Insert front page here

Det Teknisk-Naturvidenskabelige Basisår
Software
Strandvejen 12-14
Telephone 96 35 97 31
Fax 98 13 63 93
http://tnb.aau.dk

Title:
Rubik's Cube

Tema:
Network and Algorithms

Project period :
P2, spring semester 2010

Project group:
A215

Participants:

Mikael Midtgaard

Jens Mohr Mortensen

Christoffer Ilsø Vinther

Rasmus Veiergang Prentow

Dan Stenholt Møller

Mikkel Bach Klovnhøj

Alex Bondo Andersen

Advisors:
Anders Bruun
Heather Baca-Greif

Page count: 46

Appendices count and type: *TBD*

Finished: 27/5–2010

Synopsis:

Contents

Preface

Note: Remember to thank Herbert Kociemba for his e-mail. Thank Anders and Heather and Leif for help with litterature. Prerequisites for reading this report: A Rubik's Cube, knowledge of graph and group theory. Citations are with square brackets. Trademarks on stuff is omitted.

Part I

Introduction

1 Problem Analysis

Since 1977, when the Rubik's Cube was initially released for sale, the Rubik's Cube has frustrated, inspired and entertained many people. This 3x3x3 cube has so many possible settings that the solution can not just be guessed out of sheer luck. Because of this a community around solving the Rubik's Cube has emerged. The community is divided into two parts both concerning efficient solving – one efficient time-wise and the other efficient twist-wise i.e. solving in the least amount of time and solving in the least amount of twists [5].

The part concerning speed-wise efficiency, often referred to as speedcubing is the largest part of the community and the majority of the competitions held by the WCA[1] [14] revolve around speedcubing.

The first official competition was held in 1982 in Hungary and is regarded as the first World championship. Since 2002 there have been held annual world championships and plenty other events concerning speedcubing.

The part of the community concerning twist-wise efficiency is much smaller than the speedsolving part. The majority of the research in the twist-wise efficient area is published as scientific articles explaining the algorithms. Even though competitions with the goal of the least amount of twists to solve the cube are held, many of the twist-wise efficient algorithms are not useful for human solving. These algorithms rely on computer power to look through a large amount of possibilities, which is not a viable option for a human competitor.

The ultimate goal for the twist-wise efficiency community is to find the God's algorithm, which is the algorithm that solves the cube in the absolute least amount of twists from any given position. A part of finding the God's algorithm is to find the amount of moves need to perform it. The upper bound of the Rubik's Cube is the minimum number of twists that the most efficient algorithm at the moment can perform. The lower bound is the least number of twists required to solve the cube in the currently known worst case scenario. It

[1] WCA, World Cube Association, is the official organization for Rubik's Cube related competitions.

is interesting to study this part of the Rubik's Cube community because it is currently moving, proving new upper bounds.

1.1 Problem Statement

What are the current upper and lower bound of the Rubik's Cube and how have they been proven?

How can we create an application which can solve the Rubik's Cube?

- Which algorithms can be used and which is the most efficient with respect to the number of twists?

1.2 Problem Limitations

Because the amount of different algorithms for Rubik's Cube solving, not every algorithm will be covered in this project.

The Rubik's Cube solving algorithm will be primarily for technical use, meaning that usability will not be in focus.

2 Recreational Mathematics

It is important to understand what recreational mathematics is in order to get a better understanding of the Rubik's Cube. The Rubik's Cube is related to other recreational mathematical puzzles, which have inspired the Rubik's Cube and are simpler to understand at first grasp. This chapter presents a definition of recreational mathematics and a few examples of recreational mathematical puzzles other than the Rubik's Cube. Different theorems for these puzzles are presented and proved, because similar proofs are used later for the Rubik's Cube.

2.1 Definition

Recreation means to do something which is amusing or relaxing. Mathematics is somewhat harder to give a precise definition of due to the vast amount of subjects that fall under this term. Most people do however have a common idea of what mathematics is. So for the purpose a definition of mathematics, each reader may use his or her own.

Recreational mathematics is hereby defined as mathematical problems, puzzles or games which are fun and interesting to laymen as well as mathematicians. [11] [13, p. 18]

2.2 Puzzles

This project is dedicated to the Rubik's Cube and the cube will be covered in detail later in this report. This section will instead describe some puzzles related to the Rubik's cube.

2.2.1 Magic Square

A Magic Square is a square which is divided into a number of sub squares. The number of sub squares in any row or column is referred to as the "order" of the

Magic Square. In each sub square there is a positive integer. In order for the Magic Square, to be "magic", the sum of any row, column or diagonal must be the same, this sum is referred to as the magic constant. See table 2.1 below.

6	1	8	15
7	5	3	15
2	9	4	15
15	15	15	45

Table 2.1: *A Magic Square of the order 3, by adding the three numbers in any row, column or diagonal, the magic constant is seen to be 15*

The Magic Square[1] hails from ancient China. It was said that the people near the river Lo made offerings. Every time they made an offering a tortoise emerged from the river. On the back of the tortoise there was said to be a Magic Square.

The Magic Square from this tale was of the order 3. This is not the only order in which a Magic Square can be created; it is possible to make an "n" order Magic Square. Generally a Magic Square of the order n contains the numbers from 1 to n^2. It has been proven that it is not possible to make a second order Magic Square. Since simply trying all possible squares with the numbers 1, 2, 3 and 4 inside results no combination that gives the same sum on each row, column and diagonal.

In order to solve the Magic Square, it is needed to know the magic constant – the constant which every row, line and diagonal adds up to for the given order n. This constant can be computed with the formula in 2.1.

$$M(n) = \frac{n \cdot (n^2 + 1)}{2} \tag{2.1}$$

The proof of this formula is quite straight forward. As the table 2.1 illustrates, a Magic Square of the order 3 contains the numbers from 1 to 9.

The sum of the numbers of a row in a Magic Square is equal to the magic constant. If the magic constant is multiplied by the order n it would be equal to the sum of all the integers, since each number only occurs once in a Magic Square.

The equation 2.2 can be rewritten into the equation 2.3(See proof of the right hand side transcription in appendix X).

$$n \cdot M\left(n\right) = \sum_{i=1}^{n^2} i = 1 + \cdots + n^2 \tag{2.2}$$

$$n \cdot M(n) = \frac{n^2 \cdot \left(n^2+1\right)}{2} \tag{2.3}$$

$$M(n) = \frac{n \cdot \left(n^2+1\right)}{2} \tag{2.4}$$

The equation 2.4 shows the function which gives the magic constant for a Magic Square of the order n.

A Magic Cube is created from squares put on top of each other so they make up a cube form. This makes it clear that there is a connection between Magic Squares and Magic Cubes. An example of this can be seen on figure 2.1.

2.2.2 Magic Cube

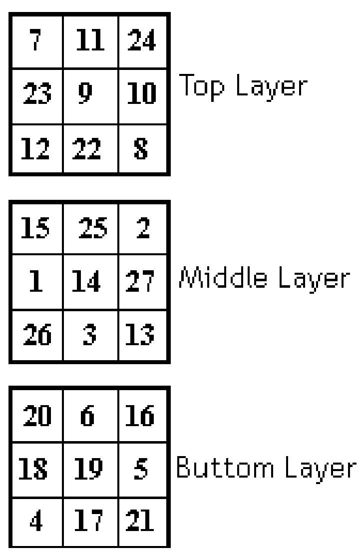


Figure 2.1: *This is a magic cube split up into 3 magic squares.*

Both a Magic Square and a Magic Cube have a magic constant, which can be the sum of each row, column and pillar. However this is where the similarity ends.

