New Features of Permutation Entropy in Ordinal Patterns Complexity Plane

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Abstract

This proposal provides a brief introduction to ordinal patterns and a literature review related to the Bandt & Pompe research article. After reviewing the literature on ordinal patterns and permutation entropy, we can conclude that ordinal patterns play an important role in conducting research. Bandt & Pompe proposed analyzing time series from the perspective of ordinal patterns to develop fast and automated methods for extracting qualitative information from nonlinear time series. They introduced the concept of permutation entropy to measure the complexity of a system underlying a time series while taking into account the ordering patterns that represent variations in the data. In this report I show how to use data to extract all ordinal patterns. First, I use the sample dataset to calculate their ordinal patterns with embedding dimension 3; I use letters to convert time series data into patterns. Secondly, I get the histogram on the data. And third, calculate the permutation entropy and statistical complexity. Next, determine the confidence interval of H without and with time dependence and finally the confidence interval of C.

Bandt and Pompe's ordinal pattern methodology has been widely used to investigate the latent dynamics of time series through their entropy, also known as permutation entropy. Nevertheless, there are no theoretical findings regarding the distribution of the permutation entropy, which needs to take the correlation effect between patterns into account. Considering that our approximation ignores the series dependence between symbols and that the asymptotic distribution of the permutation entropy is Normal We compare this result with the Multinomial sample entropy with embedding dimension 3, which assumes independence, and find that the expression

of the asymptotic variance becomes more complex as the embedding dimension increases. After that, a hypothesis test is developed and used to differentiate between the bearing fault diagnoses for four distinct rotating machines.

Contents

Chapter 1

Introduction

Time series contain valuable insights about the underlying system that generates the data. Their analysis is typically conducted using two primary approaches: time-domain and transformed-domain methods. In the context of time-domain analysis, Bandt and Pompe [5] introduced a novel methodology that is non-parametric and rooted in information theory descriptors: Ordinal Patterns symbolisation.

Bandt and Pompe [5] proposed transforming small subsets of the time series observations into symbols that encode the sorting properties of the values in these subsets. Then, they computed a histogram of those symbols. The resulting distribution is less sensitive to outliers compared to the original data, and the histogram is independent of any specific model. The proposal proceeds by computing descriptors from this histogram, and extracting information about the system from these descriptors. As a result, this approach is versatile and applicable to a wide range of scenarios.

This study explores features derived from Bandt and Pompe symbolization, specifically Shannon entropy, and represents them graphically as a point in the entropy-complexity plane under the Multinomial distribution, which is represented as a point in the $\mathbb{R}^2$ manifold. Furthermore, this proposal separately examines the permutation entropy and ordinal patterns in time series analysis, including the calculation of pattern histograms, entropy, complexity, and confidence intervals for entropy and complexity to enhance the understanding of their statistical properties. The confidence intervals for entropy and complexity under the Multinomal distribution and features for time series clustering will be discussed further. Future work will extend to other measures, such as Rényi entropy, Fisher information, and the confidence intervals for their entropy and complexity under the Multinomal distribution. Features for time series clustering with ordinal patterns for time series analysis will be further studied.

Time series analysis is widely applied across various fields, including engineering, economics, physical sciences, and more. A time series is defined as a collection of observations x_t, each representing a realized value of a particular random variable X_t, where time can be either discrete or continuous.

Examples of time series applications include finance (e.g., analyzing exchange rate movements or commodity prices), biology (e.g., modeling the growth and decline of bacterial populations), medicine (e.g., tracking the spread of diseases like COVID-19 or influenza), and geoscience (e.g., predicting wet or dry days based on past weather conditions).

The primary goal of time series analysis is to understand the nature of the phenomenon represented by the observed sequence. Time domain and frequency domain methods are the two primary approaches used in time series analysis. The temporal approach relies on concepts such as auto-correlation and regressions, where a time series' present value is analyzed in relation to its own past values or the past values of other series. This method represents time series directly as a function of time. On the other hand, the spectral approach represents time series through spectral expansions, such as wavelets or Fourier modes [48].

