3.5 B Fact: All the rows of AB are combinations of the rows of B. So the row space of AB is contained in (possibly equal to) the row space of B. **Rank** $(\boldsymbol{AB}) \leq$ **rank** $(\boldsymbol{B})$.

All columns of AB are combinations of the columns of A. So the column space of AB is contained in (possibly equal to) the column space of A. **Rank** $(\boldsymbol{AB}) \leq$ **rank** $(\boldsymbol{A})$.

If we multiply by an *invertible* matrix, the rank will not change. The rank can't drop, because when we multiply by the inverse matrix the rank can't jump back.

Problem Set 3.5

1 (a) If a 7 by 9 matrix has rank 5, what are the dimensions of the four subspaces? What is the sum of all four dimensions?

(b) If a 3 by 4 matrix has rank 3, what are its column space and left nullspace?

2 Find bases and dimensions for the four subspaces associated with A and B:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}.$$

3 Find a basis for each of the four subspaces associated with A:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4 Construct a matrix with the required property or explain why this is impossible:

(a) Column space contains $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, row space contains $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

(b) Column space has basis $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, nullspace has basis $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

(c) Dimension of nullspace $= 1 +$ dimension of left nullspace.

(d) Nullspace contains $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, column space contains $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(e) Row space $=$ column space, nullspace $\neq$ left nullspace.

5 If $\mathbf{V}$ is the subspace spanned by $(1,1,1)$ and $(2,1,0)$, find a matrix A that has $\mathbf{V}$ as its row space. Find a matrix B that has $\mathbf{V}$ as its nullspace. Multiply AB.

6 Without using elimination, find dimensions and bases for the four subspaces for

$$A = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}.$$

7 Suppose the 3 by 3 matrix A is invertible. Write down bases for the four subspaces for A, and also for the 3 by 6 matrix $B = [\, A \quad A\,]$. (The basis for $\boldsymbol{Z}$ is empty.)

8 What are the dimensions of the four subspaces for A, B, and C, if I is the 3 by 3 identity matrix and 0 is the 3 by 2 zero matrix?

$$A = \begin{bmatrix} I & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} I & I \\ 0^{\mathrm{T}} & 0^{\mathrm{T}} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 \end{bmatrix}.$$

9 Which subspaces are the same for these matrices of different sizes?

$$\text{(a)} \; \begin{bmatrix} A \end{bmatrix} \text{ and } \begin{bmatrix} A \\ A \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} A \\ A \end{bmatrix} \text{ and } \begin{bmatrix} A & A \\ A & A \end{bmatrix}.$$

Prove that all three of those matrices have the *same rank* r.

10 If the entries of a 3 by 3 matrix are chosen randomly between 0 and 1, what are the most likely dimensions of the four subspaces? What if the random matrix is 3 by 5?

11 (Important) A is an m by n matrix of rank r. Suppose there are right sides $\boldsymbol{b}$ for which $A\boldsymbol{x} = \boldsymbol{b}$ has *no solution*.

(a) What are all inequalities ($<$ or $\leq$) that must be true between m, n, and r?

(b) How do you know that $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has solutions other than $\boldsymbol{y} = \mathbf{0}$?

12 Construct a matrix with $(1, 0, 1)$ and $(1, 2, 0)$ as a basis for its row space and its column space. Why can't this be a basis for the row space and nullspace?

13 True or false (with a reason or a counterexample):

(a) If $m = n$ then the row space of A equals the column space.

(b) The matrices A and $-A$ share the same four subspaces.

(c) If A and B share the same four subspaces then A is a multiple of B.

14 Without computing A, find bases for its four fundamental subspaces:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

15 If you exchange the first two rows of A, which of the four subspaces stay the same? If $\boldsymbol{v} = (1, 2, 3, 4)$ is in the left nullspace of A, write down a vector in the left nullspace of the new matrix after the row exchange.

16 *Explain why* $\boldsymbol{v} = (1, 0, -1)$ *cannot be a row of* A *and also in the nullspace.*

17 Describe the four subspaces of $\mathbf{R}^3$ associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad I + A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

18 (Left nullspace) Add the extra column $\boldsymbol{b}$ and reduce A to echelon form:

$$\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}.$$

A combination of the rows of A has produced the zero row. What combination is it? (Look at $b_3 - 2b_2 + b_1$ on the right side.) Which vectors are in the nullspace of A^{T} and which vectors are in the nullspace of A?

19 Following the method of Problem 18, reduce A to echelon form and look at zero rows. The $\boldsymbol{b}$ column tells which combinations you have taken of the rows:

$$\text{(a)} \quad \begin{bmatrix} 1 & 2 & b_1 \\ 3 & 4 & b_2 \\ 4 & 6 & b_3 \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix}$$

From the $\boldsymbol{b}$ column after elimination, read off $m-r$ basis vectors in the left nullspace. Those $\boldsymbol{y}$'s are combinations of rows that give zero rows in the echelon form.

20 (a) Check that the solutions to $A\boldsymbol{x} = \mathbf{0}$ are perpendicular to the rows of A:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = ER.$$

(b) How many independent solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$? Why does $\boldsymbol{y}^{\mathrm{T}} = $ row 3 of E^{-1}?

21 Suppose A is the sum of two matrices of rank one: $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$.

(a) Which vectors span the column space of A?

(b) Which vectors span the row space of A?

(c) The rank is less than 2 if _____ or if _____.

(d) Compute A and its rank if $\boldsymbol{u} = \boldsymbol{z} = (1, 0, 0)$ and $\boldsymbol{v} = \boldsymbol{w} = (0, 0, 1)$.

22 Construct $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ whose column space has basis $(1, 2, 4)$, $(2, 2, 1)$ and whose row space has basis $(1, 0)$, $(1, 1)$. Write A as (3 by 2) times (2 by 2).

23 Without multiplying matrices, find bases for the row and column spaces of A:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

How do you know from these shapes that A cannot be invertible?

24 (Important) $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{d}$ is solvable when $\boldsymbol{d}$ is in which of the four subspaces? The solution $\boldsymbol{y}$ is unique when the _____ contains only the zero vector.

25 True or false (with a reason or a counterexample):

(a) A and A^{T} have the same number of pivots.

(b) A and A^{T} have the same left nullspace.

(c) If the row space equals the column space then $A^{\mathrm{T}} = A$.

(d) If $A^{\mathrm{T}} = -A$ then the row space of A equals the column space.

26 If a, b, c are given with $a \neq 0$, how would you choose d so that $\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix}$ has rank 1? Find a basis for the row space and nullspace. Show they are perpendicular!

27 Find the ranks of the 8 by 8 checkerboard matrix B and the chess matrix C:

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} r & n & b & q & k & b & n & r \\ p & p & p & p & p & p & p & p \\ & & & \multicolumn{2}{c}{\text{four zero rows}} & & & \\ p & p & p & p & p & p & p & p \\ r & n & b & q & k & b & n & r \end{bmatrix}$$

The numbers r, n, b, q, k, p are all different. Find bases for the row space and left nullspace of B and C. Challenge problem: Find a basis for the nullspace of C.

28 Can tic-tac-toe be completed (5 ones and 4 zeros in A) so that rank $(A) = 2$ but neither side passed up a winning move?

Challenge Problems

29 If $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ is a 2 by 2 matrix of rank 1, redraw Figure 3.5 to show clearly the Four Fundamental Subspaces. If B produces those same four subspaces, what is the exact relation of B to A?

30 $\mathbf{M}$ is the space of 3 by 3 matrices. Multiply every matrix X in $\mathbf{M}$ by

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}. \quad \text{Notice: } A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Which matrices X lead to $AX =$ zero matrix?

(b) Which matrices have the form AX for some matrix X?

(a) finds the "nullspace" of that operation AX and (b) finds the "column space". What are the dimensions of those two subspaces of $\mathbf{M}$? Why do the dimensions add to $(n - r) + r = 9$?

