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Edge irregular k-labeling for several classes of trees

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Abstract

Motivated by the definitions of the irregular labeling of a graph defined by Chartrand *et al.* in 1988 and Bača *et al.* in 2007, Ahmad *et al.* in 2014 defined the edge irregular k-labeling of graphs. In this paper, we prove that several types of trees, namely non-homogeneous caterpillar, homogeneous lobster and homogenous amalgamated star graph admits the edge irregular k-labeling.

Keywords: edge irregularity strength, caterpillar, lobster, amalgamated star graph
Mathematics Subject Classification : 05C78

1. INTRODUCTION

A graph $G = (V, E)$ with vertex set V and edge set E is connected, if there exists a relationship between the vertices in G. For a graph G, the degree of a vertex v is the number of incident edges and denoted as $d(v)$. A graph can be represented by a numeric number, a polynomial, a sequence of numbers or a matrix that represents the entire graph, and these representations are aimed to be uniquely defined for that graph. The basic notations and terminology are taken from [14].

A tree is also a type of graph and can be defined in terms of edges and vertices. To be precise a tree is an undirected, acyclic and connected graph. Tree structures are useful in almost all fields of sciences, particularly in computer science as coding theory for transmission and storage of data. Trees are also heavily used in chemistry for graphical representation of chemical structures. To use the trees in interdisciplinary research, vertices of tree can be labeled using mathematical

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definitions, in similar way of graph labeling. On graph labeling lot of work has been done and related script are covering the research gaps.

Chartrand *et al.* in [9] introduced edge k-labeling ϕ of a graph such that $w_\phi(x) \neq w_\phi(y)$ for all vertices $x, y \in V(G)$ with $x \neq y$, where $w_\phi(x) = \sum \phi(xz)$, the sum is over all vertices z adjacent to x. Such labelings were called *irregular assignments* and the *irregularity strength* $s(G)$ of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k. This parameter has attracted much attention [5, 6, 8, 10].

In 2007, Bača *et al.* in [7] derived two modifications of the irregularity strength of graphs, namely a *total edge irregularity strength*, denoted by $tes(G)$, and a *total vertex irregularity strength*, denoted by $tvs(G)$. Some results on total edge irregularity strength and total vertex irregularity strength can be found in [1, 2, 3, 12, 13].

Motivated by these papers, Ahmad *et al.* in [4] introduced the edge irregular labeling. A vertex k-labeling $\phi : V \to \{1, 2, \dots, k\}$ can be defined as *edge irregular k-labeling* for a graph G if for every two different edges e and f there is $w_\phi(e) \neq w_\phi(f)$, where the weight of an edge $e = xy \in E(G)$ is $w_\phi(xy) = \phi(x) + \phi(y)$. The minimum k for which the graph G has an edge irregular k-labeling is called the *edge irregularity strength* of G, denoted by $es(G)$.

A theorem proved in [4], establishes lower bound for the edge irregularity strength of a graph G.

Theorem 1.1. *[4] Let $G = (V, E)$ be a simple graph with maximum degree $\Delta = \Delta(G)$. Then*

$$es(G) \geq \max\left\{ \left\lceil \frac{|E(G)| + 1}{2} \right\rceil, \Delta(G) \right\}.$$

In this paper, we studied the three classes of trees namely non-homogeneous caterpillar, homogeneous lobster and homogeneous amalgated star graph.

2. Edge irregular k-labeling of caterpillar

Let $G_1, G_2, \dots, G_n$ be a family of disjoint stars. The tree obtained by passing a path through the central vertices of each star is called caterpillar. According to survey by Gallian [11], caterpillar is a tree with a property that if leaf vertices are removed the remaining structure will be path. Let $K_{1,m_1}, K_{1,m_2}, \dots, K_{1,m_n}$ be a family of stars with vertex sets $V(K_{1,m_i}) = \{c_i, y^i_j : 1 \leq i \leq n, 1 \leq j \leq m_i\}$. Let $CT_n(m_1, m_2, \dots, m_n)$ be a caterpillar obtained by passing a path P_n through the central vertices c_i of each star K_{1,m_i}, $1 \leq i \leq n$. If $|K_{1,m_i}| = r$, $1 \leq i \leq n$, then the caterpillar is called homogeneous and it is denoted by $CT_n(r)$, otherwise the caterpillar is called to be non-homogeneous. In the following theorem, we determine the edge irregular k-labeling of a non-homogeneous caterpillar.