However, these methods often require assumptions such as large sample sizes or normally distributed observations that are rarely met in real-world empirical data. For many statistical techniques to be valid, these assumptions must hold, but in practice, they are frequently violated.

For example, traditional approaches to time series analysis, such as time domain and frequency domain methods, rely on assumptions that are not always valid in real-world data. The time domain approach, which uses techniques like auto-correlation and regression, assumes stationarity and often struggles with nonlinear or nonstationary data. Similarly, the frequency domain approach, which represents time series through spectral expansions such as wavelets or Fourier modes, may require assumptions about periodicity and may not effectively capture short-term fluctuations.

Many statistical methods in these approaches depend on specific conditions, such as large sample sizes or normally distributed observations. However, these assumptions are often unrealistic, leading to inaccurate or biased results. When such conditions are not met, alternative methods must be considered.

As a result, alternative methods, often referred to as non-parametric techniques, must be considered. These methods rely on the rank R_t of the observations x_t rather than their actual values, making them robust and applicable to a wide range of data sets. Since non-parametric tests do not assume a normal distribution, they are highly reliable. For example, the Kruskal-Wallis H test and the Wilcoxon test are effective tools for comparing two or more population probability distributions from independent random samples. However, these techniques are not always suitable for time series data, which often require specialized methods tailored to their unique characteristics.

To address these challenges, ordinal pattern methods provide a robust alternative. Instead of analyzing the absolute values of a time series, these methods focus on the order relationships among consecutive data points.

This approach effectively captures the underlying dynamics of complex systems and offers several advantages.

The ordinal pattern-based method has become a widely used tool for characterizing complex time series. Since its introduction nearly twentythree years ago by Bandt and Pompe in their foundational paper [5], it has been successfully applied across various scientific fields, including biomedical signal processing, optical chaos, hydrology, geophysics, econophysics, engineering, and biometrics. It has also been used in the characterization of pseudo-random number generators.

The Bandt and Pompe method successfully analyzes time series by transforming them into ordinal patterns, constructing a histogram, and computing Shannon entropy, making it robust against outliers and independent of predefined models.

Later, Rosso [38] introduced an additional dimension to this analysis Statistical Complexity derived from the same histogram of causal patterns.

Introduction to Ordinal Pattern Analysis

Ordinal patterns are a non-parametric representation of real-valued time series and ordinal patterns are transformations that encode the sorting characteristics of values in $\mathbb{R}^D$ into $D!$ symbols, where D represents for the "Embedding Dimension" and usually ranges between three to six. One of the possible encoding is the set of indexes that sort the D values in non-decreasing order.

To illustrate this idea, let $\boldsymbol{x} = \{x_1, x_2, \dots, x_{(n+D-1)}\}$ be a real valued time series of length $n + D - 1$ without ties. As stated by Bandt & Pompe, if the $\boldsymbol{x}$ takes infinitely many values, it is common to replace them with a symbol sequence $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_n)$. consisting of finitely many symbols and then compute the entropy from this sequence. The corresponding symbol sequence naturally emerges from the time series without requiring any model assumptions. We compute $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_n)$ symbols from sub-sequences of embedding dimension D. There are $D!$ possible symbols: $\pi_j \in \boldsymbol{\pi} = (\pi^1, \pi^2, \dots, \pi^{D!})$. The histogram of proportions $h = (h_1, h_2, \dots, h_{D!})$ in which the bin h_ℓ is the proportion of symbols of type π^ℓ of the total number of symbols. For convenience, we will model those symbols as a k dimensional random vector where $k = D!$.

Consider a series of n independent trials in which only one of k mutually exclusive events $\pi^1, \pi^2, \ldots, \pi^k$ is observed with probability $p_1, p_2, \ldots, p_k$, respectively, where $p_\ell \geq 0$ and $\sum_{\ell=1}^{k} p_\ell = 1$. Suppose that $N = (N_1, N_2, \ldots, N_k)$ is the vector of random variables that, with $\sum_{\ell=1}^{k} N_\ell = n$, counts how many times the events $\pi^1, \pi^2, \ldots, \pi^k$ occur in the n trials. Then, the joint probability distribution of N is

$$\Pr\big(N = (n_1, n_2, \ldots, n_k)\big) = n! \prod_{\ell=1}^{k} \frac{p_\ell^{n_\ell}}{n_\ell!}, \tag{1.1}$$

where $n_\ell \geq 0$ and $\sum_{\ell=1}^{k} n_\ell = n$.