We have shown how to calculate the magic constant in a Magic Square. In a Magic Cube there is not a big difference in the formula to calculate the magic constant.

$$M(n) = \frac{n \cdot (n^3+1)}{2} \tag{2.5}$$

As shown in the formula the only difference is the power of n that is changed from 2 to 3. See appendix A.1 for en explanation.

To create a Magic Cube, there are some parts that needs to be explained. All these basics are shown on figure 2.2.

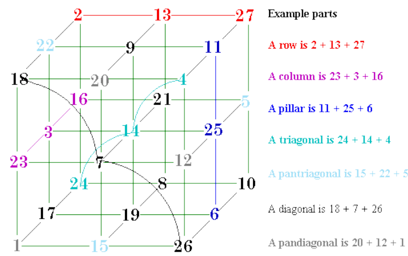


Figure 2.2: *This is a Magic Cube where the colors show all of the parts.*

Because of all these different parts there are a lot of different ways to define Magic Cubes. The simplest of them all is a simple Magic Cube. The only requirements to make such a cube is the following:

- All 9 rows, columns and pillars must be equal to the magic constant.
- All 4 triagonals must also equal the magic constant.

Figure 2.3: *This is a Rubik's Cube.*

When looking at the Rubik's Cube it is easy to see that it looks a lot like the Magic Cube. There are two differences. The first is that the Magic Cube consists of numbers whereas the Rubik's Cube has colors, which are different on each face. The other difference is that the Magic Cube has a number in the center where the Rubik's Cube does not because it is rotating around the center.

2.2.3 Magic Puzzle

The Magic Puzzle is also known as the 15-puzzle [4, pp. 48-50]. It is a puzzle that consists of a tray with 15 square tiles and an empty square arranged in a 4x4 contraption.

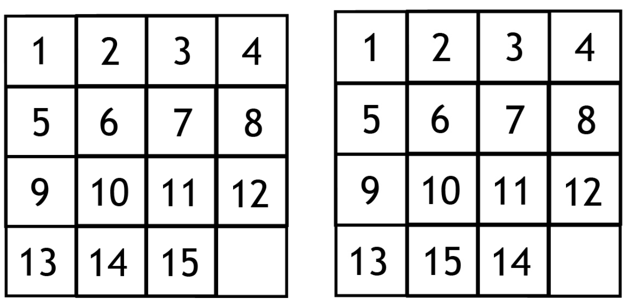
It has never been discovered who actually invented the Magic Puzzle, but Samuel Loyd who was an American chess player and puzzle author claimed that he invented the Magic Puzzle and therefore he got the credit. This is turned down by a research of Jerry Slocum. He discovered that there was a wooden version of the game already in 1865, this was manufactured by the Embossing Co. Jerry Slocum searched for the patent and found it, US 50.608 and was applied by a Henry May.

Jerry Slocum also found a patent by Ernest U. Kinsey that was published August 20th 1878. This version by Ernest U. Kinsey was a 6x6 version of the puzzle which also prevented the tiles from being lifted out.

Permutations

The tiles in a Magic Puzzle can be arranged in 16! different positions [10]. This limit can not be reached because you have to make a permutation to switch the tiles. The permutation must be an even or odd number of transpositions depending on where the position of the empty square is.

The tiles are often numbered or labeled with small pictures which when assembled correctly form a larger picture.



(a) *Figure of Magic Puzzle.* (b) *Figure of Magic Puzzle with inverse numbers.*

Figure 2.4: *Illustrations of how the legal and illegal permutations of the Magic Puzzle.*

For instance we got the figure 2.4a and want to switch the tiles to be positioned like on figuare 2.4b. This permutation requires an odd transposition of the seven pairs (1,15), (2,14), (3,13), (4,12), (5,11), (6,10) and (7,9). This permutation is not possible because it requires an even number of transpositions to get the empty square at the same position. If we color the contraption like a chess board 2.5 we can see that every odd transposition makes the empty square change color and with every even transposition the empty square lands on a square of the same color.

Therefore the number of different positions is $\frac{16!}{2}$. But if the empty square has to be in a fixed position then the possible permutations is $\frac{15!}{2}$. These permutations are almost alike the ones the Rubik's Cube use and they actually inspired Ernö Rubik into his creation of the Rubik's Cube.

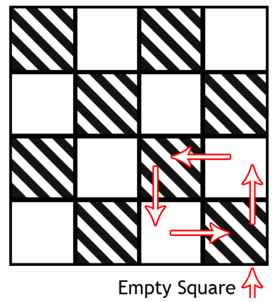


Figure 2.5: *Figure of Empty Square.*

This chapter has given a definition of recreational mathematics and shown three puzzles, which all relates to the Rubik's Cube; Magic Square, Magic Cube and Magic Puzzle. The Magic Square was the predecessor to Magic Cube, which is in turn the predecessor to the Rubik's Cube. The permutation from the Magic Puzzle inspired the creation of Rubik's Cube, which uses a similar principle for moving the pieces around.

3 Origin of the Cube

In this chapter we will describe the history behind Ernö Rubik and how he got the idea for the Rubik's Cube to get a better understanding of the Rubik's Cube. Furthermore we will look at the the development and the proble'matics with the patenting and legal issues regarding the cube are described. We will look at the patent to get a better understanding of the cubes made at the time. The purpose of this chapter is to give the reader a basic understanding of the Rubik's Cube.

3.1 Ernö Rubik

Ernö Rubik is the inventor behind the world famous Rubik's Cube. He was born in Budapest, Hungary in 1944, his father was flight engineer and his mother was poet. He graduated from the Technical University, Budapest as an architectural engineer after he graduated with a degree in architecture he stayed at the college to teach interior design.

That led to in January 1975 Rubik applied for a patent for his invention in Hungary There was made in first place to help his students. Two years later in 1977 he got the patent on the Magic Cube. He became professor with full tenure in the 80s, he started Rubik Studio, which employs a dozen people to design furniture and toys. Since Rubik has produced several other toys, including Rubik's Snake, lately the studio began developing computer game. He also became the president of the Hungarian Engineering Academy in 1990. Same Year he created the International Rubik Foundation to support especially talented young engineers and industrial designers.

