case study in our research.

2.6 Asymptotic distribution of Permutation Entropy

As we discussed earlier, real-valued time series $\mathbf{x} = \{x_1, x_2, ..., x_{n+D-1}\}$ transform into the series symbols $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ from sub-sequences of embedding dimension D, where we considered $D! = k$. Due to the overlapping of time windows, the ordinal patterns which we calculate are dependent. For $i = 1, 2, ...k$, let p_i be the probability of observing the state π_i, denote the vector probabilities, $\mathbf{p} = \{p_1, p_2, \ldots, p_k\}$ and express as $\mathbf{D_p} = \text{Diag}(p_1, p_2, \ldots, p_k)$ the diagonal matrix. The transition probability of reaching state π_j at time $t+\ell$ from the state π_i at time t, for $\ell = 1, 2, \ldots, D-1$, is denoted by $p_{ij}^{(\ell)}$. These transition probabilities can be collected in the matrix $\mathbf{Q}^{(\ell)}$ whose elements are $p_{ij}^{(\ell)}$. As describe in the Rey et al. [69] when n is sufficiently large, asymptotic variance of the Shannon entropy defined as follows.

$$
\begin{aligned}
\sigma_{\mathbf{p}}^2 = {} & \sum_{i=1}^{k} (\ln p_i + 1)^2 \left[p_i - (2D-1)p_i^2 + 2\sum_{\ell=1}^{D-1} \mathbf{Q}_{ii}^{(\ell)} \right] \\
& - 2\sum_{i=1}^{k-1} \sum_{j=i+1}^{k} (\ln p_i + 1)(\ln p_j + 1) \\
& \times \left[(2D-1)p_i p_j - \sum_{\ell=1}^{D-1} \left(\mathbf{Q}_{ij}^{(\ell)} + \mathbf{Q}_{ji}^{(\ell)} \right) \right].
\end{aligned} \tag{2.13}
$$

Asymptotic variance of the Shannon entropy of ordinal patterns considering their correlation structure is given in Equation 2.13. For practical purpose, given n is sufficiently large, we can use $H(\mathbf{p})$ has a normal distribution with mean equal to $H(\widehat{\mathbf{p}})$ and variance equal to $\sigma_{\mathbf{p}}^2/n$.

In addition to that, for $\alpha \in (0,1)$ and n sufficiently large, the $(1-\alpha)100\,\%$ confidence interval of $H(\mathbf{p})$ is given by,

$$
H(\widehat{\mathbf{p}}) \pm \frac{Z_{\alpha/2}\sigma_{\mathbf{p}}}{\sqrt{n}}, \tag{2.14}
$$

where $Z_{\alpha/2}$ is the $\alpha/2$-quantile of a standard normal random variable.

For convenience, Equation 2.14 is referred to and defined as follows.

$$
H(\widehat{\mathbf{p}}) \pm \textbf{Semi Length}, \tag{2.15}
$$

where **Semi Length** $= \dfrac{Z_{\alpha/2}\sigma_{\mathbf{p}}}{\sqrt{n}}$

2.7 Case Study of Asymptotic Distribution of the Shannon Entropy

Statistical complexity is defined as the product of two normalized quantities:

- The Shannon entropy,

- The Jensen-Shannon distance between the observed probability distribution and the uniform distribution.

In this section we discuss two key aspects with real world scenario:

1. **Significance of Asymptotic Distributions**: Why understanding large-sample behavior matters for statistical inference,
2. **Practical Formula**: A working equation for calculating the asymptotic distribution of complexity.

As a case study for our work, we consider data from the Bearing Data Center and the seeded fault test data from Case Western Reserve University, School of Engineering. The datasets includes ball bearing test data for normal bearings as well as single-point defects on the fan end and drive end. Data were collected at a rate of $48,000(48k$ drive-end) data points per second during bearing tests. Each file contains motor loads $(0, 1, 2,$ and $3)$, drive-end vibration data, and fan-end vibration data. The approximate motor speeds in RPM during testing: $1797, 1772, 1750,$ and 1730. For our case study, we consider two time series (Normal Baseline and $48k$ Drive-End) with a motor load of 0 and an RPM of 1797.

The primary objective of this study is to detect malfunctioning machinery by analyzing two time series using ordinal patterns. We introduce a distance metric based on the ordinal structure of the segments to quantify similarity. This metric facilitates the identification of faulty machines across various embedding dimensions, ranging from 3 to 6. For this case study, we employ an embedding dimension of 3 for convenience; subsequent analyses will extend to the remaining dimensions to compare results. Permutation entropy under asymptotic conditions is computed by considering the probability distribution of ordinal patterns. The results are further analyzed using the complexity–entropy plane, providing insights into the system's dynamics.

Initially, we analyzed complete datasets from two time series: one comprising 250,000 data points representing the normal baseline at motor load 0, and another containing 2,540,000 data points from the 48k drive end under the same motor load. We computed the entropy and complexity measures for these entire datasets, followed by the calculation of the asymptotic variance as defined in Section 2.6. This asymptotic variance was then used to determine the confidence interval for entropy (Equation is defined in 2.14). The calculation of the semi-length of the interval is given by Equation 2.15. The final results are presented in Table 2.1.

Entropy	Complexity	$\sigma_{\boldsymbol{p}}$	Semi Length
0.665235	0.226447	0.358893	0.000441
0.772973	0.170954	0.324376	0.001287

Table 2.1: Entropy Complexity Results

Subsequently, we segmented the data into batches of $10,000$ points, categorizing them as either 'Normal' or '48k Drive End'. We then performed a batch wise comparison of entropy and complexity metrics to identify fault data segments. The normal dataset comprises 25 batches, all corresponding to motor load 0, while the $48k$ drive end dataset includes 254 batches. Due to the extensive volume of entropy and complexity data generated, the complete results table is not included in this report. However, the entropy–complexity plane effectively illustrates both batch-wise and full-data analyses. As depicted in Figure 2.2 below, faulty machines form a distinct cluster in the entropy–complexity plane, highlighting their deviation from normal operational patterns. It is clear from the graph that there are both overlapping and non-overlapping confidence intervals. This indicates that some machines differ significantly, while others do not. The main purpose of our experiment is to identify faulty machines. Therefore, we highly recommend extending these results by increasing the embedding dimension to better understand the final outcomes. The general framework of this experiment is also provided in this chapter to clarify the main objective of the research.

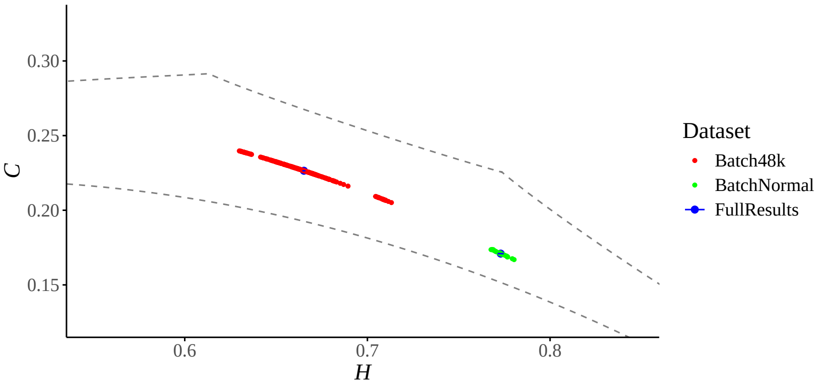

Figure 2.2: Entropy Complexity Plane

The general framework for analyzing entropy-complexity planes with confidence intervals are given as follows.

1. **Calculate Entropy (H) and Complexity (C):** appropriate estimator are Shannon entropy, statistical complexity measures
2. **Compute Confidence Intervals:** Generate multiple resampled datasets to estimate the variance of H and C.
3. **Plot on Entropy-Complexity Plane:**
   - Axes:x-axis: Entropy (H), y-axis: Statistical complexity (C)
   - Data Points: Plot individual or aggregated results.
   - Confidence Regions: Represent uncertainty
4. **Interpretation**

Region of Plane	Interpretation
High H and High C	Complex, structured systems
Low H and Low C	Simple, predictable systems
High H and Low C	Random/noisy systems
Low H and High C	Non-random systems

5. **Statistical testing:**
    - Compare confidence intervals between groups to assess significant differences.
    - Overlapping intervals $\rightarrow$ No significant difference.
    - Non-overlapping intervals $\rightarrow$ Potential significance.

