Notes on M7018M

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This is my collection of notes on the course "Applied Mathematics - M7018M" at Luleå Tekniska universitet for the autumn of 2020.

1 Lecture 1 - Equations of motion for a stretched string

Initial Value ($t = 0$):

$$\vec{r_0(s)} = \begin{bmatrix} x_0(s) \\ y_0(s) \end{bmatrix} \tag{1}$$

Where s =arclength

$$\left\| \frac{d\vec{r}}{ds} \right\| = 1 \tag{2}$$

The resting lengthdensity of the string is defined $\rho(s)$ [kg/m]

String in motion:
FIGUR 2

$$\vec{r(s)} = \begin{bmatrix} x_0(s,t) \\ y_0(s,t) \end{bmatrix} \tag{3}$$

$$\vec{r}(s,0) = \vec{r_0}(s) \tag{4}$$

The length density $\rho(s,t)$ [kg/m], $\rho(s,0) = \rho_0(s)$

Nature laws used in the derivation of the equations of motion

1. Conservation of mass
2. Newtons 2nd law: $\vec{F} = \frac{d\vec{p}}{dt}$, $\vec{F}$ = force, $\vec{p}$ = inertia
3. Eulers 2nd law: $\frac{\vec{(dL)}}{dt} = \vec{M}$, $\vec{L}$ = moment of interia $\vec{M}$ =Torque
4. Hookes law: The tension is proportional to the elongation

1.1 Conservation of mass

Consider an arbitrary short part of the string:
FIGUR 3

$$C(t):\ \vec{r}(s) \quad a \leq s \leq b$$

$$dl = \left\| \frac{d\vec{r}}{ds} \right\| ds \tag{5}$$

Total mass:

$$m = \int_{C(t)} \rho(s,t) dl \tag{6}$$

The mass is conserved:

$$0 = \frac{dm}{dt} = \frac{d}{dt} \int_{C(t)} \rho(s,t) dl \overset{(0)}{=} \frac{d}{dt} \int_{C(t)} \rho \left\| \frac{d\vec{r}}{ds} \right\| ds = \int_a^b \frac{d}{dt} (\rho \left\| \frac{d\vec{r}}{ds} \right\|) ds \tag{7}$$

The fact that a and b is chosen as an arbitrary point on the string leads to that the integrand has to be zero.

$$\frac{d}{dt} (\rho \left\| \frac{d\vec{r}}{ds} \right\|) = 0 \Rightarrow \tag{8}$$

This leads to:

$$\rho(s,t) \left\| \frac{d\vec{r}}{ds} \right\| = \rho_0(s) \tag{9}$$

1.2 Newtons 2nd law

FIGURE 4

Elongation-force $\vec{T}(s)$

Sum of forces on $C(t)$:

$$\int_{C(t)} \rho \vec{g} dl + \vec{T}(b) - \vec{T}(a) \tag{10}$$

Total inertia on $C(t)$:

$$\int_{C(t)} \rho \frac{d\vec{r}}{dt} dl \tag{11}$$

Newtons second law gives:

$$\frac{d}{dt} \int_{C(t)} \rho \frac{d\vec{r}}{dt} dl = \int_{C(t)} \rho \vec{g} dl + \vec{T}(b) - \vec{T}(a) \overset{(0)}{\Rightarrow} \int_a^b \rho \frac{d\vec{r}}{dt} \left\| \frac{d\vec{r}}{ds} \right\| ds = \int_a^b \rho \vec{g} \left\| \frac{d\vec{r}}{ds} \right\| ds + \int_a^b \frac{\partial \vec{T}}{\partial s} ds \tag{12}$$

$$\overset{(1)}{=} \int_a^b \rho_0(s) \frac{\partial^2 \vec{r}}{\partial t^2} ds = \int_a^b (\rho_0(s) \vec{g} + \frac{\partial \vec{T}}{\partial s}) ds \tag{13}$$

as a and b are arbitrary

This leads to:

$$\rho_0(s) \frac{\partial^2 \vec{r}}{\partial t^2} = \rho_0(s) \vec{g} + \frac{\partial \vec{T}}{\partial s} \tag{14}$$

