Ordinal Pattern Features: Statistical Properties and Their Applications in Time Series Clustering

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Abstract

Time series analysis plays a vital role in understanding the underlying dynamics of complex systems across various domains such as engineering, economics, and the physical sciences. Traditional approaches, namely time-domain and frequency-domain methods often rely on strong assumptions such as stationarity, large sample sizes, or normality, which are frequently violated in real-world data. As a robust alternative, Bandt and Pompe [14] introduced a non-parametric method based on ordinal pattern symbolization and information theoretic descriptors. This approach transforms local segments of the time series into rank-based symbols (the ordinal patterns), constructs a histogram of ordinal patterns, and computes Shannon entropy, offering resistance to noise and model independence. In addition to the foundational work by Bandt and Pompe, we refer to the contributions of López-Ruiz et al. [53], who introduced the concept of statistical complexity; Lamberti et al. [41], who implemented López-Ruiz's idea using the Euclidean distance; Rosso et al. [77], who proposed the joint representation of entropy and complexity in the entropy-complexity plane; and Martin et al. [55], who established the theoretical boundaries of the generalized statistical complexity measure. This proposal investigates the use of various entropy measures, such as Shannon, Tsallis, and Rényi entropy, as well as the Fisher Information Measure, statistical complexity, and associated confidence intervals for time series analysis clustering.

Contents

Chapter 1

Introduction

Time series contain valuable insights about the underlying system that generates the data. Their analysis is typically conducted using two primary approaches: time-domain and transformed-domain methods. In the context of time-domain analysis, Bandt and Pompe [14] introduced a novel methodology that is non-parametric and rooted in information theory descriptors: Ordinal Patterns symbolization.

Bandt and Pompe [14] proposed transforming small subsets of the time series observations into symbols that encode the sorting properties of the values in these subsets. Then, they computed a histogram of those symbols. The resulting distribution is less sensitive to outliers compared to the original data, and the histogram is independent of any specific model. The proposal proceeds by computing descriptors from this histogram, and extracting information about the system from these descriptors. As a result, this approach is versatile and applicable to a wide range of scenarios.

The proposal focuses on the statistical properties of features from ordinal patterns in time series clustering. It involves calculating pattern histograms, entropy, complexity, and confidence intervals to better understand the statistical properties of these tools. Additionally, the role of confidence intervals in entropy and complexity, along with their applications in time-series clustering, will be explored. Future work will expand to include alternative measures, such as Rényi entropy, Tsallis entropy and Fisher information, with a focus on deriving confidence intervals for their entropy and complexity under the Multinomial model.

Time series analysis is widely applied across various fields, including engineering, economics, physical sciences, and more. A time series is defined as a collection of observations x_t, each representing a realized value of a particular random variable X_t, where time can be either discrete or continuous.

Examples of time series applications include finance (e.g., analyzing exchange rate movements or commodity prices), biology (e.g., modeling the growth and decline of bacterial populations), medicine (e.g., tracking the spread of diseases like COVID-19 or influenza), and geoscience (e.g., predicting wet or dry days based on past weather conditions).

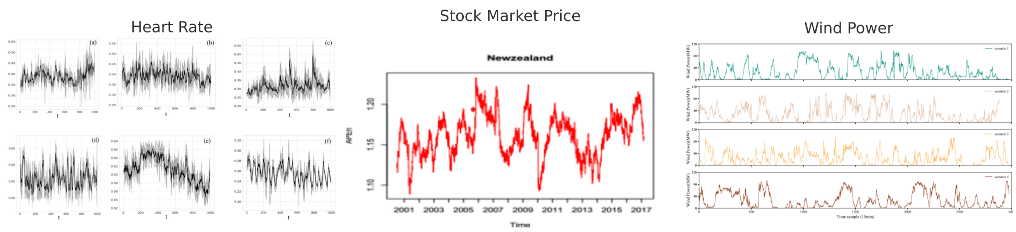

Figure 1.1: Examples for Biology, Finance, and Geoscience: sources from Silva et.al., Patra et.al. and Meng et.al. [27, 67, 57]

The primary goal of time series analysis is to understand the nature of the phenomenon represented by the observed sequence. Time domain and frequency domain methods are the two primary approaches used in time series analysis. The temporal approach relies on concepts such as auto-correlation and regressions, where a time series' present value is analyzed in relation to its own past values or the past values of other series. This method represents time series directly as a function of time. On the other hand, the spectral approach represents time series through spectral expansions, such as wavelets or Fourier modes [94].

However, these methods often require assumptions such as large sample sizes or normally distributed observations that are rarely met in real-world empirical data. For many statistical techniques to be valid, these assumptions must hold, but in practice, they are frequently violated.

For example, traditional approaches to time series analysis, such as time domain and frequency domain methods, rely on assumptions that are not always valid in real-world data. The time domain approach, which uses techniques like auto-correlation and regression, assumes stationary and often struggles with nonlinear or non-stationary data. Similarly, the frequency domain approach, which represents time series through spectral expansions such as wavelets or Fourier modes, may require assumptions about periodicity and may not effectively capture short-term fluctuations.

Many statistical methods in these approaches depend on specific conditions, such as large sample sizes or normally distributed observations. However, these assumptions are often unrealistic, leading to inaccurate or biased results. When such conditions are not met, alternative methods must be considered.

As a result, alternative methods, commonly known as non-parametric techniques, are often considered. These approaches do not rely on the actual numerical values of the observations x_t, but rather on their ranks R_t, making them more robust and less sensitive to outliers and applicable to a wide range of data sets. Since non-parametric tests do not assume any specific distribution, such as normality, they are considered highly reliable for a range of data types.

However, while these techniques are powerful for general statistical analysis, they are not always well-suited for time series data.

To address these challenges, ordinal pattern methods provide a robust alternative. Instead of analyzing the absolute values of a time series, these methods focus on the order relationships among consecutive data points.

This approach effectively captures the underlying dynamics of complex systems and offers several advantages.

The ordinal pattern-based method has become a widely used tool for characterizing complex time series. Since its introduction nearly twenty-three years ago by Bandt and Pompe in their foundational paper [14], it has been successfully applied across various scientific fields, including biomedical signal processing, optical chaos, hydrology, geophysics, econophysics, engineering, and biometrics. It has also been used in the characterization of pseudo-random number generators.

The Bandt and Pompe method successfully analyzes time series by transforming them into ordinal patterns, constructing a histogram, and computing Shannon entropy, making it robust against outliers and independent of predefined models.

1.1 Introduction to Ordinal Pattern Analysis

Ordinal patterns are a non-parametric representation of real-valued time series by transforming small subsets of observations into symbols based on their relative order, rather than looking into actual values. This approach maps each segment of the time series in $\mathbb{R}^D$ into a finite set of $D!$ distinct symbols, where D represents for the "Embedding Dimension" and usually ranges between three to six. One of the possible encoding is the set of indexes that sort the D values in non-decreasing order.

To illustrate this idea, let $\boldsymbol{x} = \{x_1, x_2, \dots, x_{n+D-1}\}$ be a real valued time series of length $n + D - 1$ without ties. The corresponding symbol sequence naturally emerges from the time series without requiring any model assumptions. We compute $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_n)$ symbols from subsequences of embedding dimension D. There are $D!$ possible symbols: $\pi_j \in \{\pi^1, \pi^2, \dots, \pi^{D!}\}$, for each $1 \leq j \leq n$. The histogram of proportions $h = (h_1, h_2, \dots, h_{D!})$ in which the bin h_ℓ is the proportion of symbols of type π^ℓ of the total number of symbols. For convenience, we will model those symbols as a k dimensional random vector where $k = D!$.

1.2 Problem Statement

To illustrate this concept, imagine tracking the mean monthly humidity in Wellington. You want to analyze how humidity changes throughout the year. By examining this data, you can uncover interesting patterns that highlight the variations in humidity across different months.

Month	Mean of relative humidity
January	77.3
February	81.0
March	82.4
April	81.7
May	83.6
June	85.6
July	84.4
August	83.1
September	78.8
October	79.6
November	78.2
December	78.8

Table 1.1: Mean monthly humidity variations in Wellington throughout the year

The mean monthly humidity in Wellington is shown in Figure 1.2.

We can convert this actual data into ordinal patterns. To do this, for each month, we determine the order of the humidity values rather than their actual magnitudes. Each three-time-point sequence (which can be adjusted based on preference) is converted into an ordinal pattern. This "embedding dimension" usually varies between 3 and 6, but any dimension

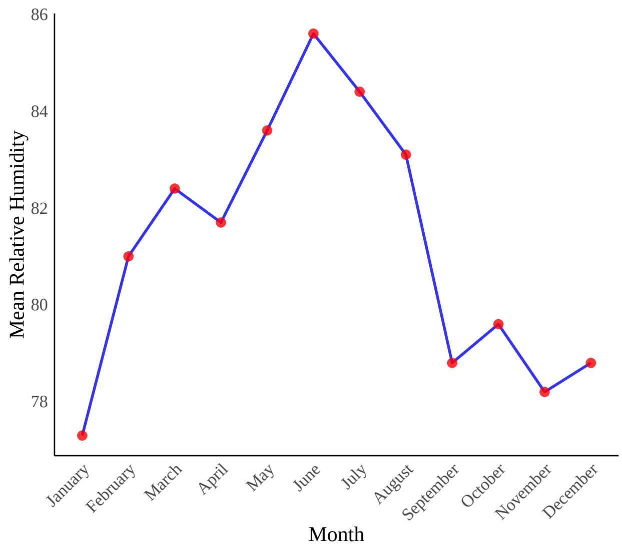

Figure 1.2: Mean Monthly humidity in Wellington

is possible. The conversion can be made in any way that uniquely maps the sorting properties of the sub sequence into a symbol. For example, consider the time series presented in Table 1.1. We can transform this series into ordinal patterns as follows. Assume we use patterns of length $D = 3$. The first overlapping window $(77.3, 81, 82.4)$ corresponds to the pattern $(1, 2, 3)$, where type is considering with the order of the real time series data. Here 77.3 is the smallest value and is assigned rank 1; 81 is the next highest and is assigned rank 2; and 82.4 is the largest, assigned rank 3. As another example, consider the overlapping window $(83.1, 78.8, 79.6)$. In this case, 78.8 is the smallest value and is assigned rank 1; 79.6 is the next highest, assigned rank 2; and 83.1 is the largest, assigned rank 3. Therefore, the pattern for this window is $(3, 1, 2)$. Table 1.2 has shown this scenario.

