1 Graph Theory

This chapter concerns the graph theory in relation to the Rubik's Cube. The shortest path of a graph and the calculation of the diameter of a graph will be described. This can be used as a method to prove how many twists are needed to solve the Rubik's Cube in the worst case, also called the upper bound. The chapter will conclude with an example using a much smaller Rubik's Cube graph; the Middle movement graph.

A graph consists of a set of vertices, V [3, p. 592], and a set of edges, E. There are several types of graphs, but in this chapter only simple graphs are described. A vertex is a point. A vertex can be connected to other vertices by an edge. An edge is a line between two vertices(an edge can in some graphs connect a vertex to it self, called a loop, but such graphs are not considered here). Figure 1.1 illustrates the basic principles of a graph.

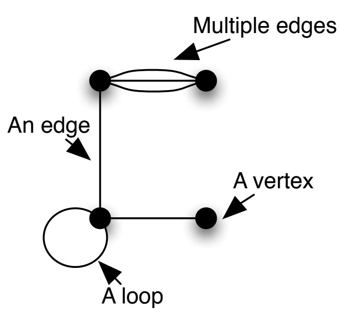
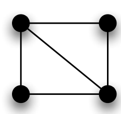


Figure 1.1: *This is a graph with multiple lines and a loop. The graph is connected, but not simple.*

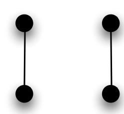
If there is only one edge between any two vertices in a graph, then the graph is said to be simple. A graph is said to be connected if you can visit every vertex by starting at an arbitrary vertex and then reach every other vertex by traveling along the edges of the graph. Figure 1.2b illustrates an unconnected graph while figure 1.2a illustrates a connected graph.

Graphs can be directed. this means the edges has a direction and travel can only be made along an edge in one direction. Figure 1.2c illustrates a directional graph.

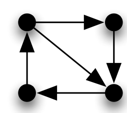
A graph can be weighted, which means that every edge will have a weight. This weight can represent different things, distance between vertices, price for traveling between vertices or what ever makes sense for the given weighted graph.



(a) *A simple connected graph.*



(b) *A simple unconnected graph.*



(c) *A simple directed graph*

Figure 1.2: *Two simple graphs and one simple directed graph.*

Graphs can be used to illustrate many different things such as relationships, geographical maps, tournaments etc.[3, pp. 592-593]

For a graph of the Rubik's Cube it would be ideal to make vertices represent positions and edge represent twists. The weight of the edges would then be the number of twists required to get from one of the vertex to the other – hence the weight for every edge would be 1 if only edges representing a single twist is included in the graph.

The Rubik's Cube graph can have different sizes depending on the allowed twist, e.g. a graph only considering Rm2, Um2 and Fm2 twists can be made into a relatively small graph compared to the full Rubik's Cube graph(see 1.3 for illustration), which contains $4.33 \cdot 10^{19}$ vertices.

1.1 Shortest Path

The shortest path from one vertex to another in a graph can be found by checking the length of each possible path. This is easy for small graph such as the The Middle Movement Graph 1.3.1, which will be described later in this chapter. For a bigger graph such as the full Rubik's Cube graph this is practically impossible. The shortest path between two vertices can be found with Dijkstra's Algorithm [3, p. 651]. The description of the algorithm is omitted for brevity. Dijkstra's Algorithm takes an weighted graph and two vertices as input. Because of this the Rubik's Cube graph has to be weighted. Since each twist contributes to the total number of twists by the same number, one, each edge is given the weight 1. The weight will be omitted in every illustration in this chapter because they are all 1.

1.2 Solving the Diameter

To find the diameter of a graph is to find the longest shortest path between any two vertices in the given graph i.e. the shortest path with the highest value. This can be done by using Dijkstra's Algorithm on every set of vertices in the graph. This can be applied to the Rubik's Cube graph as well in order to find the maximum number of twists to solve the Rubik's Cube in the worst case scenario.

1.3 Describing the Cube as a Graph

In order to describe the Rubik's Cube as a graph it is necessary to determine the edges and the vertices. For the Rubik's Cube graph edges are defined to represent twists and vertices to represent the positions. The full Rubik's Cube graph is indeed a simple graph. A move can be reversed, hence no directions. In order to get from a position a to an adjacent position b one can only get there by a single move. Unless a detour is taken trough one or several other positions. The full Rubik's Cube graph consist of approximately $4.33 \cdot 10^{19}$ vertices and all having 18 edges. It is practically impossible to draw this graph. Therefore the graph will be explained with a much simpler graph; the Middle Movement graph.

1.3.1 The Middle Movement Graph

This graph is a Rubik's Cube graph consisting of only the moves that twist the middle sections. This is Rm2 Fm2 Um2. Note: Rm2 = R2 L2. This graph is fairly small, since it only consist of eight vertices and 12 edges. See figure 1.3. [4, pp. 158-167]

Because of this graphs relatively small size the computation of the diameter is a some what simple task. It is possible to calculate the distance from all vertices to all other vertices, but since the graph is clearly symmetrical many calculations can be omitted. It is easy to see that the diameter must go from one corner to an opposite corner e.g. from the solved state to the pons asinorum[1]. The diameter here is 3.

In the full Rubik's Cube group it is believed that the analogous position to the pons asinorum, the superflip position, is the position which has the longest shortest path [1]. This has never been proven. This shortest path from the superflip to the solved position is 20[2].

A short general description of graph theory has been presented in this chapter along with a way to calculate the diameter of a graph. For better understanding of the theory, an example on applying the theory on the Rubik's Cube has been shown.

[1] Pons asinorum is obtained from the solved state of a Rubik's Cube by the move sequence Rm2 Fm2 Um2.

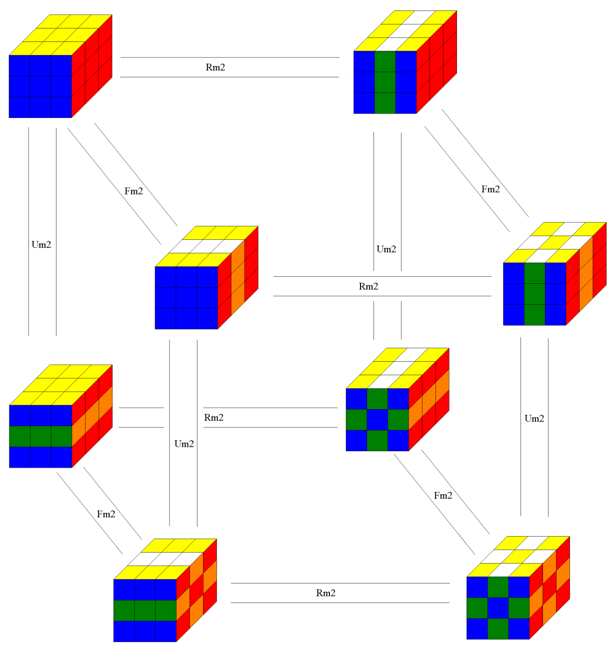


Figure 1.3: *The graph of the middle movement positions.*

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