Problem Statement

To illustrate this concept, imagine tracking the mean monthly humidity in Wellington. You want to analyze how humidity changes throughout the year. By examining this data, you can uncover interesting patterns that highlight the variations in humidity across different months.

Mean monthly humidity in Wellington is shown in Figure 1.1

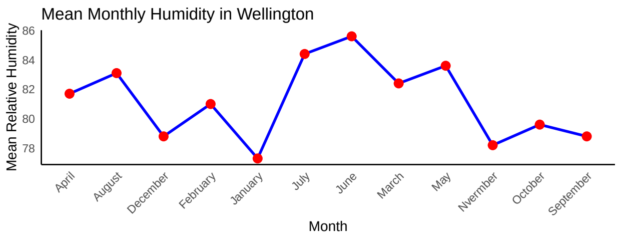

Figure 1.1: Mean Monthly humidity in Wellington

We can convert this actual data into ordinal patterns. To do this, for each month, we determine the order of the humidity values rather than their actual magnitudes. Each three-time-point sequence (which can be adjusted based on preference) is converted into an ordinal pattern. This

Month	Mean of relative humidity
January	77.3
February	81
March	82.4
April	81.7
May	83.6
June	85.6
July	84.4
August	83.1
September	78.8
October	79.6
November	78.2
December	78.8

Table 1.1: Mean monthly humidity variations in Wellington throughout the year

"embedding dimension" usually varies between 3 and 6, but any dimension is possible. The conversion can be made in any way that uniquely maps the sorting properties of the subsequence into a symbol.

t	Mean Humidity sequence	Ordinal Pattern
1	(77.3,81,82.4)	$(123) = \pi^1$
2	(81,82.4,81.7)	$(132) = \pi^2$
3	(82.4,81.7,83.6)	$(213) = \pi^3$
4	(81.7,83.6,85.6)	$(123) = \pi^1$
5	(83.6,85.6,84.4)	$(132) = \pi^2$
6	(85.6,84.4,83.1)	$(321) = \pi^6$
7	(84.4,83.1,78.8)	$(321) = \pi^6$
8	(83.1,78.8,79.6)	$(312) = \pi^5$
9	(78.8,79.6,78.2)	$(231) = \pi^4$
10	(79.6,78.2,78.8)	$(312) = \pi^5$

Table 1.2: Ordinal Patterns

As shown in Table 1.2, we have six mutually exclusive events, $(\pi^1, \pi^2, \ldots, \pi^6) = (123), (132), (213), (231), (312), (321)$ respectively. The probability distribution of the mean humidity is calculated based on ordinal patterns as given below.

$$\hat{p_i} = \frac{\#\{\pi_j \in \boldsymbol{\pi} : \pi_j = \pi^i\}}{n}; 1 \leq i \leq 6 \tag{1.2}$$

where $\hat{\boldsymbol{p}} = (\hat{p_1}, \ldots, \hat{p_6})$

Notation	Probability
$p(\pi^1)$	$\frac{2}{10}$
$p(\pi^2)$	$\frac{2}{10}$
$p(\pi^3)$	$\frac{1}{10}$
$p(\pi^4)$	$\frac{1}{10}$
$p(\pi^5)$	$\frac{2}{10}$
$p(\pi^6)$	$\frac{2}{10}$

Table 1.3: Probability function

We construct the histogram of proportions $h = (h_1, h_2, h_3, h_4, h_5, h_6)$, where each bin h_ℓ represents the proportion of symbols of type π^ℓ out of the total six symbols. The histogram graph is shown Figure 1.2.