t	Mean Humidity sequence	Ordinal Pattern
1	(77.3,81,82.4)	(123) $= \pi^1$
2	(81,82.4,81.7)	(132) $= \pi^2$
3	(82.4,81.7,83.6)	(213) $= \pi^3$
4	(81.7,83.6,85.6)	(123) $= \pi^1$
5	(83.6,85.6,84.4)	(132) $= \pi^2$
6	(85.6,84.4,83.1)	(321) $= \pi^6$
7	(84.4,83.1,78.8)	(321) $= \pi^6$
8	(83.1,78.8,79.6)	(312) $= \pi^5$
9	(78.8,79.6,78.2)	(231) $= \pi^4$
10	(79.6,78.2,78.8)	(312) $= \pi^5$

Table 1.2: Ordinal Patterns

As shown in Table 1.2, we have six mutually exclusive events which we denote as $\{\pi^1, \pi^2, \ldots, \pi^6\} = \{(123), (132), (213), (231), (312), (321)\}$. The probability distribution of the mean humidity is calculated based on ordinal patterns as given below.

$$\hat{p}_i = \frac{\#\{\pi_j \in \boldsymbol{\pi} : \pi_j = \pi^i\}}{n}; 1 \leq i \leq 6, \tag{1.1}$$

where $\hat{\boldsymbol{p}} = (\hat{p}_1, \ldots, \hat{p}_6)$.

Notation	Probability
$p(\pi^1)$	$\frac{2}{10}$
$p(\pi^2)$	$\frac{2}{10}$
$p(\pi^3)$	$\frac{1}{10}$
$p(\pi^4)$	$\frac{1}{10}$
$p(\pi^5)$	$\frac{2}{10}$
$p(\pi^6)$	$\frac{2}{10}$

Table 1.3: Probability function

We construct the histogram of proportions $h = (h_1, h_2, h_3, h_4, h_5, h_6)$, where each bin h_ℓ represents the proportion of symbols of type π^ℓ out of the total six symbols. The histogram graph is shown Figure 1.3.

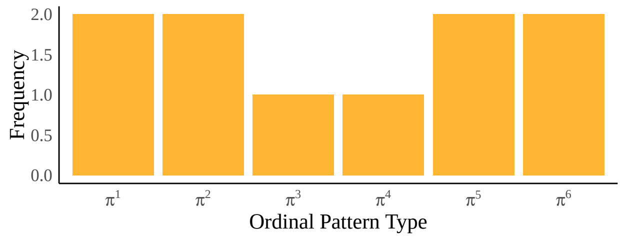

Figure 1.3: Histogram of proportions of the observed patterns according to Table 1.3.

This example explains how time series data can be converted into ordinal patterns and how the probability distribution function can be calculated from these patterns. Chapter 2 will review the literature on ordinal pattern analysis. Chapter 3 will expand on ordinal pattern concept by exploring the characterization of time series and cover the computation of two key descriptors entropy and complexity from the resulting histograms. Additionally, Chapter 4 will outline the main ideas and objectives of this research project.

Chapter 2

Literature Review

The analysis of complex time series has long relied on both time-domain and frequency-domain techniques. However, traditional methods often fall short in capturing nonlinear dynamics or are limited by strict assumptions such as stationarity and Gaussianity.

The ordinal symbolic approach introduced by Bandt and Pompe in 2002 marked a significant theoretical advance by enabling robust, model-free characterization of time series. Their approach, rooted in information theory, involves converting segments of time series data into symbols based on the ordinal (rank) relationships among the data points. These symbols are called "ordinal patterns." After computing all the symbols, their relative frequencies are used to estimate the probability distribution of ordinal patterns.

From this distribution estimate, two key descriptors entropy and complexity are calculated to characterize the time series: the scaled Shannon entropy, now widely known as permutation entropy, and the statistical complexity.

This chapter is divided into four main sections. Section 2.1 presents a brief overview of the area, focusing on what we consider the four seminal papers. Section 2.2 discusses the research question and the motivation for conducting the bibliometric analysis. The final section, Section 2.3, highlights the importance of bibliometric analysis and presents the results obtained from references that cite the Bandt and Pompe methodology and other related topics based on our research focus. Section 2.8 discusses the statistical properties of features from ordinal patterns based on the literature review.

2.1 The Onset of the Entropy-Complexity Plane

This section discusses the emergence of the entropy-complexity plane. This topic is presented as a central theme because it reflects the foundation of our main research focus and illustrates how it has evolved over time into the current approach to time series analysis based on the concept of ordinal patterns.

López-Ruiz et al. [53] to capture the structure of a system: the product between the entropy and a distance between the estimated model and a non-informative model is an interesting way of measuring complexity. Lamberti et al. [41], using that idea, proposed using the Euclidean distance between the measured probability function and the uniform distribution. Rosso et al. [77] discussed using other distances, proposed the Jensen-Shannon distance, and used it jointly with the scaled Shannon entropy to form a bivariate feature. They mapped this feature into the so-called "Entropy-Complexity Plane," devising a powerful diagnostic tool to distinguish between different dynamical regimes, such as chaos, noise, and periodicity. Further, Martin et.al. [55] discussed the boundaries of this generalized statistical complexity measure.

In the following, we present the research question and the motivation for continuing this research.

2.2 Research Question and Motivation

The primary research question guiding this study is:

> *How can confidence intervals for generalized entropy measures (Shannon, Tsallis, Rényi, Fisher information measure) and their associated complexity metrics be used to improve the robustness and discriminative power of time series clustering techniques?*

We are motivated to conduct a literature review to confirm the relevance of our research areas in relation to the research question. Our aim is to determine whether other researchers are engaging with similar types of questions. Additionally, we seek to verify whether there is a strong focus on practical applications within this topic.

2.3 The Bibliometric Analysis: data collection, tools and background

Bibliometric analyses provide a quantitative approach to reviewing and mapping the intellectual structure of a research field. By systematically analyzing citation patterns, author collaborations, and keyword co-occurrences, they help identify key themes, research trends, and emerging topics. This method ensures objectivity, reveals key contributions, and offers a structured overview of intellectual development, making it a valuable tool for systematic literature reviews.

In this study, we conducted a bibliometric analysis using the Bibliometrix package in R and its user-friendly web interface Biblioshiny [11] focusing on literature related to ordinal patterns, permutation entropy, and complexity measures in time series analysis.

Section 2.3.1 discusses the data extraction process using the Bibliometrix package in R.

2.3.1 Conceptual Structure: Data extraction and Summary Statistics

Scopus-indexed references that cited the seminal work by Bandt and Pompe, along with other references relevant to our research topic, were collected on June 9, 2025. Based on these reference files, we analyzed a dataset consisting of 4125 reference files spanning the years 1993 to 2025. The descriptive analysis of the dataset revealed a total of 4063 usable documents (out of 4125; the others had missing data and were removed), sourced from 1317 publication sources. The dataset shows an annual growth rate of 18.15 %, involving 7254 authors, 123 single-authored documents, 27.32 % international co-authorship, with an average of 4.28 co-authors per document. The author keywords totaled 7667, with an average document age of 5.7 years, and 22.6 citations per document.

2.3.2 Thematic Map Analysis

Thematic map helps to understand the research direction and the relevant topics for the future studies. Therefore, we motivated to analyse it. The axes of the thematic map depicts the strength of their internal (density), which reflects inter-cluster growth, and external (connectivity) relevance or significance of the study in a particular area (centrality).

Figure 2.1 illustrates the thematic map derived from author keywords.

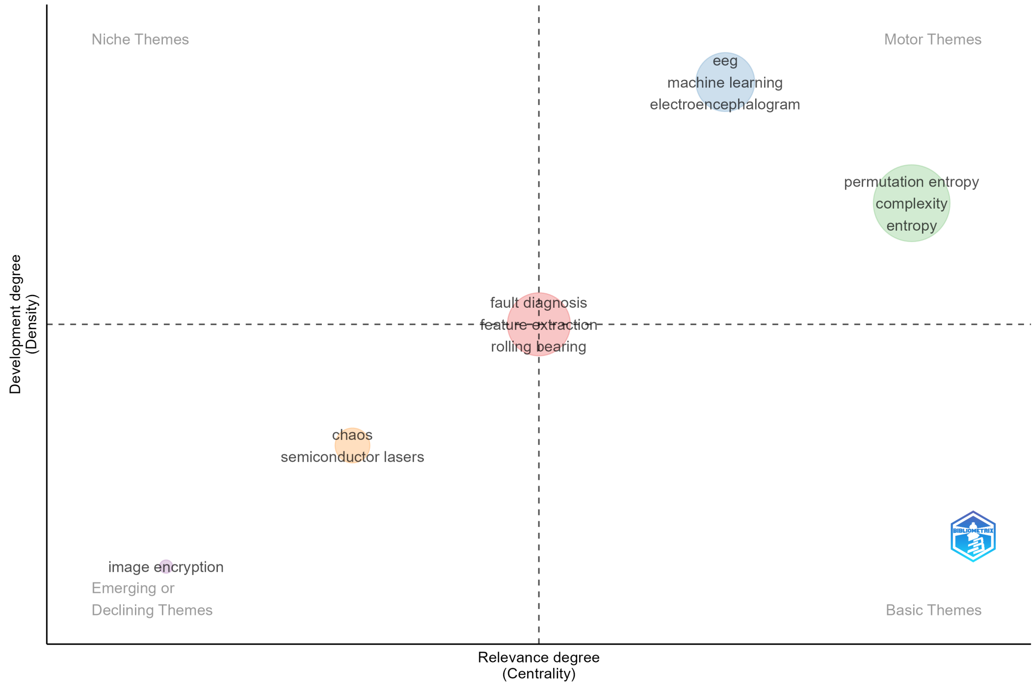

Figure 2.1: The Thematic Map generated by Bibliometrix.