3.2 Rubik's Cube(Magic Cube)

In the 70s Ernö Rubik was teaching Interior Design at Academy of Applied Arts and Crafts and he was trying to find a tool to help his students to understand

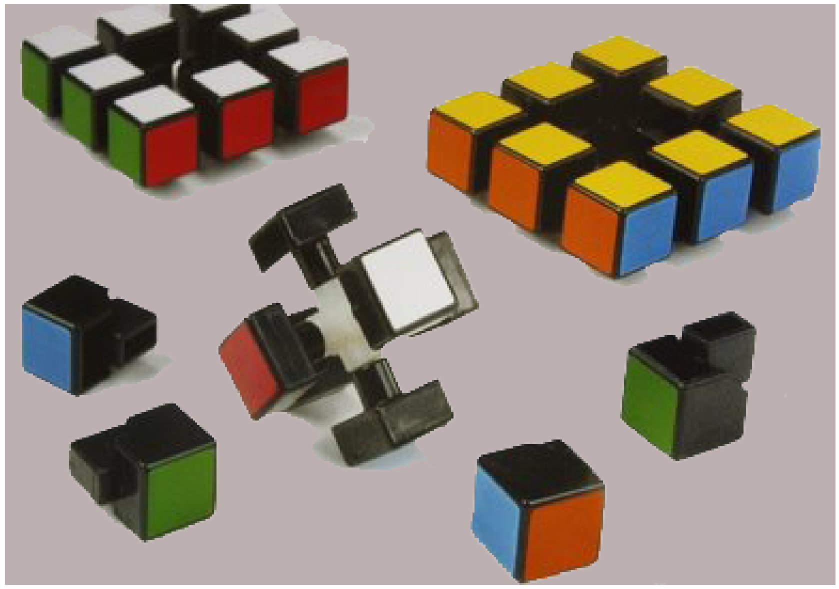


Figure 3.1: *Figure of Rubik's Cube.*

three dimensional objects as result he made the *Magic Cube*n 1974 and obtained a Hungarian patent HU170062. Rubik got the idea for the cube when he wanted to make a three dimensional design with blocks that could move individually but many at the same time. Rubik initially tried to make a cube that was held together with rubber bands but failed. Then he got the idea that the cubes had to hold each other in place, which resulted in a 3x3x3 cube that could twist each face individually. Rubik got the inspiration for the cube from the Magic Puzzle (see chapter 2).

Rubik described that some of the most important features behind the cube were that the parts of the cube stay together, which many other puzzles do not. He also pointed out that you can move several pieces at once. Also that it is three dimensional.

In the end of 70s a Hungarian Businessman showed the Magic Cube at the Nuremberg toy fair and made it popular in Europe. The company Ideal Toy bought exclusive rights for the Magic Cube, but changed the name of the cube to Rubik's Cube within a year in order to get trademark protection.

At that time there were also two others applying for patent for products similar to the Rubik's Cube. One of them was an American named Doctor Larry D. Nichols, and his cube was a 2x2x2 cube which was held together with magnets. The other one who applied for patent was a Japanese man named Terutoshi Ishige. He applied for patent a year after Rubik. Terutoshi Ishige's cube was almost identically to the Rubik's Cube.

Ideal Toy Company were bought by CBS Toy Company in 1982 and the trademark surpassed with it, but they sold the rights to Rubik's Cube to Seven

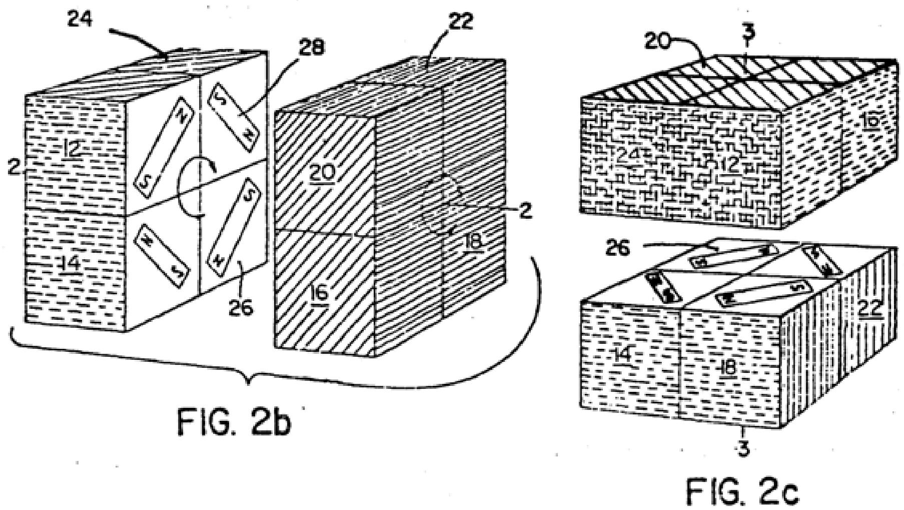


Figure 3.2: *Figure of Nichols Patent.*

Towns which is a Toy Company in Great Britain, they are still producing The Rubik's Cube today.

3.3 The Nichols Cube Puzzle

Dr. Larry D. Nichols studied chemistry at DePauw University in Greencastle, Indiana, before moving to Massachusetts to attend Harvard Graduate School. He is a lifelong puzzle enthusiast and inventor who began developing a twist cube puzzle with six colored faces in 1957. It was made of eight smaller cubes assembled to a 2x2x2 cube. The eight cubes were held together by magnets.

Give me a picture!

On April 11, 1972 he was granted U.S. Patent 3,655,201 on behalf of Moleculon Research Corp. U.S. Patent 3,655,201 covered Nichols Cube and the possibility for making larger versions later. This was two years before Ernö Rubik took out the patent for his Rubik's Cube in Hungary.

In 1982 Moleculon Research corp. Sued Ideal Toy Company that had the U.S. Patent 4,378,116 for Rubik's Cube because they believed that Ideal Toy Company violated their patent, but the U.S. District Court ruled in Ideal Toy Company favor. In 1986 the Court of Appeals ruled that the Pocket Rubik's Cube 2x2x2 was guilty of infringement but not the 3x3x3 Rubik's Cube.

In this chapter it has been described how Ernö Rubik got the idea for the cube.

We also stated that Ernö Rubik was not the only one at that time, that came with the invention of cubes. Ernö Rubik's cube was special since the blocks hold each other together, which is different from the one Doctor Larry D. Nichols applied patent for which is hold together with magnets.

Part II

Theory

4 Terminology

In order to fully understand the theoretical part of this report it is necessary to know the terminology there will be used

---

4.1 General cube terminology

- **Cuber**: The self reference for people who are devoted to the community of the Rubik's Cube.
- **Face**: A face is an entire side of the cube. A Rubik's Cube has 6 faces.
- **Facelet**: The small stickers on the cube. Each face has 9 facelets.
- **Corner**: Corner pieces have 3 facelets and are placed at the corners.
- **Edge**: Edge pieces have 2 facelets and are placed at the edges of each face.
- **Center**: Center pieces have 1 facelet and are placed at the center of each face and are immovable unless the cube is turned.
- **Turn**: A turn of the cube is equal to rotating the whole cube 90 degrees – changing view angle.
- **Twist**: A rotation of a face.
- **Move sequence**: The same as a sequence of twists.

4.2 Movement notation terminology

A cube consists of 6 faces and the notations of these are the following.

- **Front face – F**: This face faces the cuber.

- **Left face – L**: This face faces the left hand side of the cuber.
- **Right face – R**: This face faces the right hand side of the cuber.
- **Up face – U**: This face faces up.
- **Down face – D**: This face faces down.
- **Back face – B**: This face faces away from the cuber.
- **General move – M**: Can be any of the above faces.