Chapter 3

Literature Review

The analysis of complex time series has long relied on both time-domain and frequency-domain techniques. However, traditional methods often fall short in capturing nonlinear dynamics or are limited by strict assumptions such as stationarity and Gaussianity.

The ordinal symbolic approach introduced by Bandt and Pompe in 2002 marked a significant theoretical advance by enabling robust, model-free characterization of time series. Their approach, rooted in information theory, involves converting segments of time series data into symbols based on the ordinal (rank) relationships among the data points. These symbols are called ”ordinal patterns.” After computing all the symbols, their relative frequencies are used to estimate the probability distribution of ordinal patterns.

From this distribution estimate, two key descriptors entropy and complexity are calculated to characterize the time series: the scaled Shannon entropy, now widely known as permutation entropy, and the statistical complexity.

This chapter is divided into three main sections. Section 3.1 presents a brief overview of the area, focusing on what we consider the four seminal papers. Section 3.2 discusses the research question and the motivation for conducting the bibliometric analysis. The final section, Section 3.3, highlights the importance of bibliometric analysis and presents the results obtained from references that cite the Bandt and Pompe methodology and other related topics based on our research focus.

3.1 The Onset of the Entropy-Complexity Plane

This section discusses the emergence of the entropy-complexity plane. This topic is presented as a central theme because it reflects the foundation of our main research focus and illustrates how it has evolved over time into the current approach to time series analysis based on the concept of ordinal patterns.

López-Ruiz et al. [51] to capture the structure of a system: the product between the entropy and a distance between the estimated model and a non-informative model is an interesting way of measuring complexity. Lamberti et al. [41], using that idea, proposed using the Euclidean distance between the measured probability function and the uniform distribution. Rosso et al. [73] discussed using other distances, proposed the Jensen-Shannon distance, and used it jointly with the scaled Shannon entropy to form a bivariate feature. They mapped this feature into the so-called "Entropy-Complexity Plane," devising a powerful diagnostic tool to distinguish between different dynamical regimes, such as chaos, noise, and periodicity. Further, Martin et.al. [53] discussed the boundaries of this generalized statistical complexity measure.

In the following, we present the research question and the motivation for continuing this research.

3.2 Research Question and Motivation

The primary research question guiding this study is:

> *How can confidence intervals for generalized entropy measures (Shan-*

non, Tsallis, Rényi, Fisher information measure) and their associated complexity metrics be used to improve the robustness and discriminative power of time series clustering techniques?

We are motivated to conduct a literature review to confirm the relevance of our research areas in relation to the research question. Our aim is to determine whether other researchers are engaging with similar types of questions. Additionally, we seek to verify whether there is a strong focus on practical applications within this topic.

3.3 The Bibliometric Analysis: data collection, tools and background

Bibliometric analyses provide a quantitative approach to reviewing and mapping the intellectual structure of a research field. By systematically analyzing citation patterns, author collaborations, and keyword co-occurrences, they help identify key themes, research trends, and emerging topics. This method ensures objectivity, reveals key contributions, and offers a structured overview of intellectual development, making it a valuable tool for systematic literature reviews.

In this study, we conducted a bibliometric analysis using the Bibliometrix package in R and its user-friendly web interface Biblioshiny [11] focusing on literature related to ordinal patterns, permutation entropy, and complexity measures in time series analysis.

Section 3.3.1 discusses the data extraction process using the Bibliometrix package in R.

3.3.1 Conceptual Structure: Data extraction and Summary Statistics

Scopus-indexed references that cited the seminal work by Bandt and Pompe, along with other references relevant to our research topic, were collected on June 9, 2025. Based on these reference files, we analyzed a dataset consisting of 4125 reference files spanning the years 1993 to 2025. The descriptive analysis of the dataset revealed a total of 4063 usable documents (out of 4125; the others had missing data and were removed), sourced from 1317 publication sources. The dataset shows an annual growth rate of 18.15 %, involving 7254 authors, 123 single-authored documents, 27.32 % international co-authorship, with an average of 4.28 co-authors per document. The author keywords totaled 7667, with an average document age of 5.7 years, and 22.6 citations per document.

3.3.2 Thematic Map Analysis

Thematic map helps to understand the research direction and the relevant topics for the future studies. Therefore, we motivated to analyse it. The axes of the thematic map depicts the strength of their internal (density), which reflects inter-cluster growth, and external (connectivity) relevance or significance of the study in a particular area (centrality).

Figure 3.1 illustrates the thematic map derived from author keywords.

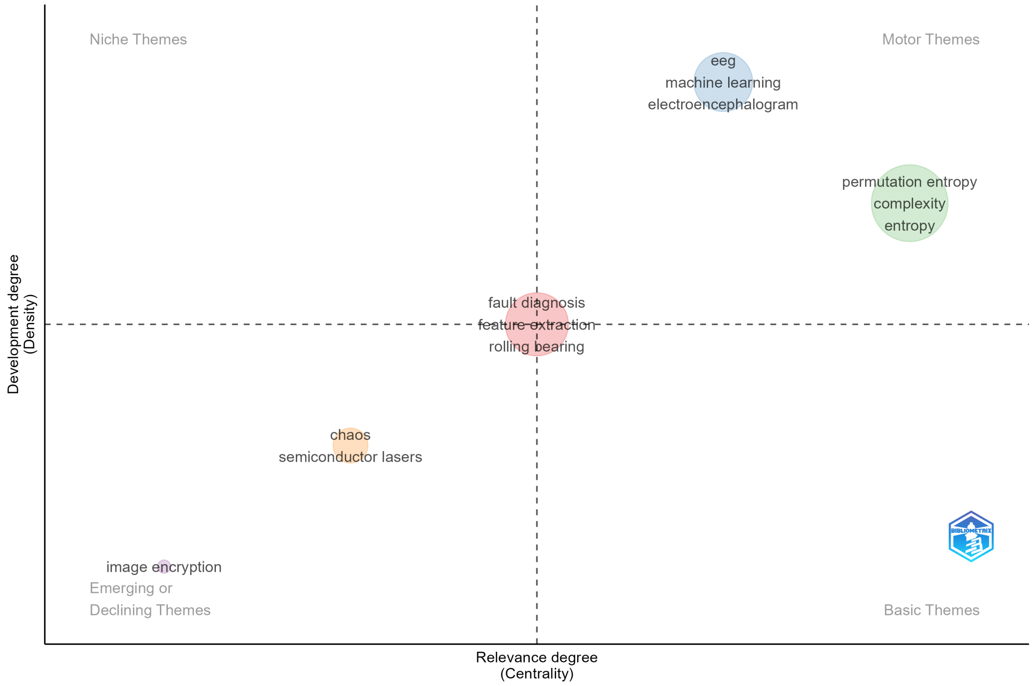

Figure 3.1: The Thematic Map generated by Bibliometrix.

The map is divided into four quadrants based on two dimensions: centrality (relevance) and density (development). A right upper quadrant representing motor themes that are both well developed and highly interconnected areas of research, such as EEG, machine learning, permutation entropy, complexity, and other types of entropy. This theme is also at the core of current research, particularly in areas such as biomedical signal processing and nonlinear time series analysis. The consistent presence of entropy-based measures and machine learning highlights the interface between theory and practice.

Emerging or declining themes like image encryption reflect peripheral or potentially declining research interests, while chaos and semiconductor lasers hold theoretical interest, but its practical integration appears limited.

A right down quadrant depicts basic themes indicates that opportunities for further theoretical and methodological advancement. Research fields such as fault diagnosis, feature extraction and rolling bearings are at the center of the map. Its moderate centrality and density mean that it is still active and is subject to evolving research fields. These topics are closely linked to engineering and diagnostic applications. These are important and growing areas which require further methodological refinement and integration.

The size of each circle further represents the frequency of the topic based on keyword occurrences associated with the publications. The clustering structure shows that entropy-related measures (such as the entropy of a permutation and the complexity of a system) are gaining ground not only in theory but also in practical applications such as EEG and machine learning.

