W4260: Problem Set #4

Due: Tuesday, March 3

In this problem, we will solve a set of differential equations to model the structure of solid planets. We assume that the planet is perfectly spherical, so that the pressure $P(r)$ and density $\rho(r)$ at any point of the star depends only on the distance to the center of the star (the radius r). If we consider an spherical shell of the planet to be in mechanical equilibrium, with pressure balanced against gravity, we obtain the equation of hydrostatic equilibrium:

$$\frac{dP(r)}{dr} = -\frac{G\rho(r)m(r)}{r^2} \tag{1}$$

where G is the gravitational constant, $m(r)$ is the mass of the planet interior to r and the minus sign indicates the pressure increases as r decreases.

In addition, there is a differential equation for the enclosed mass $m(r)$:

$$\frac{dm(r)}{dr} = 4\pi\rho(r)r^2 \tag{2}$$

These two equations are two coupled first order differential equations that determine the structure of the star for a given equation of state which relates the density ρ to the pressure P. Example equations of state include the familiar ideal gas relation; however, terrestrial ("rocky") planet interiors are not well described by this relation (although it is appropriate for some gas giants). Instead, we will use a fit to experimental data from Seager et al (2007, ApJ 669, 1279), which is of the form:

$$\rho(P) = \rho_0 + cP^n \tag{3}$$

where ρ_0, c and n are constants and depend on the material being modeled. For example, for Iron (Fe), $\rho_0 = 8300$ kg m^{-3}, $c = 0.00349$ kg m^{-3}Pa^{-n} and $n = 0.528$. Other materials can be found in Table 3 of the above paper.

The values of the dependent variables at $r = 0$ are: $P = P_c$ (the central pressure), and $m = 0$. Integrating outward in r gives the density profile, the outer radius R, being determined by the point at which the pressure goes to zero. The total mass of the star is then $M = m(R)$. Since both the mass and the radius depend on the central pressure, P_c, variation of this parameter allows planets of different masses to be studied. To do the integration, you will need to choose a radial step Δr – this should be some small fraction of an earth radius (since we expect "rocky" planets to be approximately earth sized). You should experiment with Δr to make sure your answer is reasonably accurate (i.e doesn't change much as you decrease Δr). Finally, note that you will not be able to evaluate dP/dr exactly at $r = 0$, so you will need to start the integration at some small r (say, $r = \Delta r$).

Problem 1

Write an ODE solver which uses the fourth-order Runga-Kutta technique, and use it to solve this set of ODE's. Apply it to the case $P_c = 10^{12}$ Pa, and output (or better yet, plot), the variation of density vs. radius and report your final radius and mass (in terms of the earth's radius and mass).

Problem 2

Repeat this for a range of central pressures to find the relation between planet total mass M and the outer radius R (where the pressure goes to zero). Pick a range of central pressure such that you cover the range from 0.1 to 100 earth masses. Plot the resulting radius - mass relation (in units of earth radii and earth masses, using a logarithm scale).

Problem 3

Repeat problem 2, but use different materials such as H_20 or $MgSi0_3$ (see Table 3 of the paper referenced above). Plot the resulting M-R relations and also plot points for the solar system's terrestrial planets (Mercury, Venus, Earth and Mars), and some exoplanet data – see, for example, Table 1 of arxiv.org/pdf/1312.0936v4.pdf (you don't have to plot all of those exoplanets – a small selection is fine).