A face can be twisted in two directions – clockwise and counterclockwise. When twisting a face the direction is determined as if you were facing the face. A twist in the clockwise direction has the same name as the face. i.e. a clockwise turn of the right face is notated “R” and pronounced “right”. A counterclockwise twist of the right face is notated “R” and pronounced “right prime”. This goes for all the faces. A 180 degree twist of a face is denoted M2. Beside the normal face twist there are middle twists. These are denoted Mm Mm' and Mm2 e.g. Rm2 is a twist of the middle section looking from the R face. This twist is equal to the two twist R2 and L2.

A turn of the cube can be done in six directions. Clockwise and counterclockwise around each of the three axes.

4.3 Algorithm terminology

- S: The 18 standard twists.
- s: A specific position of the Rubik's Cube.
- $|s|$: The number of twists of a move sequence that transform e into s.
- H: Positions which obey the following:
  - Every piece is correctly oriented.
  - Every edge piece not containing either an up facelet or a down facelet is positioned in the middle layer.
- A: Set containing the following twists: U, U', U2 ,F2, R2, B2 ,L2, D, D' and D2.
- G: All positions which can be obtained without disassembling the Rubik's Cube.
- $r(s)$: The relabeling of the position s.
- S^n: Set of move sequences consisting every possible sequence of max n moves in S.
- S^*: Every combination of twists in S.

- $d(s)$: Distance, the shortest $|s|$.
- R: set of $r(G)$.
- M: 48 color permutation turn and mirror.
- e: The solved state of the Rubik's Cube(unit cube).

4.3.1 Kociemba

Here are the terms which are specifically used section 7.2. Note that positions can be considered as the shortest move sequence that transform it into e.

- a: A position in G.
- b: Path from a to a position in H using move sequences in $S*$.
- c: Path from b to e using move sequences in $A*$.
- ab: Combination of the to move sequences a and b, in that order.
- d: Distance for phase 1.
- $d2$: Table lookup for the distance from ab to e.

By reading this chapter the basement of understanding this report is laid.

5 Group Theory (Not done)

In this chapter we will explain group theory and how it can be used to solve the Rubik's Cube. When we have a good understand of the group theory we will be able to use it later for our own program.

5.1 Permutations

In this section we will use the same move notations as in 4.2. To calculate how many positions the cube can be placed in, we have to look at the general cube terminology 4.1. As stated in general cube terminology there is 8 corners piece and the first corner you place can be placed in 8 positions, and after that is placed the next corner piece can be placed in one of the 7 positions left, since we already use 1, and so on. So that means that the corner pieces can be placed in $8 * 7 * 6 * 5 * 4 * 3 * 2 * 1 = 8!$ Now the 8 corner piece is placed at the right position they might not have the right orientation since a corner piece have three different colors and therefor 3 different orientations, so there is 3^8 orientations of the 8 corner piece. That means there is $8! * 3^8$ ways the corner piece can be placed. As stated in the terminology there is 12 edge piece. These edge pieces can be positioned in 12 different positions and every edge piece can be oriented in two different ways. So there is 2^12 different orientations and 12! different positions of the edge piece. This gives us $2^{12} * 12!$ different edge permutations and a total of $3^8 * 2^{12} * 12! * 8!$

5.2 Definition of the Rubik's Cube group

To fully understand if the Rubik's Cube can be described as group theory, we will have to understand what a group is. The group we have choosen to look at is the (G, *) group. This group consists of a set G and an operation *. (ref til gruppen (G, *) her, for bedre at forstaa den)

To describe the Rubik's Cube with group theory we will take a set of moves and make them into a group, which we will call (G, *). G is the possible moves of the Rubik's Cube.

G defines all the moves of the Rubik's Cube that is possible. Group operations can be defined the following way: M1 is a move and M2 is a move, so therefore M1*M2 is a move where you have to do the M1 move first, and then the M2 move. To prove that the Rubik's Cube is a group there is four points that must be correct.

- The element G is underneath * because M1 and M2 are moves and M1*M2 is also a move.
- e is a empty move (which does not change the configuration of the Rubik's Cube), So if you have to do the move M*e that basicly means that you have the move M and then do nothing, so that means that $M * e = M$
- If M is a move then it is possible to reverse this move, this moved is called M'. Therefore $M * M' = e$, so every elements in G has a reverse move.
- To prove * is associative it is important to remember that the moves made on the Rubik's Cube can be defined on the changes it makes to the configuration of the Rubik's Cube. If c is an oriented rubik's cubie, M(c) will be the orientation c for the oriented cubicle c ends in after the move is applied. Example the move R wll move the ur cubie to the br cubicle, so therefore $R(ur) = br$. If there is more than one move sequence then the operation will look like this $B'(R(ur)) = db$. If there is another move the cubie will be oriented in the M2(M1(c)), therefore $(M1 * M2)(c) = M2(M1(c))$.

The multiplications operator is used because the Rubik's Cube movements is not commutative and the addition operator is used with commutative elements which is the reason that can not be used. (evt matrix eksempel) KILDE HER.

* is associative (samme som not commutative?) because $(M1 * M2) * M3 = M1 * (M2 * M3)$ for any moves M1, M2 and M3. $(M1 * M2) * M3$ and $M1 * (M2 * M3)$ does the same operation to every cubie. This is the same as saying $[(M1*M2)*M3](C) = [M1*(M2*M3)](C) = M3(M2(M1(C)))$ for any cubie C. Therefore * is associative.

5.3 Subgroup

As said in the Permutation section, there is $3^8 * 2^{12} * 12! * 8!$ possible configurations.

5.4 The Symmetric Group

instead of than just looking at configurations of 8 cubies, the configurations of the cube can be seen as n objects. these objects be named $1, 2, ..., n$, these names are arbitrary. the arranging of these objects can be seen as putting them into n slots. If the slots is numberet $1, 2, ..., n$, can it be define as a function $\sigma : 1, 2, ..., n \to 1, 2, ..., n$ by letting $\sigma(i)$ be the number put into slot i.

5.4.1 Example 5.1.

Tag the objects $1, 2, 3$ in the order 312. So, it corresponds to the function $\sigma : 1, 2, 3 \to 1, 2, 3$ defined by $\sigma(1) = 3$,$\sigma(2) = 1$, and $\sigma(3) = 2$.

5.4.2 Lemma 5,2

Imagine that x $\neq$ y. Since a number cannot be in more than one slot, if $x \neq y$, slots x and y must contain different numbers. That is, $\sigma(x) \neq \sigma(y)$. Therefore, σ is one-to-one.

Any number $y \in 1, 2, ..., n$ must lie in some slot, say slot x. Then, $\sigma(x) = y$.

On the other hand, if $\sigma : 1, ..., n \to 1, ..., n$ is a bijection, then σ defines an arrangement of the n objects: just put object $\sigma(i)$ in slot i. So, the set of possible arrangements is really the same as the set of bijections $1, ..., n \to 1, ..., n$. Therefore, instead of studying possible arrangements, we can study these bijections.

6 Graph Theory

This chapter concerns the graph theory in relation to the Rubik's Cube. The shortest path of a graph and the calculation of the diameter of a graph will be described. This can be used as a method to prove how many twists are needed to solve the Rubik's Cube in the worst case, also called the upper bound. The chapter will conclude with an example using a much smaller Rubik's Cube graph; the Middle movement graph.