From these results, we conclude that our research focus on entropy and complexity of permutation is relevant for many studies. These are basic concepts which are widely used in the literature, but which offer considerable potential for further development. Moreover, this thematic map shows that research is strongly focused on entropy-based methods, machine learning and biomedical applications, with fault diagnosis and feature extraction emerging as promising intermediate topics for further discuss.

3.3.3 Factorial Analysis

To identify the broad overview of the main research topics, we analyze the conceptual structure map. In this case we considered all keywords which are automatically generated by indexing databases. The shaded polygon in Figure 3.2 outlines the conceptual space defined by the most distinctive keywords. The X-axis (Dim 1) and the Y-axis (Dim 2) are the first two dimensions of the factorial space, and explain the largest differences in the co-occurrence of the keywords.

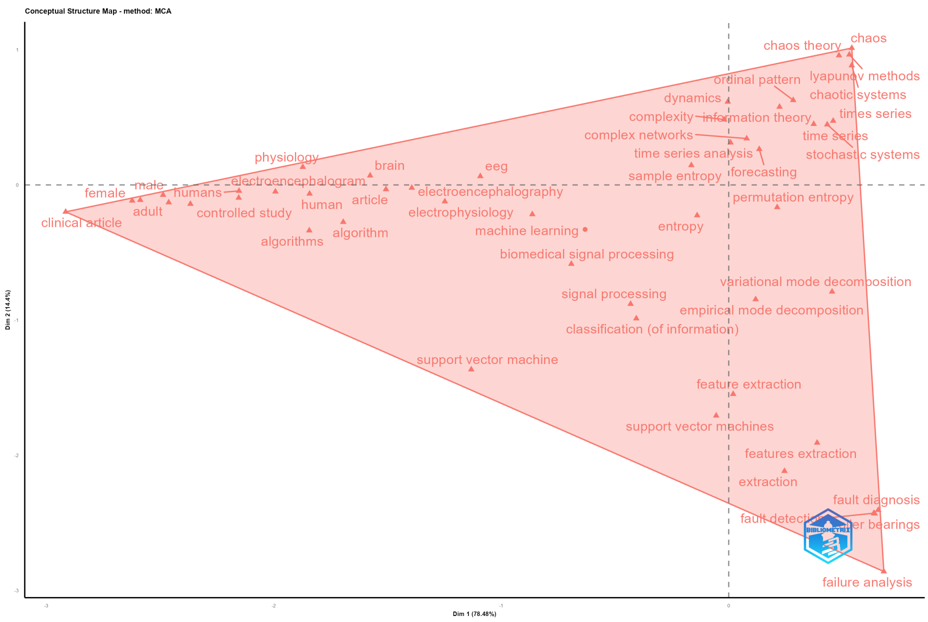

Figure 3.2: Conceptual Structure map generated by Bibliometrix.

The conceptual structure map reveals four major clusters:

The first cluster (upper right) includes theoretical concepts such as chaos theory, chaotic systems, Lyapunov methods, ordinal pattern, information theory, complex networks, and permutation entropy, forming the theoretical core of the research area.

The second cluster (lower right) includes practical applications such as feature extraction, empirical mode decomposition, variational mode decomposition, fault diagnosis, fault detection, and failure analysis, emphasizing the applied relevance of complexity-based time series analysis.

The third cluster (center) comprises terms related to biomedical signal processing and machine learning, including biomedical signal processing, EEG, support vector machine, classification, and machine learning, indicating the multidisciplinary applications of entropy measures.

A fourth, smaller cluster (left) includes clinical and physiological study keywords such as clinical article, controlled study, adult, male, female, humans, and physiology.

The conceptual map thus provides further evidence of the diverse applications and theoretical development surrounding ordinal patterns and complexity measures, supporting the originality and relevance of the present research.

3.3.4 Conclusion and Justification of Research Focus

Based on the thematic map and factorial analysis, research topics are categorized into five clusters and we will structure our review of the literature based on this clusters:

- Permutation entropy, Complexity and other types of Entropy (section 3.4);
- EEG and Machine learning approach (section 3.5);
- Fault diagnosis and failure analysis (section 3.6);
- Chaos other research works (section 3.7);

3.4 Permutation entropy, Complexity and other types of Entropy

This cluster is associated with the keywords time series analysis, statistical complexity, nonlinear dynamics, ordinal patterns, entropy, and information theory. Although ordinal pattern based research have gained wide recognition as powerful tools for nonlinear time series analysis, with applications ranging from biomedical signal processing to cyber-physical systems, their theoretical and practical challenges remain unsolved [37, 97].

Li et.al. [45] asses the complexity of short-term heartbeat interval series by using distribution entropy. They found that sample entropy (SampEn) or fuzzy entropy (FuzzyEn) quantifies essentially the randomness, which may not be uniformly identical to complexity. Zhang et.al. [98] uses the ordinal pattern technique to analyze dynamic behaviors such as regular, chaotic, or random patterns. The paper highlights the application of ordinal patterns in various fields, including ecology, finance, and physiology, where assessing variability and complexity is essential. It also emphasizes the advantages of using ordinal models to identify structural differences in time series that may not be detectable through traditional statistical methods.

Several studies have formalized the statistical behaviour of entropy and complexity measures derived from ordinal patterns, and have provided insights into their asymptotic distribution [68, 69] and sensitivity under the Multinomial law [70]. Despite these advances, the reliability of these measures in finite sampling conditions, in particular in non-stationary and noisy environments, remains limited [20].

Applications such as Internet of Things(IoT) botnet detection [20] and synthetic aperture radar(SAR) structure classification [24] have demonstrated the discriminability of ordinal-based features, particularly when combined with multiscale analysis; however, these methods often depend on the optimisation of parameters (dimension, delay) and may lack generalizability to a wide range of data. Similarly, the integration of entropy-complexity representations of class separation in time series dynamics [21], although effective in a structured environment, poses problems when extended to real-time or data-scarce scenarios.

The conceptual works linking ordinal complexity to broader ideas, such as the technological singularity [56] and the development of artistic expression [83] illustrate the richness of the framework, but these studies are often qualitative and lack empirical rigour.

Entropy-based clustering techniques have revealed evolving efficiency patterns in cryptocurrency markets [82]. The reliance on sensitive parameters and lack of standardized benchmarks highlight the need for more robust and interpretable methods to track market maturation reliably.

Moreover, white noise testing using entropy-complexity plane [25] and the use of ordinal properties in compressor signal diagnostics [16] demonstrate the methodological versatility of ordinal approaches. , the lack of uniform and reliable reference points prevents cross-domain comparison. Therefore, while ordinal methods provide a mathematically elegant and computationally efficient basis for obtaining information from complex signals, further research is needed to improve their statistical robustness, interpretability and adaptability to the challenges of the real world.

Although ordinal patterns are widely used in time series analysis, particularly in deriving entropy measures, studying their asymptotic distribution under the multinomial law, and analyzing the behavior of permutation entropy, their integration with confidence intervals remains widely unexplored. A systematic literature review reveals that while many studies investigate the statistical properties, asymptotic behavior, and robustness of ordinal pattern-based measures, no work has addressed the derivation or application of confidence intervals to assess the reliability or significance of time series clustering. This lack of statistical property limits interpretability and inference, especially when ordinal patterns are applied in data-driven algorithms. Therefore, this study aims to fill this gap by investigating how confidence intervals can be constructed for Shannon, Tsallis, and Renyi entropies, Fisher information measure, complexities, and how they may enhance time series clustering.

3.5 EEG and Machine learning approach

This cluster is based on multidisciplinary studies involving entropy measures applied to areas such as EEG, epilepsy, classification, heart rate variability, deep learning methods, and nonlinear analysis. An analysis of the author keywords indicates that many of these studies are centered on applications in biomedical signal processing. Acharya et al. [1, 2, 3, 4, 5, 6, 7] have made significant contributions to biomedical signal processing by developing and evaluating advanced automated diagnostic tools across a wide range of clinical applications. Their work includes the use of entropy measures to detect epilepsy and heart disease from EEG and ECG signals, as well as the use of nonlinear dynamics to enhance the detection of sleep stages and the characterisation of focal EEG signals. In the cardiovascular field, they have performed comparative studies on the localization of myocardial infarction using various ECG leads and have developed empirical decomposition methods to identify congestive heart failure from cardiac signals. Another study by Lajnef et.al. [40] revealed that an automated approach to the classification of sleep stages using a multi-class support vector machine (SVM) based decision tree approach. The proposed method uses physiological signals (such as EEG, EOG, and EMG) to effectively classify the different stages of sleep.