Theorem 2.1. *Let $T = CT_n(m_1, m_2, \dots, m_n)$ a non-homogeneous caterpillar with $m_{i+1} = m_i + 1$ for $1 \leq i \leq n-1$ and $m_1 = 1$. Then T admits the edge irregular $k = \lceil \frac{|T|}{2} \rceil$-labeling.*

Proof. Let us consider the vertex set and the edge set of T are

$$\begin{aligned} V(T) =& \{c_1, c_2, \dots, c_n\} \cup \{y^i_j : 1 \leq i \leq n, 1 \leq j \leq m_i\} \\ E(T) =& \{c_i c_{i+1} :: 1 \leq i \leq n-1\} \cup \{c_i y^i_j : 1 \leq i \leq n, 1 \leq j \leq m_i\}. \end{aligned}$$

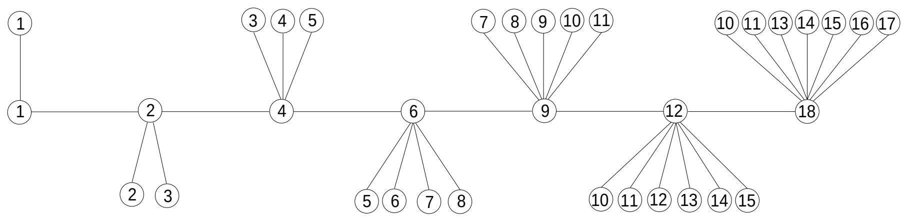

Figure 1. An edge irregular k-labeling of non-homogeneous caterpillar $T = CT_7(1, 2, \ldots, 7)$

It is easy to see that the maximum degree of T is $m_n + 1$, the order of T is $\frac{n(n+3)}{2}$ i.e $|T| = \frac{n(n+3)}{2}$ and the size of T is $\frac{n(n+3)}{2} - 1$. According to the Theorem 1.1, $es(T) \geq \max\left\{\left\lceil \frac{|T|}{2} \right\rceil, m_n + 1\right\} = \left\lceil \frac{|T|}{2} \right\rceil = k$ (let). For the converse, we define the vertex labeling $\phi : V(T) \to \{1, 2, 3, \ldots, k\}$ as follows:

$$\phi(c_i) = \begin{cases} (\frac{i+1}{2})^2, & \text{for } 1 \leq i < n \text{ and } i \text{ odd} \\ \frac{i}{2}(\frac{i}{2} + 1), & \text{for } 2 \leq i < n \text{ and } i \text{ even} \end{cases}$$

$$\phi(y_j^i) = \begin{cases} \frac{i^2-1}{4} + j, & \text{for } 2 \leq i < n,\ 1 \leq j \leq m_i \text{ and } i \text{ odd} \\ (\frac{i}{2})^2 + j, & \text{for } 2 \leq i < n,\ 1 \leq j \leq m_i \text{ and } i \text{ even} \end{cases}$$

$$\phi(x_n) = k.$$

$$\phi(y_j^n) = \begin{cases} k - n - 1 + j - \frac{n-1 \pmod 4}{2}, & \text{for } 1 \leq j \leq \lfloor \frac{n+2}{4} \rfloor, \text{ when } n \text{ odd} \\ k - n + j - \frac{n-1 \pmod 4}{2}, & \text{for } \lfloor \frac{n+2}{4} \rfloor + 1 \leq j \leq m_n, \text{ when } n \text{ odd} \\ k - n - 1 + j - \frac{n \pmod 4}{2}, & \text{for } 1 \leq j \leq \lceil \frac{n}{4} \rceil, \text{ when } n \text{ even} \\ k - n + j - \frac{n \pmod 4}{2}, & \text{for } \lceil \frac{n}{4} \rceil + 1 \leq j \leq m_n, \text{ when } n \text{ even} \end{cases}$$

It is observed that all vertex labels are at most k and the edge weights form the set of different integers, namely $\{2, 3, 4, \ldots, |T|\}$. Thus the labeling ϕ is satisfying an edge irregular k-labeling. This completes the proof. $\square$

The example of edge irregular k-labeling for a non-homogeneous caterpillar $T = CT_7(1, 2, \ldots, 7)$ is shown in Figure 1.