A graph consists of a set of vertices, V [8, p. 592], and a set of edges, E. There are several types of graphs, but in this chapter only simple graphs are described. A vertex is a point. A vertex can be connected to other vertices by an edge. An edge is a line between two vertices(an edge can in some graphs connect a vertex to it self, but such graphs are not considered here). If there is only one edge between any two vertices in a graph, then the graph is said to be simple. A graph is said to be connected if you can visit every vertex by starting at an arbitrary vertex and then reach every other vertex by traveling along the edges of the graph.

A graph can be weighted, which means that every edge will have a weight. This weight can represent different things, distance between vertices, price for traveling between vertices or what ever makes sense for the given weighted graph. See figure 6.1 for an example of a graph.

Graphs can be used to illustrate many different things such as relationships, geographical maps, tournaments etc.[8, pp. 592-593]

For a graph of the Rubik's Cube it would be ideal to make vertices represent positions and edge represent twists. The weight of the edges would then be the number of twists required to get from one of the edges to the other – hence the weight for every edge would be 1 if only edges representing a single twist is included in the graph.

The Rubik's Cube graph can have different sizes depending on the allowed twist, e.g. a graph only considering Rm2, Um2 and Fm2 twists can be made into a relatively small graph compared to the full Rubik's Cube graph(see 6.2 for illustration), which contains $4.33 \cdot 10^{19}$ vertices.

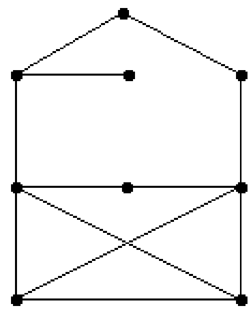


Figure 6.1: *A simple connected graph with nine vertices. The black circles are vertices and the lines are edges.*

6.1 Shortest Path

The shortest path from one vertex to another in a graph can be found by checking the length of each possible path. This is easy for small graph such as the The Middle Movement Graph 6.3.1, which will be described later in this chapter. For bigger graph such as the full Rubik's Cube graph this is practically impossible. The shortest path between two vertices can be found with Dijkstra's Algorithm [8, p. 651]. The description of the algorithm is omitted for brevity. Dijkstra's Algorithm takes an weighted graph and two vertices as input. Because of this the Rubik's Cube graph has to be weighted. Since each twist contributes to the total number of twists by the same number, one, each edge is given the weight 1. The weight will be omitted in every illustration in this chapter because they are all 1.

6.2 Solving the Diameter

To find the diameter of a graph is to find the longest shortest path between any two vertices in the given graph i.e. the shortest path with the highest value. This can be done by using Dijkstra's Algorithm on every set of vertices in the graph. This can be applied to the Rubik's Cube graph as well in order to find the maximum number of twists to solve the Rubik's Cube in the worst case scenario.

6.3 Describing the Cube as a Graph

In order to describe the Rubik's Cube as a graph it is necessary to determine the edges and the vertices. For the Rubik's Cube graph edges are defined to represent twists and vertices to represent the positions. The full Rubik's Cube graph is indeed a simple graph. A move can be reversed, hence no directions. In order to get from a position a to an adjacent position b one can only get there by a single move. Unless a detour is taken trough one or several other positions. The full Rubik's Cube graph consist of approximately $4.33 \cdot 10^{19}$ vertices and all having 18 edges. It is practically impossible to draw this graph. Therefore the graph will be explained with a much simpler graph; the Middle Movement graph.

6.3.1 The Middle Movement Graph

This graph is a Rubik's Cube graph consisting of only the moves that twist the middle sections. This is Rm2 Fm2 Um2. Note: Rm2 = R2 L2. This graph is fairly small, since it only consist of eight vertices and 12 edges. See figure 6.2. [9, pp. 158-167]

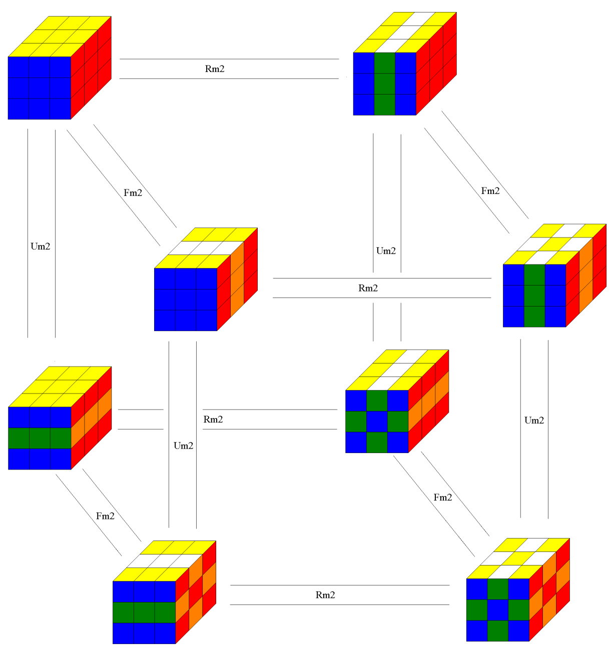


Figure 6.2: *The graph of the middle movement positions.*

Because of this graphs relatively small size the computation of the diameter is a some what simple task. It is possible to calculate the distance from all vertices to all other vertices, but since the graph is clearly symmetrical many calculations can be omitted. It is easy to see that the diameter must go from one corner to an opposite corner e.g. from the solved state to the pons asinorum[1]. The diameter here is 3.

In the full Rubik's Cube group it is believed that the analogous position to the pons asinorum, the superflip position, is the position which has the longest

[1] Pons asinorum is obtained from the solved state of a Rubik's Cube by the move sequence Rm2 Fm2 Um2.

shortest path [6]. This has never been proven. This shortest path from the superflip to the solved position is 20[7].

A short general description of graph theory has been presented in this chapter along with a way to calculate the diameter of a graph. For better understanding of the theory, an example on applying the theory on the Rubik's Cube has been shown.

7 Solving Strategies

This chapter will present two different algorithms which can be used for solving a Rubik's Cube. These algorithms will used as a foundation for the implementation part, which makes this chapter essential for the report.

7.1 The Beginner's Algorithm

This section describes the easiest algorithm to remember for a human solver. The algorithm is divided into 5 different steps. Once the algorithm has been memorized, it is easy to recognize which step of the algorithm has been reached, so the correct move sequence can be applied. It is only necessary to remember a single move sequence for each step. The beginner's algorithm can however be made more efficient by remembering another somewhat similar move sequence for a few of the steps. Despite of this the beginner's algorithm is twist-wise quite inefficient, because purpose of the moves is to reach the next step instead of reaching the solved state and thereby taking a solving detour.

7.1.1 Step 1 - getting the cross

The first step of the beginner's algorithm [2] is to get a cross on any face. Getting a cross on a face means to align the facelets next to the center facelet, so that all of the aligned facelets are of the same color, while at the same time the used edge pieces have the same color of the center facelets on each of the two faces on which they are.