31 Suppose the m by n matrices A and B have *the same four subspaces*. If they are both in row reduced echelon form, prove that F must equal G:

$$A = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} I & G \\ 0 & 0 \end{bmatrix}.$$

Chapter 4

Orthogonality

4.1 Orthogonality of the Four Subspaces

1 Orthogonal vectors have $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$. Then $||\boldsymbol{v}||^2 + ||\boldsymbol{w}||^2 = ||\boldsymbol{v}+\boldsymbol{w}||^2 = ||\boldsymbol{v}-\boldsymbol{w}||^2$.

2 Subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$ are orthogonal when $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$ for every $\boldsymbol{v}$ in $\boldsymbol{V}$ and every $\boldsymbol{w}$ in $\boldsymbol{W}$.

3 The row space of A is orthogonal to the nullspace. The column space is orthogonal to $\boldsymbol{N}(A^{\mathrm{T}})$.

4 One pair of dimensions adds to $r + (n-r) = n$. The other pair has $r + (m-r) = m$.

5 Row space and nullspace are orthogonal *complements*: Every $\boldsymbol{x}$ in $\mathbf{R}^n$ splits into $\boldsymbol{x}_{\text{row}} + \boldsymbol{x}_{\text{null}}$.

6 Suppose a space $\boldsymbol{S}$ has dimension d. Then every basis for $\boldsymbol{S}$ consists of d vectors.

7 If d vectors in $\boldsymbol{S}$ are independent, they span $\boldsymbol{S}$. If d vectors span $\boldsymbol{S}$, they are independent.

Two vectors are orthogonal when their dot product is zero: $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$. This chapter moves to **orthogonal subspaces** and **orthogonal bases** and **orthogonal matrices**. The vectors in two subspaces, and the vectors in a basis, and the column vectors in Q, all pairs will be orthogonal. Think of $a^2 + b^2 = c^2$ for a *right triangle* with sides $\boldsymbol{v}$ and $\boldsymbol{w}$.

Orthogonal vectors	$\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$	and	$\lVert\boldsymbol{v}\rVert^2 + \lVert\boldsymbol{w}\rVert^2 = \lVert\boldsymbol{v}+\boldsymbol{w}\rVert^2$.

The right side is $(\boldsymbol{v}+\boldsymbol{w})^{\mathrm{T}}(\boldsymbol{v}+\boldsymbol{w})$. This equals $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v} + \boldsymbol{w}^{\mathrm{T}}\boldsymbol{w}$ when $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = \boldsymbol{w}^{\mathrm{T}}\boldsymbol{v} = 0$.

Subspaces entered Chapter 3 to throw light on $A\boldsymbol{x} = \boldsymbol{b}$. Right away we needed the column space and the nullspace. Then the light turned onto A^{T}, uncovering two more subspaces. Those four fundamental subspaces reveal what a matrix really does.

A matrix multiplies a vector: A *times* $\boldsymbol{x}$. At the first level this is only numbers. At the second level $A\boldsymbol{x}$ is a combination of column vectors. The third level shows subspaces. But I don't think you have seen the whole picture until you study Figure 4.2.

The subspaces fit together to show the hidden reality of A times $\boldsymbol{x}$. The $90°$ angles between subspaces are new—and we can say now what those right angles mean.

The row space is perpendicular to the nullspace. Every row of A is perpendicular to every solution of $A\boldsymbol{x} = \mathbf{0}$. That gives the $90°$ angle on the left side of the figure. This perpendicularity of subspaces is Part 2 of the Fundamental Theorem of Linear Algebra.

The column space is perpendicular to the nullspace of A^{T}. When $\boldsymbol{b}$ is outside the column space—when we want to solve $A\boldsymbol{x} = \boldsymbol{b}$ and can't do it—then this nullspace of A^{T} comes into its own. It contains the error $\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{x}$ in the "least-squares" solution. Least squares is the key application of linear algebra in this chapter.

Part 1 of the Fundamental Theorem gave the dimensions of the subspaces. The row and column spaces have the same dimension r (they are drawn the same size). The two nullspaces have the remaining dimensions $n - r$ and $m - r$. Now we will show that ***the row space and nullspace are orthogonal subspaces inside*** $\mathbf{R}^n$.

DEFINITION Two subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$ of a vector space are ***orthogonal*** if every vector $\boldsymbol{v}$ in $\boldsymbol{V}$ is perpendicular to every vector $\boldsymbol{w}$ in $\boldsymbol{W}$:

Orthogonal subspaces	$\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$ ***for all*** $\boldsymbol{v}$ ***in*** $\boldsymbol{V}$ ***and all*** $\boldsymbol{w}$ ***in*** $\boldsymbol{W}$.

Example 1 The floor of your room (extended to infinity) is a subspace $\boldsymbol{V}$. The line where two walls meet is a subspace $\boldsymbol{W}$ (one-dimensional). Those subspaces are orthogonal. Every vector up the meeting line of the walls is perpendicular to every vector in the floor.

Example 2 Two walls look perpendicular but those two subspaces are not orthogonal! The meeting line is in both $\boldsymbol{V}$ and $\boldsymbol{W}$—and this line is not perpendicular to itself. Two planes (dimensions 2 and 2 in $\mathbf{R}^3$) cannot be orthogonal subspaces.

When a vector is in two orthogonal subspaces, it *must* be zero. It is perpendicular to itself. It is $\boldsymbol{v}$ and it is $\boldsymbol{w}$, so $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v} = 0$. This has to be the zero vector.

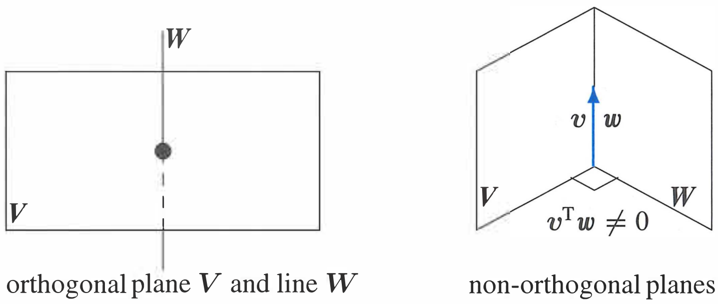


Figure 4.1: Orthogonality is impossible when $\dim \boldsymbol{V} + \dim \boldsymbol{W} > \dim$ (whole space).

The crucial examples for linear algebra come from the four fundamental subspaces. Zero is the only point where the nullspace meets the row space. More than that, the **nullspace and row space of** A **meet at** $\mathbf{90°}$. This key fact comes directly from $A\boldsymbol{x} = \mathbf{0}$:

Every vector $\boldsymbol{x}$ in the nullspace is perpendicular to every row of A, because $A\boldsymbol{x} = \mathbf{0}$. ***The nullspace $N(A)$ and the row space $C(A^{\mathrm{T}})$ are orthogonal subspaces of $\mathbf{R}^n$.***

To see why $\boldsymbol{x}$ is perpendicular to the rows, look at $A\boldsymbol{x} = \mathbf{0}$. Each row multiplies $\boldsymbol{x}$:

$$A\boldsymbol{x} = \begin{bmatrix} \textbf{row 1} \\ \vdots \\ \textbf{row } m \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{x} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \longleftarrow & (\textbf{row 1}) \cdot \boldsymbol{x} \text{ is zero} \\ \\ \longleftarrow & (\textbf{row } m) \cdot \boldsymbol{x} \text{ is zero} \end{matrix} \qquad (1)$$

The first equation says that row 1 is perpendicular to $\boldsymbol{x}$. The last equation says that row m is perpendicular to $\boldsymbol{x}$. *Every row has a zero dot product with* $\boldsymbol{x}$. Then $\boldsymbol{x}$ is also perpendicular to every *combination* of the rows. The whole row space $\boldsymbol{C}(A^{\mathrm{T}})$ is orthogonal to $\boldsymbol{N}(A)$.