3. Edge irregular *k*-labeling of lobster

According to Gallian survey [11], a lobster graph, lobster tree, or simply lobster, is a tree with a property that if leaf vertices are removed the remaining structure will be caterpillar. In this section, we discuss the a special type of lobster $Lob(n, p)$ and for illustration the combinatorics of $Lob(n, p)$ are shown in Figure 2.

Let S_p be a star graph of order p. According to definition of star graph S_p there exist one internal vertex having degree $p - 1$ is known as central vertex and remaining $p - 1$ vertices have degree exactly 1 are called leaf vertices. Consider $2n$ isomorphic copies of star graph S_p. If

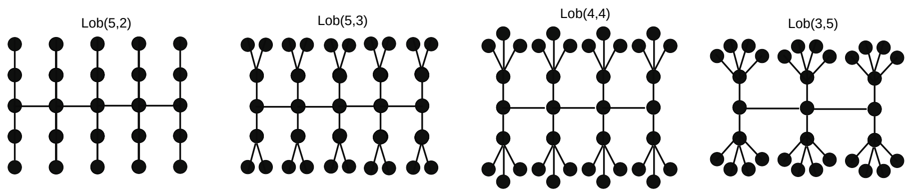

Figure 2. Combinatorics of lobster graphs

internal vertices of two isomorphic copies of star graph S_p are connected with each vertex of path graph P_n, the newly obtained structure will become lobster $Lob(n,p)$. If we consider the non-isomorphic copies of star graphs, then the lobster is called non-homogeneous. Due the symmetrical arrangements of vertices in $Lob(n,p)$, the number of can be calculated by using the formula $n(2p+1)$. In the following theorem, we determine the edge irregular k-labeling of a homogeneous lobster $Lob(n,p)$.

Theorem 3.1. *For $n,p \geq 2$, the homogeneous lobster $Lob(n,p)$ admits the edge irregular k-labeling, where $k = \lceil \frac{n(2p+1)}{2} \rceil$.*

Proof. Let us consider the vertex set and the edge set of lobster $Lob(n,p)$ are

$$\begin{aligned} V(Lob(n,p)) =& \{x_i, z_i, t_i : 1 \leq i \leq n\} \cup \{y_j^i, w_j^i : 1 \leq i \leq n, 1 \leq j \leq p-1\} \\ E(Lob(n,p)) =& \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i z_i, x_i t_i : 1 \leq i \leq n\} \\ & \cup \{z_i y_j^i, t_i w_j^i : 1 \leq i \leq n, 1 \leq j \leq p-1\}. \end{aligned}$$

Let the maximum degree of lobster $Lob(n,p)$ is Δ. The order of $Lob(n,p)$ is $n(2p+1)$ and the size of $Lob(n,p)$ is $n(2p+1)-1$. According to the Theorem 1.1, $es(Lob(n,p)) \geq \max\left\{\left\lceil \frac{n(2p+1)}{2} \right\rceil, \Delta\right\} = \left\lceil \frac{n(2p+1)}{2} \right\rceil = k$ (let). For the converse, we define the vertex labeling $\phi : V(Lob(n,p)) \to \{1,2,3,\ldots,k\}$ as follows:

$\phi(z_1) = 1$, $\phi(t_1) = p+1$, $\phi(y_1^j) = \phi(w_1^j) = j+1$ where $1 \leq j < p$.

