Projecteuler 323

Solution

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Let us define a discrete stochastic process X_t = number of 1-s in base 2 after t steps. We know that $X_0 = 0$ and it takes values from 0 to 32 and 32 is an absorbing state.

Generating a random 32 bit integer means flipping a coin 32 times, i.e. each bit has 1/2 probability of being 1 or 0. That's why the probabilities for the first step are given by Bin(32, 0.5) and each consecutive step only depends on the previous step

$$X_i \sim \text{Bin}(32 - X_{i-1},\ 0.5). \tag{1}$$

Let $T = \min\{n \geq 0 :\ X_n = 32\}$ and we wish to find $E(T \mid X_0 = 0)$ with first step analysis. Let $v_i = E(T \mid X_0 = i)$ for $i = 0, 1, 2, \ldots, 32$, then $v_{32} = 0$ and

$$v_i = 1 + \sum_{j=0}^{32} v_j P_{ij}, \tag{2}$$

where

$$P_{ij} = P(X_{t+1} = j \mid X_t = i) = \binom{32-i}{j-i} 2^{i-32}$$

because of (1). Solving (2) for $i = 0$ gives us

$$\begin{aligned} v_0 &= 1 + \sum_{j=0}^{32} v_j P_{0j} \\ &= 1 + 2^{-32} \sum_{j=0}^{32} v_j \binom{32}{j} \\ &= 1 + v_0 2^{-32} + 2^{-32} \sum_{j=1}^{31} v_j \binom{32}{j} \\ &= \frac{1 + 2^{-32} \sum_{j=1}^{31} v_j \binom{32}{j}}{1 - 2^{-32}}. \end{aligned}$$

In order to find v_0, we need to find v_i, $i = 1, 2, \ldots, 31$ first

$$v_i = \frac{1 + 2^{i-32} \sum_{j=i+1}^{31} v_j \binom{32-i}{j-i}}{1 - 2^{i-32}}.$$

Solution is

$$v_0 \approx 6.3551758451.$$