The map is divided into four quadrants based on two dimensions: centrality (relevance) and density (development). A right upper quadrant representing motor themes that are both well developed and highly interconnected areas of research, such as EEG, machine learning, permutation entropy, complexity, and other types of entropy. This theme is also at the core of current research, particularly in areas such as biomedical signal processing and nonlinear time series analysis. The consistent presence of entropy-based measures and machine learning highlights the interface between theory and practice.

Emerging or declining themes like image encryption reflect peripheral or potentially declining research interests, while chaos and semiconductor lasers hold theoretical interest, but its practical integration appears limited.

A right down quadrant depicts basic themes indicates that opportunities for further theoretical and methodological advancement. Research fields such as fault diagnosis, feature extraction and rolling bearings are at the center of the map. Its moderate centrality and density mean that it is still active and is subject to evolving research fields. These topics are closely linked to engineering and diagnostic applications. These are important and growing areas which require further methodological refinement and integration.

The size of each circle further represents the frequency of the topic based on keyword occurrences associated with the publications. The clustering structure shows that entropy-related measures (such as the entropy of a permutation and the complexity of a system) are gaining ground not only in theory but also in practical applications such as EEG and machine learning.

From these results, we conclude that our research focus on entropy and complexity of permutation is relevant for many studies. These are basic concepts which are widely used in the literature, but which offer considerable potential for further development. Moreover, this thematic map shows that research is strongly focused on entropy-based methods, machine learning and biomedical applications, with fault diagnosis and feature extraction emerging as promising intermediate topics for further discuss.

2.3.3 Factorial Analysis

To identify the broad overview of the main research topics, we analyze the conceptual structure map. In this case we considered all keywords which are automatically generated by indexing databases. The shaded polygon in Figure 2.2 outlines the conceptual space defined by the most distinctive keywords. The X-axis (Dim 1) and the Y-axis (Dim 2) are the first two dimensions of the factorial space, and explain the largest differences in the co-occurrence of the keywords.

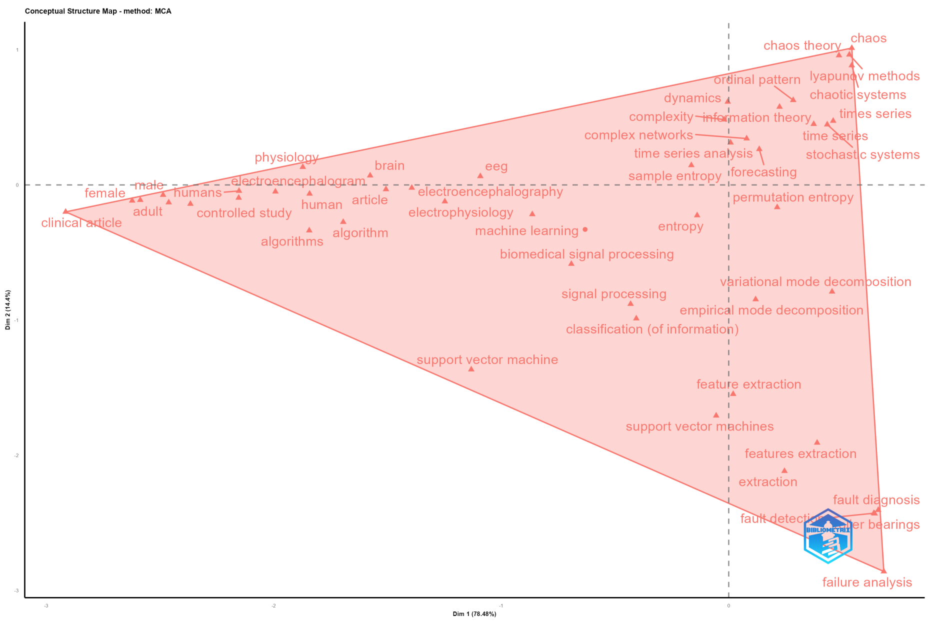

Figure 2.2: Conceptual Structure map generated by Bibliometrix.

The conceptual structure map reveals four major clusters:

The first cluster (upper right) includes theoretical concepts such as chaos theory, chaotic systems, Lyapunov methods, ordinal pattern, information theory, complex networks, and permutation entropy, forming the theoretical core of the research area.

The second cluster (lower right) includes practical applications such as feature extraction, empirical mode decomposition, variational mode decomposition, fault diagnosis, fault detection, and failure analysis, emphasizing the applied relevance of complexity-based time series analysis.

The third cluster (center) comprises terms related to biomedical signal processing and machine learning, including biomedical signal processing, EEG, support vector machine, classification, and machine learning, indicating the multidisciplinary applications of entropy measures.

A fourth, smaller cluster (left) includes clinical and physiological study keywords such as clinical article, controlled study, adult, male, female, humans, and physiology.

The conceptual map thus provides further evidence of the diverse applications and theoretical development surrounding ordinal patterns and complexity measures, supporting the originality and relevance of the present research.

2.3.4 Conclusion and Justification of Research Focus

Based on the thematic map and factorial analysis, research topics are categorized into four clusters and we will structure our review of the literature based on this clusters:

- Permutation entropy, Complexity and other types of Entropy (section 2.4);
- EEG and Machine learning approach (section 2.5);
- Fault diagnosis and failure analysis (section 2.6);
- Chaos other research works (section 2.7);

2.4 Permutation entropy, Complexity and other types of Entropy

This cluster is associated with the keywords time series analysis, statistical complexity, nonlinear dynamics, ordinal patterns, entropy, and information theory. Although ordinal pattern based research have gained wide recognition as powerful tools for nonlinear time series analysis, with applications ranging from biomedical signal processing to cyber-physical systems, their theoretical and practical challenges remain unsolved [37, 102].

Li et.al. [47] asses the complexity of short-term heartbeat interval series by using distribution entropy. They found that sample entropy (SampEn) or fuzzy entropy (FuzzyEn) quantifies essentially the randomness, which may not be uniformly identical to complexity. Zhang et.al. [103] uses the ordinal pattern technique to analyze dynamic behaviors such as regular, chaotic, or random patterns. The paper highlights the application of ordinal patterns in various fields, including ecology, finance, and physiology, where assessing variability and complexity is essential. It also emphasizes the advantages of using ordinal models to identify structural differences in time series that may not be detectable through traditional statistical methods.

Several studies have formalized the statistical behaviour of entropy and complexity measures derived from ordinal patterns, and have provided insights into their asymptotic distribution [74, 73] and sensitivity under the Multinomial law [75]. Despite these advances, the reliability of these measures in finite sampling conditions, in particular in non-stationary and noisy environments, remains limited [19].

Applications such as Internet of Things(IoT) botnet detection [19] and synthetic aperture radar(SAR) structure classification [23] have demonstrated the discriminability of ordinal-based features, particularly when combined with multiscale analysis; however, these methods often depend on the optimisation of parameters (dimension, delay) and may lack generalizability to a wide range of data. Similarly, the integration of entropy-complexity representations of class separation in time series dynamics [20], although effective in a structured environment, poses problems when extended to real-time or data-scarce scenarios.

The conceptual works linking ordinal complexity to broader ideas, such as the technological singularity [59] and the development of artistic expression [88] illustrate the richness of the framework, but these studies are often qualitative and lack empirical rigour.

Entropy-based clustering techniques have revealed evolving efficiency patterns in cryptocurrency markets [87]. The reliance on sensitive parameters and lack of standardized benchmarks highlight the need for more robust and interpretable methods to track market maturation reliably.

Moreover, white noise testing using entropy-complexity plane [24] and the use of ordinal properties in compressor signal diagnostics [15] demonstrate the methodological versatility of ordinal approaches. , the lack of uniform and reliable reference points prevents cross-domain comparison. Therefore, while ordinal methods provide a mathematically elegant and computationally efficient basis for obtaining information from complex signals, further research is needed to improve their statistical robustness, interpretability and adaptability to the challenges of the real world.

Although ordinal patterns are widely used in time series analysis, particularly in deriving entropy measures, studying their asymptotic distribution under the multinomial law, and analyzing the behavior of permutation entropy, their integration with confidence intervals remains widely unexplored. A systematic literature review reveals that while many studies investigate the statistical properties, asymptotic behavior, and robustness of ordinal pattern-based measures, no work has addressed the derivation or application of confidence intervals to assess the reliability or significance of time series clustering. This lack of statistical property limits interpretability and inference, especially when ordinal patterns are applied in data-driven algorithms. Therefore, this study aims to fill this gap by investigating how confidence intervals can be constructed for Shannon, Tsallis, and Renyi entropies, Fisher information measure, complexities, and how they may enhance time series clustering.

2.5 EEG and Machine learning approach

This cluster is based on multidisciplinary studies involving entropy measures applied to areas such as EEG, epilepsy, classification, heart rate variability, deep learning methods, and nonlinear analysis. An analysis of the author keywords indicates that many of these studies are centered on applications in biomedical signal processing. Acharya et al. [1, 2, 3, 4, 5, 6, 7] have made significant contributions to biomedical signal processing by developing and evaluating advanced automated diagnostic tools across a wide range of clinical applications. Their work includes the use of entropy measures to detect epilepsy and heart disease from EEG and ECG signals, as well as the use of nonlinear dynamics to enhance the detection of sleep stages and the characterisation of focal EEG signals. In the cardiovascular field, they have performed comparative studies on the localization of myocardial infarction using various ECG leads and have developed empirical decomposition methods to identify congestive heart failure from cardiac signals. Another study by Lajnef et.al. [40] revealed that an automated approach to the classification of sleep stages using a multi-class support vector machine (SVM) based decision tree approach. The proposed method uses physiological signals (such as EEG, EOG, and EMG) to effectively classify the different stages of sleep.

2.6 Fault diagnosis and failure analysis

This section primarily relates to the engineering applications of complexity-based time series analysis. The author keywords most commonly used to categorize this cluster include fault diagnosis, feature extraction, rolling bearing, support vector machine, variational mode decomposition, multiscale permutation entropy, dispersion entropy, rotating machinery, and empirical mode decomposition.