The face on which the cross is being assembled is set to be the top face. An edge piece that consists of the same colors as the center piece of the top face and the center piece of the front face is placed in the bottom of the front face. With two twists of the front face the edge piece is positioned correctly in the cross. If the edge piece is oriented correctly the cube is turned (noted y) and the process is repeated until the cross is assembled. However if the piece is oriented in the wrong way the following move sequence will change it's orientation without ruining any part of the cross that may already be assembled:

F' U L' U' or F U' R U

7.1.2 Step 2 - completing the first layer

Figure 7.1: *A first layer cross on the green face*

When the cross is completed the next step is to position the corner pieces of the first layer correctly. The first layer is set as the down face (D). A corner with a facelet of the color of the down face is positioned directly above it's correct position. The correct position is between the three faces that have the same colors as the three facelets of the corner piece. Once the piece is above it's correct position, the cube should be viewed in such an angle that the piece is in the upper right corner of the front face, the following move sequence is repeated until the corner piece is oriented and positioned correctly:

R' D' R D

If the piece is above the correct position the algorithm twists the corner clock-wise and positions it in the correct position. If the piece is in the correct position the algorithm positions the piece above the correct position. The maximum number of repetitions until the piece is oriented and positioned correctly is five, because the piece can be two twists away from it's correct orientation. The move sequence can be performed inverted which twists the corner counter clock-wise and looks as follows:

D' R' D R

If the correct move sequence is used the maximum number of repetitions is three. If number of twists and time used is not of importance it is only necessary to remember one of them.

7.1.3 Step 3 - completing the second layer

The purpose of this step is to position the four edges belonging to the second layer correctly. The move sequence used in this step either swaps the top edge piece of the front face with the left or right edge piece. That is why the edge piece that needs to be moved down to the second layer must be positioned at the top of the face that is next to the face of one of the edges colors, where the second color is the same as the front face's color. There is however a difference in which of the two faces is used as the front face, because if the wrong face is used as front face the piece will only be positioned correctly but not oriented correctly. That facelet of the edge towards the front face must be of the same color as the front face. If the edge piece is to be swapped with the right edge piece of the front face the following move sequence is used:

U R U' R' U' F' U F

If the piece is to be swapped with the left edge piece of the front face the following move sequence is used:

U' L' U L U F U' F'

If none of the edge pieces who belong to the second layer are in the third layer and the first two layers is still not completed. This occurs if the edges are all in the second layer but not positioned or oriented correctly. One of the move sequences can be used to swap an edge piece from the third layer with one of the edges in the second layer which is not positioned or oriented correctly. Now the edge piece belonging to the second layer can be moved to it's correct position.

Figure 7.2: *The first layer completed*

7.1.4 Step 4 - getting the last layer cross

Solving the last layer is divided into four steps. The order of these steps can be different and still yield the same result. We start out by getting the cross in the last layer, which is the same as the cross on the first layer. The move sequence used is however different. The only move sequence used in this step is the following:

F R U R' U' F'

Besides remembering the move sequence it is also important to know how the cube should be oriented.

If the up face colors form a line. The cube can be turned until the line forms a horizontal line. If the move sequence then is performed, the cross will be formed.

Figure 7.3: *The first two layers completed*

(a) *The up face colors form the opposite L*

(b) *The up face colors form the horizontal line*

(c) *The up face colors form the cross*

Figure 7.4: *The steps in the completion of the last layer cross*

If the up face colors form an opposite L character with the up face colors the move sequence will form the line. The cube must be oriented with the opposite L in the back left corner of the cube.

If the up face colors do not form the opposite L the line or the cross, the move sequence will form the opposite L.

To orient the cross correctly there is one move sequence to remember:

R U R' U R 2U R'

Figure 7.5: *The correct oriented last layer cross.*

Again it is important to know how to orient the cube. By twisting the upper layer it is possible to position the upper layer so it either has two edge pieces next to each other or directly across from each other.

If the correct edges piece are next to each other. The cube must be oriented with a correct edge piece on the back face and a complete face on the right face. The move sequence will then make it possible to twist the up layer so that all the edge pieces are correctly positioned and oriented.

If the correct edge pieces are across each other the move sequence can be performed a single time to get two edge pieces next to each other.

7.1.5 Step 5 - completing the last layer

The purpose of this step is to position and orient the corners correctly. Firstly the corners must be positioned correctly. To do so there are two move sequences to remember – one for rotating three corners clockwise and one for rotating them counter-clockwise. It is however only necessary to remember one of them and repeat that one until the corners are positioned correctly, if the number of twists is unimportant to the solver.

U R U' L' U R' U' L
This move sequence will rotate the corners counter-clockwise.

U' L' U R U' L U R'
This move sequence will rotate the corners clockwise.

The orientation is again important to position the corners. If one of the corners already is positioned correctly. That corner is chosen not to be moved.

If the three other corners need to be moved counter-clockwise. The correct corner is positioned as the front right corner. The move sequence will then position the corners correctly.

If the three other corners need to be moved clockwise. The correct corner is positioned as the front left corner. The move sequence will then position the corners correctly.

If there are no correct corners. One of the two move sequences above performed once will make yield a correcly positioned corner.

To orient the corners correctly the move sequence from orienting the corners in the first layer is used:

R' D' R D

But in this step when one corner is completed. Instead of turning the whole cube to the next corner, the cube is locked on one face. Then when going to solve the next corner the upper layer is twisted until the incorrect corner is positioned at the front right corner of the cube and the move sequence is repeated.

Figure 7.6: *The solved cube – e*

7.2 Kociemba's Optimal Solver

Kociamba's optimal solver is an algorithm created with the purpose to find the twist-wise optimal solution to any scrambled Rubik's Cube[3] [12]. The algorithm consists of two phases. The first phase is based on a principle called relabeling, which will be defined at first.

7.2.1 Relabeling

The relabeling process starts with choosing an up face with a corresponding down face. Then choosing a front face with a corresponding back face. Each facelet with the color of the up or down face is marked with the letters "UD". Every edge piece that is not labeled with "'UD"' is labeled with "'FB"' on the front and back face. Figure 7.7 shows an example. When relabeling a Rubik's

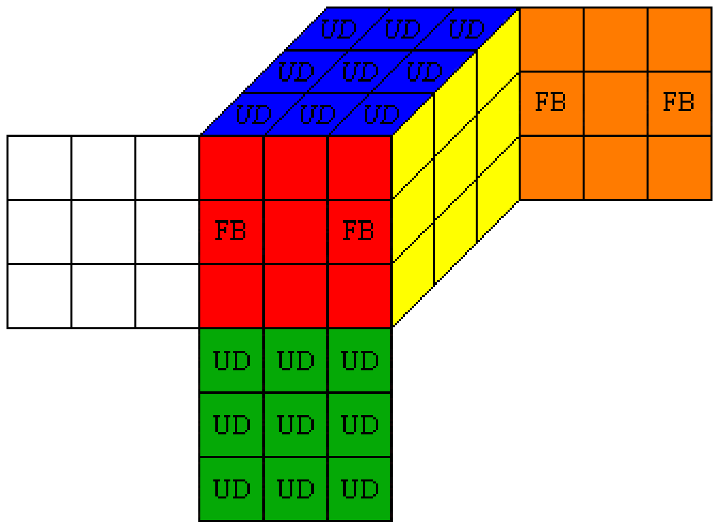


Figure 7.7: *A relabeled Rubik's Cube with the up and down faces as blue and green and red and orange as front and back.*

Cube in the position s it is written as $r(s)$. A Rubik's Cube in any position, s, is said to be in the set of positions called H(See subsection 7.2.2) if and only if $r(s) = r(e)$.