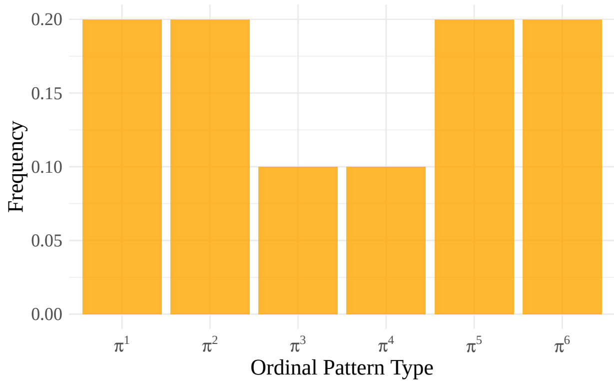

Figure 1.2: Histogram of proportions of the observed patterns according to Table 1.3.

This example explains how time series data can be converted into ordinal patterns and how the probability distribution function can be calculated from these patterns. Chapter Two will review the literature on ordinal pattern analysis, while Chapter Three will expand on this concept by exploring the characterization of time series. It will also cover the computation of two key descriptors entropy and complexity from the resulting histograms. Additionally, Chapter Three will outline the main ideas and objectives of this research project.

Chapter 2

Literature Review

More about the methodology

In this concept first, the time series data is convert into order patterns and then calculate the sequence of probabilities from the patterns. Third, estimate the normalized Shannon entropy from the probability sequence. After that ordinal patterns complexity plane is use as a visual representation of the relationship between permutation entropy and the complexity measures.

The outline of this proposal is as follows: After the introduction, in Chapter 2 we have literature review, followed by more explanation of the details of this project in Chapter 3. Finally, we describe future plans for this project in Chapter 4.

2.1 Preliminaries

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Chapter 3

The Research Project

In this chapter, we outline the main ideas and objectives of this research project. Section One discusses complexity analysis in time series, highlighting its advantages and limitations. Section Two examines the limitations of the Bandt and Pompe method. Section Three provides background knowledge on the statistical complexity plane. Section Four explores the entropy-complexity plane for a broad class of time series. Section five provides the asymptotic distribution of the entropy. Finally, the chapter concludes with the objectives of the research project and a case study related to our work.

Complexity Analysis in Time Series: Advantages and Limitations

While this method effectively captures ordinal relationships between data points, it has notable drawbacks. First, it ignores amplitude information in the time series. Second, classifying data via the complexity-entropy plane becomes challenging (or even misleading) in high-dimensional chaotic systems, as both deterministic chaotic time series and stochastic surrogate data may occupy overlapping regions within the plane.

Despite these limitations, integrating statistical complexity measures such as those derived from the complexity-entropy plane offers a robust framework to enhance ordinal pattern analysis. These measures quantify deviations from uniform ordinal pattern distributions, enabling a more nuanced characterization of dynamical processes. By combining permutation entropy with statistical complexity, researchers gain a refined tool to differentiate stochastic signals from deterministic chaos, thereby revealing intricate structural patterns and degrees of randomness in time series data.

The Bandt and Pompe Method: A Robust Approach

The concept of ordinal patterns in time series can be effectively demonstrated through real world examples. Traditionally, numerous algorithms, techniques, and heuristics have been employed to estimate complexity measures from real world data. However, these methods often perform well only for low dimensional dynamical systems and struggle when noise is introduced.

The Bandt and Pompe method overcomes this limitation by providing a robust approach that remains reliable even in noisy environments. In time series analysis, key complexity measures such as entropy, fractal dimension, and Lyapunov exponents play a crucial role in comparing neighboring values and uncovering the underlying structure and dynamics of the data.

The advantages of Bandt & Pompe methods:

- Simplicity
- Extremely fast calculation
- Robutness
- Invariance to nonlinear monotonous transformations

This method exhibits low sensitivity to noise and naturally accounts for the causal order of elements in a time series. As a result, it can be applied to various real-world problems, particularly in differentiating between chaotic and stochastic signals.

Despite its limitations, researchers have developed extensions to the original method to address its shortcomings and enhance its applicability to a broader range of complex systems.