3.6 Fault diagnosis and failure analysis

This section primarily relates to the engineering applications of complexity-based time series analysis. The author keywords most commonly used to categorize this cluster include fault diagnosis, feature extraction, rolling bearing, support vector machine, variational mode decomposition, multiscale permutation entropy, dispersion entropy, rotating machinery, and empirical mode decomposition.

Recent studies in bearing fault diagnosis often use entropy-based methods. Multiscale permutation entropy (MPE) and dispersion entropy are two popular techniques. These methods help detect complex changes in signals under different working conditions. For example, using variational mode decomposition with weighted entropy features helps extract useful information from non-stationary vibration signals. This improves how well faults can be classified [43]. The weighted multiscale entropy method also works well by focusing on important frequency parts of the signal [55]. Self-adaptive hierarchical multiscale fuzzy entropy is also applied in bearing fault diagnosis. It makes fault detection easier without needing many manual settings [94]. Composite multiscale fluctuation dispersion entropy can detect small fault signs even in noisy signals [34]. Some methods combine data decomposition with multiscale permutation entropy to better handle complex, changing systems [96]. These improved entropy methods are also used in medical signal analysis, such as ECG or EEG, showing they work in other areas [13, 36]. Dispersion entropy is known for being fast and good at finding small signal changes [75, 76], whereas multiscale permutation entropy still has issues. It can be affected by the length of the signal, noise, and it can be slow to compute [36, 99]. Multiscale Permutation Entropy (MPE) with the Natural Visibility Graph (NVG) to enhance the fault diagnosis of rolling bearings by capturing both the dynamic complexity and structural features of time series method is proposed by Ma et.al. [52]. However, the method may still face limitations related to computational cost, parameter sensitivity, and the requirement for relatively long and noise-free signals to ensure reliable multiscale analysis.Therefore, more research is needed to make these methods faster, better with noise, and easier to use in different fault diagnosis tasks.

3.7 Chaos and Other research works

Research works related to chaos, semiconductor lasers, and image encryption are discussed in this category. A self-synchronous chaotic stream cipher, designed to resist active attacks and limit error propagation during image transmission, is a novel technique for image encryption. [32]. The 2D discrete wavelet transform, Arnold mapping, and a four-dimensional hyper-chaotic system with positive Lyapunov exponents are used to enhance the security and complexity of the encryption method. The advancement of chaos-based encryption and intelligent video security techniques in modern information systems has been demonstrated through a successful hardware implementation that transforms non-chaotic systems into chaotic ones, significantly enhancing unpredictability for secure communication. In addition, temporal action segmentation for video encryption has been analyzed to optimize computational resources and improve data protection [35, 50]

Chapter 4

Future Works

As a short summary, we have completed the necessary preliminaries studies on various topics such as:

- Entropy
- Complexity
- Entropy Complexity Plane
- Confidence interval

We have also examined key research articles by Bandt and Pompe [15], along with an overview of the area focusing on four seminal works. These include:

- López-Ruiz et al. [51], who introduced the concept of statistical complexity;
- Lamberti et al. [41], who applied López-Ruiz's idea using the Euclidean distance;
- Rosso et al. [73], who proposed the entropy-complexity plane as a diagnostic tool; and

- Martin et al. [53], who defined the theoretical boundaries of this generalized statistical complexity measure.

In addition, we reviewed recent work by Rey et al. [67, 69, 70], which investigates the statistical properties of entropy derived from ordinal patterns, including the asymptotic distribution under the Multinomial law and the behavior of permutation entropy.

As a case study, we computed the Shannon entropy, statistical complexity, and their associated asymptotic variances based on the probability distribution of ordinal patterns. Using these results, we derived confidence intervals for both entropy and complexity. The analysis was further visualized using the entropy–complexity plane, offering insights into the underlying system dynamics. All computations were performed using two large-sample datasets under the asymptotic distribution, as detailed in Chapter 2, Section 2.7.

The formulas and procedures used to analyze the case study are summarized as follows:

- Calculate the Shannon entropy of the time series.
- Calculate the statistical complexity.
- Estimate the asymptotic variance for Shannon entropy.
- Construct confidence intervals for entropy.
- Plot the results in the entropy–complexity plane.
- Divide the data into batches (batch size = 10,000).
- Repeat the above calculations for each batch.
- Graphically represent the results of the two time series across batches in the entropy–complexity plane.
- Finally, the results are analyzed for time series clustering, as shown in the final output in Figure 2.2

Asymptotic distribution of normalized Shannon entropy $H(\mathbf{p})$ was derived under the assumption of independent ordinal patterns, following the Multinomial law. As a foundational step, we use the normalized Shannon entropy formula:

$$H(\mathbf{p}) = -\frac{1}{\log k}\sum_{\ell=1}^{k} p_\ell \ln p_\ell. \tag{4.1}$$

Where, $k = D!$ is the number of possible ordinal patterns. To evaluate statistical complexity, we compute the Jensen–Shannon divergence between the histogram of proportion $\mathbf{p}$ and the uniform probability function $\mathbf{u} = (1/k, 1/k, \dots, 1/k)$, defined by:

$$Q'(\mathbf{p}, \mathbf{u}) = \sum_{\ell=1}^{k} p_\ell \log \frac{p_\ell}{u_\ell} + u_\ell \log \frac{u_\ell}{p_\ell}. \tag{4.2}$$

This disequilibrium measure is normalized using:

$$Q = \frac{Q'}{\max{(Q')}}, \tag{4.3}$$

where $\max(Q')$ is defined as follows

$$\max(Q') = -2\left[\frac{k+1}{k}\log(k+1) - 2\log(2k) + \log k\right]. \tag{4.4}$$

The statistical complexity is then calculated as:

$$C = HQ, \tag{4.5}$$

where both H and Q are normalized quantities, therefore C is also normalized.

Then the entropy-complexity plane, which is a two-dimensional representation used to graphically represent the results.

As a key component of our research, we also calculated the asymptotic variance of the Shannon entropy estimator. The estimated normalized entropy based on sample proportions $\widehat{\boldsymbol{p}}$ is:

$$H_s(\widehat{\boldsymbol{p}}) = -\frac{1}{\log k}\sum_{\ell=1}^{k} \widehat{p}_\ell \log \widehat{p}_\ell. \tag{4.6}$$

The corresponding asymptotic variance under the Multinomial model is given by:

$$\widehat{\sigma}_p^2 = \frac{1}{n}\sum_{\ell=1}^{k} p_\ell(1-p_\ell)(\log p_\ell+1)^2 - \frac{2}{n}\sum_{j=1}^{k-1}\sum_{\ell=j+1}^{k} p_\ell p_j(\log p_\ell+1)(\log p_j+1). \quad (4.7)$$

where n is the sample size. From this variance, we derive confidence intervals for entropy, which are used to assess uncertainty in the entropy-complexity plane. The asymptotic distribution of statistical complexity under the Multinomial law is:

$$C[\widehat{\boldsymbol{p}}] = H[\widehat{\boldsymbol{p}}]Q[\widehat{\boldsymbol{p}}]. \quad (4.8)$$

This approach will be further analyzed, as described in the following objectives, to evaluate the accuracy of the results.

This proposal has three objectives in order to continue this research work.

- Define a data base of time series for clustering, i.e., finding similar time series.
- Extract all the features we know from their Bandt & Pompe symbolization (Shannon, Tsallis and Renyi entropies, Fisher information measure, complexities, and the available confidence intervals)
- Use those features for time series clustering

Chapter 5

Statistical Properties of Features from Ordinal Patterns

Although ordinal pattern based methods, such as permutation entropy, have been widely used for nonlinear time series analysis, the statistical properties of the features derived from these patterns, such as their distribution, variance, and confidence intervals remain under-explored and require further theoretical and empirical investigation. Therefore, the purpose of this chapter is to investigate the researchers who worked related to ordinal patterns, what kind of statistical properties of features used for their research work. Table 5.1 provides more information about the research articles and the test statistics or distributions they used.

