Optimization Notes

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1. Linear Programming Problems

Disposition

- Linear Programming Problems
- Standard form
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- Simplex
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  - (Degeneracy)

Examples

$$z = 0 + x_1 + x_2$$

$$1 \geq x_1 + x_2$$

$$2 \geq x_1 + x_2$$

$$\begin{array}{cc} \max/\min & z = c^T x \\ s.t. & Ax \begin{pmatrix} \leq \\ \geq \\ = \end{pmatrix} b \\ & x \geq 0 \end{array}$$

Definitions

- **Objective Function:** A Linear Function, over all variables, to be maximized.
- **Polytope:** The geometric shape formed by the constraints.
- **Solution:** Any possible values that can be assigned to the variables, ignoring constraints.
- **Feasible Solution:** Any *Solution*, satisfying all constraints.
- **Basic Solution:** A *Feasible Solution* which lies in the geometric corner of the polytope.
- **Optimal Solution:** A *Feasible Solution* that maximizes the target function.

Linear Programming Problems

- Problems of the form: $z = \sum_{i=0}^{n} c_i x_i$ where we want to maximize or minimize z
- Must have constraints

Standard form

1. Must be maximization problem
2. All constraints must be $\geq$

Fundemental Theorem of Linear Programming

1. If no optimal solution exists, the problem is infeasible or unbounded
2. If *1.* and there exists a feasible solution there exists a Basic Feasible Solution
3. If there exists an optimal solution, there exists a Basic Optimal Solution

- **TLDR:** If an optimal solution exists, there must be one in a corner of the convex polytope.

Simplex

- Takes a Linear Programming Problem in Standard Form,
- Returns the optimal solution
- Simplex Tableu:
$$\begin{bmatrix} 1 & -\vec{c}^T & 0 \\ 0 & A & \vec{b} \end{bmatrix}$$
- **Slack variales:** Constraints of the form $c \leq c_1x_1 + c_2x_2 \to x_{n+1} = c - (c_1x_1 + c_2x_2)$
- **Cycles:** Suppose we have some Dictionary D_0, and we pivot some number of times to get Dictionary D_k, where we have already seen D_k:
  - $D_0 \to D_1 \cdots D_k : D_k \in [D_0, D_{k-1}]$

- **Bland's Rule:**
  - Why?
    - Prevents cycles
  - How?
    - Start by choosing the left-most non-basic variable with a positive coefficient

Example

We start with:

$$z = 0 + 10x_1 + 22x_2$$
$$11 \geq 3x_1 + 4x_2$$
$$15 \geq 5x_1 + 20x_2$$

Introducing slack variables:

$$z = 0 + 10x_1 + 22x_2$$
$$x_3 = 11 - 3x_1 - 4x_2$$
$$x_4 = 15 - 5x_1 - 20x_2$$

1. We need to choose the *Entering Variable*, using Bland's rule we choose x_1.
2. Then we need to choose an *Exiting Variable*, using Bland's rule, $x_3 : 11/3 = 3 + \frac{2}{3}$ and $x_4 : 15/5 = 3$, x_4 has the smallest non-negative value, so x_4 is the exiting variable.
3. Isolate the entering variable, (x_1), from the definition of our exiting variable (x_4).
4. Repeat 1-3 until all coefficients in the objective function are non-negative (We don't need to repeat for this example):

$$x_4 = 15 - 5x_1 - 20x_2$$
$$x_4 + 5x_1 = 15 - 20x_2$$
$$5x_1 = 15 - 20x_2 - x_4$$
$$x_1 = 3 - 4x_2 - \frac{1}{5}x_4$$

Inserting x_1 in z:

$$z = 0 + 10x_1 + 22x_2$$
$$z = 0 + (30 - 2x_4 - 40x_2) + 22x_2$$

$$z = 30 - 2x_4 - 18x_2$$

Because we are done, we don't actually need to do it for x_3, but for completeness, we finish step 3:

$$x_3 = 11 - 3x_1 - 4x_2$$
$$x_3 = 11 - (9 - 12x_2 - \frac{3}{5}x_4) - 4x_2$$
$$x_3 = 2 + 8x_2 + \frac{3}{5}x_4$$

So our final dictionary:

$$z = 30 - 2x_4 - 18x_2$$
$$x_1 = 3 - 4x_2 - \frac{1}{5}x_4$$
$$x_3 = 2 + 8x_2 + \frac{3}{5}x_4$$

This means that we have found our maximum, 30. If we want to find the necessary variables to produce 30. We know $x_2 = 0$, to isolate x_4:

$$x_1 = 3 - 4x_2 - \frac{1}{5}x_4$$
$$x_1 = 3 - 4 \cdot 0 - \frac{1}{5} \cdot 0$$
$$x_1 = 3$$