Statistical Complexity measures

After Bandt and Pompe introduced a successful method for analyzing time series, Rosso [38] expanded on this approach by incorporating statistical complexity derived from the same histogram of causal patterns. This method, which utilizes the complexity-entropy plane, has been successfully applied to various dynamic regimes including, system parameter change [9, 4, 16, 55, 39, 17, 56, 12], optical chaos [45, 54, 47, 51, 24], hydrology [19, 43, 46], geophysics [11, 41, 44], engineering [50, 3, 2, 33], biometrics [40], characterization of pseudo-random number generators [13, 14], biomedical signal analysis [52, 22, 21, 32, 31, 20, 28, 29, 23, 27, 30], econophysics [52, 58, 57, 6, 7, 8, 53].

After computing all symbols as described in 1, the histogram proportions are used to estimate the probability distribution of ordinal patterns. From this distribution, two key descriptors are calculated to characterize the time series:

1. Entropy (a measure of system disorder)
2. Statistical complexity

The most common metric for the first descriptor is the normalized Shannon entropy, defined as:

$$H(\mathbf{p}) = -\frac{1}{\log k}\sum_{j=1}^{k} p_j \ln p_j \tag{3.1}$$

Here, $k = D!$ represents the total number of possible permutation patterns, and terms in the summation where p_j are excluded by convention. This entropy is bounded within the unit interval.

- It reaches its minimum value $(H = 0)$ when a single pattern dominates for some $p_j = 1$ for some j
- It achieves its maximum $(H = 1)$ under uniform probability $p_j = 1/k$ for all j for all.

This normalized entropy is often termed permutation entropy in time series analysis.

While normalized Shannon entropy is a powerful tool for quantifying disorder, it fails to fully characterize complex dynamics. To address this limitation, López-Ruiz et al ([25]) introduced the disequilibrium Q concept, which quantifies the deviation of a probability distribution $\mathbf{p}$ from a uniform (non-informative) equilibrium state. Specifically, disequilibrium measures the Euclidean distance between $\mathbf{p}$ and the uniform distribution, providing a complementary metric to Shannon entropy for assessing structural complexity in systems.

The Jensen-Shannon distance (Q') between histogram of proportion $\mathbf{p}$ and the uniform probability function $\mathbf{u} = u_1, u_2, \ldots, u_k$, where $k = D!$ corresponds to the number of possible permutation patterns—provides a robust metric for quantifying deviations from uniformity. This distance measure, derived from the symmetric Jensen-Shannon divergence, is particularly suited for analyzing ordinal pattern distributions due to its ability to capture both structural differences and statistical equilibrium in time series data. Here is the formula for Q':

$$Q'(\mathbf{p}, \mathbf{u}) = \sum_{\ell=1}^{k} p_\ell \log \frac{p_\ell}{u_\ell} + u_\ell \log \frac{u_\ell}{p_\ell} \tag{3.2}$$

Lamberti et al. [18] proposed Jensen-Shannon distance as a symmetric metric rooted in the Jensen-Shannon divergence. As the reference model, most works consider the uniform distribution $\mathbf{u} = (1/k, 1/k, \ldots, 1/k)$. The normalized disequilibrium is defined as follows

$$Q = \frac{Q'}{max(Q')} \tag{3.3}$$

where $max(Q')$ is defined as follows

$$max(Q') = -2\left[\frac{k+1}{k}\log(k+1) - 2\log(2k) + \log k\right] \tag{3.4}$$

After this concept, Lamberti et al. [18] proposed **Statistical Complexity Measure** which is defined as

$$C = HQ \tag{3.5}$$

where both H and Q are normalized quantities, therefore C is also normalized.

3.1 The Entropy Complexity Plane

The complexity-entropy plane is a two-dimensional representation where time series are mapped based on their permutation entropy and statistical complexity. These metrics are derived from ordinal pattern distributions obtained through time-delay embedding—a technique involving:

- Embedding dimension D : Determines the number of permutation patterns $D!$ used to construct histograms.
- Time delay (τ) : Often optimized for specific applications (e.g., mutual information minimization)

3.1.1 Key Dynamics in the plane

1. Highly Ordered Systems

   Example: Strictly monotonic time series.