Here is a second proof of that orthogonality for readers who like matrix shorthand. The vectors in the row space are combinations $A^{\mathrm{T}}\boldsymbol{y}$ of the rows. Take the dot product of $A^{\mathrm{T}}\boldsymbol{y}$ with any $\boldsymbol{x}$ in the nullspace. *These vectors are perpendicular*:

Nullspace orthogonal to row space $\qquad \boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y}) = (A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}^{\mathrm{T}}\boldsymbol{y} = 0. \qquad (2)$

We like the first proof. You can see those rows of A multiplying $\boldsymbol{x}$ to produce zeros in equation (1). The second proof shows why A and A^{T} are both in the Fundamental Theorem.

Example 3 The rows of A are perpendicular to $\boldsymbol{x} = (1, 1, -1)$ in the nullspace:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 2 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{gives the dot products} \quad \begin{matrix} 1 + 3 - 4 = 0 \\ 5 + 2 - 7 = 0 \end{matrix}$$

Now we turn to the other two subspaces. In this example, the column space is all of $\mathbf{R}^2$. The nullspace of A^{T} is only the zero vector (orthogonal to every vector). The column space of A and the nullspace of A^{T} are always orthogonal subspaces.

Every vector $\boldsymbol{y}$ in the nullspace of A^{T} is perpendicular to every column of A. ***The left nullspace $N(A^{\mathrm{T}})$ and the column space $C(A)$ are orthogonal in $\mathbf{R}^m$.***

Apply the original proof to A^{T}. The nullspace of A^{T} is orthogonal to the row space of A^{T}—and the row space of A^{T} is the column space of A. Q.E.D.

For a visual proof, look at $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. Each column of A multiplies $\boldsymbol{y}$ to give 0:

$$\boldsymbol{C}(\boldsymbol{A}) \perp \boldsymbol{N}(\boldsymbol{A}^{\mathrm{T}}) \qquad A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} (\textbf{column 1})^{\mathrm{T}} \\ \cdots \\ (\textbf{column } n)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{y} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ 0 \end{bmatrix}. \qquad (3)$$

The dot product of $\boldsymbol{y}$ with every column of A is zero. Then $\boldsymbol{y}$ in the left nullspace is perpendicular to each column of A—and to the whole column space.

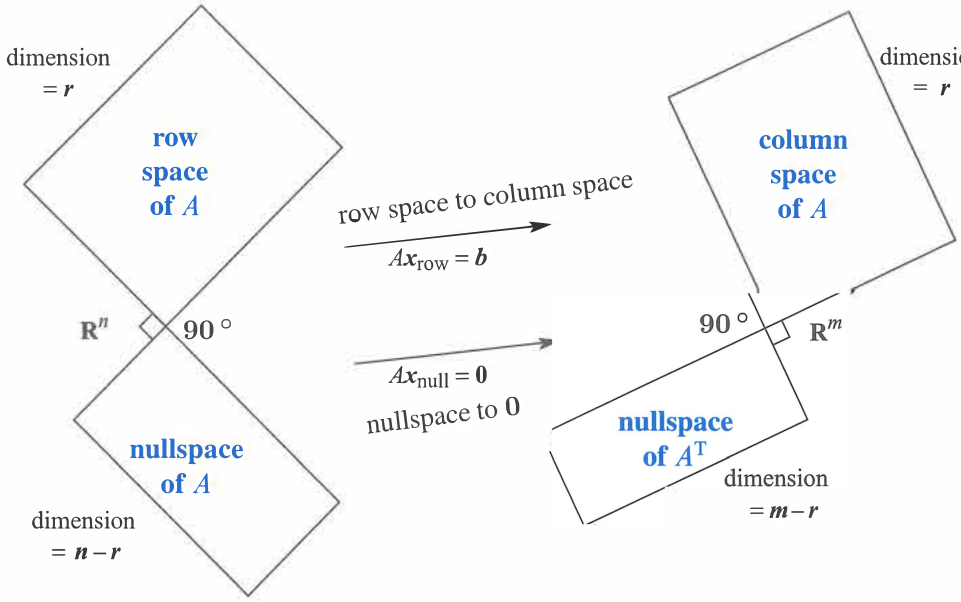


Figure 4.2: Two pairs of orthogonal subspaces. The dimensions add to n and add to m. **This is the Big Picture**—two subspaces in $\mathbf{R}^n$ and two subspaces in $\mathbf{R}^m$.

Orthogonal Complements

Important The fundamental subspaces are more than just orthogonal (in pairs). Their dimensions are also right. Two lines could be perpendicular in $\mathbf{R}^3$, **but those lines *could not be* the row space and nullspace of a 3 by 3 matrix**. The lines have dimensions 1 and 1, adding to 2. But the correct dimensions r and $n - r$ must add to $n = 3$.

The fundamental subspaces of a 3 by 3 matrix have dimensions 2 and 1, or 3 and 0. Those pairs of subspaces are not only orthogonal, they are *orthogonal complements*.

DEFINITION The ***orthogonal complement*** of a subspace $\boldsymbol{V}$ contains ***every*** vector that is perpendicular to $\boldsymbol{V}$. This orthogonal subspace is denoted by $\boldsymbol{V}^{\perp}$ (pronounced "$\boldsymbol{V}$ perp").

By this definition, the nullspace is the orthogonal complement of the row space. ***Every*** $\boldsymbol{x}$ that is perpendicular to the rows satisfies $A\boldsymbol{x} = \mathbf{0}$, and lies in the nullspace.

The reverse is also true. ***If $\boldsymbol{v}$ is orthogonal to the nullspace, it must be in the row space***. Otherwise we could add this $\boldsymbol{v}$ as an extra row of the matrix, without changing its nullspace. The row space would grow, which breaks the law $r + (n - r) = n$. We conclude that the nullspace complement $\boldsymbol{N}(A)^{\perp}$ is exactly the row space $\boldsymbol{C}(A^{\mathrm{T}})$.

In the same way, the left nullspace and column space are orthogonal in $\mathbf{R}^m$, and they are orthogonal complements. Their dimensions r and $m - r$ add to the full dimension m.

***Fundamental Theorem of Linear Algebra*, Part 2**

$N(A)$ is the orthogonal complement of the row space $C(A^{\mathrm{T}})$ (in $\mathbf{R}^n$).

$N(A^{\mathrm{T}})$ is the orthogonal complement of the column space $C(A)$ (in $\mathbf{R}^m$).

Part 1 gave the dimensions of the subspaces. Part 2 gives the 90° angles between them. The point of "complements" is that every $\boldsymbol{x}$ can be split into a *row space component* $\boldsymbol{x}_r$ and a *nullspace component* $\boldsymbol{x}_n$. When A multiplies $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$, Figure 4.3 shows what happens to $A\boldsymbol{x} = A\boldsymbol{x}_r + A\boldsymbol{x}_n$:

The nullspace component goes to zero: $A\boldsymbol{x}_n = \mathbf{0}$.

The row space component goes to the column space: $A\boldsymbol{x}_r = A\boldsymbol{x}$.

Every vector goes to the column space! Multiplying by A cannot do anything else. More than that: *Every vector* $\boldsymbol{b}$ *in the column space comes from one and only one vector* $\boldsymbol{x}_r$ *in the row space*. Proof: If $A\boldsymbol{x}_r = A\boldsymbol{x}'_r$, the difference $\boldsymbol{x}_r - \boldsymbol{x}'_r$ is in the nullspace. It is also in the row space, where $\boldsymbol{x}_r$ and $\boldsymbol{x}'_r$ came from. This difference must be the zero vector, because the nullspace and row space are perpendicular. Therefore $\boldsymbol{x}_r = \boldsymbol{x}'_r$.

There is an r by r invertible matrix hiding inside A, if we throw away the two nullspaces. ***From the row space to the column space, A is invertible***. The "pseudoinverse" will invert that part of A in Section 7.4.

Example 4 Every matrix of rank r has an r by r invertible submatrix:

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{contains the submatrix} \quad \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}.$$

The other eleven zeros are responsible for the nullspaces. The rank of B is also $r = 2$:

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 6 \end{bmatrix} \quad \text{contains} \quad \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \quad \text{in the pivot rows and columns.}$$

Every matrix can be diagonalized, when we choose the right bases for $\mathbf{R}^n$ and $\mathbf{R}^m$. This ***Singular Value Decomposition*** has become extremely important in applications.