We can make a last sanity check:

$$z = 0 + 10 \cdot 3 + 22 \cdot 0$$
$$z = 0 + 30 + 0$$
$$z = 30$$

2. Duality

Disposition

- Duality
  - Motivation
  - Geometric intuition
  - Strong & Weak Duality Theorems
  - Complimentary Slackness
- Matrix Games

- Example
- Nash Equilibrium
- Fair Game
- Principle of Indifference

Examples

General Example

Primal:

$$
\begin{array}{cc}
\multicolumn{2}{l}{\textbf{P:}} \\
\max & z = c^T x \\
s.t. & Ax \leq b \\
 & x \geq 0
\end{array}
$$

Dual:

$$
\begin{array}{cc}
\multicolumn{2}{l}{\textbf{D:}} \\
\min & w = b^T y \\
s.t. & A^T y \geq c \\
 & y \geq 0
\end{array}
$$

Rock Paper Scissors

Our matrix A:

	Rock	Paper	Scissors
Rock	0	1	−1
Paper	−1	0	1
Scissors	1	−1	0

Primal (Column Player):

$$
\begin{array}{cc}
\multicolumn{2}{l}{\textbf{P:}} \\
\max & z = v \\
s.t. & A\overrightarrow{p} \leq \overrightarrow{v} \\
 & \overrightarrow{p} \geq \overrightarrow{0} \\
 & \overrightarrow{p}\,\overrightarrow{1} = 1
\end{array}
$$

Dual (Row Player):

$$
\begin{array}{cc}
\multicolumn{2}{l}{\textbf{D:}} \\
\max & w = u \\
s.t. & A^T\overrightarrow{q} \geq \overrightarrow{u} \\
 & \overrightarrow{q} \geq \overrightarrow{0} \\
 & \overrightarrow{q}\,\overrightarrow{1} = 1
\end{array}
$$

Duality theorems/properties

- For any feasible solution $p \in P$, and any feasible solution $d \in D$, $p \leq d$
- **Weak Duality Theorem:**
  - $p \leq d$
- **Strong Duality Theorem:**
  - $p = d \Leftrightarrow p = \text{optimal}(P) \wedge d = \text{optimal}(D)$
- If P is unbounded then D is infeasible and vice versa

Matrix game

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

Integer Linear Programming

Lecture 7 graph geometry

Duality example

P (max)

$$z = c_1x_1 + c_2x_2 + c_3x_3$$

$$b_1 \geq a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3$$

$$b_2 \geq a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3$$

D (min)

$$w = b_1y_1 + b_2y_2$$

$$c_1 \leq a_{1,1}y_1 + a_{2,1}y_2$$

$$c_2 \leq a_{1,2}y_1 + a_{2,2}y_2$$

$$c_3 \leq a_{1,3}y_1 + a_{2,3}y_2$$

3. Network Flows

Disposition

- Network Flow
- Minimum-cost Flow problem
- Max cut-min flow theorem
- Ford-Fulkerson example

Network Flow

Balances

- A flow network: $D = (N, A)$
- Outgoing flow from node i: $\sum_{ij \in A} x_{ij}$
- Ingoing flow to node i: $\sum_{ji \in A} x_{ji}$
- Balance at node i: $b_i = \text{out} - \text{in} = \sum_{ij \in A} x_{ij} - \sum_{ji \in A} x_{ji}$
- Balance restriction: $\sum_{i \in N} b_i = 0$
- If $b_i > 0$ then node i is a source, if $b_i < 0$ then node i is a sink

Arc Constraints

- Lower (l_{ij}) and upper (u_{ij}) bound for flows in nodes: $l_{ij} \leq x_{ij} \leq u_{ij}$
- Assumption: $0 \leq l_{ij} \leq u_{ij}$

The Maximum (s, t)-flow Problem

- One source (s)
- One sink (t)
- Flow conservation restriction: $b_i = 0 \mid i \in N \backslash \{s, t\}$
- Flow is feasible if:
  - No negative flows
  - Flow conservation restriction
  - Must satisfy arc constraints
- Convert maximum (s, t)-flow problem $(D = (N, A))$ to minimum flow problem:
  - New edge s to t is added to D' with cost -1 and upper bound ∞
  - All other edges has cost 0
  - All other nodes has balance +
  - Feasible flows x in D is feasible flows $x' \in D'$ where cost of $x' = -x$