1.3 Eulers 2nd law

$\frac{d\vec{L}}{dt} = \vec{M}$ (For a fixed value of t)

$$\frac{d}{dt} \int_{C(t)} \vec{r} \times (\rho \frac{d\vec{r}}{dt}) d\vec{l} = \int_{C(t)} \vec{r} \times (\rho \vec{g}) dl + (\vec{r} \times \vec{T}) \Big|_{s=b} - (\vec{r} \times \vec{T}) \Big|_{s=a} \overset{(0),(1)}{\Rightarrow} \int_a^b \rho_0 \frac{d}{dt} (\vec{r} \times \frac{d\vec{r}}{dt}) ds = \int_a^b \rho_0 (\vec{r} \times \vec{g}) + \frac{\partial}{\partial s} (\vec{r} \times \vec{T}) \text{a,b arbitrary.}$$

$$\rho_0 \frac{d}{dt} (\vec{r} \times \frac{d\vec{r}}{dt}) = \rho_0 (\vec{r} \times \vec{g}) + \frac{\partial}{\partial s} (\vec{r} \times \vec{T}) \overset{(2)}{\Rightarrow} \frac{\partial \vec{r}}{\partial s} \times \vec{T} = \vec{0}.$$

The strings tension σ and it's unit-tangent $\hat{\tau} = \frac{1}{\left\| \frac{\partial \vec{r}}{\partial s} \right\|} \frac{\partial \vec{r}}{\partial s}$ gives the result:

$$\boxed{\vec{T} = \sigma\hat{\tau}} \qquad (15)$$

1.4 Hookes law

Assume that the strings tension at rest ($t = 0$) is σ_0 [N]
Hookes law: $\sigma - \sigma_0 = E\epsilon$ with E =Young Modulus and ϵ =elongation

$$\epsilon = \frac{dl - ds}{ds} = \frac{dl}{ds} - 1 \overset{(0)}{=} \left\| \frac{d\vec{r}}{ds} \right\| - 1 \overset{(1)}{=} \frac{\rho_0}{\rho} - 1 \Rightarrow .$$

$$\boxed{\sigma = \sigma_0 + E(\frac{\rho_0}{\rho} - 1)} \qquad (16)$$

With $\sigma_0 = \sigma_0(s)$, E constant.

1.5 The equation of motion of the string

$$\vec{T} = \sigma\hat{\tau} \overset{(3)}{\Rightarrow} \frac{\sigma}{\left\|\frac{\partial\vec{r}}{\partial s}\right\|}\frac{\partial\vec{r}}{\partial s} \overset{(4)}{=} \frac{\sigma_0 + E(\frac{\rho_0}{\rho} - 1)}{\frac{\rho_0}{\rho}}\frac{\partial\vec{r}}{\partial s}.$$

In equation (2) this gives the equations of motion

$$\rho_0\frac{\partial^2\vec{r}}{\partial t^2} = \rho_0\vec{g} + \frac{\partial}{\partial s}(\frac{\sigma_0 + E(\frac{\rho_0}{\rho} - 1)}{\frac{\rho_0}{\rho}}\frac{\partial\vec{r}}{\partial s}) \qquad (17)$$

$$\rho\left\|\frac{\partial\vec{r}}{\partial s}\right\| = \rho_0 \qquad (18)$$

Where eq.17 can be split into to equations, one for movement in x-direction and one in y-direction
Example: Tranversal motion in a "stiff" string
Some assumptions:
$\vec{r_0}(s) = \begin{bmatrix} s \\ 0 \end{bmatrix}$, i.e. s=x
The string can only move along the y-axis, i.e. $\vec{r}(x,t) = \begin{bmatrix} x \\ y(x,t) \end{bmatrix}$ and $y(x,0) = 0$
"Stiff string", i.e. $\rho \approx \rho_0$

$$0 = \frac{\partial\sigma}{\partial x} \Rightarrow \ \sigma_0 \text{ is constant.}$$

$$\rho_0\frac{\partial^2 y}{\partial t^2} = -\rho_0 g + \sigma_0\frac{\partial^2 y}{\partial x_0^2}.$$

$$\sqrt{1 + (\frac{dy}{dx})^2} \approx 1 \Rightarrow \frac{\partial y}{\partial x} \approx 0.$$