   - Produces a single ordinal pattern ($H = 0$).
   - Maximal disequilibrium (distance from uniform distribution).
   - Maps to $(0, 0)$, indicating minimal complexity.

2. Perfectly Random Systems

   Example: White noise

   - Uniform ordinal pattern distribution ($H = 1$).
   - Disequilibrium vanishes (distance $= 0$).
   - Maps to $(1, 0)$, reflecting maximal entropy without structural complexity.

The two extreme values are proved by Anteneodo & Plastino [1].Expressions for the boundaries, derived using geometrical arguments within space configurations, were proposed by Martin et al. [26]. These formulations provide a structured approach to understanding and analyzing the spatial behavior of specific systems or models. The lower boundary is characterized by a smooth curve, whereas the upper boundary consists of $D! - 1$ distinct segments. As the embedding dimension D approaches infinity, the upper boundary gradually converges into a smooth curve. Example for the entropy complexity plane is shown in Figure 3.1

3.2 Asymptotic Distribution of the Shannon Entropy under the Mulinomal Distribution

The Multinomial distribution describes how observations fall into categories when an adequate model is available. It is similar to the multi-

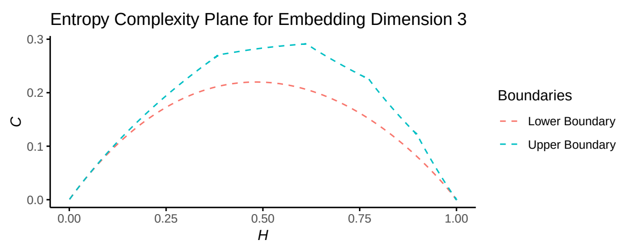

Figure 3.1: Entropy Complexity Plane for Embedding dimension 3

variate normal distribution, which is one of the continuous Multivariate distributions. Furthermore, it has received considerable attention from researchers, both in theoretical studies and in applications related to discrete multivariate distributions. The normalized Shannon entropy, often employed in applications like permutation entropy, can be rigorously connected to its asymptotic distribution through the lens of statistical estimation theory. When estimating entropy from finite data, the plug-in estimator (computed directly from observed frequencies) converges to a normal distribution as the sample size $N \longrightarrow \infty$ even for dependent processes such as Markov chains. This asymptotic normality, demonstrated by Martin et al [37]. and Chagas et al [10]., ensures that the estimator's fluctuations around the true entropy value are Gaussian-distributed, with variance and bias determined by the underlying system's dynamics. For normalized entropy $H \in [1]$, this asymptotic behavior persists, enabling statistical inference such as confidence intervals or hypothesis tests to validate entropy-based hypotheses (e.g., distinguishing chaos from noise). Crucially, the convergence rate and limiting distribution depend on the system's correlation structure deviating from the Multinomial case but remain tractable via spectral analysis of the transition matrix or ordinal pattern distributions [37, 10]. This connection underscores the reliability of normalized entropy measures in large data regimes while highlighting the need to account for dependence structures in finite sample applications.

However, despite the widespread use of ordinal patterns to study the latent dynamics of time series through permutation entropy, there are no established theoretical results concerning the distribution of permutation entropy that account for the correlation effects between patterns. Nonetheless, Rey et al. [35] demonstrated that the asymptotic distribution of permutation entropy is normal. They compared their findings with those of Multinomial sample entropy, which assumes independence between patterns. Notably, the expression for the asymptotic variance becomes increasingly complex as the embedding dimension increases.