Ford-Fulkerson Algorithm

See the following video

4. P, NP and Cook's theorem

Disposition

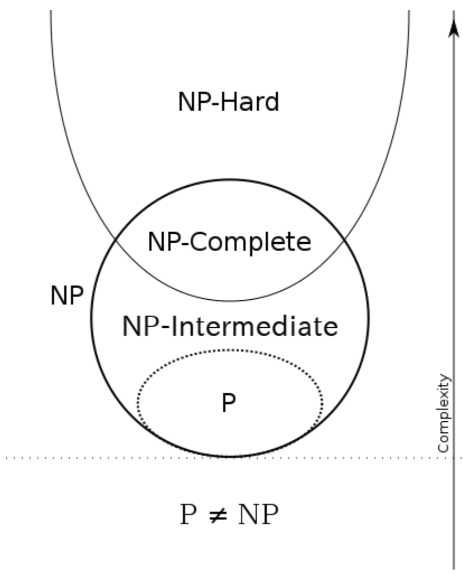
- Decision Problems
- P, NP, NPC, NPH
- Cook's theorem

NP completeness teori

- Model definition:

- Operates on bits and bites
  - Input size is equal to number of bits in input
  - Time complexity of an algorithm is number of bit operations done.
- Decision problems, yes-no answers from input
- Inputs are bits
- Pair function:
  - $\langle x, y \rangle = x_1 0 \cdots x_n 0 \quad 11 \quad y_1 0 \cdots y_n 0$
- Church-Turing Thesis:
  - **Alanzo Church:** *No computational procedure will be considered as an algorithm unless it can be represented as a Turing Machine.*
  - **Polynomial Church-Turing thesis:** A decision problem can be solved in polynomial time **iff** it can be solved in polynomial time by a turing machine.

Language Complexity



- **P:** All decision problems that can be solved by a deterministic Turing machine in polynomial time.
- **NP:** All decision problems, where the solutions that evaluates to "yes," can be verified in polynomial time.
- **NPC:** A set of decision problems that can all be reduced into one another.
- **NPI:** All `NP` decision problems that are *not* in `NPC` or `NP`.
- **NP-hard:** All decision problems that are *at least* as hard as `NPC`.
- $L_1 \leq L_2 \Leftrightarrow (x \in L_1 \Leftrightarrow r(x) \in L_2)$

SAT & CSAT

- For every boolean function $f(x)$, an equivilant curcuit $C(x)$ exists.
  - **Lemma:** For every boolean function $f: \{0,1\}^n \to \{0,1\}^m,\ \exists C\ s.t.\ \forall x \in \{0,1\}^n, C(x) = f(x)$
- **Literals:**
  - **Positive Literal:** An atom x
  - **Negative Literal:** A negation of an atom $\neg x$
- **Clause:** Collection of literals and logical connectives.
- **CNF:** Conjunctive Normal Form is a conjunction (`AND`'s) of clauses.
- **DNF:** Disjunctive Normal Form is a disjunction (`OR`'s) of clauses.
- **SAT:** Can the variables of a given CNF be replaced by either TRUE or FALSE such that the CNF evaluates to TRUE
- **Cook's Theorem:** SAT $\in$ NPC

5. NP-Complete Problems

Disposition

- P, NP, NPC, NPH
- CSAT
- 3SAT
- Clique

6. Approximation Algorithms and Search Heuristics

Disposition

- P, NP, NPC, NPH
- What is Approximation Algorithms?
- Max Cut Deterministic
- Max Cut Randomized

Deterministic Max-Cut Example

Given a graph $G = (V, E)$

Algorithm 1: Deterministic Max-Cut Algorithm

$S := \emptyset$, $T := \emptyset$
for $v \in V$ **do**
    if $w(\{v\}, S) > w(\{v\}, T)$ **then**
        $T := T \cup \{v\}$
    else
        $S := S \cup \{v\}$
    end
end
return (S, T)

Most optimal case is where all edges cross S and T, in short, the sum of all weights in V:

$$OPT = w(V)$$

Now to derive ρ for the deterministic algorithm:

$$\begin{aligned} w(S,T) &\geq w(S,S) + w(T,T) \\ w(S,T) + w(S,T) &\geq w(S,T) + w(S,S) + w(T,T) \\ 2w(S,T) &\geq w(V) \\ w(S,T) &\geq \frac{w(V)}{2} \end{aligned}$$

We chose our C to be the worst case scenario that our algorithm can come up with:

$$C = \frac{w(V)}{2}$$

Finding approximation ratio:

$$\rho = \frac{OPT}{C} = \frac{w(V)}{\frac{w(V)}{2}} = 2 \cdot \frac{w(V)}{w(V)} = 2$$

Randomized Max-Cut Example

Given a graph $G = (V, E)$

Algorithm 2: Randomized Max-Cut Algorithm

$S := \emptyset$, $T := \emptyset$
for $v \in V$ **do**
  Let $b \in_R \{0, 1\}$
  if $b = 1$ **then**
    $T := T \cup \{v\}$
  else
    $S := S \cup \{v\}$
  end
end
return (S, T)