Table 5.1: The test statistics used by the research articles for hypothesis testing

Paper Title/Reference	Distribution	Brief Description
A non-parametric independence test using permutation entropy. Matilla-García et.al. [54]	Empirical (permutation) distribution	Tests independence by comparing the observed permutation entropy-based statistic to its distribution under random shuffling (permutation) of the time series.
Asymptotic distribution of certain types of entropy under the multinomial law. Rey at.at. [70]	Normal, Chi-squared (χ^2)	Analyzes the asymptotic distribution (normal and chi-squared) of entropy estimators under the multinomial law.
Asymptotic distribution of entropies and Fisher information measure of ordinal patterns with applications. Reyet.al. [68]	Normal, Chi-squared (χ^2)	Provides asymptotic distributions for entropy and Fisher information measures of ordinal patterns.
Asymptotic distribution of the permutation entropy. Rey at.al. [69]	Normal	Derives the asymptotic normal distribution for permutation entropy estimators.
Asymptotic distribution of the statistical complexity under the multinomial law. Rey at.al. [67]	Normal, Chi-squared (χ^2)	Studies the asymptotic distribution of statistical complexity measures derived from ordinal patterns under the multinomial law.
Assessing serial dependence in ordinal patterns processes using chi-squared tests with application to EEG data. Yamashita et.al. [92]	Chi-squared (χ^2)	Applies chi-squared tests to assess serial dependence in ordinal pattern processes, with application to EEG data.

Continued on next page

Table 5.1 – continued from previous page

Paper Title/Reference	Distribution	Brief Description
Bearing fault diagnosis based on Alpha-stable distribution feature extraction and SVM classifier. Chouri et.al. [26]	Alpha-stable	Uses alpha-stable distribution parameters for feature extraction and SVM for classification in bearing fault diagnosis.
Belief permutation entropy of time series: A natural transition in analytical framework from probability theory to evidence theory. Xie et. al. [91]	Belief functions, Evidence theory basis	Introduces belief permutation entropy (BPE) using evidence theory (belief functions and mass assignments) and Deng entropy instead of classical probability distributions.
Markov modeling via ordinal partitions: An alternative paradigm for network-based time-series analysis. Sakellariou et.al. [78]	Markov (transition) probabilities	Uses Markov models constructed from ordinal partitions to analyze time series as networks.
On the statistical properties of Multiscale Permutation Entropy: Characterization of the estimator's variance. Dávalos et.al. [31]	Normal (asymptotic)	Shows that the multiscale permutation entropy estimator is asymptotically normally distributed, allowing inference on variance.
Price predictability at ultra high frequency:Entropy based randomness test. Shternshis et.al. [81]	Chi-squared (χ^2)	Proposes a randomness test for time series based on entropy, with asymptotic chi-squared distribution for the test statistic.
Statistical properties of the entropy from ordinal patterns. Chagas et.al. [23]	Normal	Investigates statistical properties and asymptotic normality of entropy estimators from ordinal patterns.

Continued on next page

Table 5.1 – continued from previous page

Paper Title/Reference	**Distribution**	**Brief Description**
The asymptotic distribution of the permutation entropy. Rey at.al. [69]	Normal	Derives the asymptotic normal distribution for permutation entropy estimators.
The modified permutation entropy-based independence test of time series. Ashtari et.al. [12]	Empirical (permutation) distribution	Tests independence by comparing the observed permutation entropy-based statistic to its distribution under random shuffling (permutation) of the time series.
White Noise Test from ordinal patterns in the entropy-complexity plane. Chagas et.al. [25]	Empirical Distribution	Used in white noise tests by repeatedly shuffling the time series to estimate the null distribution of the test statistic.

Bibliography

[1] ACHARYA, U. R., BHAT, S., FAUST, O., ADELI, H., CHUA, E. C.-P., LIM, W. J. E., AND KOH, J. E. W. Nonlinear dynamics measures for automated eeg-based sleep stage detection. *European Neurology 74*, 5-6 (2015), 268 – 287. Cited by: 105; All Open Access, Bronze Open Access.

[2] ACHARYA, U. R., FUJITA, H., SUDARSHAN, V. K., BHAT, S., AND KOH, J. E. Application of entropies for automated diagnosis of epilepsy using eeg signals: A review. *Knowledge-Based Systems 88* (2015), 85 – 96. Cited by: 417.

[3] ACHARYA, U. R., FUJITA, H., SUDARSHAN, V. K., LIH OH, S., MUHAMMAD, A., KOH, J. E. W., HONG TAN, J., CHUA, C. K., POO CHUA, K., AND SAN TAN, R. Application of empirical mode decomposition (emd) for automated identification of congestive heart failure using heart rate signals. *Neural Computing and Applications 28*, 10 (2017), 3073 – 3094. Cited by: 55.

[4] ACHARYA, U. R., FUJITA, H., SUDARSHAN, V. K., OH, S. L., ADAM, M., KOH, J. E., TAN, J. H., GHISTA, D. N., MARTIS, R. J., CHUA, C. K., POO, C. K., AND TAN, R. S. Automated detection and localization of myocardial infarction using electrocardiogram: A comparative study of different leads. *Knowledge-Based Systems 99* (2016), 146 – 156. Cited by: 206.

[5] ACHARYA, U. R., HAGIWARA, Y., DESHPANDE, S. N., SUREN, S., KOH, J. E. W., OH, S. L., ARUNKUMAR, N., CIACCIO, E. J., AND LIM, C. M. Characterization of focal eeg signals: A review. *Future Generation Computer Systems 91* (2019), 290 – 299. Cited by: 207.

[6] ACHARYA, U. R., HAGIWARA, Y., KOH, J. E., TAN, J. H., BHANDARY, S. V., RAO, A. K., AND RAGHAVENDRA, U. Automated screening tool for dry and wet age-related macular degeneration (armd) using pyramid of histogram of oriented gradients (phog) and nonlinear features. *Journal of Computational Science 20* (2017), 41 – 51. Cited by: 24.

[7] ACHARYA, U. R., HAGIWARA, Y., KOH, J. E. W., OH, S. L., TAN, J. H., ADAM, M., AND TAN, R. S. Entropies for automated detection of coronary artery disease using ecg signals: A review. *Biocybernetics and Biomedical Engineering 38*, 2 (2018), 373 – 384. Cited by: 89.

[8] ANTENEODO, C., AND PLASTINO, A. R. Some features of the lópez-ruiz-mancini-calbet (lmc) statistical measure of complexity. *Physics Letters A 223*, 5 (1996), 348–354.

[9] AQUINO, A. L., RAMOS, H. S., FRERY, A. C., VIANA, L. P., CAVALCANTE, T. S., AND ROSSO, O. A. Characterization of electric load with information theory quantifiers. *Physica A: Statistical Mechanics and its Applications 465* (2017), 277 – 284. Cited by: 27; All Open Access, Bronze Open Access, Green Open Access.

[10] AQUINO, A. L. L., CAVALCANTE, T. S. G., ALMEIDA, E. S., FRERY, A. C., AND ROSSO, O. A. Characterization of vehicle behavior with information theory. *European Physical Journal B 88*, 10 (2015). Cited by: 21.

[11] ARIA, M., AND CUCCURULLO, C. bibliometrix: An r-tool for comprehensive science mapping analysis. *Journal of Informetrics 11*, 4 (2017), 959–975.

[12] ASHTARI NEZHAD, E., WAGHEI, Y., MOHTASHAMI BORZADARAN, G., NILLI SANI, H., AND ALIZADEH NOUGHABI, H. The modified permutation entropy-based independence test of time series. *Communications in Statistics: Simulation and Computation 48*, 10 (2019), 2877 – 2897. Cited by: 3.

[13] AZAMI, H., ROSTAGHI, M., ABASOLO, D., AND ESCUDERO, J. Refined composite multiscale dispersion entropy and its application to biomedical signals. *IEEE Transactions on Biomedical Engineering 64*, 12 (2017), 2872 – 2879. Cited by: 257; All Open Access, Green Open Access.