Imagine a sequence of n independent trials, each resulting in precisely one outcome from a set of k distinct possibilities labeled $\pi^1, \pi^2, \ldots, \pi^k$ and so on. These outcomes are mutually exclusive, meaning only one can occur per trial, with respective probabilities $\boldsymbol{p} = \{p_1, p_2, \ldots, p_k\}$, such that $p_\ell \geq 0$ and $\sum_{\ell=1}^{k} p_\ell = 1$. The random vector $\boldsymbol{N} = (N_1, N_2, \ldots, N_k)$ counts the number of occurrences of the events $\pi^1, \pi^2, \ldots, \pi^k$ in the n trials, with $N_\ell \geq 0$ and $\sum_{\ell=1}^{k} N_\ell = n$. A $\boldsymbol{n}$ is a sample from $\boldsymbol{N}$ and it has a $k-$variate vector of integer values $\boldsymbol{n} = (n_1, n_2, \ldots, n_k)$. Then the joint distribution of $\boldsymbol{N}$ is

$$Pr(\boldsymbol{N} = \boldsymbol{n}) = Pr(N_1 = n_1, N_2 = n_2, \ldots, N_k = n_k) = n! \prod_{\ell=1}^{k} \frac{{p_\ell}^{n_\ell}}{n_\ell!} \tag{3.6}$$

This situation is denoted as $\boldsymbol{N} \sim \text{Mult}(n, \boldsymbol{p})$. [36]

In practical applications, the true probability distribution $\boldsymbol{p}$ governing a multinomial system is typically unknown. Instead, estimators $\widehat{p_l}$, are derived empirically by calculating the observed frequency of each event π^l within the set of k possible outcomes $\boldsymbol{\pi} = \pi^1, \pi^2, \ldots, \pi^k$ across n independent trials. These frequencies approximate the underlying probabilities, enabling inference about the system's behavior. This maximum likelihood estimator (MLE) aligns with the empirical estimator derived from first-moment matching of the distribution. Due to its consistency, asymptotic normality, and computational tractability under regularity conditions, it remains the predominant choice in applied statistical modeling.

Shannon entropy quantifies the level of disorder within a system. When the system's behavior is entirely predictable, the Shannon entropy reaches its minimum, indicating complete knowledge of future observations. Conversely, when the system follows a uniform distribution where all possible outcomes have equal probability—the entropy is maximized, reflecting minimal knowledge about the system's behavior. Chagas et al. [10] have analyzed the asymptotic distribution of Shannon entropy in their study.

Shannon entropy H, in the distribution of $H(\boldsymbol{p})$ indexed by $\widehat{\boldsymbol{p}}$, the maximum likelihood estimator of $\boldsymbol{p}$. The distribution of $H(\widehat{\boldsymbol{p}})$ are defined as follows.

$$H_s(\widehat{\boldsymbol{p}}) = -\sum_{\ell=1}^{k} \widehat{p_\ell} \log \widehat{p_\ell} \tag{3.7}$$

Moreover, other types of descriptors, such as Rényi entropy[34], Tsallis entropy[49], and Fisher information [15], have been proposed to extract additional information that is not captured by Shannon entropy. From these entropy measures, Fisher information has garnered more attention due to its unique properties. Fisher information is defined as the average logarithmic derivative of a continuous probability density function.

For discrete probability distributions, Fisher information can be approximated by calculating the differences between probabilities of consecutive distribution elements. A key distinction between Shannon entropy and Fisher information lies in their focus: Shannon entropy quantifies the overall unpredictability of a system, while Fisher information measures the rate of change between consecutive observations, making it more sensitive to small changes and perturbations.

The following equations define Tsallis entropy $(H_T^q(\widehat{\boldsymbol{p}}))$, Rényi entropy $(H_R^q(\widehat{\boldsymbol{p}}))$, and Fisher information measures $(H_F(\widehat{\boldsymbol{p}}))$ [42] :

$$H_T^q(\widehat{\boldsymbol{p}}) = \sum_{\ell=1}^{k} \frac{\widehat{p_\ell} - \widehat{p_\ell}^{q}}{q-1}, \tag{3.8}$$

where the index $q \in \mathbb{R}\backslash\{1\}$

$$H_R^q(\widehat{\boldsymbol{p}}) = \frac{1}{1-q} \log \sum_{\ell=1}^{k} \widehat{p_\ell}^q, \tag{3.9}$$

where the index $q \in \mathbb{R}^+\backslash\{1\}$

$$H_F(\widehat{\boldsymbol{p}}) = F_0 \sum_{\ell=1}^{k-1} (\sqrt{\widehat{p}_{\ell+1}} - \sqrt{\widehat{p_\ell}})^2 \tag{3.10}$$

where the re-normalization coefficient is $F_0 = 4$ [42]

The main results of the asymptotic distribution of the Shannon, Tsallis, Rényi entropy, and Fisher information are defined as follows [36].