$$\phi(x_i) = \begin{cases} \frac{(2p+1)i-(2p-1)}{2}, & \text{for } 1 \leq i \leq n-1 \text{ and } i \text{ odd} \\ \frac{(2p+1)i}{2}, & \text{for } 1 \leq i \leq n \text{ and } i \text{ even} \\ \frac{(2p+1)n+1}{2}, & \text{for } i = n \text{ odd} \end{cases}$$

$$\phi(z_i) = \begin{cases} \frac{(2p+1)i-1}{2}, & \text{for } 2 \leq i \leq n-1 \text{ and } i \text{ odd} \\ \frac{(2p+1)i}{2} - p, & \text{for } 2 \leq i \leq n \text{ and } i \text{ even} \\ \frac{(2p+1)(n-1)-2}{2}, & \text{for } i = n \text{ odd} \end{cases}$$

$$\phi(t_i) = \begin{cases} \frac{(2p+1)(i+1)-2}{2}, & \text{for } 2 \leq i \leq n-1 \text{ and } i \text{ odd} \\ \frac{(2p+1)i}{2}, & \text{for } 2 \leq i \leq n \text{ and } i \text{ even} \\ \frac{(2p+1)n-1}{2}, & \text{for } i = n \text{ odd} \end{cases}$$

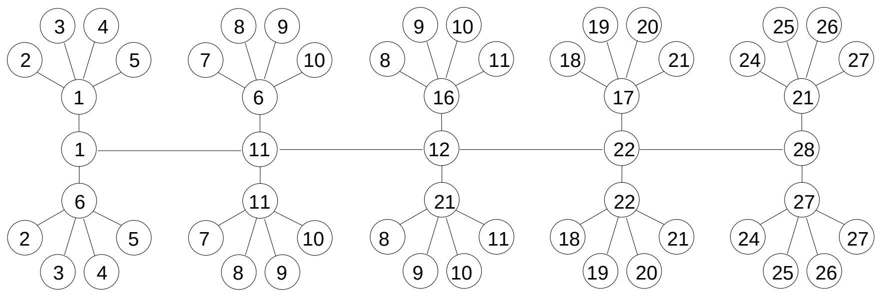

Figure 3. An edge irregular k-labeling for a homogeneous lobster $Lob(5, 5)$

$$\phi(y_j^i) = \phi(w_j^i) = \begin{cases} \frac{(2p+1)(i-1)}{2} - p + 1 + j, & \text{for } 1 \le j \le p-1, 2 \le i \le n-1 \text{ and } i \text{ odd} \\ \frac{(2p+1)i}{2} - p + j, & \text{for } 1 \le j \le p-1, 2 \le i \le n-1 \text{ and } i \text{ even} \end{cases}$$

It is a routine matter to verify that all vertex labels are at most k and the edge weights form the set of different integers, namely $\{2, 3, 4, \ldots, n(2p+1)\}$. Thus the labeling ϕ is satisfying edge irregular k-labeling. This completes the proof. $\square$

The example of edge irregular k-labeling for a homogeneous lobster $Lob(5, 5)$ is shown in Figure 3.

4. Edge irregular k-labeling of amalgamated stars

Let $S_{p_1}, S_{p_2}, \ldots, S_{p_n}$ be family of disjoint stars. For $n \ge 2$ the amalgamation of stars is denoted as $S_{n;p_1,p_2,\ldots,p_n}$ is obtained by joining the central vertex of each star with an additional vertex. If the the order of each star graph $S_{p_i} = K_{1,p-1} = S_p$ for $1 \le i \le n$, then it is called homogenous amalgamation of star graph and it is denoted by $S_{n;p}$. In the next theorem, we determine the edge irregular k-labeling of homogenous amalgamated star graph $S_{n;p}$ for $p = 3$ and $n \ge 3$.