These are the same condition that was solved in the course Mathematical Physics at an earlier point in the Y-programme.

2 Lecture 2 - Dimensional Analysis and Scaling

A very useful tool in mathematical modeling are the use of dimensional analysis and scaling.

2.1 Dimensional Analysis

There are 7 (or more) ground quantities that can be used to build up all the SI-units and different units that are used in models. These are:

Table 1: Basic units

Quantity	Symbol	SI-unit
Length	L	[m]
Time	T	[s]
Mass	M	[kg]
Temperature	Θ	[°C]
Electrical Current	J	[A]
Light Strength		[cd]
Amount of Substance		[mol]

All other units can be created by combining these basic units.

Table 2: Units

Quantity	Expression	Unit
Force	LMT^{-2}	$N = kg \cdot m \cdot s^2$
Energy	$L^2MT{-}2$	$J = N \cdot m$
Power	$L^2MT{-}3$	$W = J \cdot s^{-1}$
Volumedensity	ML^{-3}	$\rho = kg \cdot m^{-3}$
Electrical voltage	$L^2MT^{-3}J^{-1}$	$V = W * A{-}1$

Dimensionless variables, e.g.:

$$\frac{x}{L} = \hat{x}.$$

if x has dimension L the $\hat{x}$ is dimensionless.

Buckinghams Pi-theorem
For every physical law

$$f(q_1, q_2, ..., q_n) = 0.$$

that relates the quantities $q_1, q_2, ..., q_n$ there is an equivalent law

$$g(\pi_1, \pi_2, ..., \pi_m) = 0.$$

where $\pi_1, \pi_2, ..., \pi_m$ is the least possible independant dimensionless quantities that can be made with $q_1, q_2, ..., q_n$.

Example: A ball is falling freely in a visquos fluid. After some time it gets the constant velocity U $[\frac{m}{s}]$. How to determine U?
FIGUR

The sum of the forces are 0 as $U = constant$

$$D = m \cdot g = \frac{4\pi a^3}{3}\rho_0 g.$$

Assume there are a physical relation
$$f(a, \rho_0, U, \mu, D) = 0.$$
Dimensional analysis of chosen quantities.
$$\begin{aligned} a &\sim L \\ \rho_0 &\sim ML^{-3} \\ U &\sim LT^{-1} \\ \mu &\sim ML^{-1}T^{-1} \\ D &\sim MLT^{-2} \end{aligned}$$