Optimal solution same as in the deterministic:

$$OPT = w(V)$$

The probability that an edge will connect S and T:

$$P(w_{(i,j)\in(S,T)}) = \frac{1}{2}$$

$$E[\ |E_{\in(S,T)}|\ ] = |E_{\in G}|\ P(w_{(i,j)\in(S,T)})$$
$$E[\ |E_{\in(S,T)}|\ ] = \frac{|E_{\in G}|}{2})$$

The expected value of a randomly chosen edge:

$$E[e \in E] = \frac{w(V)}{|E_{\in G}|}$$

To find C:

$$C = E[\texttt{RAN}]$$
$$C = E[e \in E] \cdot E[\ |E_{\in(S,T)}|\ ])$$
$$C = \frac{w(V)}{|E|} \cdot \frac{|E|}{2}$$
$$C = \frac{w(V)}{2}$$

To find ρ:

$$\rho = \frac{OPT}{C} = \frac{OPT}{E[\texttt{RAN}]} = \frac{w(V)}{\frac{w(V)}{2}} = 2 \cdot \frac{w(V)}{w(V)} = 2$$

Appendix

Logic gates to CNF proofs

NOT

$$z \leftrightarrow \neg x$$
$$(\overline{z} + \overline{x})(z + \overline{\overline{x}})$$
$$(\overline{x} + \overline{z})(x + z)$$

COPY

$$z \leftrightarrow x$$
$$(\overline{z} + x)(z + \overline{x})$$
$$(x + \overline{z})(\overline{x} + z)$$

AND

$$z \leftrightarrow xy$$
$$(z + \overline{(x \cdot y)})(z + xy)$$
$$(z + \overline{x} + \overline{y})(z + xy)$$
$$(z + \overline{x} + \overline{y})(z + x)(z + y)$$

OR

$$z \leftrightarrow xy$$
$$(z + \overline{(x + y)})(\overline{z} + (x + y))$$
$$(z + (\overline{x} \cdot \overline{y}))(\overline{z} + x + y))$$
$$(\overline{x} + z)(\overline{y} + z)(x + y + \overline{z})$$

XOR

$$z \leftrightarrow x \oplus y$$
$$z \leftrightarrow (\overline{x} + \overline{y})(x + y)$$
$$(\overline{z} + (\overline{x} + \overline{y})(x + y)) \cdot (z + \overline{((\overline{x} + \overline{y})(x + y))})$$

We start with the left side:

$$\overline{z} + ((\overline{x} + \overline{y})(x + y))$$
$$(\overline{x} + \overline{y} + \overline{z})(x + y + \overline{z})$$

Then the right:

$$z + (\overline{(\overline{x} + \overline{y})} + \overline{(x + y)})$$
$$z + ((xy) + (\overline{x}\overline{y}))$$
$$z + (((xy) + \overline{x}) \cdot ((xy) + \overline{y}))$$
$$z + ((x + \overline{x})(y + \overline{x})(x + \overline{y})(y + \overline{y}))$$
$$z + (1 \cdot (y + \overline{x})(x + \overline{y}) \cdot 1)$$
$$z + ((\overline{x} + y)(x + \overline{y}))$$
$$((\overline{x} + y + z)(x + \overline{y} + z))$$

Finally giving us:

$$(\overline{x} + \overline{y} + \overline{z})(x + y + \overline{z})(\overline{x} + y + z)(x + \overline{y} + z)$$

EQ

$$z \leftrightarrow x \odot y$$
$$z \leftrightarrow (x + \overline{y})(\overline{x} + y)$$

We start with the left side:

$$\overline{z} + ((x + \overline{y})(\overline{x} + y))$$
$$(x + \overline{y} + \overline{z})(\overline{x} + y + \overline{z})$$

Then the right:

$$z + (\overline{(x + \overline{y})} + \overline{(\overline{x} + y)})$$
$$z + ((x\overline{y}) + (\overline{x}y))$$
$$z + (((x\overline{y}) + \overline{x}) \cdot ((x\overline{y}) + y))$$
$$z + ((x + \overline{x})(\overline{y} + \overline{x})(x + y)(\overline{y} + y))$$
$$z + (1 \cdot (\overline{y} + \overline{x})(x + y) \cdot 1)$$
$$z + ((\overline{x} + \overline{y})(x + y))$$
$$(\overline{x} + \overline{y} + z)(x + y + z)$$

Leaving us with:

$$(x + \overline{y} + \overline{z})(\overline{x} + y + \overline{z})(\overline{x} + \overline{y} + z)(x + y + z)$$