Let me repeat one clear fact. A row of A can't be in the nullspace of A (except for a zero row). The only vector in two orthogonal subspaces is the zero vector.

If a vector v is orthogonal to itself then v is the zero vector.

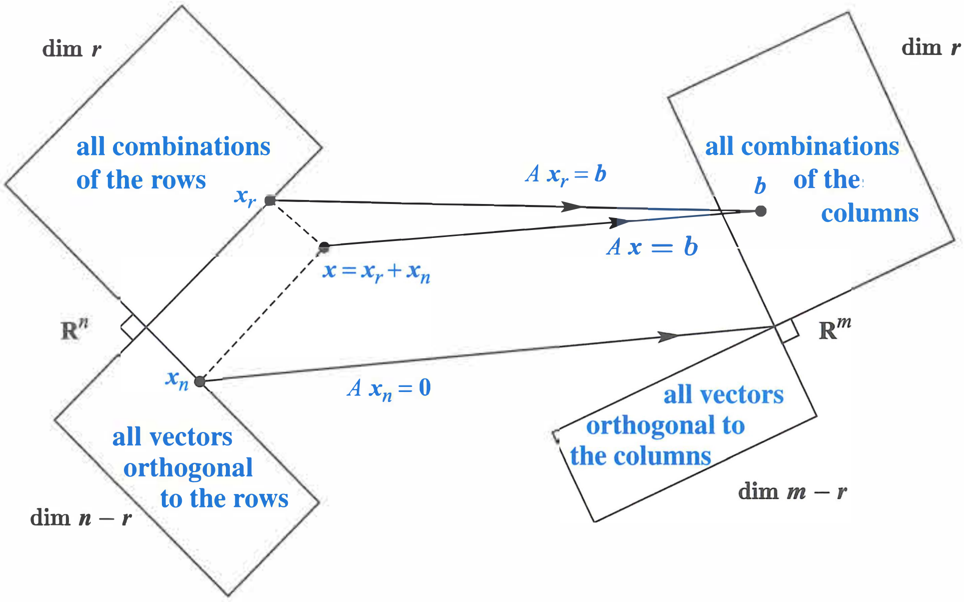


Figure 4.3: This update of Figure 4.2 shows the true action of A on $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$. Row space vector $\boldsymbol{x}_r$ to column space, nullspace vector $\boldsymbol{x}_n$ to zero.

Drawing the Big Picture

I don't know the best way to draw the four subspaces in Figures 4.2 and 4.3. This big picture has to show the orthogonality of those subspaces. I can see a possible way to do it when a line meets a plane—maybe Figure 4.4 also shows that those spaces are infinite, more clearly than the rectangles in Figure 4.3. But how do I draw a pair of two-dimensional subspaces in $\mathbf{R}^4$, to show they are orthogonal to each other? Good ideas are welcome.

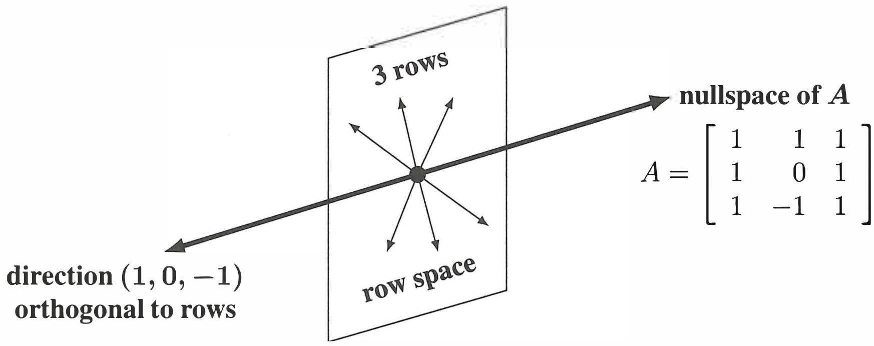


Figure 4.4: Row space of A = plane. Nullspace = orthogonal line. Dimensions $2 + 1 = 3$.

Combining Bases from Subspaces

What follows are some valuable facts about bases. They were saved until now—when we are ready to use them. After a week you have a clearer sense of what a basis is (*linearly independent* vectors that *span the space*). Normally we have to check both properties. When the count is right, one property implies the other:

> Any n independent vectors in $\mathbf{R}^n$ must span $\mathbf{R}^n$. So they are a basis.
> Any n vectors that span $\mathbf{R}^n$ must be independent. So they are a basis.

Starting with the correct number of vectors, one property of a basis produces the other. This is true in any vector space, but we care most about $\mathbf{R}^n$. When the vectors go into the columns of an n by n *square* matrix A, here are the same two facts:

> If the n columns of A are independent, they span $\mathbf{R}^n$. So $A\boldsymbol{x} = \boldsymbol{b}$ is solvable.
> If the n columns span $\mathbf{R}^n$, they are independent. So $A\boldsymbol{x} = \boldsymbol{b}$ has only one solution.

Uniqueness implies existence and existence implies uniqueness. ***Then** A **is invertible***. If there are no free variables, the solution $\boldsymbol{x}$ is unique. There must be n pivot columns. Then back substitution solves $A\boldsymbol{x} = \boldsymbol{b}$ (the solution exists).

Starting in the opposite direction, suppose that $A\boldsymbol{x} = \boldsymbol{b}$ can be solved for every $\boldsymbol{b}$ (*existence of solutions*). Then elimination produced no zero rows. There are n pivots and no free variables. The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ (*uniqueness* of solutions).

With bases for the row space and the nullspace, we have $r + (n - r) = n$ vectors. This is the right number. Those n vectors are independent.[2] *Therefore they span* $\mathbf{R}^n$.

Each $\boldsymbol{x}$ is the sum $\boldsymbol{x}_r + \boldsymbol{x}_n$ of a row space vector $\boldsymbol{x}_r$ and a nullspace vector $\boldsymbol{x}_n$.

The splitting in Figure 4.3 shows the key point of orthogonal complements—the dimensions add to n and all vectors are fully accounted for.

Example 5 For $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ split $\boldsymbol{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ into $\boldsymbol{x}_r + \boldsymbol{x}_n = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

The vector $(2, 4)$ is in the row space. The orthogonal vector $(2, -1)$ is in the nullspace. The next section will compute this splitting for any A and $\boldsymbol{x}$, by a projection.

[2] If a combination of all n vectors gives $\boldsymbol{x}_r + \boldsymbol{x}_n = \mathbf{0}$, then $\boldsymbol{x}_r = -\boldsymbol{x}_n$ is in both subspaces. So $\boldsymbol{x}_r = \boldsymbol{x}_n = \mathbf{0}$. All coefficients of the row space basis and of the nullspace basis must be zero. This proves independence of the n vectors together.

■ REVIEW OF THE KEY IDEAS ■

1. Subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$ are orthogonal if every $\boldsymbol{v}$ in $\boldsymbol{V}$ is orthogonal to every $\boldsymbol{w}$ in $\boldsymbol{W}$.
2. $\boldsymbol{V}$ and $\boldsymbol{W}$ are "orthogonal complements" if $\boldsymbol{W}$ contains **all** vectors perpendicular to $\boldsymbol{V}$ (and vice versa). Inside $\mathbf{R}^n$, the dimensions of complements $\boldsymbol{V}$ and $\boldsymbol{W}$ add to n.
3. The nullspace $\boldsymbol{N}(A)$ and the row space $\boldsymbol{C}(A^{\mathrm{T}})$ are orthogonal complements, with dimensions $(n-r)+r=n$. Similarly $\boldsymbol{N}(A^{\mathrm{T}})$ and $\boldsymbol{C}(A)$ are orthogonal complements with $(m-r)+r=m$.
4. Any n independent vectors in $\mathbf{R}^n$ span $\mathbf{R}^n$. Any n spanning vectors are independent.