Theorem 4.1. *For $n \ge 3$, the homogenous amalgamated star graph $S_{n;3}$ admits the edge irregular $\left\lceil \frac{3n+1}{2} \right\rceil = k$-labeling.*

Proof. Let us consider the vertex set and the edge set of lobster $S_{n;3}$ are

$$\begin{aligned} V(S_{n;3}) = &\{x_c\} \cup \{x_i, y_j^i : 1 \le i \le n, j = 1, 2\} \\ E(S_{n;3}) = &\{x_c x_i : 1 \le i \le n\} \cup \{x_i y_1^i, x_i y_2^i : 1 \le i \le n\}. \end{aligned}$$

Let the maximum degree of homogenous amalgamated star graph $S_{n;3}$ is n. The order of $S_{n;3}$ is $3n + 1$ and the size of $S_{n;3}$ is $3n$. According to the Theorem 1.1, $es(S_{n;3}) \ge \max\left\{\left\lceil \frac{3n+1}{2} \right\rceil, n\right\} = \left\lceil \frac{3n+1}{2} \right\rceil = k$. For the converse, we define the vertex labeling $\phi : V(S_{n;3}) \to \{1, 2, 3, \ldots, k\}$ as follows: Let us distinguish into two cases:

$\phi(x_c) = 1$ **Case 1:** $n \equiv 0, 2, 3 \pmod 4$

$$\phi(x_i) = \begin{cases} 3i-2, & \text{for } 1 \leq i \leq \lceil \frac{n}{4} \rceil + 1 \\ 2\lceil \frac{n}{4} \rceil + i, & \text{for } \lceil \frac{n}{4} \rceil + 2 \leq i \leq n \end{cases}$$

$$\phi(y_j^i) = \begin{cases} j+1, & \text{for } 1 \leq i \leq \lceil \frac{n}{4} \rceil \text{ and } j = 1, 2 \\ n+i+j-1-2\lceil \frac{n}{4} \rceil, & \text{for } \lceil \frac{n}{4} \rceil + 1 \leq i \leq n \text{ and } j = 1, 2 \end{cases}$$

Case 2: $n \equiv 1 \pmod 4$

$$\phi(x_i) = \begin{cases} 3i-2, & \text{for } 1 \leq i \leq \lceil \frac{n}{4} \rceil \\ 2\lceil \frac{n}{4} \rceil + i - 1, & \text{for } \lceil \frac{n}{4} \rceil + 1 \leq i \leq n \end{cases}$$

$$\phi(y_j^i) = \begin{cases} j+1, & \text{for } 1 \leq i \leq \lceil \frac{n}{4} \rceil - 1 \text{ and } j = 1, 2 \\ 2, & \text{for } i = \lceil \frac{n}{4} \rceil \text{ and } j = 1 \\ n - \lceil \frac{n}{4} \rceil + 3, & \text{for } i = \lceil \frac{n}{4} \rceil \text{ and } j = 2 \\ n+i+j-2\lceil \frac{n}{4} \rceil, & \text{for } \lceil \frac{n}{4} \rceil + 1 \leq i \leq n \text{ and } j = 1, 2 \end{cases}$$

It is a routine matter to verify that all vertex labels are at most k and the edge weights form the set of different integers, namely $\{2, 3, 4, \ldots, 3n+1\}$. Thus the labeling ϕ is satisfying edge irregular k-labeling. This completes the proof. □

The example of edge irregular k-labeling for homogeneous amalgamated of star $S_{8;3}$ is shown in Figure 4.

5. Concluding Remarks

The problem studied in this paper is about edge irregular k-labeling for three classes of tree graphs, where basic structure is star with different or same orders. All three problems solved in this study further support that the lower bound of Theorem 1.1 is tight by providing concrete results.

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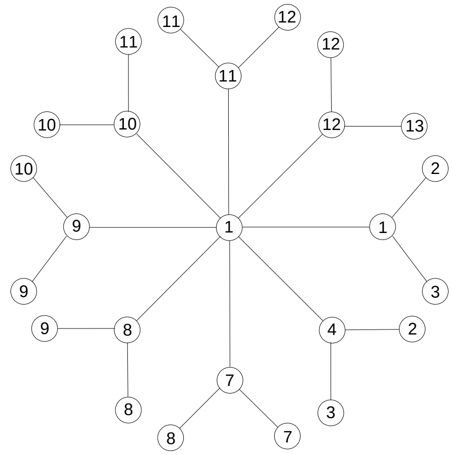

Figure 4. An edge irregular k-labeling of homogeneous amalgamated of star $S_{8;3}$

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