Question: How many dimensionless quantites can be made with a, ρ, U, μ, D? E.g. how many independant dimensionless quantities
$$q = a^{\alpha_1}\rho^{\alpha_2}U^{\alpha_3}\mu^{\alpha_4}D^{\alpha_5}.$$
can be made?
$q \sim L^{\alpha_1} \cdot (ML^{-3})^{\alpha_2} \cdot (LT^{-1})^{\alpha_3} \cdot (ML^{-1}T^{-1})^{\alpha_4} \cdot (MLT^{-2})^{\alpha_5} = L^{\alpha_1 - 3\alpha_2 + \alpha_3 - \alpha_4 + \alpha_5} T^{-\alpha_3 - \alpha_4 - 2\alpha_5} M^{\alpha_2 + \alpha_4 + \alpha_5}.$
$$\begin{cases} \alpha_1 - 3\alpha_2 + \alpha_3 - \alpha_4 + \alpha_5 = 0 \\ -\alpha_3 - \alpha_4 - 2\alpha_5 = 0 \\ \alpha_2 + \alpha_4 + \alpha_5 = 0 \end{cases} \sim \begin{pmatrix} 1 & -3 & 1 & -1 & 1 \\ 0 & 0 & -1 & -1 & -2 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \sim$$
$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} \alpha_4 = s \\ \alpha_5 = t \\ \alpha_3 = -\alpha_4 - 2\alpha_5 = -s - 2t \\ \alpha_2 = \alpha_4 - \alpha_5 = -s - t \\ \alpha_1 = alpha_4 - 2\alpha_5 \end{cases} \iff \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} = s \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$
Thus maximum of two independant dimensionless variables, e.g.
$$\begin{aligned} \pi_1 &= a^1\rho^1U^1\mu^{-1} = Re \\ \pi_2 &= a^{-2}\rho^{-1}U^{-2}D^1 = \frac{D}{\rho a^2 U^2} \end{aligned}$$

$\pi-$theorem gives a dimensionless physical relationship
$$0 = \Phi(\pi_1, \pi_2) = \Phi(Re, \frac{D}{\rho a^2 U^2}).$$
Implicit functiontheorem gives
$$\frac{D}{\rho a^2 U^2} = f(Re).$$
$$D = \rho a^2 U^2 f(Re) = \mu a U \frac{\rho a U}{\mu} f(Re) (\overset{(\text{def})}{=} f(\hat{R}e) = \frac{\rho a U}{\mu} f(Re)) = \mu a U f(\hat{R}e.$$
Approximative analytical solution of Naiver-Stokes equations gives
$$f(\hat{R}e) = 6\pi(1 + \frac{3}{8}Re + \frac{9}{40}Re^2\log(Re) + O(Re^2)).$$
For $Re << 1$ we get
$$D \approx 6\pi\mu a U.$$
$$U \approx \frac{D}{6\pi\mu a} = \frac{4\pi a^3 \rho_0 g}{3} \frac{1}{6\pi\mu a} = \frac{2a^2\rho_0 g}{9\mu}.$$
Answer $U \approx \frac{2a^2\rho_0 g}{9\mu}$
If u check units
$$U \sim \frac{L^2 \cdot M \cdot L^{-3} \cdot L \cdot T^{-2}}{M \cdot L^{-1} \cdot T^{-}1} = L \cdot T^{-1}.$$

2.2 Scaling

FIGUR PENDEL

- l [m] the pendulum length
- g [$\frac{m}{s^2}$] gravitational acceleration
- θ [-] angle
- m [kg] mass
- t [s] time

Total energy

$$E = T + V = \frac{1}{2}m(l\dot{\theta})^2 + mg(l - \cos(\theta)).$$

$$0 = \frac{dE}{dt} = \frac{1}{2}ml^2 2\ddot{\theta}\dot{\theta} + mgl\sin(\theta)\dot{\theta} = ml\dot{\theta}(l\ddot{\theta} + g\sin(\theta)).$$

Initial value problem (IVP)

$$\begin{cases} l\ddot{\theta} + \sin\theta = 0 \\ \theta(0) = \theta_0 \\ \dot{\theta} = \omega_0 \end{cases}$$

Parameters:

- l $\sim L$
- g $\sim L \cdot T^{-2}$
- $\theta_0 \sim 1$
- $\omega_0 \sim T^{-1}$

Characteristic time: $\frac{1}{\omega_0} \sim T$ or $\tau = \sqrt{\frac{l}{g}} \sim (\frac{L}{LT^{-2}})^{\frac{1}{2}} = T$