■ WORKED EXAMPLES ■

4.1 A Suppose $\boldsymbol{S}$ is a six-dimensional subspace of nine-dimensional space $\mathbf{R}^9$.

(a) What are the possible dimensions of subspaces orthogonal to $\boldsymbol{S}$?

(b) What are the possible dimensions of the orthogonal complement $\boldsymbol{S}^\perp$ of $\boldsymbol{S}$?

(c) What is the smallest possible size of a matrix A that has row space $\boldsymbol{S}$?

(d) What is the smallest possible size of a matrix B that has nullspace $\boldsymbol{S}^\perp$?

Solution

(a) If $\boldsymbol{S}$ is six-dimensional in $\mathbf{R}^9$, subspaces orthogonal to $\boldsymbol{S}$ can have dimensions $0, 1, 2, 3$.

(b) The complement $\boldsymbol{S}^\perp$ is the largest orthogonal subspace, with dimension 3.

(c) The smallest matrix A is 6 by 9 (its six rows will be a basis for $\boldsymbol{S}$).

(d) This is the same as question (c) !

If a new row 7 of B is a combination of the six rows of A, then B has the same row space as A. It also has the same nullspace. The special solutions $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3$ to $A\boldsymbol{x} = \mathbf{0}$. will be the same for $B\boldsymbol{x} = \mathbf{0}$. Elimination will change row 7 of B to all zeros.

4.1 B The equation $x - 3y - 4z = 0$ describes a plane $\boldsymbol{P}$ in $\mathbf{R}^3$ (actually a subspace).

(a) The plane $\boldsymbol{P}$ is the nullspace $\boldsymbol{N}(A)$ of what 1 by 3 matrix A? *Ans*: $A = [1\ -3\ -4]$.

(b) Find a basis $\boldsymbol{s}_1, \boldsymbol{s}_2$ of special solutions of $x - 3y - 4z = 0$ (these would be the columns of the nullspace matrix N). *Answer*: $\boldsymbol{s}_1 = (3, 1, 0)$ and $\boldsymbol{s}_2 = (4, 0, 1)$.

(c) Find a basis for the line $\boldsymbol{P}^\perp$ that is perpendicular to $\boldsymbol{P}$. *Answer*: $(1, -3, -4)$**!**

Problem Set 4.1

Questions 1–12 grow out of Figures 4.2 and 4.3 with four subspaces.

1 Construct any 2 by 3 matrix of rank one. Copy Figure 4.2 and put one vector in each subspace (and put two in the nullspace). Which vectors are orthogonal?

2 Redraw Figure 4.3 for a 3 by 2 matrix of rank $r = 2$. Which subspace is $\boldsymbol{Z}$ (zero vector only)? The nullspace part of any vector $\boldsymbol{x}$ in $\mathbf{R}^2$ is $\boldsymbol{x}_n =$ ____ .

3 Construct a matrix with the required property or say why that is impossible:

(a) Column space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, nullspace contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(b) Row space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, nullspace contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(c) $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has a solution and $A^{\mathrm{T}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(d) Every row is orthogonal to every column (A is not the zero matrix)

(e) Columns add up to a column of zeros, rows add to a row of 1's.

4 If $AB = 0$ then the columns of B are in the ____ of A. The rows of A are in the ____ of B. With $AB = 0$, why can't A and B be 3 by 3 matrices of rank 2?

5 (a) If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution and $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$, *is* $(\boldsymbol{y}^{\mathrm{T}}\boldsymbol{x} = 0)$ *or* $(\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 0)$?

(b) If $A^{\mathrm{T}}\boldsymbol{y} = (1, 1, 1)$ has a solution and $A\boldsymbol{x} = \boldsymbol{0}$, then ____ .

6 This system of equations $A\boldsymbol{x} = \boldsymbol{b}$ has *no solution* (they lead to $0 = 1$):

$$\begin{aligned} x + 2y + 2z &= 5 \\ 2x + 2y + 3z &= 5 \\ 3x + 4y + 5z &= 9 \end{aligned}$$

Find numbers y_1, y_2, y_3 to multiply the equations so they add to $0 = 1$. You have found a vector $\boldsymbol{y}$ in which subspace? Its dot product $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b}$ is 1, so no solution $\boldsymbol{x}$.

7 Every system with no solution is like the one in Problem 6. There are numbers $y_1, \ldots, y_m$ that multiply the m equations so they add up to $0 = 1$. This is called **Fredholm's Alternative**:

Exactly one of these problems has a solution

$$A\boldsymbol{x} = \boldsymbol{b} \quad \textbf{OR} \quad A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0} \quad \text{with} \quad \boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 1.$$

If $\boldsymbol{b}$ is not in the column space of A, it is not orthogonal to the nullspace of A^{T}. Multiply the equations $x_1 - x_2 = 1$ and $x_2 - x_3 = 1$ and $x_1 - x_3 = 1$ by numbers y_1, y_2, y_3 chosen so that the equations add up to $0 = 1$.

8 In Figure 4.3, how do we know that $A\boldsymbol{x}_r$ is equal to $A\boldsymbol{x}$? How do we know that this vector is in the column space? If $A = \left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$ and $\boldsymbol{x} = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ what is $\boldsymbol{x}_r$?

9 If $A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{0}$ then $A\boldsymbol{x} = \boldsymbol{0}$. Reason: $A\boldsymbol{x}$ is in the nullspace of A^{T} and also in the _____ of A and those spaces are _____. *Conclusion*: *$A^{\mathrm{T}}A$ has the same nullspace as A. This key fact is repeated in the next section.*

10 Suppose A is a symmetric matrix ($A^{\mathrm{T}} = A$).

(a) Why is its column space perpendicular to its nullspace?

(b) If $A\boldsymbol{x} = \boldsymbol{0}$ and $A\boldsymbol{z} = 5\boldsymbol{z}$, which subspaces contain these "eigenvectors" $\boldsymbol{x}$ and $\boldsymbol{z}$? **Symmetric matrices have perpendicular eigenvectors** $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{z} = 0$.

11 (Recommended) Draw Figure 4.2 to show each subspace correctly for

$$A = \begin{bmatrix}1 & 2\\3 & 6\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}.$$

12 Find the pieces $\boldsymbol{x}_r$ and $\boldsymbol{x}_n$ and draw Figure 4.3 properly if

$$A = \begin{bmatrix}1 & -1\\0 & 0\\0 & 0\end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix}2\\0\end{bmatrix}.$$

Questions 13–23 are about orthogonal subspaces.

13 Put bases for the subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$ into the columns of matrices V and W. Explain why the test for orthogonal subspaces can be written $V^{\mathrm{T}}W =$ zero matrix. This matches $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$ for orthogonal vectors.

14 The floor $\boldsymbol{V}$ and the wall $\boldsymbol{W}$ are not orthogonal subspaces, because they share a nonzero vector (along the line where they meet). No planes $\boldsymbol{V}$ and $\boldsymbol{W}$ in $\mathbf{R}^3$ can be orthogonal! Find a vector in the column spaces of both matrices:

$$A = \begin{bmatrix}1 & 2\\1 & 3\\1 & 2\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}5 & 4\\6 & 3\\5 & 1\end{bmatrix}$$

This will be a vector $A\boldsymbol{x}$ and also $B\widehat{\boldsymbol{x}}$. Think 3 by 4 with the matrix $[\,A \;\; B\,]$.

15 Extend Problem 14 to a p-dimensional subspace $\boldsymbol{V}$ and a q-dimensional subspace $\boldsymbol{W}$ of $\mathbf{R}^n$. What inequality on $p + q$ guarantees that $\boldsymbol{V}$ intersects $\boldsymbol{W}$ in a nonzero vector? These subspaces cannot be orthogonal.

16 Prove that every $\boldsymbol{y}$ in $\boldsymbol{N}(A^{\mathrm{T}})$ is perpendicular to every $A\boldsymbol{x}$ in the column space, using the matrix shorthand of equation (2). Start from $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$.