[14] BANDT, C. Ordinal time series analysis. *Ecological Modelling 182*, 3-4 (2005), 229 – 238. Cited by: 123.

[15] BANDT, C., AND POMPE, B. Permutation entropy: A natural complexity measure for time series. *Phys. Rev. Lett. 88* (Apr 2002), 174102.

[16] BARBOSA, K., FRERY, A., AND CAVALCANTI, G. Analysis of signals from air conditioner compressors with ordinal patterns and machine learning. *Journal of Low Frequency Noise Vibration and Active Control* (2024). cited By 0.

[17] BARIVIERA, A. F., GUERCIO, M. B., MARTINEZ, L. B., AND ROSSO, O. A. The (in)visible hand in the libor market: an information theory approach. *European Physical Journal B 88*, 8 (2015). Cited by: 24; All Open Access, Green Open Access.

[18] BARIVIERA, A. F., GUERCIO, M. B., MARTINEZ, L. B., AND ROSSO, O. A. A permutation information theory tour through different interest rate maturities: The libor case. *Philosophical Transactions of*

the Royal Society A: Mathematical, Physical and Engineering Sciences 373, 2056 (2015). Cited by: 34; All Open Access, Green Open Access.

[19] BARIVIERA, A. F., GUERCIO, M. B., MARTINEZ, L. B., AND ROSSO, O. A. Libor at crossroads: Stochastic switching detection using information theory quantifiers. *Chaos, Solitons and Fractals 88* (2016), 172 – 182. Cited by: 9; All Open Access, Green Open Access.

[20] BORGES, J., MEDEIROS, J., BARBOSA, L., RAMOS, H., AND LOUREIRO, A. Iot botnet detection based on anomalies of multiscale time series dynamics. *IEEE Transactions on Knowledge and Data Engineering 35*, 12 (2023), 12282–12294. cited By 8.

[21] BORGES, J., RAMOS, H., AND LOUREIRO, A. A classification strategy for internet of things data based on the class separability analysis of time series dynamics. *ACM Transactions on Internet of Things 3*, 3 (2022). cited By 7.

[22] CAO, Y., TUNG, W.-W., GAO, J., PROTOPOPESCU, V., AND HIVELY, L. Detecting dynamical changes in time series using the permutation entropy. *Physical Review E - Statistical, Nonlinear, and Soft Matter Physics 70*, 4 2 (2004), 046217–1–046217–7. Cited by: 385.

[23] CHAGAS, E., FRERY, A., GAMBINI, J., LUCINI, M., RAMOS, H., AND REY, A. Statistical properties of the entropy from ordinal patterns. *Chaos 32*, 11 (2022). cited By 4.

[24] CHAGAS, E., FRERY, A., ROSSO, O., AND RAMOS, H. Analysis and classification of sar textures using information theory. *IEEE Journal of Selected Topics in Applied Earth Observations and Remote Sensing 14* (2021), 663–675. cited By 13.

[25] CHAGAS, E., QUEIROZ-OLIVEIRA, M., ROSSO, O., RAMOS, H., FREITAS, C., AND FRERY, A. White noise test from ordinal patterns in

the entropy–complexity plane. *International Statistical Review 90*, 2 (2022), 374–396. cited By 6.

[26] CHOURI, B., FABRICE, M., DANDACHE, A., EL AROUSSI, M., AND SAADANE, R. Bearing fault diagnosis based on alpha-stable distribution feature extraction and svm classifier. In *International Conference on Multimedia Computing and Systems -Proceedings* (2014), p. 1545 – 1550. Cited by: 13.

[27] CONSOLINI, G., AND DE MICHELIS, P. Permutation entropy analysis of complex magnetospheric dynamics. *Journal of Atmospheric and Solar-Terrestrial Physics 115-116* (2014), 25 – 31. Cited by: 19.

[28] DE MICCO, L., FERNÁNDEZ, J. G., LARRONDO, H. A., PLASTINO, A., AND ROSSO, O. A. Sampling period, statistical complexity, and chaotic attractors. *Physica A: Statistical Mechanics and its Applications 391*, 8 (2012), 2564 – 2575. Cited by: 32.

[29] DE MICCO, L., GONZÁLEZ, C., LARRONDO, H., MARTIN, M., PLASTINO, A., AND ROSSO, O. Randomizing nonlinear maps via symbolic dynamics. *Physica A: Statistical Mechanics and its Applications 387*, 14 (2008), 3373 – 3383. Cited by: 55.

[30] DE MICCO, L., LARRONDO, H., PLASTINO, A., AND ROSSO, O. Quantifiers for randomness of chaotic pseudo-random number generators. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences 367*, 1901 (2009), 3281 – 3296. Cited by: 29; All Open Access, Green Open Access.

[31] DÁVALOS, A., JABLOUN, M., RAVIER, P., AND BUTTELLI, O. On the statistical properties of multiscale permutation entropy: Characterization of the estimator's variance. *Entropy 21*, 5 (2019). Cited by: 18; All Open Access, Gold Open Access, Green Open Access.

[32] FAN, C., AND DING, Q. A novel image encryption scheme based on self-synchronous chaotic stream cipher and wavelet transform. *Entropy 20*, 6 (2018). Cited by: 19; All Open Access, Gold Open Access, Green Open Access.

[33] FRIEDEN, B. R. *Science from Fisher information: a unification*. Cambridge University Press, 2004.

[34] GAN, X., LU, H., AND YANG, G. Fault diagnosis method for rolling bearings based on composite multiscale fluctuation dispersion entropy. *Entropy 21*, 3 (2019). Cited by: 24; All Open Access, Gold Open Access, Green Open Access.

[35] GAO, S., IU, H.-C., MOU, J., ERKAN, U., LIU, J., WU, R., AND TANG, X. Temporal action segmentation for video encryption. *Chaos, Solitons and Fractals 183* (2024). cited By 21.

[36] HUMEAU-HEURTIER, A., WU, C.-W., AND WU, S.-D. Refined composite multiscale permutation entropy to overcome multiscale permutation entropy length dependence. *IEEE Signal Processing Letters 22*, 12 (2015), 2364 – 2367. Cited by: 87.

[37] KELLER, K., MANGOLD, T., STOLZ, I., AND WERNER, J. Permutation entropy: New ideas and challenges. *Entropy 19*, 3 (2017). Cited by: 67; All Open Access, Gold Open Access, Green Open Access.

[38] KOWALSKI, A., MARTÍN, M., PLASTINO, A., AND ROSSO, O. Bandt-pompe approach to the classical-quantum transition. *Physica D: Nonlinear Phenomena 233*, 1 (2007), 21 – 31. Cited by: 81.

[39] KOWALSKI, A., MARTÍN, M., PLASTINO, A., AND ROSSO, O. Fisher information description of the classicalquantal transition. *Physica A: Statistical Mechanics and its Applications 390*, 12 (2011), 2435 – 2441. Cited by: 2; All Open Access, Green Open Access.

[40] LAJNEF, T., CHAIBI, S., RUBY, P., AGUERA, P.-E., EICHENLAUB, J.-B., SAMET, M., KACHOURI, A., AND JERBI, K. Learning machines and sleeping brains: Automatic sleep stage classification using decision-tree multi-class support vector machines. *Journal of Neuroscience Methods 250* (2015), 94 – 105. Cited by: 265.

[41] LAMBERTI, P., MARTIN, M., PLASTINO, A., AND ROSSO, O. Intensive entropic non-triviality measure. *Physica A: Statistical Mechanics and its Applications 334*, 1-2 (2004), 119–131.

[42] LANGE, H., ROSSO, O., AND HAUHS, M. Ordinal pattern and statistical complexity analysis of daily stream flow time series. *European Physical Journal: Special Topics 222*, 2 (2013), 535 – 552. Cited by: 29.

[43] LEI, N., HUANG, F., AND LI, C. Rolling bearing fault diagnosis based on variational mode decomposition and weighted multidimensional feature entropy fusion. *Journal of Vibroengineering 26*, 3 (2024), 590–614. cited By 2.

[44] LI, J., YAN, J., LIU, X., AND OUYANG, G. Using permutation entropy to measure the changes in eeg signals during absence seizures. *Entropy 16*, 6 (2014), 3049 – 3061. Cited by: 99; All Open Access, Gold Open Access, Green Open Access.