Recent studies in bearing fault diagnosis often use entropy-based methods. Multiscale permutation entropy (MPE) and dispersion entropy are two popular techniques. These methods help detect complex changes in signals under different working conditions. For example, using variational mode decomposition with weighted entropy features helps extract useful information from non-stationary vibration signals. This improves how well faults can be classified [45]. The weighted multiscale entropy method also works well by focusing on important frequency parts of the signal [58]. Self-adaptive hierarchical multiscale fuzzy entropy is also applied in bearing fault diagnosis. It makes fault detection easier without needing many manual settings [99]. Composite multiscale fluctuation dispersion entropy can detect small fault signs even in noisy signals [34]. Some methods combine data decomposition with multiscale permutation entropy to better handle complex, changing systems [101]. These improved entropy methods are also used in medical signal analysis, such as ECG or EEG, showing they work in other areas [12, 36]. Dispersion entropy is known for being fast and good at finding small signal changes [79, 80], whereas multiscale permutation entropy still has issues. It can be affected by the length of the signal, noise, and it can be slow to compute [36, 104]. Multiscale Permutation Entropy (MPE) with the Natural Visibility Graph (NVG) to enhance the fault diagnosis of rolling bearings by capturing both the dynamic complexity and structural features of time series method is proposed by Ma et.al. [54]. However, the method may still face limitations related to computational cost, parameter sensitivity, and the requirement for relatively long and noise-free signals to ensure reliable multiscale analysis.Therefore, more research is needed to make these methods faster, better with noise, and easier to use in different fault diagnosis tasks.

2.7 Chaos and Other research works

Research works related to chaos, semiconductor lasers, and image encryption are discussed in this category. A self-synchronous chaotic stream cipher, designed to resist active attacks and limit error propagation during image transmission, is a novel technique for image encryption. [32]. The 2D discrete wavelet transform, Arnold mapping, and a four-dimensional hyper-chaotic system with positive Lyapunov exponents are used to enhance the security and complexity of the encryption method. The advancement of chaos-based encryption and intelligent video security techniques in modern information systems has been demonstrated through a successful hardware implementation that transforms non-chaotic systems into chaotic ones, significantly enhancing unpredictability for secure communication. In addition, temporal action segmentation for video encryption has been analyzed to optimize computational resources and improve data protection [35, 52]

2.8 Statistical Properties of Features from Ordinal Patterns

Although ordinal pattern based methods, such as permutation entropy, have been widely used for nonlinear time series analysis, the statistical properties of the features derived from these patterns, such as their distribution, variance, and confidence intervals remain under-explored and require further theoretical and empirical investigation. Therefore, based on the literature review we investigate the researchers who worked related to ordinal patterns, what kind of statistical properties of features used for their research work. Table 2.1 provides more information about the research articles and the test statistics or distributions they used.

Table 2.1: The test statistics used by the research articles for hypothesis testing

Paper Title/Reference	**Distribution**	**Brief Description**
A non-parametric independence test using permutation entropy. Matilla-García et.al. [56]	Empirical (permutation) distribution	Tests independence by comparing the observed permutation entropy-based statistic to its distribution under random shuffling (permutation) of the time series.
Asymptotic distribution of certain types of entropy under the multinomial law. Rey at.at. [75]	Normal, Chi-squared (χ^2)	Analyzes the asymptotic distribution (normal and chi-squared) of entropy estimators under the multinomial law.

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Table 2.1 – continued from previous page

Paper Title/Reference	**Distribution**	**Brief Description**
Asymptotic distribution of entropies and Fisher information measure of ordinal patterns with applications. Reyet.al. [74]	Normal, Chi-squared (χ^2)	Provides asymptotic distributions for entropy and Fisher information measures of ordinal patterns.
Asymptotic distribution of the permutation entropy. Rey at.al. [73]	Normal	Derives the asymptotic normal distribution for permutation entropy estimators.
Asymptotic distribution of the statistical complexity under the multinomial law. Rey at.al. [72]	Normal, Chi-squared (χ^2)	Studies the asymptotic distribution of statistical complexity measures derived from ordinal patterns under the multinomial law.
Assessing serial dependence in ordinal patterns processes using chi-squared tests with application to EEG data. Yamashita et.al. [97]	Chi-squared (χ^2)	Applies chi-squared tests to assess serial dependence in ordinal pattern processes, with application to EEG data.
Bearing fault diagnosis based on Alpha-stable distribution feature extraction and SVM classifier. Chouri et.al. [25]	Alpha-stable	Uses alpha-stable distribution parameters for feature extraction and SVM for classification in bearing fault diagnosis.
Belief permutation entropy of time series: A natural transition in analytical framework from probability theory to evidence theory. Xie et. al. [96]	Belief functions, Evidence theory basis	Introduces belief permutation entropy (BPE) using evidence theory (belief functions and mass assignments) and Deng entropy instead of classical probability distributions.

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Table 2.1 – continued from previous page

Paper Title/Reference	**Distribution**	**Brief Description**
Markov modeling via ordinal partitions: An alternative paradigm for network-based time-series analysis. Sakellariou et.al. [82]	Markov (transition) probabilities	Uses Markov models constructed from ordinal partitions to analyze time series as networks.
On the statistical properties of Multiscale Permutation Entropy: Characterization of the estimator's variance. Dávalos et.al. [31]	Normal (asymptotic)	Shows that the multiscale permutation entropy estimator is asymptotically normally distributed, allowing inference on variance.
Price predictability at ultra high frequency:Entropy based randomness test. Shternshis et.al. [85]	Chi-squared (χ^2)	Proposes a randomness test for time series based on entropy, with asymptotic chi-squared distribution for the test statistic.
Statistical properties of the entropy from ordinal patterns. Chagas et.al. [22]	Normal	Investigates statistical properties and asymptotic normality of entropy estimators from ordinal patterns.
The asymptotic distribution of the permutation entropy. Rey at.al. [73]	Normal	Derives the asymptotic normal distribution for permutation entropy estimators.
The modified permutation entropy-based independence test of time series. Ashtari et.al. [65]	Empirical (permutation) distribution	Tests independence by comparing the observed permutation entropy-based statistic to its distribution under random shuffling (permutation) of the time series.

Continued on next page

Table 2.1 – continued from previous page

Paper Title/Reference	**Distribution**	**Brief Description**
White Noise Test from ordinal patterns in the entropy-complexity plane. Chagas et.al. [24]	Empirical Distribution	Used in white noise tests by repeatedly shuffling the time series to estimate the null distribution of the test statistic.
Variance of entropy for testing time-varying regimes with an application to meme stocks Shternshis et.al. [86]	Chi-squared (χ^2)	Introduces a rigorous hypothesis testing framework featuring an unbiased analytic approximation of sample Shannon entropy's variance and an optimal rolling-window selection method.

Chapter 3

The Research Project

In this chapter, we outline the main ideas and objectives of this research project. Section 3.1 discusses entropy and complexity analysis in time series, highlighting its advantages and limitations. Section 3.2 examines the advantages and limitations of the Bandt and Pompe method. Section 3.3 provides background knowledge on the entropy-complexity plane. Section 3.4 explores the entropy-complexity plane for a broad class of time series. Section 3.5 provides the asymptotic distribution of the entropy. Finally, the chapter concludes with the objectives of the research project and a case study related to our work.

3.1 Entropy and Complexity Analysis in Time Series: Advantages and Limitations

Entropy and complexity analysis provides powerful tools for characterizing the unpredictability and structural richness of dynamical systems, which evolve over time. Entropy measures, such as Shannon entropy (quantifying uncertainty in a probability distribution) and permutation entropy (which measures the order structure of a time series through ordinal patterns), are widely used to assess randomness and disorder. The main distinction is that permutation entropy computes Shannon entropy on the ordinal patterns extracted from a time series data.

Complexity measures complement entropy by evaluating the balance between order and chaos. Together, entropy and complexity are particularly effective for detecting nonlinear patterns, relationships captured by nonlinear models, where inputs and outputs are not proportional and small changes can produce large or unpredictable effects. Such behavior frequently appears in biological, financial, or climate systems.

While these methods reveal hidden structures and irregular dynamics beyond the reach of traditional linear approaches, they also require careful preprocessing, appropriate parameter selection (e.g., embedding dimension and time delay), and sufficient domain knowledge. Without this, interpretations may be misleading.

Despite these challenges, entropy and complexity remain essential in modern time series analysis. Unlike linear techniques, such as autocorrelation, regression, or Fourier analysis, that assume proportional and stationary relationships, entropy and complexity measures are designed to capture irregularities, pattern changes, and chaotic dynamics. This makes them invaluable for studying complex and nonlinear systems where conventional tools often fall short.

3.2 The Bandt and Pompe Method: A Robust Approach

The concept of ordinal patterns in time series can be effectively studied through real world examples. Traditionally, numerous algorithms, techniques, and heuristics have been employed to estimate complexity measures from real world data.

However, these methods often perform well only for low-dimensional dynamical systems and struggle when noise is introduced. Low-dimensional dynamical systems are systems whose behavior can be described using a small number of variables or equations, typically two or three, such as the logistic map, or pendulum. These systems exhibit rich and often chaotic dynamics but remain mathematically tractable and easier to analyze using entropy and complexity measures. Because of their limited dimensionality, the patterns within the data are more distinct, making it easier to extract meaningful information.

The Bandt and Pompe method overcomes this limitation by providing a robust approach that remains reliable even in noisy environments. In time series analysis, key complexity parameters such as entropy, fractal dimension, and Lyapunov exponents play a crucial role in comparing neighboring values and uncovering the underlying structure and dynamics of the data. A Lyapunov exponent measures the average rate at which nearby trajectories in a dynamical system diverge or converge. It provides deeper understanding of system's behavior.

The advantages of Bandt & Pompe methods:

- Simplicity
- Extremely fast calculation
- Robustness
- Invariance to nonlinear monotonous transformations

This method exhibits low sensitivity to noise and naturally accounts for the causal order of elements in a time series. As a result, it can be applied to various real-world problems, particularly in differentiating between chaotic and stochastic signals.

Despite its limitations, researchers have developed extensions to the original method to address its shortcomings and enhance its applicability to a broader range of complex systems.

3.3 Statistical Complexity measures

Bandt and Pompe introduced a highly effective method for analyzing time series within this framework. They calculated Shannon entropy based on the histogram of causal patterns and successfully identified chaotic components in sequences of words, among other applications.