17 If S is the subspace of $\mathbf{R}^3$ containing only the zero vector, what is $S^\perp$? If S is spanned by $(1,1,1)$, what is $S^\perp$? If S is spanned by $(1,1,1)$ and $(1,1,-1)$, what is a basis for $S^\perp$?

18 Suppose S only contains two vectors $(1,5,1)$ and $(2,2,2)$ (not a subspace). Then $S^\perp$ is the nullspace of the matrix $A =$ ____. $S^\perp$ is a subspace even if S is not.

19 Suppose L is a one-dimensional subspace (a line) in $\mathbf{R}^3$. Its orthogonal complement $L^\perp$ is the ____ perpendicular to L. Then $(L^\perp)^\perp$ is a ____ perpendicular to $L^\perp$. In fact $(L^\perp)^\perp$ is the same as ____.

20 Suppose V is the whole space $\mathbf{R}^4$. Then $V^\perp$ contains only the vector ____. Then $(V^\perp)^\perp$ is ____. So $(V^\perp)^\perp$ is the same as ____.

21 Suppose S is spanned by the vectors $(1,2,2,3)$ and $(1,3,3,2)$. Find two vectors that span $S^\perp$. This is the same as solving $A\boldsymbol{x} = \mathbf{0}$ for which A?

22 If P is the plane of vectors in $\mathbf{R}^4$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$, write a basis for $P^\perp$. Construct a matrix that has P as its nullspace.

23 If a subspace S is contained in a subspace V, prove that $S^\perp$ contains $V^\perp$.

Questions 24–30 are about perpendicular columns and rows.

24 Suppose an n by n matrix is invertible: $AA^{-1} = I$. Then the first column of A^{-1} is orthogonal to the space spanned by which rows of A?

25 Find $A^{\mathrm{T}}A$ if the columns of A are unit vectors, all mutually perpendicular.

26 Construct a 3 by 3 matrix A with no zero entries whose columns are mutually perpendicular. Compute $A^{\mathrm{T}}A$. Why is it a diagonal matrix?

27 The lines $3x+y=b_1$ and $6x+2y=b_2$ are ____. They are the same line if ____. In that case (b_1,b_2) is perpendicular to the vector ____. The nullspace of the matrix is the line $3x+y=$ ____. One particular vector in that nullspace is ____.

28 Why is each of these statements false?

(a) $(1,1,1)$ is perpendicular to $(1,1,-2)$ so the planes $x+y+z=0$ and $x+y-2z=0$ are orthogonal subspaces.

(b) The subspace spanned by $(1,1,0,0,0)$ and $(0,0,0,1,1)$ is the orthogonal complement of the subspace spanned by $(1,-1,0,0,0)$ and $(2,-2,3,4,-4)$.

(c) Two subspaces that meet only in the zero vector are orthogonal.

29 Find a matrix with $\boldsymbol{v} = (1,2,3)$ in the row space and column space. Find another matrix with $\boldsymbol{v}$ in the nullspace and column space. Which pairs of subspaces can $\boldsymbol{v}$ *not* be in?

Challenge Problems

30 Suppose A is 3 by 4 and B is 4 by 5 and $AB = 0$. So $\boldsymbol{N}(A)$ contains $\boldsymbol{C}(B)$. Prove from the dimensions of $\boldsymbol{N}(A)$ and $\boldsymbol{C}(B)$ that $\operatorname{rank}(A) + \operatorname{rank}(B) \leq 4$.

31 The command $N = \mathsf{null}(A)$ will produce a basis for the nullspace of A. Then the command $B = \mathsf{null}(N')$ will produce a basis for the ____ of A.

32 Suppose I give you four nonzero vectors $\boldsymbol{r}, \boldsymbol{n}, \boldsymbol{c}, \boldsymbol{l}$ in $\mathbf{R}^2$.

(a) What are the conditions for those to be bases for the four fundamental subspaces $C(A^{\mathrm{T}}), N(A), C(A), N(A^{\mathrm{T}})$ of a 2 by 2 matrix?

(b) What is one possible matrix A?

33 Suppose I give you eight vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{l}_1, \boldsymbol{l}_2$ in $\mathbf{R}^4$.

(a) What are the conditions for those pairs to be bases for the four fundamental subspaces of a 4 by 4 matrix?

(b) What is one possible matrix A?

4.2 Projections

1 The projection of a vector $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ is the closest point $\boldsymbol{p} = \boldsymbol{a}(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}\,/\,\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})$.

2 The error $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ is perpendicular to $\boldsymbol{a}$: Right triangle $\boldsymbol{b}\,\boldsymbol{p}\,\boldsymbol{e}$ has $||\boldsymbol{p}||^2 + ||\boldsymbol{e}||^2 = ||\boldsymbol{b}||^2$.

3 The **projection** of $\boldsymbol{b}$ onto a subspace $\boldsymbol{S}$ is the closest vector $\boldsymbol{p}$ in $\boldsymbol{S}$; $\boldsymbol{b} - \boldsymbol{p}$ is orthogonal to $\boldsymbol{S}$.

4 $A^{\mathrm{T}}A$ is invertible (and symmetric) only if A has independent columns: $\boldsymbol{N}(A^{\mathrm{T}}A) = \boldsymbol{N}(A)$.

5 Then the projection of $\boldsymbol{b}$ onto the column space of A is the vector $\boldsymbol{p} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}$.

6 The **projection matrix** onto $\boldsymbol{C}(A)$ is $\boxed{P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}.}$ It has $\boldsymbol{p} = P\boldsymbol{b}$ and $P^2 = P = P^{\mathrm{T}}$.

May we start this section with two questions? (In addition to that one.) The first question aims to show that projections are easy to visualize. The second question is about "projection matrices"—symmetric matrices with $P^2 = P$. *The projection of* $\boldsymbol{b}$ *is* $P\boldsymbol{b}$.

1 What are the projections of $\boldsymbol{b} = (2, 3, 4)$ onto the z axis and the xy plane?

2 What matrices P_1 and P_2 produce those projections onto a line and a plane?

When $\boldsymbol{b}$ is projected onto a line, ***its projection*** $\boldsymbol{p}$ ***is the part of*** $\boldsymbol{b}$ ***along that line***. If $\boldsymbol{b}$ is projected onto a plane, $\boldsymbol{p}$ is the part in that plane. ***The projection*** $\boldsymbol{p}$ ***is*** $P\boldsymbol{b}$.

The projection matrix P multiplies $\boldsymbol{b}$ to give $\boldsymbol{p}$. This section finds $\boldsymbol{p}$ and also P.

The projection onto the z axis we call $\boldsymbol{p}_1$. The second projection drops straight down to the xy plane. The picture in your mind should be Figure 4.5. Start with $\boldsymbol{b} = (2, 3, 4)$. The projection across gives $\boldsymbol{p}_1 = (0, 0, 4)$. The projection down gives $\boldsymbol{p}_2 = (2, 3, 0)$. Those are the parts of $\boldsymbol{b}$ along the z axis and in the xy plane.

The projection matrices P_1 and P_2 are 3 by 3. They multiply $\boldsymbol{b}$ with 3 components to produce $\boldsymbol{p}$ with 3 components. Projection onto a line comes from a rank one matrix. Projection onto a plane comes from a rank two matrix:

Projection matrix Onto the z axis: $P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix}$ Onto the xy plane: $P_2 = \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

P_1 picks out the z component of every vector. P_2 picks out the x and y components. To find the projections $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ of $\boldsymbol{b}$, multiply $\boldsymbol{b}$ by P_1 and P_2 (small $\boldsymbol{p}$ for the vector, capital P for the matrix that produces it):

$$\boldsymbol{p}_1 = P_1\boldsymbol{b} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{z} \end{bmatrix} \qquad \boldsymbol{p}_2 = P_2\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \mathbf{0} \end{bmatrix}.$$

In this case the projections p_1 and p_2 are perpendicular. The xy plane and the z axis are ***orthogonal subspaces***, like the floor of a room and the line between two walls.