For any $k-$dimensional multivariate normal distribution $\boldsymbol{Z} \sim N(\boldsymbol{\mu}, \sum)$, with $\mu \in \mathbb{R}^k$ and covariance matrix $\sum = (\sigma_{\ell j})$, holds that the distribution of $W = \boldsymbol{a}^T \boldsymbol{Z}$, with $\boldsymbol{a} \in \mathbb{R}^k$, is $N(\boldsymbol{a}^T \boldsymbol{\mu}, \sum_{\ell=1}^{k} a_\ell^2 \sigma_{\ell\ell} + 2 \sum_{j=1}^{k-1} \sum_{\ell=j+1}^{k} a_\ell a_j \sigma_{\ell j})$.

$$H_s(\widehat{\boldsymbol{p}}) = -\sum_{\ell=1}^{k} \widehat{p_\ell} \log \widehat{p_\ell} \tag{3.11}$$

$$\widehat{\sigma}^2 = \frac{1}{n} \sum_{\ell=1}^{k} p_\ell (1-p_\ell)(\log p_\ell + 1)^2 - \frac{2}{n} \sum_{j=1}^{k-1} \sum_{\ell=j+1}^{k} p_\ell p_j (\log p_\ell + 1)(\log p_j + 1) \tag{3.12}$$

3.3 Case Study of Asymptotic Distribution of the Complexity

Statistical complexity is defined as the product of two normalized quantities:

- The Shannon entropy
- The Jensen-Shannon distance between the observed probability distribution and the uniform distribution

In this section we discuss three key aspects with real world scenario:

1. **Significance of Asymptotic Complexity Distributions**: Why understanding large-sample behavior matters for statistical inference

2. **Multinomial Model Framework**: Derivation of the asymptotic distribution for statistical complexity under multinomial assumptions

3. **Practical Formula**: A working equation for calculating the asymptotic distribution of complexity

As a case study for our work, we consider data from the Bearing Data Center and the seeded fault test data from Case Western Reserve University, School of Engineering. The dataset includes ball bearing test data for normal bearings as well as single-point defects on the fan end and drive end. Data were collected at a rate of 48,000 (48k drive-end) data points per second during bearing tests. Each file contains motor rotational speed (0, 1, 2, and 3), drive-end vibration data, and fan-end vibration data. The approximate motor speeds in RPM during testing: 1797, 1772, 1750, and 1730. For our case study, we consider two time series (Normal Baseline and 48k Drive-End) with a motor load of 0 and an RPM of 1797.

The goal of this study is to locate malfunctioning machinery. We use ordinal patterns to analyze the two time series. Based on the ordinal structure of the segments, we introduce distance as a metric of similarity. This metric can be used to identify malfunctioning machines with certain embedding dimension ranging from 3 to 6. The permutation entropy with asymptotic condition were computed taking into account the ordinal patter probability distribution. The results are analyzed using the complexity plane.

Chapter 4

Future Works

As a short summary, we have completed the necessary preliminaries studies on various topics such as:

- mmmmmmmmmmmmmmmmm
- mmmmmmmmmmmmmmmmm
- mmmmmmmmmmmmmmmm

and also we have examined the relevant research articles by Bandt & Pompe [5], Rey et.al [35, 36].

This proposal has three objectives in order to continue this research work.

- Define a data base of time series for clustering, i.e., finding similar time series.
- Extract all the features we know from their Bandt & Pompe symbolization (Shannon, Tsallis and Renyi entropies, Fisher information measure, complexities, and the available confidence intervals)
- Use those features for time series clustering

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