Later, Rosso et al. [77] expanded this analysis by introducing an additional dimension: the statistical complexity derived from the same histogram of causal patterns. The authors have contributed to a wide range of applications. This approach, which utilizes the entropy-complexity plane, has been successfully applied to the visualization and characterization of different dynamical regimes as system parameters change [13, 21, 28, 38, 39, 76, 107, 108], as well as to optical chaos [51, 91, 93, 100, 106], hydrology [42, 84, 92], geophysics [26, 81, 90], engineering [9, 10, 70, 98], biometrics [78], characterization of pseudo-random number generators [29, 30], biomedical signal analysis [46, 48, 49, 50, 60, 61, 62, 63, 64, 66, 68, 69, 102], and econophysics [16, 17, 18, 102, 105, 109, 110], to name a few.

After computing all symbols as described in Chapter 1, the histogram proportions are used to estimate the probability distribution of ordinal patterns. From this distribution, two key descriptors are calculated to characterize the time series:

1. Entropy
2. Statistical complexity

The most common metric for the first descriptor is the normalized Shannon entropy, defined as:

$$H(\mathbf{p}) = -\frac{1}{\log k}\sum_{\ell=1}^{k} p_\ell \ln p_\ell. \tag{3.1}$$

Here, $k = D!$ represents the total number of possible permutation patterns. This entropy is bounded within the unit interval:

- It reaches its minimum value ($H = 0$) when a single pattern dominates, i.e., $p_\ell = 1$ for some ℓ.
- It achieves its maximum ($H = 1$) under uniform probability $p_\ell = 1/k$ for all ℓ.

This normalized entropy is often termed permutation entropy in time series analysis.

While normalized Shannon entropy is a powerful tool for quantifying disorder, it fails to fully characterize complex dynamics. To address this limitation, López-Ruiz et al. [53] introduced the disequilibrium Q concept, which quantifies the deviation of a probability distribution $\mathbf{p}$ from a uniform (non-informative) equilibrium state. López-Ruiz and the team employed the Euclidean distance between $\mathbf{p}$ and the uniform distribution, providing a complementary metric to Shannon entropy for assessing structural complexity in systems.

The Jensen-Shannon distance between histogram of proportion $\mathbf{p}$ and the uniform probability function $\mathbf{u} = (1/k, 1/k, \dots, 1/k)$, where $k = D!$ corresponds to the number of possible permutation patterns provides a robust metric for quantifying deviations from uniformity. This distance measure, derived from the symmetric Jensen-Shannon divergence, is particularly suited for analyzing ordinal pattern distributions due to its ability to capture both structural differences and statistical disequilibrium in time series data. It is defined as:

$$Q'(\mathbf{p}, \mathbf{u}) = \sum_{\ell=1}^{k} p_\ell \log \frac{p_\ell}{u_\ell} + u_\ell \log \frac{u_\ell}{p_\ell}. \tag{3.2}$$

Lamberti et al. [41] proposed Jensen-Shannon distance as a symmetric metric rooted in the Jensen-Shannon divergence. As the reference model, most works consider the uniform distribution $\mathbf{u} = (1/k, 1/k, \dots, 1/k)$. The normalized disequilibrium is defined as follows

$$Q = \frac{Q'}{\max(Q')}, \tag{3.3}$$

where $\max(Q')$ is defined as follows

$$\max(Q') = -2\left[\frac{k+1}{k}\log(k+1) - 2\log(2k) + \log k\right]. \tag{3.4}$$

With this, Lamberti et al. [41] proposed complexity as a measure of the statistical complexity of the underlying dynamics, which is defined as

$$C = HQ, \tag{3.5}$$

where both H and Q are normalized quantities, therefore C is also normalized.

3.4 The Entropy Complexity Plane

The entropy-complexity plane is a two-dimensional representation where time series are mapped based on their entropy and statistical complexity. These metrics are derived from ordinal pattern distributions obtained through embedding dimension D that are mapped on histograms of $D!$ bins.

3.4.1 Key Dynamics in the plane

1. Highly Ordered Systems, where the behavior is very predictable, structured, and often repeats in a regular pattern over time.

   Example: Strictly monotonic time series.

   - Produces a single ordinal pattern ($H = 0$).
   - Maximal disequilibrium (distance from uniform distribution).
   - Maps to $(0, 0)$, indicating minimal complexity.

2. Perfectly Random Systems

   Example: White noise

- Uniform ordinal pattern distribution ($H = 1$).
- Disequilibrium vanishes (distance $= 0$).
- Maps to $(1, 0)$, reflecting maximal entropy without structural complexity.

The two extreme values are proved by Anteneodo & Plastino [8]. Expressions for the boundaries, derived using geometrical arguments within space configurations, were proposed by Martin et al. [55]. These formulations provide a structured approach to understanding and analyzing the spatial behavior of specific systems or models. The lower boundary is characterized by a smooth curve, whereas the upper boundary consists of $D! - 1$ distinct segments. As the embedding dimension D approaches infinity, the upper boundary gradually converges into a smooth curve. Example for the entropy complexity plane is shown in Figure 3.1. Further, ten time series and their points in the $H \times C$ plane for embedding dimension 6 according to our application is shown in Figure 3.2.

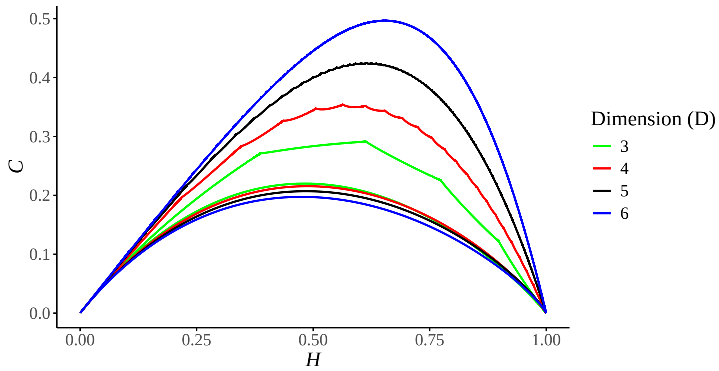

Figure 3.1: Entropy Complexity Plane for Embedding dimension 3, 4,5, and 6

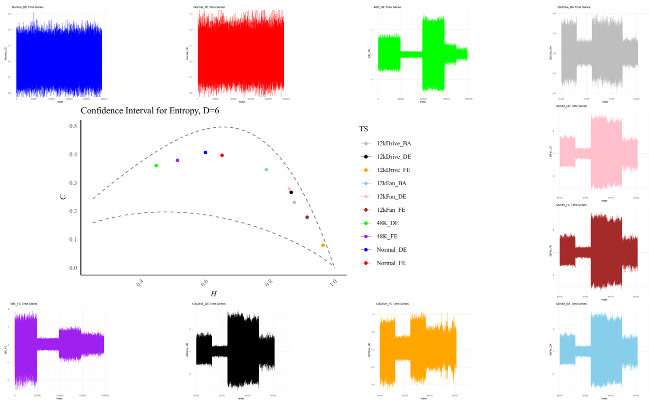

Figure 3.2: Time series plots and their points in the $H \times C$ plane for Embedding dimension 6

3.5 Asymptotic Distribution of the permutation entropy

Ordinal patterns are a robust symbolic transformation method that enables the unveiling of latent dynamics in time series data. This approach involves constructing histograms of patterns and calculating both entropy and statistical complexity. According to the literature, determining the exact distribution of features derived from ordinal patterns is challenging; therefore, researchers have extensively investigated this distribution and its related statistical properties. Two types of statistical distributions are discussed in the literature: the empirical distribution and the asymptotic distribution.

The empirical distribution is used in conjunction with knowledge of the expected variability of entropy and complexity, allowing hypothesis tests to be performed for a wide variety of models according to the underlying dynamics. Results in this direction can be found in the literature [24, 29, 43, 44]. In this approach, researchers construct empirical distributions directly from observed data, which reflect the actual frequencies of patterns or outcomes.

Furthermore, two types of asymptotic distributions are discussed: the first assumes that patterns are independent and identically distributed (multinomial model), while the second accounts for dependent patterns.

3.5.1 Asymptotic Distribution of the Shannon Entropy under the Multinomial Model

The multinomial distribution models counts of observations in k mutually exclusive categories $\pi^1, \pi^2, \ldots, \pi^k$ from n independent trials, with probability vector $\mathbf{p} = (p_1, p_2, \ldots, p_k)$, $p_\ell \geq 0$, $\sum_{\ell=1}^{k} p_\ell = 1$. Let $\mathbf{N} = (N_1, \ldots, N_k)$ denote category counts, $\sum_{\ell=1}^{k} N_\ell = n$. Its probability mass function is

$$\Pr(\mathbf{N} = \mathbf{n}) = n! \prod_{\ell=1}^{k} \frac{p_\ell^{n_\ell}}{n_\ell!}, \quad \mathbf{N} \sim \mathrm{Mult}(n, \mathbf{p}),$$

with moments

$$E(N_\ell) = np_\ell, \quad \mathrm{Var}(N_\ell) = np_\ell(1 - p_\ell), \quad \mathrm{Cov}(N_\ell, N_j) = -np_\ell p_j.$$

The maximum likelihood (ML) estimator $\widehat{p}_\ell = N_\ell/n$ satisfies $n\widehat{\mathbf{p}} \sim \mathrm{Mult}(n, \mathbf{p})$. For any smooth function $g(\mathbf{p})$, the plug-in estimator $\widehat{g}(\mathbf{p}) = g(\widehat{\mathbf{p}})$ is also ML, enabling the asymptotic distribution of Shannon entropy

$$H(\mathbf{p}) = -\sum_{\ell=1}^{k} p_\ell \log p_\ell, \tag{3.6}$$

to be derived from the asymptotic normality of $\widehat{\mathbf{p}}$ as $n \to \infty$. The Shannon's entropy of a multinomial distributed random variable is bounded between 0 and $\ln k$. The minimum is attained when $p_\ell = 1$ for some $1 \leq \ell \leq k$ and $p_j = 0$ for every $j \neq \ell$. The expression is maximized by $p_\ell = 1/k$ for every $1 \leq \ell \leq k$.