7.2.2 The subgroup H

The moves U, U', U2, D, D', D2, R2, L2, F2 and B2 is the set of moves A. Using moves from A on a position in H, will always result in a Rubik's Cube in H. The reason for this can easily be tested with a Rubik's Cube. If using one of the three up face moves or one of the three down face moves, the "FB" labels are not moved and the "UD" labels are simply rotated on the face that is being twisted, see figure 7.8a. The last four moves are all very similar to each other. If any side face – not up or down – is twisted 180 degrees, the three facelets on the up face is moved to the down face and vice versa and thereby keeping all the "UD" labels on the up and down face and keeping the orientation correct. The two remaining relabeled facelets are swapped which keeps the "FB" labels placed and oriented correctly, see figure 7.8b.

7.2.3 Overall description

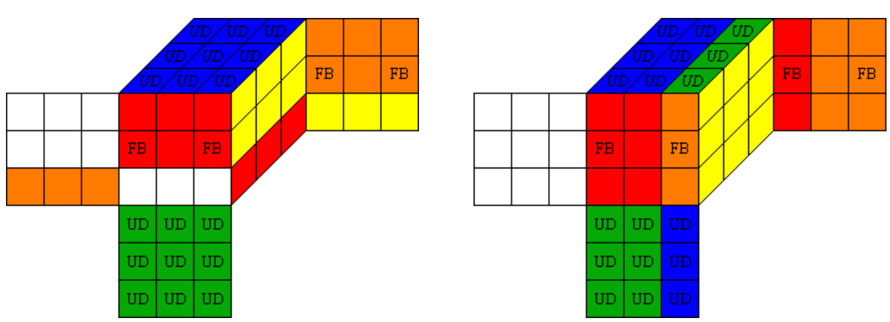
The first phase takes a scrambled cube a and relabels it $r(a)$ then it finds a move sequence b which will transform the relabeled cube into the subgroup H. This move will be denoted: $r(a) \cdot b \in H$. The second phase will then determine the length from the position ab to the unit position e by a table lookup.

Algorithm 1 Kociemba's Algorithm [7]

$d = 0$
$l = \infty$
while $d < l$ **do**
    for $b \in S^d$ **do**
        if $r(ab) \in H$ **then**
            if $d + d_2(ab) < l$ **then**
                $l = d + d_2(ab)$
            end if
        end if
    end for
    $d = d + 1$
end while
b is now the optimal solution

The search distance in the algorithm (d), see figure 7.10 is initially set to zero and the total length (l) is set to infinite. The total length is the amount of moves used to get from a to e, while d is the limited search length to find a move sequence that transforms the Rubik's Cube into a position in H. The **while** loop will run as long as d is less than l.

The **for** loop runs through all move sequences in the range d – recall that S^d is the set containing every move sequence that uses d twists. This search of moves in S^d is called the first phase and is further described in 7.2.4. Then an



(a) *A relabeled cube that has been permutated with the move D.*

(b) *A relabeled cube that has been permutated with the move R2.*

Figure 7.8: *Two positions which have been permuted with a move in A.*

if-statement checks if the move sequence transforms the cube into the subgroup H. If so the algorithm checks whether $d + d_2(ab)$ is less than the length of the last solution, if so, $d + d_2(ab)$ is the new shortest move sequence, l, to e. The lookup table denoted $d_2(ab)$ returns the numbers of twist required to transform a position in H to the position e. This is done only by using moves in A. The lookup table is the second phase and is further described in the subsection 7.2.5. The first time this **if**-statement is executed it will return true and this will be the new length of the solution. The **while** loop will end when d is incremented to l. Figure 7.9 illustrates the algorithm.

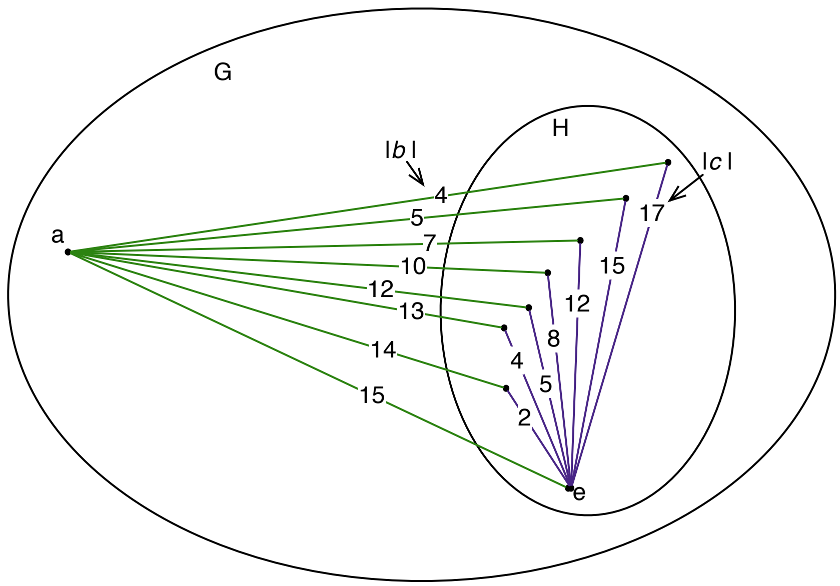


Figure 7.9: *Here the move sequences the leads to a shorter length in the algorithm is denoted. The lines going from a to a point in H is the move sequence denoted b. The lines from points in H to the point e is the move sequence denoted c . The numbers beside the lines are the number of moves. Note that the moves in c is decreasing as the numbers of moves in b is increasing.*

7.2.4 First phase

The first phase finds a move sequence, b, from a position a that transforms the *Rubik's Cube*nto the subgroup H this is done by going through all possible moves with a sequence of the length d. This is a breadth-first search algorithm [8, pp. 729-731]. In figure 7.10 the amount of elements in S^d to a specific d is illustrated. An example of a search from a position a is given. The algorithm starts by searching in S^d, where $d = 0$. This will give the position a itself. The distance d is then incremented to 1. This will result in 18 new possible move sequences all one twist away from a. Thereafter the d is incremented and the search now moves 18 moves from the previous positions obtained in the search where $d = 1$. This allows a possibility for optimization of the algorithm since some of the moves will eventually be the same. e.g. the two move sequences R2 and R' R' is of length 1 and 2 respectively, but the result is the same transformed cube. The algorithm checks whether the transformed cube is in H by relabeling it and checking if $r(ab) = r(e)$. The actual implementation of the algorithm may vary and will be omitted for now.

d continues to be incremented until it reaches the length l. Since l starts with the value infinite, it has to be changed in order to stop the loop. This happens in the second phase.