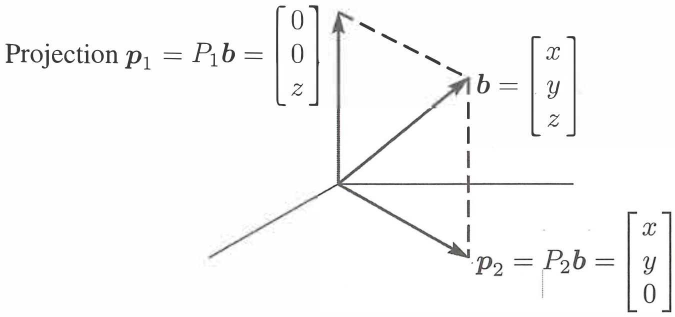


Figure 4.5: The projections $\boldsymbol{p}_1 = P_1\boldsymbol{b}$ and $\boldsymbol{p}_2 = P_2\boldsymbol{b}$ onto the z axis and the xy plane.

More than just orthogonal, the line and plane are orthogonal ***complements***. Their dimensions add to $1 + 2 = 3$. Every vector $\boldsymbol{b}$ in the whole space is the sum of its parts in the two subspaces. The projections $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are exactly those two parts of $\boldsymbol{b}$:

$$\text{The vectors give } \boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{b}. \qquad \text{The matrices give } P_1 + P_2 = I. \tag{1}$$

This is perfect. Our goal is reached—for this example. We have the same goal for any line and any plane and any n-dimensional subspace. The object is to find the part $\boldsymbol{p}$ in each subspace, and the projection matrix P that produces that part $\boldsymbol{p} = P\boldsymbol{b}$. Every subspace of $\mathbf{R}^m$ has its own m by m projection matrix. To compute P, we absolutely need a good description of the subspace that it projects onto.

The best description of a subspace is a basis. We put the basis vectors into the columns of A. ***Now we are projecting onto the column space of A***! Certainly the z axis is the column space of the 3 by 1 matrix A_1. The xy plane is the column space of A_2. That plane is *also* the column space of A_3 (a subspace has many bases). So $\boldsymbol{p}_2 = \boldsymbol{p}_3$ and $P_2 = P_3$.

$$A_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 0 & 0 \end{bmatrix}.$$

Our problem is ***to project any b onto the column space of any m by n matrix***. Start with a line (dimension $n = 1$). The matrix A will have only one column. Call it $\boldsymbol{a}$.

Projection Onto a Line

A line goes through the origin in the direction of $\boldsymbol{a} = (a_1, \ldots, a_m)$. Along that line, we want the point $\boldsymbol{p}$ closest to $\boldsymbol{b} = (b_1, \ldots, b_m)$. The key to projection is orthogonality: ***The line from b to p is perpendicular to the vector a***. This is the dotted line marked $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ for the error on the left side of Figure 4.6. We now compute $\boldsymbol{p}$ by algebra.

The projection $\boldsymbol{p}$ will be some multiple of $\boldsymbol{a}$. Call it $\boldsymbol{p} = \widehat{x}\boldsymbol{a} =$ "x hat" times $\boldsymbol{a}$. Computing this number $\widehat{x}$ will give the vector $\boldsymbol{p}$. Then from the formula for $\boldsymbol{p}$, we will read off the projection matrix P. These three steps will lead to all projection matrices: **find $\widehat{x}$, then find the vector p, then find the matrix P.**

The dotted line $\boldsymbol{b} - \boldsymbol{p}$ is the "error" $\boldsymbol{e} = \boldsymbol{b} - \widehat{x}\boldsymbol{a}$. It is perpendicular to $\boldsymbol{a}$—this will determine $\widehat{x}$. Use the fact that $\boldsymbol{b} - \widehat{x}\boldsymbol{a}$ **is perpendicular to** $\boldsymbol{a}$ when their dot product is zero:

Projecting $\boldsymbol{b}$ onto $\boldsymbol{a}$ with error $\boldsymbol{e} = \boldsymbol{b} - \widehat{x}\boldsymbol{a}$

$\boldsymbol{a} \cdot (\boldsymbol{b} - \widehat{x}\boldsymbol{a}) = 0$ or $\boldsymbol{a} \cdot \boldsymbol{b} - \widehat{x}\boldsymbol{a} \cdot \boldsymbol{a} = 0$

$$\widehat{x} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{a} \cdot \boldsymbol{a}} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}. \tag{2}$$

The multiplication $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}$ is the same as $\boldsymbol{a} \cdot \boldsymbol{b}$. Using the transpose is better, because it applies also to matrices. Our formula $\widehat{x} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ gives the projection $\boldsymbol{p} = \widehat{x}\boldsymbol{a}$.

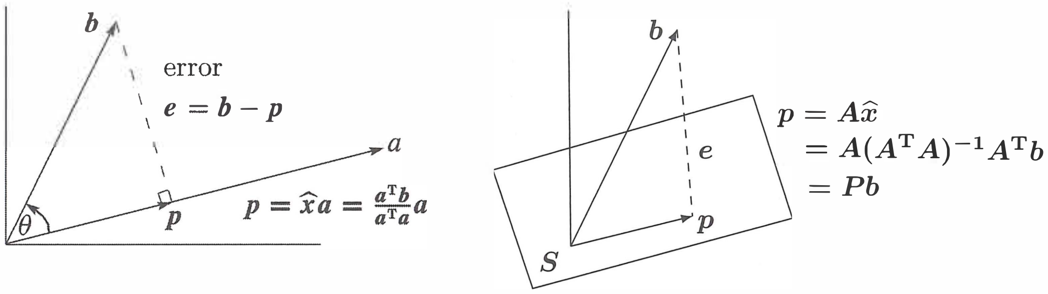


Figure 4.6: The projection $\boldsymbol{p}$ of $\boldsymbol{b}$ onto a line and onto $\boldsymbol{S}$ = column space of A.

The projection of b onto the line through a is the vector $\boldsymbol{p} = \widehat{x}\boldsymbol{a} = \dfrac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a}$.

Special case 1: If $\boldsymbol{b} = \boldsymbol{a}$ then $\widehat{x} = 1$. The projection of $\boldsymbol{a}$ onto $\boldsymbol{a}$ is itself. $P\boldsymbol{a} = \boldsymbol{a}$.

Special case 2: If $\boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$ then $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = 0$. The projection is $\boldsymbol{p} = \boldsymbol{0}$.

Example 1 Project $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto $\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ to find $\boldsymbol{p} = \widehat{x}\boldsymbol{a}$ in Figure 4.6.

Solution The number $\widehat{x}$ is the ratio of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = 5$ to $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = 9$. So the projection is $\boldsymbol{p} = \frac{5}{9}\boldsymbol{a}$.

The error vector between $\boldsymbol{b}$ and $\boldsymbol{p}$ is $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$. Those vectors $\boldsymbol{p}$ and $\boldsymbol{e}$ will add to $\boldsymbol{b} = (\mathbf{1}, \mathbf{1}, \mathbf{1})$:

$$\boldsymbol{p} = \frac{5}{9}\boldsymbol{a} = \left(\frac{5}{9}, \frac{10}{9}, \frac{10}{9}\right) \quad \text{and} \quad \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \left(\frac{4}{9}, -\frac{1}{9}, -\frac{1}{9}\right).$$

The error $\boldsymbol{e}$ should be perpendicular to $\boldsymbol{a} = (1, 2, 2)$ and it is: $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{a} = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$.