Further, asymptotic distribution can be described as follows. Let $X_n = (X_{1n}, X_{2n}, \ldots, X_{kn})$ be a sequence of independent and identical distributed random vectors, with distribution Mult$(n, \mathbf{p})$. If $\widehat{\mathbf{p}}$ denotes the vector of sample proportions, and

$$\mathbf{Y}_n = \sqrt{n}(\widehat{\mathbf{p}} - \mathbf{p})$$

then

$$E(\mathbf{Y}_n) = \mathbf{0},$$
$$\text{Cov}(\mathbf{Y}_n) = \mathbf{D}_\mathbf{p} - \mathbf{p}\mathbf{p}^T,$$

where $\mathbf{D}_\mathbf{p} = \text{Diag}(p_1, p_2, \ldots, p_k)$, and the superscript T denotes transposition. The asymptotic distribution of $\mathbf{Y}_n$ is multivariate normal with mean vector $\mathbf{0}$ and covariance matrix $\mathbf{D}_\mathbf{p} - \mathbf{p}\mathbf{p}^T$, denoted as

$$\mathbf{Y}_n \xrightarrow{\mathscr{D}} N(\mathbf{0}, \mathbf{D}_\mathbf{p} - \mathbf{p}\mathbf{p}^T). \tag{3.7}$$

Our focus is on the statistical properties of $H(\mathbf{p})$ when $\mathbf{p}$ is replaced by its maximum likelihood estimate $\widehat{\mathbf{p}} = (\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_k)$. The problem thus reduces to determining the distribution of $H(\widehat{\mathbf{p}})$.

$$\begin{aligned} H(\widehat{\mathbf{p}}) &= -\sum_{\ell=1}^{k} \widehat{p}_\ell \log \widehat{p}_\ell \\ &= -\sum_{\ell=1}^{k} \frac{N_\ell}{n} \log \frac{N_\ell}{n} \\ &= \log n - \frac{1}{n} \sum_{\ell=1}^{k} N_\ell \log N_\ell, \end{aligned} \tag{3.8}$$

under $N = (N_1, N_2, \ldots, N_k) \sim$ Mult$(n, \mathbf{p})$

For the asymptotic distribution case, we refer to the Delta Method theorems and their multivariate version.

Theorem 1 (Delta Method, univariate). *Let X_n be a sequence of independent and identically distributed random variables such that $\sqrt{n}(X_n - \theta) \xrightarrow{\mathscr{D}} N(0, \sigma^2)$. Consider a function h such that $h'(\theta)$ exists and $h'(\theta) \neq 0$. Then,*

$$\sqrt{n}\,[h(X_n) - h(\theta)] \xrightarrow{\mathscr{D}} N\big(0,\, \sigma^2[h'(\theta)]^2\big)\,.$$

Theorem 2 (Delta Method, multivariate)**.** *Let $\mathbf{X}_n = (X_{1n}, X_{2n}, \ldots, X_{kn})$ be a sequence of independent and identically distributed random vectors such that*

$$\sqrt{n}\,(\mathbf{X}_n - \boldsymbol{\theta}) \xrightarrow{\mathscr{D}} N_k(\mathbf{0}, \Sigma),$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ and Σ is the covariance matrix. Suppose that $h\colon \mathbb{R}^k \to \mathbb{R}^m$ is continuously differentiable in a neighborhood of $\boldsymbol{\theta}$, with Jacobian matrix

$$B = \left(\frac{\partial h_i}{\partial \theta_j}\right)_{i,j=1}^{k},$$

which is non-singular at $\boldsymbol{\theta}$. Then,

$$\sqrt{n}\,\big(h(\mathbf{X}_n) - h(\boldsymbol{\theta})\big) \xrightarrow{\mathscr{D}} N_m\big(\mathbf{0},\, B\Sigma B^T\big)\,.$$

Further, covariance matrix of Equation (3.7) can be expressed as

$$(\mathbf{D}_{\mathbf{p}} - \mathbf{p}\mathbf{p}^T)_{\ell j} = \begin{cases} p_\ell(1 - p_\ell), & \text{if } \ell = j, \\ -\,p_\ell p_j, & \text{if } \ell \neq j, \end{cases} \tag{3.9}$$

for $1 \leq \ell, j \leq k$. Even for dependent processes (e.g., Markov chains), normalized Shannon entropy converges to a normal distribution with variance determined by the covariance structure of $\widehat{\mathbf{p}}$.

Rey et. al. [72] derived the asymptotic distribution of statistical complexity, defined as normalized Shannon entropy times normalized Jensen–Shannon divergence from the uniform distribution under the multinomial model. They demonstrated convergence to normality, with variance and bias reflecting the system's dynamics. Numerical studies confirm robustness even when the multinomial model is approximate, such as in Bandt–Pompe ordinal patterns.

We refer the asymptotic equation for the mean and variance provided by Rey et. al. [72] for our research work.

$$\mu_{n,\widehat{\mathbf{p}}} = H(\widehat{\mathbf{p}}) = -\sum_{\ell=1}^{k} \widehat{p}_\ell \log \widehat{p}_\ell \tag{3.10}$$

The estimator in (3.10) is normally distributed with mean $H(\mathbf{p})$ and variance $\sigma^2_{n,\mathbf{p}}$, where

$$\sigma^2_{n,\mathbf{p}} = \frac{1}{n}\sum_{\ell=1}^{k} p_\ell(1-p_\ell)(\log p_\ell + 1)^2 - \frac{2}{n}\sum_{j=1}^{k-1}\sum_{\ell=j+1}^{k} p_\ell p_j(\log p_\ell + 1)(\log p_j + 1). \tag{3.11}$$

3.5.2 Asymptotic distribution of Permutation Entropy under the pattern dependence

As we discussed earlier, real-valued time series $\mathbf{x} = \{x_1, x_2, \ldots, x_{n+D-1}\}$ transform into the series symbols $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ from sub-sequences of embedding dimension D, where we considered $D! = k$. Due to the overlapping of time windows, the ordinal patterns which we calculate are dependent. For $i = 1, 2, \ldots k$, let p_i be the probability of observing the state π_i, denote the vector probabilities, $\mathbf{p} = \{p_1, p_2, \ldots, p_k\}$ and express as $\mathbf{D_p} = \text{Diag}(p_1, p_2, \ldots, p_k)$ the diagonal matrix. The transition probability of reaching state π_j at time $t+r$ from the state π_i at time t, for $r = 1, 2, \ldots, D-1$, is denoted by $p_{ij}^{(r)}$. These transition probabilities can be collected in the matrix $\mathbf{Q}^{(r)}$ whose elements are $p_{ij}^{(r)}$. As describe in the Rey et al. [73] when n is sufficiently large, asymptotic variance of the Shannon entropy defined as follows:

$$\begin{aligned}\widehat{\nu}^2_{\widehat{\mathbf{p}}} = \widehat{\sigma}^2_{\widehat{\mathbf{p}}} + &\sum_{i=1}^{k}(\log p_i + 1)^2\left[(2D-2)p_i^2 + 2\sum_{r=1}^{D-1}\mathbf{Q}_{ii}^{(r)}\right] \\ &- 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(\log p_i + 1)(\log p_j + 1) \\ &\times\left[(2D-1)p_i p_j - \sum_{r=1}^{D-1}\left(\mathbf{Q}_{ij}^{(r)} + \mathbf{Q}_{ji}^{(r)}\right)\right].\end{aligned} \tag{3.12}$$

For practical purposes, given n sufficiently large, the estimator in (3.6) can be approximated by a normal distribution with mean $H(\widehat{\mathbf{p}})$ and variance $\widehat{\nu}^2_{\widehat{\mathbf{p}}}/n$. In addition, for $\alpha \in (0,1)$ and sufficiently large n, the $(1-\alpha)100\,\%$ confidence interval for the estimated entropy is given below.

$$H(\widehat{\mathbf{p}}) \pm \frac{Z_{\alpha/2}\widehat{\nu}_{\widehat{\mathbf{p}}}}{\sqrt{n}}, \tag{3.13}$$

where $Z_{\alpha/2}$ is the $\alpha/2$-quantile of a standard normal random variable.

For convenience, Equation 3.13 is referred to and defined as follows.

$$H(\widehat{\mathbf{p}}) \pm \textbf{Semi Length}, \tag{3.14}$$

where **Semi Length** $= \dfrac{Z_{\alpha/2}\widehat{\nu}_{\widehat{\mathbf{p}}}}{\sqrt{n}}$

3.5.3 Asymptotic Variance of Statistical Complexity $C(\widehat{\mathbf{p}})$

The asymptotic variance of the statistical complexity $C(\widehat{\mathbf{p}})$, under both the multinomial model and dependence, can be expressed using formulas derived for permutation entropy, the Jensen-Shannon divergence, and their joint asymptotics. The key approach follows Rey et al. [72] and recent research work by Silbernagel et al. [89] , employing the multivariate Delta method for functions of multinomial proportions.

The variance of the complexity, $C(\widehat{\mathbf{p}})$, has two estimators:

- **Multinomial model variance**, denoted as $\widehat{\omega}^2_{\widehat{\mathbf{p}}}$, which assumes independent ordinal patterns.
- **Serial dependence variance**, denoted as $\widehat{\eta}^2_{\widehat{\mathbf{p}}}$.