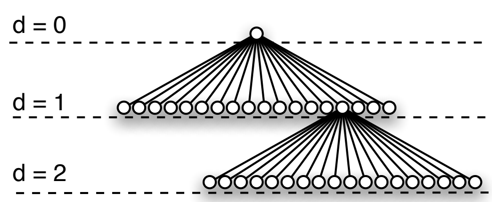


Figure 7.10: *As the distance of the search d increases, the possible move sequences expands exponentially. For each vertex on the graph there is 18 child vertices. The amount of leaves for each search depth would be* 18^d.

7.2.5 Second Phase

The goal of the second phase is to find the length of the shortest move sequence to transform the Rubik's Cube from a position in H to e. A way to solve this problem is by having a lookup table. This table has to be very large, considering the amount of positions in H.

The amount of positions can be calculated by imagining that one were to assemble the Rubik's Cube, starting with a disassembled Rubik's Cube. The first corner piece can be placed in eight different places. The next corner has seven possible positions etc. Note that all pieces in a cube in H has the correct orientation. This gives $8! = 40320$ possibilities for the corners. The eight edges of the top and down layer can be placed similarly and also yields $8!$. The four edges of the middle layer can be placed in four different places this yield $4! = 24$. Since it is impossible to swap two corners without swapping any edges the amount of possibilities is halved. The final result is $\frac{4! \cdot (8!)^2}{2} \approx 19.508 \cdot 10^9$ elements in H. The actual implementation of such a table will be discussed in section(HUSK REF).

The two algorithms described in this chapter will be used in the implementation part.

Part III

Appendix

A Proofs

A.1 Proof of Magic Constant

Theorem 1 (**Magic Constant**)**.** A hyper cube is a term that covers both the Magic Square and the Magic Cube. In theory the numbers of dimensions of a hyper cube can be any positive integer, the illustration of a hyper cube of any dimension higher than 3 has to be an abstraction. It is still possible to compute the magic constant of a hyper cube of any dimension d of the order n using the function in equation A.1:

$$M(n,d) = \frac{n^d \cdot (n+1)}{2} \tag{A.1}$$

Proof. The proof of this function resembles that of the function for 2 dimensions – which is the magic Square(See section 2.2.1).

First of all a hyper cube of d dimensions and the order n, contains the integers from 1 through n^d.

The magic constant of the given hyper cube can be obtain by calculating the sum of any line of numbers (be that a row, column, pilar or any other line that is appropriate for the given dimension). This sum can than be multiplied by n^{d-1} which is the same as adding all the numbers in the hyper cube together, since you add one dimension's magic constant's together every time n is multiplied. Therefore we can write:

$$
\begin{aligned}
n^{d-1} \cdot M(n) &= \sum_{i=1}^{n^d} i = 1 + \cdots + n^d \\
n^{d-1} \cdot M(n) &= \frac{n^d \cdot \left(n^d + 1\right)}{2} \\
M(n) &= \frac{n \cdot \left(n^d + 1\right)}{2} \qquad \square
\end{aligned}
$$

B

E-mail Correspondence with H. Kociemba

```
From: Herbert Kociemba [kociemba@t-online.de]
Sent: 16. marts 2010 16:44
To: Alex Bondo Andersen
Subject: Re: Rubik's Cube study at Aalborg University

Hello,

it is quite unusual to give away private informations to some "strangers", but ok,
here is some information.
I studied mathematics and physics (for "Lehramt an Gymnasien") at the Technische
Universität Darmstadt http://www.tu-darmstadt.de/ from
1974-1979 and am teacher for mathematics and physics since then at a Gymnasium. I
still live in Darmstadt. I am interested in Rubik's Cube since the beginning in
1980 - the same time were personal computers came up - and was immediately
interested to solve the cube algorithmically.
But it was not before 1990 that the PC- power was big enough to develop the ideas
and implementation for the two-phase-algorithm. I used an Atari St with 1 MB of
main memory for the first implementation and already got average solutions lengths
of about 21 moves....

If you have some other specific question, let me know.

Best regards

Herbert Kociemba
>
> Good day Herbert Kociemba,
>
> My name is Alex Bondo Andersen, I am attending Aalborg University in
```

```
> Denmark. My university group and I are working on a paper about the
> Rubik's Cube and are interested in using your webpage:
> http://kociemba.org/cube.htm as reference for a solving algorithm.
>
> If you are okay with us using your webpage as reference we would like
> to know a little about you in order to verify you as a credible
> source. So if you have the time it would be appreciated if you wrote
> which schools you have attended, where you live, which jobs you have
> had and why you are interested in the Rubik's Cube.
>
> In advance I would like to thank you for your time.
>
> Best Regards
>
> Group A215, Alex Bondo Andersen
>
```

Bibliography

[1] Hardeep Aiden. Anything but square: from magic squares to sudoku. WWW, March 2006. URL `http://plus.maths.org/issue38/features/aiden/`. Last viewed: 16/2. 2.2.1

[2] Alan Chang. Learn2cube. WWW, February 2009. URL `http://www.learn2cube.com`. Last viewed: 18/3. 7.1.1

[3] Herbert Kociemba. Cube explorer 4.64. WWW, 2009. URL `http://kociemba.org/cube.htm`. 7.2

[4] Mogens Esrom Larsen. *Rubiks terning.* Nyt Nordisk Forlag Arnold Busck, 1981. 2.2.3

[5] Jelsoft Enterprises Ltd. Speedsolving the rubik's cube & other puzzles. WWW, 2010. URL `http://www.speedsolving.com/forum/index.php`. Last viewed: 18/3. 1

[6] Jelsoft Enterprises Ltd. Superflip - speedsolving.com wiki. WWW, January 2010. URL `http://www.speedsolving.com/wiki/index.php/Superflip`. Last viewed: 23/3. 6.3.1

[7] Tomas Rocicki. Twenty-two moves suffice for rubik's cube. *Mathematical Entertainments*, November 2009. 6.3.1, 1

[8] Kenneth H. Rosen. *Discrete Mathematics And Its Applications.* McGraw-Hill, sixth edition, 2007. ISBN-13: 978-007-124474-9. 6, 6, 6.1, 7.2.4

[9] Ernö Rubik, Tamás Varga, Cerzson Kéri, György Marx, and Tamás Vekerdy. *Rubik's Cubic Compendium.* Oxford University Press, 1987. ISBN: 0-19-853202-4. 6.3.1

[10] Jaap Scherphuis. Jaap's puzzle page. WWW, 2010. URL `http://www.jaapsch.net/puzzles/`. Last viewed: 16/2. 2.2.3

[11] David Singmaster. The unreasonable utility of recreational mathematics. WWW, December 1998. URL `http://www.eldar.org/~problemi/singmast/ecmutil.html`. Last viewed: 15/2. 2.1

[12] Dik t. Winter. Kociembas algorithm. Emails, December 1992. URL `http://www.math.ucf.edu/~reid/Rubik/Cubelovers/cube-mail-8`. Last viewed: 15/4. 7.2

[13] Charles W. Trigg. What is recreational mathematics? *Mathematics Magazine*, 51:18 – 21, Januar 1978. URL `http://www.jstor.org/stable/2689642?seq=1&cookieSet=1`. 2.1

[14] Ron van Bruchem, Tyson Mao, and Masayuki Akimoto. World cube association. WWW, February 2010. URL `http://www.worldcubeassociation.org/`. Last viewed: 16/2. 1