Dimensionless variables: $\hat{t} = \frac{t}{\tau}$, $\hat{\theta} = \hat{\theta}(\hat{t}) = \frac{\theta(t)}{\theta_0}$ i.e. $\theta = \theta_0\hat{\theta}$
let $\dot{\hat{\theta}} = \frac{\partial\hat{\theta}}{\partial\hat{t}}$

$$\dot{\theta} = \theta_0 \, , \frac{d\hat{\theta}}{dt} = \theta_0 \, , \frac{\partial\hat{\theta}}{\partial\hat{t}}\frac{\partial\hat{l}}{\partial t} = \frac{\theta_0}{\tau}\dot{\hat{\theta}}.$$

$$\ddot{\theta} = \frac{\theta_0}{\tau^2}\ddot{\hat{\theta}}.$$

A new scaled IVP can be written:

$$\begin{cases} \theta_0\ddot{\hat{\theta}} + \sin(\theta_0\hat{\theta}) = 0 \\ \hat{\theta}(0) = \frac{\theta(0)}{\theta_0} = 1 \\ \dot{\hat{\theta}}(0) = \frac{\omega_0\tau}{\theta_0} \end{cases}.$$

If $\theta_0 = \epsilon$ and $\omega_0 = 0$ we get:

$$\begin{cases} \ddot{\hat{\theta}} + \epsilon^{-1}\sin(\epsilon\hat{\theta}) = 0 \\ \hat{\theta}(0) = 1 \\ \dot{\hat{\theta}}(0) = 0 \end{cases}.$$

3 Perturbation Theory

Solutions of an algebraic equation $f(x, \epsilon = 0$ by assuming that x can be written as an power series expansion of ϵ:

$$x(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n x_n = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...$$

Ex Find a real solution to

$$\begin{cases} f(x,\epsilon) = x^3 + \epsilon x - 1 = 0 \\ f(0,\epsilon) = -1 \\ f(1,\epsilon) = \epsilon \\ \frac{\partial}{\partial x} f(x,\epsilon) = 3x^2 + \epsilon > 0 \end{cases}.$$

We want a solution on the form

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + ...$$

$$0 = f(x,\epsilon) = (x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + ...)^3 - \epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + ...) - 1.$$

First term of eq.

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + ...)^3 = x_0^3 + 3\epsilon x_0^2 x_1 + \epsilon^2(3x_0^2 x_2 + 3x_0 x_1^2) + \epsilon^3(3x_0^2 x_3 + 6x_0 x_1 x_2 + x_1^3) + \mathcal{O}(\epsilon^4).$$

In the first equation put every coefficent (to ϵ^k) equal to 0.

$$\epsilon^0 : x_0^3 - 1 = 0 \Rightarrow x_0 = 1.$$

$$\epsilon : 3x_0^2 x_1 + x_0 = 0 \Rightarrow 3x_1 + 1 = 0 \Rightarrow x_1 = -\frac{1}{3}.$$

$$\epsilon^2 : 3x_0^2 x_2 + 3x_0 x_1^2 + x_1 = 0 \Rightarrow 3x_2 + 3\frac{1}{9} - \frac{1}{3} = 0 \ \Rightarrow x_2 = 0.$$

$$\epsilon^3 : 3x_0^2 x_3 + 6x_0 x_1 x_2 + x_1^3 + x_2 = 0 \Rightarrow 3x_3 - \frac{1}{27} = 0 \Rightarrow x_3 = \frac{1}{81}.$$

Thus $x = 1 - \frac{\epsilon}{3} + \frac{\epsilon^3}{81} + \mathcal{O}(\epsilon^4)$

For $a = 1 - \frac{\epsilon}{3} + \frac{\epsilon^3}{81}$ and $f(x,\epsilon) = x^3 + \epsilon x - 1$ we get.

$$f(a,\epsilon) = (1 - \frac{\epsilon}{3} + \frac{\epsilon^3}{81})^3 + \epsilon(1 - \frac{\epsilon}{3} + \frac{\epsilon^3}{81}) - 1 = -\frac{\epsilon^4}{81} + \frac{\epsilon^5}{243} + \frac{\epsilon^6}{2187} - \frac{\epsilon^7}{6561} + \frac{\epsilon^9}{531441}.$$