[45] LI, P., LIU, C., LI, K., ZHENG, D., LIU, C., AND HOU, Y. Assessing the complexity of short-term heartbeat interval series by distribution entropy. *Medical and Biological Engineering and Computing 53*, 1 (2015), 77 – 87. Cited by: 213.

[46] LI, X., CUI, S., AND VOSS, L. J. Using permutation entropy to measure the electroencephalographic effects of sevoflurane. *Anesthesiology 109*, 3 (2008), 448 – 456. Cited by: 185; All Open Access, Bronze Open Access.

[47] LI, X., OUYANG, G., AND RICHARDS, D. A. Predictability analysis of absence seizures with permutation entropy. *Epilepsy Research 77*, 1 (2007), 70 – 74. Cited by: 254.

[48] LIANG, Z., WANG, Y., SUN, X., LI, D., VOSS, L. J., SLEIGH, J. W., HAGIHIRA, S., AND LI, X. Eeg entropy measures in anesthesia. *Frontiers in Computational Neuroscience 9*, JAN (2015). Cited by: 239; All Open Access, Gold Open Access, Green Open Access.

[49] LIU, H., REN, B., ZHAO, Q., AND LI, N. Characterizing the optical chaos in a special type of small networks of semiconductor lasers using permutation entropy. *Optics Communications 359* (2016), 79 – 84. Cited by: 25.

[50] LIU, W., SUN, K., WANG, H., AND LI, B. Delayed feedback chaotification model and its hardware implementation. *IEEE Transactions on Industrial Electronics 71*, 10 (2024), 13002–13011. cited By 8.

[51] LOPEZ-RUIZ, R., MANCINI, H. L., AND CALBET, X. A statistical measure of complexity. *Physics letters A 209*, 5-6 (1995), 321–326.

[52] MA, P., LIANG, W., ZHANG, H., WANG, C., AND LI, X. Multiscale permutation entropy based on natural visibility graph and its application to rolling bearing fault diagnosis. *Structural Health Monitoring 24*, 1 (2025), 313–326. cited By 3.

[53] MARTIN, M., PLASTINO, A., AND ROSSO, O. Generalized statistical complexity measures: Geometrical and analytical properties. *Physica A: Statistical Mechanics and its Applications 369*, 2 (2006), 439 – 462. Cited by: 293.

[54] MATILLA-GARCÍA, M., AND RUIZ MARÍN, M. A non-parametric independence test using permutation entropy. *Journal of Econometrics 144*, 1 (2008), 139 – 155. Cited by: 82; All Open Access, Green Open Access.

[55] MINHAS, A., KANKAR, P., KUMAR, N., AND SINGH, S. Bearing fault detection and recognition methodology based on weighted multiscale entropy approach. *Mechanical Systems and Signal Processing 147* (2021). cited By 54.

[56] MODIS, T. Links between entropy, complexity, and the technological singularity. *Technological Forecasting and Social Change 176* (2022), 121457.

[57] MONTANI, F., BARAVALLE, R., MONTANGIE, L., AND ROSSO, O. A. Causal information quantification of prominent dynamical features of biological neurons. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences 373*, 2056 (2015). Cited by: 27; All Open Access, Bronze Open Access, Green Open Access.

[58] MONTANI, F., DELEGLISE, E. B., AND ROSSO, O. A. Efficiency characterization of a large neuronal network: A causal information approach. *Physica A: Statistical Mechanics and its Applications 401* (2014), 58 – 70. Cited by: 29; All Open Access, Green Open Access.

[59] MONTANI, F., AND ROSSO, O. A. Entropy-complexity characterization of brain development in chickens. *Entropy 16*, 8 (2014), 4677 – 4692. Cited by: 30; All Open Access, Gold Open Access, Green Open Access.

[60] MONTANI, F., ROSSO, O. A., MATIAS, F. S., BRESSLER, S. L., AND MIRASSO, C. R. A symbolic information approach to determine anticipated and delayed synchronization in neuronal circuit models. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences 373*, 2056 (2015). Cited by: 25; All Open Access, Green Open Access.

[61] MORABITO, F. C., LABATE, D., LA FORESTA, F., BRAMANTI, A., MORABITO, G., AND PALAMARA, I. Multivariate multi-scale per-

mutation entropy for complexity analysis of alzheimer's disease eeg. *Entropy 14*, 7 (2012), 1186 – 1202. Cited by: 231; All Open Access, Gold Open Access, Green Open Access.

[62] PARLITZ, U., BERG, S., LUTHER, S., SCHIRDEWAN, A., KURTHS, J., AND WESSEL, N. Classifying cardiac biosignals using ordinal pattern statistics and symbolic dynamics. *Computers in Biology and Medicine 42*, 3 (2012), 319 – 327. Cited by: 160.

[63] PERINELLI, A., AND RICCI, L. Stationarity assessment of resting state condition via permutation entropy on EEG recordings. *Scientific Reports 15*, 1 (2025). cited By 0.

[64] PERINELLI, A., TABARELLI, D., MINIUSSI, C., AND RICCI, L. Dependence of connectivity on geometric distance in brain networks. *Scientific Reports 9*, 1 (2019). cited By 17.

[65] REDELICO, F. O., TRAVERSARO, F., OYARZABAL, N., VILABOA, I., AND ROSSO, O. A. Evaluation of the status of rotary machines by time causal information theory quantifiers. *Physica A: Statistical Mechanics and its Applications 470* (2017), 321 – 329. Cited by: 7; All Open Access, Green Open Access.

[66] RÉNYI, A. On measures of entropy and information. In *Proceedings of the fourth Berkeley symposium on mathematical statistics and probability, volume 1: contributions to the theory of statistics* (1961), vol. 4, University of California Press, pp. 547–562.

[67] REY, A., FRERY, A. C., AND GAMBINI, J. Asymptotic distribution of the statistical complexity under the multinomial law. *Chaos, Solitons & Fractals 193* (2025), 116085.

[68] REY, A., FRERY, A. C., GAMBINI, J., AND LUCINI, M. Asymptotic distribution of entropies and fisher information measure of ordi-

nal patterns with applications. *Chaos, Solitons & Fractals 188* (2024), 115481.

[69] REY, A. A., FRERY, A. C., GAMBINI, J., AND LUCINI, M. M. The asymptotic distribution of the permutation entropy. *Chaos: An Interdisciplinary Journal of Nonlinear Science 33*, 11 (11 2023), 113108.

[70] REY, A. A., FRERY, A. C., LUCINI, M., GAMBINI, J., CHAGAS, E. T. C., AND RAMOS, H. S. Asymptotic distribution of certain types of entropy under the multinomial law. *Entropy 25*, 5 (2023).

[71] RICCI, L. Asymptotic distribution of sample shannon entropy in the case of an underlying finite, regular markov chain. *Phys. Rev. E 103* (Feb 2021), 022215.

[72] ROSSO, O. A., DE MICCO, L., PLASTINO, A., AND LARRONDO, H. A. Info-quantifiers' map-characterization revisited. *Physica A: Statistical Mechanics and its Applications 389*, 21 (2010), 4604 – 4612. Cited by: 30.

[73] ROSSO, O. A., MARTIN, M. T., FIGLIOLA, A., KELLER, K., AND PLASTINO, A. EEG analysis using wavelet-based information tools. *Journal of Neuroscience Methods 153* (2006), 163–182.

[74] ROSSO, O. A., OSPINA, R., AND FRERY, A. C. Classification and verification of handwritten signatures with time causal information theory quantifiers. *PLoS ONE 11*, 12 (2016). Cited by: 30; All Open Access, Gold Open Access, Green Open Access.

[75] ROSTAGHI, M., ASHORY, M. R., AND AZAMI, H. Application of dispersion entropy to status characterization of rotary machines. *Journal of Sound and Vibration 438* (2019), 291 – 308. Cited by: 96.

[76] ROSTAGHI, M., AND AZAMI, H. Dispersion entropy: A measure for time-series analysis. *IEEE Signal Processing Letters 23*, 5 (2016), 610 – 614. Cited by: 636.