Formula for Asymptotic Variance of Statistical Complexity

Let:

- $\mathbf{p} = (p_1, \ldots, p_k)$: ordinal patterns probability vector
- $\widehat{\mathbf{p}} = (\widehat{p}_1, \ldots, \widehat{p}_k)$: empirical probability vector
- n : sample size

- $H(\widehat{\mathbf{p}})$: normalized Shannon entropy
- $Q(\widehat{\mathbf{p}})$: normalized Jensen Shannon divergence from the uniform distribution
- $C(\widehat{\mathbf{p}}) = H(\widehat{\mathbf{p}}) \times Q(\widehat{\mathbf{p}})$: normalized statistical complexity

1. **Multinomial Model (Independence)**

   The estimator is asymptotically normal:

   $$\sqrt{n}(C(\widehat{\mathbf{p}}) - C(\mathbf{p})) \xrightarrow{\mathscr{D}} N(\mathbf{0}, \widehat{\omega}^2_{\widehat{\mathbf{p}}}),$$

   where the asymptotic variance is

   $$\widehat{\omega}^2_{\widehat{\mathbf{p}}} = \nabla C(\mathbf{p})^T \Sigma \nabla C(\mathbf{p}).$$

   - $\Sigma = \mathbf{D}_\mathbf{p} - \mathbf{p}\mathbf{p}^T$ is the covariance matrix of sample proportions under independence.
   - $\nabla C(\mathbf{p})$ is the gradient (vector of partial derivatives) of C with respect to $\mathbf{p}$:
     $$\frac{\partial C}{\partial p_\ell} = Q(\mathbf{p})\frac{\partial H}{\partial p_\ell} + H(\mathbf{p})\frac{\partial Q}{\partial p_\ell},$$
     where:
   - $\dfrac{\partial H}{\partial p_\ell} = -(\log p_\ell + 1),$
   - The exact partial derivative of Jensen-Shannon divergence $Q(\mathbf{p})$ with respect to depends on the definition but can be explicitly calculated.

2. **Serial Dependence Case**

   For dependent ordinal patterns, the asymptotic variance increases:

   $$\widehat{\eta}^2_{\widehat{\mathbf{p}}} = a \times \widehat{\omega}^2_{\widehat{\mathbf{p}}},$$

   where:

- $a = \dfrac{\widehat{\nu}^2_{\widehat{\mathbf{p}}}}{\widehat{\sigma}^2_{\widehat{\mathbf{p}}}}$,
- $\widehat{\nu}^2_{\widehat{\mathbf{p}}}$ is the variance of entropy under serial dependence,
- $\widehat{\sigma}^2_{\widehat{\mathbf{p}}}$ is the variance of entropy under independence.

Bringing these results together:

$$\text{Var}(C(\widehat{\mathbf{p}})) = \widehat{\eta}^2_{\widehat{\mathbf{p}}} \approx a\nabla C(\mathbf{p})^T \Sigma \nabla C(\mathbf{p})$$

3.5.4 Other types of Entropy

Moreover, other types of descriptors, such as Rényi entropy [71], Tsallis entropy [95], and Fisher information [33], have been proposed to extract additional information that is not captured by Shannon entropy. From these entropy measures, Fisher information has garnered more attention due to its unique properties. Fisher information is defined as the average logarithmic derivative of a continuous probability density function.

For discrete probability distributions, Fisher information can be approximated by calculating the differences between probabilities of consecutive distribution elements. A key distinction between Shannon entropy and Fisher information lies in their focus: Shannon entropy quantifies the overall unpredictability of a system, while Fisher information measures the rate of change between consecutive observations, making it more sensitive to small changes and perturbations.

The following equations define Tsallis entropy $(H^q_T(\widehat{\mathbf{p}}))$, Rényi entropy $(H^q_R(\widehat{\mathbf{p}}))$, and Fisher information measures $(H_F(\widehat{\mathbf{p}}))$ [83] :

$$H^q_T(\widehat{\mathbf{p}}) = \sum_{\ell=1}^{k} \frac{\widehat{p_\ell} - \widehat{p_\ell}^q}{q-1}, \tag{3.15}$$

where the index $q \in \mathbb{R}\backslash\{1\}$

$$H^q_R(\widehat{\mathbf{p}}) = \frac{1}{1-q} \log \sum_{\ell=1}^{k} \widehat{p_\ell}^q, \tag{3.16}$$

where the index $q \in \mathbb{R}^+ \setminus \{1\}$

$$H_F(\widehat{\mathbf{p}}) = F_0 \sum_{\ell=1}^{k-1} \left(\sqrt{\widehat{p}_{\ell+1}} - \sqrt{\widehat{p}_\ell} \right)^2, \tag{3.17}$$

where the re-normalization coefficient is $F_0 = 4$ [83]

Rényi entropy, Tsallis entropy, and Fisher information are alternative statistical measures that reveal specific aspects of time-series structure and variability not captured by Shannon entropy. Rey et al. demonstrated that, when computed from empirical ordinal pattern distributions, these measures become increasingly reliable as sample size grows. Specifically, their sample estimates approach well-defined asymptotic distributions that can be used for statistical inference and hypothesis testing.

For the multinomial model, where ordinal patterns are assumed independent and drawn according to fixed probabilities, the sample entropies and Fisher information converge in distribution as n increases: the central limit theorem applies, yielding asymptotic normality for Rényi and Tsallis entropy estimators, as well as for Fisher information. Explicit formulas for mean and variance are available, and confidence intervals can be constructed accordingly. If the empirical histogram is used, these asymptotic results remain valid under large sample size and mild regularity. In all cases, normality holds for entropy-type measures in the multinomial setting, and specific corrections or other distributions (like chi-squared for quadratic forms) may arise for more complex dependency structures. Thus, under the multinomial and empirical frameworks, these advanced entropy and information measures possess robust, predictable properties that enable their practical use in time-series analysis and statistical hypothesis testing.

3.6 Case Study of Asymptotic Distribution of the Shannon Entropy under pattern dependence and independence

Statistical complexity is defined as the product of two normalized quantities:

- The Shannon entropy,
- The Jensen-Shannon distance between the observed probability distribution and the uniform distribution.

In this section we discuss two key aspects with real world scenario:

1. **Significance of Asymptotic Distributions**: Why understanding large-sample behavior matters for statistical inference,
2. **Practical Formula**: A working equation for calculating the asymptotic distribution of complexity.

As a case study for our work, we consider data from the Bearing Data Center and the seeded fault test data from Case Western Reserve University, School of Engineering. The datasets includes ball bearing test data for normal bearings as well as single-point defects on the fan end and drive end. Data were collected at a rate of $48,000$($48k$ drive-end) data points per second during bearing tests. Each file contains motor loads ($0, 1, 2$, and 3), drive-end vibration data, and fan-end vibration data. The approximate motor speeds in RPM during testing: $1797, 1772, 1750$, and 1730. For our case study, we consider two time series (Normal Baseline and $48k$ Drive-End) with a motor load of 0 and an RPM of 1797.

The primary objective of this study is to detect malfunctioning machinery by analyzing two time series using ordinal patterns. We introduce a distance metric based on the ordinal structure of the segments to quantify similarity. This metric facilitates the identification of faulty machines across various embedding dimensions, ranging from 3 to 6. For this case study, we employ an embedding dimension of 3 for convenience; subsequent analyses will extend to the remaining dimensions to compare results. Permutation entropy under asymptotic conditions is computed by considering the probability distribution of ordinal patterns. The results are further analyzed using the complexity–entropy plane, providing insights into the system's dynamics.

Initially, we analyzed complete datasets from two time series: one comprising 250,000 data points representing the normal baseline at motor load 0, and another containing 2,540,000 data points from the 48k drive end under the same motor load. We computed the entropy and complexity measures for these entire datasets, followed by the calculation of the asymptotic variance as defined in Section 3.5.2. This asymptotic variance with pattern independence was then used to determine the confidence interval for entropy (Equation is defined in 3.13). The calculation of the semi-length of the interval is given by Equation 3.14. The final results are presented in Table 3.1.

Entropy	Complexity	$\widehat{\sigma}_{\widehat{\mathbf{p}}}$	Semi Length
0.665235	0.226447	0.358893	0.000441
0.772973	0.170954	0.324376	0.001287

Table 3.1: Entropy Complexity Results

Subsequently, we segmented the data into batches of $10,000$ points, categorizing them as either 'Normal' or '48k Drive End'. We then performed a batch wise comparison of entropy and complexity metrics to identify fault data segments. The normal dataset comprises 25 batches, all corresponding to motor load 0, while the $48k$ drive end dataset includes 254 batches. Due to the extensive volume of entropy and complexity data generated, the complete results table is not included in this report. However, the entropy–complexity plane effectively illustrates both batch-wise and full-data analyses. As depicted in Figure 3.3 below, faulty machines form a distinct cluster in the entropy–complexity plane, highlighting their deviation from normal operational patterns. It is clear from the graph that there are both overlapping and non-overlapping confidence intervals. This indicates that some machines differ significantly, while others do not. The main purpose of our experiment is to identify faulty machines. Therefore, we highly recommend extending these results by increasing the embedding dimension to better understand the final outcomes. The general framework of this experiment is also provided in this chapter to clarify the main objective of the research.

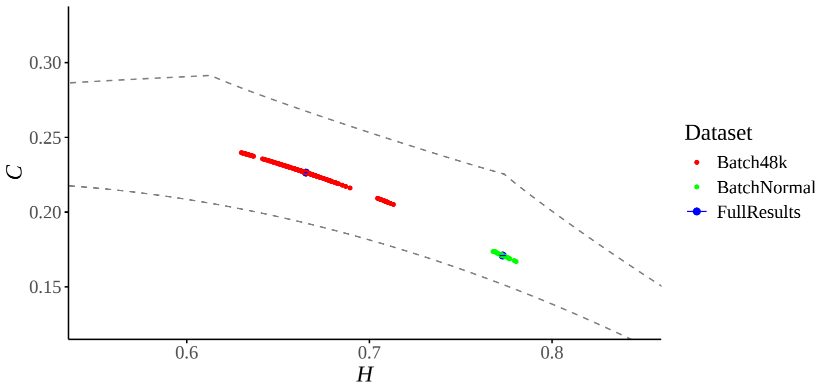

Figure 3.3: Entropy Complexity Plane

In addition to this we analyze the full data results for higher embedding dimension $D = 6$.

Because the original case study involved a large sample size, we computed entropy and statistical complexity for smaller sample sizes of 100, 1000, and 2000. These values were then analyzed using both the Multinomial and Serial Dependence models. The analysis clearly demonstrates the confidence intervals for both entropy and complexity.

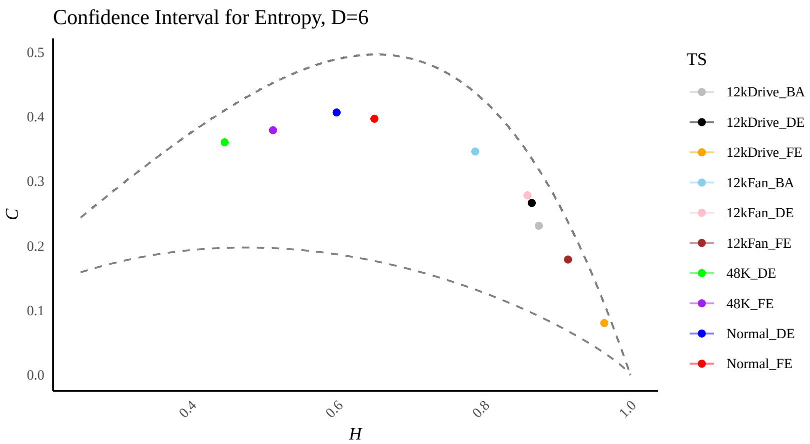

Figure 3.4: Entropy Complexity Plane for $D = 6$

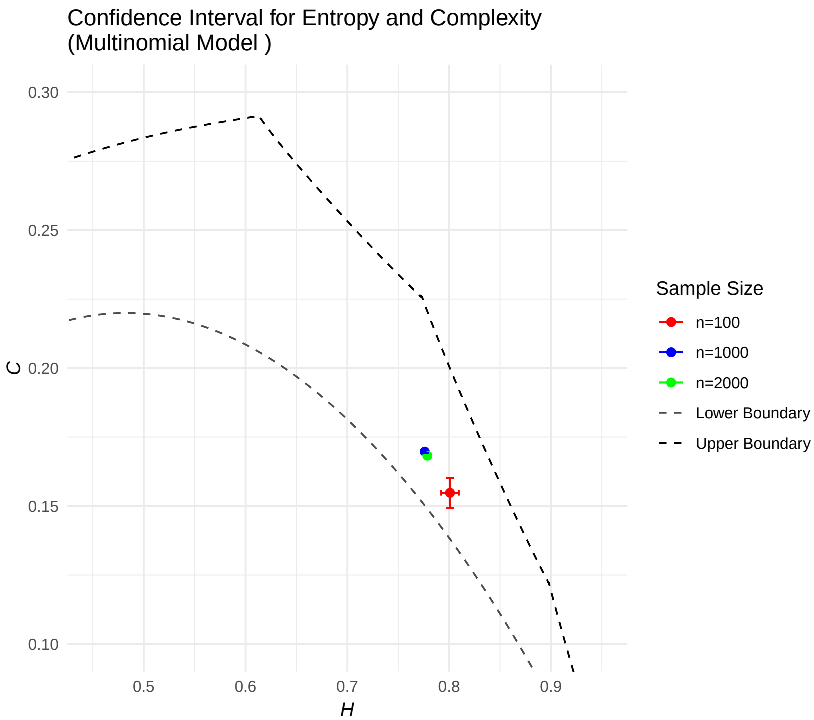

Figure 3.5: Confidence Interval for Multinomial Model

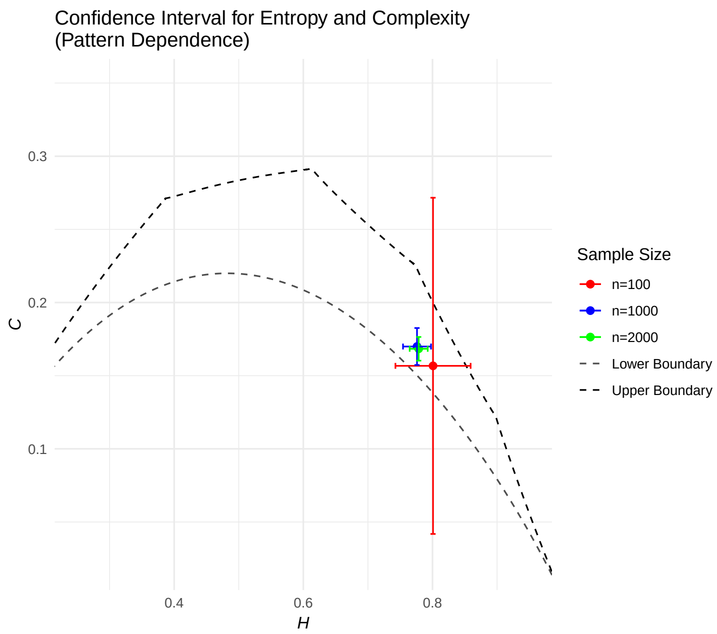

Figure 3.6: Confidence Interval Pattern Dependence model

The general framework for analyzing entropy-complexity planes with confidence intervals are given as follows.

1. **Calculate Entropy (H) and Complexity (C):** appropriate estimator are Shannon entropy, statistical complexity measures

2. **Compute Confidence Intervals:** Generate multiple resampled datasets to estimate the variance of H and C.

3. **Plot on Entropy-Complexity Plane:**

   - Axes: x-axis: Entropy (H), y-axis: Statistical complexity (C)
   - Data Points: Plot individual or aggregated results.
   - Confidence Regions: Represent uncertainty

4. **Interpretation**

Region of Plane	Interpretation
High H and High C	Complex, structured systems
Low H and Low C	Simple, predictable systems
High H and Low C	Random/noisy systems
Low H and High C	Non-random systems

5. **Statistical testing:**

   - Compare confidence intervals between groups to assess significant differences.
   - Overlapping intervals $\rightarrow$ No significant difference.
   - Non-overlapping intervals $\rightarrow$ Potential significance.

Chapter 4

Future Works

As a short summary, we have completed the necessary preliminaries studies on various topics such as:

- Entropy
- Complexity
- Entropy Complexity Plane
- Confidence interval

We have also examined key research articles by Bandt and Pompe [14], along with an overview of the area focusing on four seminal works. These include:

- López-Ruiz et al. [53], who introduced the concept of statistical complexity;
- Lamberti et al. [41], who applied López-Ruiz's idea using the Euclidean distance;
- Rosso et al. [77], who proposed the entropy-complexity plane as a diagnostic tool; and

- Martin et al. [55], who defined the theoretical boundaries of this generalized statistical complexity measure.

In addition, we reviewed recent work by Rey et al. [72, 73, 75], which investigates the statistical properties of entropy derived from ordinal patterns, including the asymptotic distribution under the Multinomial law and the behavior of permutation entropy.

As a case study, we computed the Shannon entropy, statistical complexity, and their associated asymptotic variances based on the probability distribution of ordinal patterns. Using these results, we derived confidence intervals for both entropy and complexity. The analysis was further visualized using the entropy–complexity plane, offering insights into the underlying system dynamics. All computations were performed using two large-sample datasets under the asymptotic distribution, as detailed in Chapter 3, Section 3.6.

The formulas and procedures used to analyze the case study are summarized as follows:

- Calculate the Shannon entropy of the time series.
- Calculate the statistical complexity.
- Estimate the asymptotic variance for Shannon entropy.
- Construct confidence intervals for entropy.
- Plot the results in the entropy–complexity plane.
- Divide the data into batches (batch size = 10,000).
- Repeat the above calculations for each batch.
- Graphically represent the results of the two time series across batches in the entropy–complexity plane.
- Finally, the results are analyzed for time series clustering, as shown in the final output in Figure 3.3

Asymptotic distribution of normalized Shannon entropy $H(\mathbf{p})$ was derived under the assumption of independent ordinal patterns, following the Multinomial law. As a foundational step, we use the normalized Shannon entropy formula:

$$H(\mathbf{p}) = -\frac{1}{\log k}\sum_{\ell=1}^{k} p_\ell \ln p_\ell. \tag{4.1}$$

Where, $k = D!$ is the number of possible ordinal patterns. To evaluate statistical complexity, we compute the Jensen–Shannon divergence between the histogram of proportion $\mathbf{p}$ and the uniform probability function $\mathbf{u} = (1/k, 1/k, \dots, 1/k)$, defined by:

$$Q'(\mathbf{p}, \mathbf{u}) = \sum_{\ell=1}^{k} p_\ell \log\frac{p_\ell}{u_\ell} + u_\ell \log\frac{u_\ell}{p_\ell}. \tag{4.2}$$

This disequilibrium measure is normalized using:

$$Q = \frac{Q'}{\max(Q')}, \tag{4.3}$$

where $\max(Q')$ is defined as follows

$$\max(Q') = -2\left[\frac{k+1}{k}\log(k+1) - 2\log(2k) + \log k\right]. \tag{4.4}$$

The statistical complexity is then calculated as:

$$C = HQ, \tag{4.5}$$

where both H and Q are normalized quantities, therefore C is also normalized.

Then the entropy-complexity plane, which is a two-dimensional representation used to graphically represent the results.

As a key component of our research, we also calculated the asymptotic variance of the Shannon entropy estimator. The estimated normalized entropy based on sample proportions $\widehat{\boldsymbol{p}}$ is:

$$H_s(\widehat{\boldsymbol{p}}) = -\frac{1}{\log k}\sum_{\ell=1}^{k} \widehat{p}_\ell \log \widehat{p}_\ell. \tag{4.6}$$

The corresponding asymptotic variance under the Multinomial model is given by:

$$\widehat{\sigma}_p^2 = \frac{1}{n}\sum_{\ell=1}^{k} p_\ell(1-p_\ell)(\log p_\ell+1)^2 - \frac{2}{n}\sum_{j=1}^{k-1}\sum_{\ell=j+1}^{k} p_\ell p_j(\log p_\ell+1)(\log p_j+1). \quad (4.7)$$

where n is the sample size. From this variance, we derive confidence intervals for entropy, which are used to assess uncertainty in the entropy-complexity plane. The asymptotic distribution of statistical complexity under the Multinomial law is:

$$C[\widehat{\boldsymbol{p}}] = H[\widehat{\boldsymbol{p}}]Q[\widehat{\boldsymbol{p}}]. \quad (4.8)$$

This approach will be further analyzed, as described in the following objectives, to evaluate the accuracy of the results.

This proposal has three objectives in order to continue this research work.

- Define a data base of time series for clustering, i.e., finding similar time series.
- Extract all the features we know from their Bandt & Pompe symbolization (Shannon, Tsallis and Renyi entropies, Fisher information measure, complexities, and the available confidence intervals)
- Use those features for time series clustering

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