[77] SACO, P. M., CARPI, L. C., FIGLIOLA, A., SERRANO, E., AND ROSSO, O. A. Entropy analysis of the dynamics of el niño/southern oscillation during the holocene. *Physica A: Statistical Mechanics and its Applications 389*, 21 (2010), 5022 – 5027. Cited by: 70.

[78] SAKELLARIOU, K., STEMLER, T., AND SMALL, M. Markov modeling via ordinal partitions: An alternative paradigm for network-based time-series analysis. *Physical Review E 100*, 6 (2019). Cited by: 26; All Open Access, Green Open Access.

[79] SÁNCHEZ-MORENO, P., YÁNEZ, R., AND DEHESA, J. Discrete densities and fisher information. In *Proceedings of the 14th International Conference on Difference Equations and Applications. Difference Equations and Applications. Istanbul, Turkey: Bahçesehir University Press* (2009), pp. 291–298.

[80] SERINALDI, F., ZUNINO, L., AND ROSSO, O. A. Complexity–entropy analysis of daily stream flow time series in the continental united states. *Stochastic Environmental Research and Risk Assessment 28*, 7 (2014), 1685 – 1708. Cited by: 53.

[81] SHTERNSHIS, A., AND MARMI, S. Price predictability at ultra-high frequency: Entropy-based randomness test. *Communications in Nonlinear Science and Numerical Simulation 141* (2025), 108469.

[82] SIGAKI, H., PERC, M., AND RIBEIRO, H. Clustering patterns in efficiency and the coming-of-age of the cryptocurrency market. *Scientific Reports 9*, 1 (2019). cited By 67.

[83] SIGAKI, H. Y., PERC, M., AND RIBEIRO, H. V. History of art paintings through the lens of entropy and complexity. *Proceedings of the National Academy of Sciences of the United States of America 115*, 37 (2018), E8585 – E8594. Cited by: 97; All Open Access, Bronze Open Access, Green Open Access.

[84] SIPPEL, S., LANGE, H., MAHECHA, M. D., HAUHS, M., BODESHEIM, P., KAMINSKI, T., GANS, F., AND ROSSO, O. A. Diagnosing the dynamics of observed and simulated ecosystem gross primary productivity with time causal information theory quantifiers. *PLoS ONE 11*, 10 (2016). Cited by: 19; All Open Access, Gold Open Access, Green Open Access.

[85] SORIANO, M. C., ZUNINO, L., ROSSO, O. A., FISCHER, I., AND MIRASSO, C. R. Time scales of a chaotic semiconductor laser with optical feedback under the lens of a permutation information analysis. *IEEE Journal of Quantum Electronics 47*, 2 (2011), 252 – 261. Cited by: 170; All Open Access, Green Open Access.

[86] STOSIC, T., TELESCA, L., DE SOUZA FERREIRA, D. V., AND STOSIC, B. Investigating anthropically induced effects in streamflow dynamics by using permutation entropy and statistical complexity analysis: A case study. *Journal of Hydrology 540* (2016), 1136 – 1145. Cited by: 43.

[87] TOOMEY, J., AND KANE, D. Mapping the dynamic complexity of a semiconductor laser with optical feedback using permutation entropy. *Optics Express 22*, 2 (2014), 1713 – 1725. Cited by: 94; All Open Access, Gold Open Access.

[88] TREITEL, S. Spectral analysis for physical applications: multitaper and conventional univariate techniques. *American Scientist 83*, 2 (1995), 195–197.

[89] TSALLIS, C. Possible generalization of boltzmann-gibbs statistics. *Journal of statistical physics 52* (1988), 479–487.

[90] WANG, T., KHOO, S., ONG, Z., SIOW, P., AND WANG, T. Distance similarity entropy: A sensitive nonlinear feature extraction method for rolling bearing fault diagnosis. *Reliability Engineering and System Safety 255* (2025). cited By 1.

[91] XIE, J., XU, G., CHEN, X., ZHANG, X., CHEN, R., YANG, Z., LI, B., AND ZHANG, S. Belief permutation entropy of time series: A natural transition in analytical framework from probability theory to evidence theory. *Information Sciences 718* (2025), 122352.

[92] YAMASHITA RIOS DE SOUSA, A., AND HLINKA, J. Assessing serial dependence in ordinal patterns processes using chi-squared tests with application to eeg data analysis. *Chaos 32*, 7 (2022). cited By 7.

[93] YAN, R., LIU, Y., AND GAO, R. X. Permutation entropy: A nonlinear statistical measure for status characterization of rotary machines. *Mechanical Systems and Signal Processing 29* (2012), 474 – 484. Cited by: 375.

[94] YAN, X., XU, Y., AND JIA, M. Intelligent fault diagnosis of rolling-element bearings using a self-adaptive hierarchical multiscale fuzzy entropy. *Entropy 23*, 9 (2021). cited By 16.

[95] YANG, L., PAN, W., YAN, L., LUO, B., AND LI, N. Mapping the dynamic complexity and synchronization in unidirectionally coupled external-cavity semiconductor lasers using permutation entropy. *Journal of the Optical Society of America B: Optical Physics 32*, 7 (2015), 1463 – 1470. Cited by: 4.

[96] YASIR, M. N., AND KOH, B.-H. Data decomposition techniques with multi-scale permutation entropy calculations for bearing fault diagnosis. *Sensors (Switzerland) 18*, 4 (2018). Cited by: 32; All Open Access, Gold Open Access, Green Open Access.

[97] ZANIN, M., ZUNINO, L., ROSSO, O. A., AND PAPO, D. Permutation entropy and its main biomedical and econophysics applications: A review. *Entropy 14*, 8 (2012), 1553 – 1577. Cited by: 522; All Open Access, Gold Open Access, Green Open Access.

[98] ZHANG, Y., SHANG, P., AND SUN, Z. Diversity analysis based on ordered patterns. *Physica A: Statistical Mechanics and its Applications 506* (2018), 1126 – 1133. Cited by: 0.

[99] ZHENG, J., CHENG, J., AND YANG, Y. Multiscale permutation entropy based rolling bearing fault diagnosis. *Shock and Vibration 2014* (2014). Cited by: 63; All Open Access, Gold Open Access, Green Open Access.

[100] ZUNINO, L., BARIVIERA, A. F., GUERCIO, M. B., MARTINEZ, L. B., AND ROSSO, O. A. Monitoring the informational efficiency of european corporate bond markets with dynamical permutation min-entropy. *Physica A: Statistical Mechanics and its Applications 456* (2016), 1 – 9. Cited by: 24; All Open Access, Green Open Access.

[101] ZUNINO, L., ROSSO, O. A., AND SORIANO, M. C. Characterizing the hyperchaotic dynamics of a semiconductor laser subject to optical feedback via permutation entropy. *IEEE Journal on Selected Topics in Quantum Electronics 17*, 5 (2011), 1250 – 1257. Cited by: 69; All Open Access, Green Open Access.

[102] ZUNINO, L., SORIANO, M., FISCHER, I., ROSSO, O., AND MIRASSO, C. Permutation-information-theory approach to unveil delay dynamics from time-series analysis. *Physical Review E - Statistical, Nonlinear, and Soft Matter Physics 82*, 4 (2010). Cited by: 198; All Open Access, Green Open Access.

[103] ZUNINO, L., SORIANO, M., AND ROSSO, O. Distinguishing chaotic and stochastic dynamics from time series by using a multiscale symbolic approach. *Physical Review E - Statistical, Nonlinear, and Soft Matter Physics 86*, 4 (2012). Cited by: 182; All Open Access, Green Open Access.

[104] ZUNINO, L., ZANIN, M., TABAK, B. M., PÉREZ, D. G., AND ROSSO, O. A. Forbidden patterns, permutation entropy and stock market inefficiency. *Physica A: Statistical Mechanics and its Applications 388*, 14 (2009), 2854 – 2864. Cited by: 199.

[105] ZUNINO, L., ZANIN, M., TABAK, B. M., PÉREZ, D. G., AND ROSSO, O. A. Complexity-entropy causality plane: A useful approach to quantify the stock market inefficiency. *Physica A: Statistical Mechanics and its Applications 389*, 9 (2010), 1891 – 1901. Cited by: 183; All Open Access, Green Open Access.