Look at the right triangle of $\boldsymbol{b}$, $\boldsymbol{p}$, and $\boldsymbol{e}$. The vector $\boldsymbol{b}$ is split into two parts—its component along the line is $\boldsymbol{p}$, its perpendicular part is $\boldsymbol{e}$. Those two sides $\boldsymbol{p}$ and $\boldsymbol{e}$ have length $||\boldsymbol{p}|| = ||\boldsymbol{b}|| \cos\theta$ and $||\boldsymbol{e}|| = ||\boldsymbol{b}|| \sin\theta$. Trigonometry matches the dot product:

$$\boldsymbol{p} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a} \quad \text{has length} \quad ||\boldsymbol{p}|| = \frac{||\boldsymbol{a}||\,||\boldsymbol{b}|| \cos\theta}{||\boldsymbol{a}||^2}||\boldsymbol{a}|| = ||\boldsymbol{b}|| \cos\theta. \tag{3}$$

The dot product is a lot simpler than getting involved with $\cos\theta$ and the length of $\boldsymbol{b}$. The example has square roots in $\cos\theta = 5/3\sqrt{3}$ and $||\boldsymbol{b}|| = \sqrt{3}$. There are no square roots in the projection $\boldsymbol{p} = 5\boldsymbol{a}/9$. The good way to $5/9$ is $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$.

Now comes the ***projection matrix***. In the formula for $\boldsymbol{p}$, what matrix is multiplying $\boldsymbol{b}$? You can see the matrix better if the number $\widehat{x}$ is on the right side of $\boldsymbol{a}$:

Projection matrix P $\quad \boldsymbol{p} = \boldsymbol{a}\widehat{x} = \boldsymbol{a}\dfrac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = P\boldsymbol{b}$ ***when the matrix is*** $P = \dfrac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}.$

P is a column times a row! The column is $\boldsymbol{a}$, the row is $\boldsymbol{a}^{\mathrm{T}}$. Then divide by the number $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$. The projection matrix P is m by m, but ***its rank is one***. We are projecting onto a one-dimensional subspace, the line through $\boldsymbol{a}$. *That line is the column space of P.*

Example 2 Find the projection matrix $P = \dfrac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}$ onto the line through $\boldsymbol{a} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{2} \end{bmatrix}$.

Solution Multiply column $\boldsymbol{a}$ times row $\boldsymbol{a}^{\mathrm{T}}$ and divide by $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = 9$:

$$\text{Projection matrix} \qquad P = \frac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \frac{1}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}.$$

This matrix projects *any* vector $\boldsymbol{b}$ onto $\boldsymbol{a}$. Check $\boldsymbol{p} = P\boldsymbol{b}$ for $\boldsymbol{b} = (1, 1, 1)$ in Example 1:

$$\boldsymbol{p} = P\boldsymbol{b} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 5 \\ 10 \\ 10 \end{bmatrix} \quad \text{which is correct.}$$

If the vector $\boldsymbol{a}$ is doubled, the matrix P stays the same! It still projects onto the same line. If the matrix is squared, P^2 equals P. ***Projecting a second time doesn't change anything***, **so $\boldsymbol{P^2 = P}$**. The diagonal entries of P add up to $\frac{1}{9}(1 + 4 + 4) = 1$.

The matrix $I-P$ should be a projection too. It produces the other side $\boldsymbol{e}$ of the triangle—the perpendicular part of $\boldsymbol{b}$. Note that $(I-P)\boldsymbol{b}$ equals $\boldsymbol{b}-\boldsymbol{p}$ which is $\boldsymbol{e}$ in the left nullspace.

When P projects onto one subspace, $I-P$ projects onto the perpendicular subspace. Here $I-P$ projects onto the plane perpendicular to $\boldsymbol{a}$.

Now we move beyond projection onto a line. Projecting onto an n-dimensional subspace of $\mathbf{R}^m$ takes more effort. The crucial formulas will be collected in equations (5)–(6)–(7). Basically you need to remember those three equations.

Projection Onto a Subspace

Start with n vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ in $\mathbf{R}^m$. Assume that these $\boldsymbol{a}$'s are linearly independent.

Problem: Find the combination $\boldsymbol{p} = \widehat{x}_1\boldsymbol{a}_1 + \cdots + \widehat{x}_n\boldsymbol{a}_n$ closest to a given vector $\boldsymbol{b}$. We are projecting each $\boldsymbol{b}$ in $\mathbf{R}^m$ onto the subspace spanned by the $\boldsymbol{a}$'s.

With $n=1$ (one vector $\boldsymbol{a}_1$) this is projection onto a line. The line is the column space of A, which has just one column. In general the matrix A has n columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.

The combinations in $\mathbf{R}^m$ are the vectors $A\boldsymbol{x}$ in the column space. We are looking for the particular combination $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$ (***the projection***) that is closest to $\boldsymbol{b}$. The hat over $\widehat{\boldsymbol{x}}$ indicates the *best* choice $\widehat{\boldsymbol{x}}$, to give the closest vector in the column space. That choice is $\widehat{x} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ when $n=1$. For $n>1$, the best $\widehat{\boldsymbol{x}} = (\widehat{x}_1, \ldots, \widehat{x}_n)$ is to be found now.

We compute projections onto n-dimensional subspaces in three steps as before: ***Find the vector $\widehat{\boldsymbol{x}}$, find the projection $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$, find the projection matrix P.***

The key is in the geometry! The dotted line in Figure 4.6 goes from $\boldsymbol{b}$ to the nearest point $A\widehat{\boldsymbol{x}}$ in the subspace. ***This error vector $\boldsymbol{b} - A\widehat{\boldsymbol{x}}$ is perpendicular to the subspace.*** The error $\boldsymbol{b} - A\widehat{\boldsymbol{x}}$ makes a right angle with all the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ in the base. The n right angles give the n equations for $\widehat{\boldsymbol{x}}$:

$$\begin{aligned} \boldsymbol{a}_1^{\mathrm{T}}(\boldsymbol{b} - A\widehat{\boldsymbol{x}}) &= 0 \\ &\vdots \\ \boldsymbol{a}_n^{\mathrm{T}}(\boldsymbol{b} - A\widehat{\boldsymbol{x}}) &= 0 \end{aligned} \qquad \text{or} \qquad \boxed{\begin{bmatrix} - \boldsymbol{a}_1^{\mathrm{T}} - \\ \vdots \\ - \boldsymbol{a}_n^{\mathrm{T}} - \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{b} - A\widehat{\boldsymbol{x}} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \mathbf{0} \\ \\ \end{bmatrix}.} \tag{4}$$

The matrix with those rows $\boldsymbol{a}_i^{\mathrm{T}}$ is A^{T}. The n equations are exactly $A^{\mathrm{T}}(\boldsymbol{b} - A\widehat{\boldsymbol{x}}) = \mathbf{0}$.

Rewrite $A^{\mathrm{T}}(\boldsymbol{b} - A\widehat{\boldsymbol{x}}) = \mathbf{0}$ in its famous form $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. This is the equation for $\widehat{\boldsymbol{x}}$, and the coefficient matrix is $A^{\mathrm{T}}A$. Now we can find $\widehat{\boldsymbol{x}}$ and $\boldsymbol{p}$ and P, in that order.

The combination $\boldsymbol{p} = \widehat{x}_1\boldsymbol{a}_1 + \cdots + \widehat{x}_n\boldsymbol{a}_n = A\widehat{\boldsymbol{x}}$ that is closest to $\boldsymbol{b}$ comes from $\widehat{\boldsymbol{x}}$:

$$\textbf{Find } \widehat{\boldsymbol{x}}\ (n \times 1) \qquad \boxed{A^{\mathrm{T}}(\boldsymbol{b} - A\widehat{\boldsymbol{x}}) = \mathbf{0} \quad \text{or} \quad A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}.} \tag{5}$$

This symmetric matrix $A^{\mathrm{T}}A$ is n by n. It is invertible if the $\boldsymbol{a}$'s are independent. The solution is $\widehat{\boldsymbol{x}} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}$. The ***projection*** of $\boldsymbol{b}$ onto the subspace is $\boldsymbol{p}$:

$$\textbf{Find } \boldsymbol{p}\ (m \times 1) \qquad \boxed{\boldsymbol{p} = A\widehat{\boldsymbol{x}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}.} \tag{6}$$

The next formula picks out the ***projection matrix*** that is multiplying $\boldsymbol{b}$ in (6):

$$\textbf{Find } \boldsymbol{P}\ (m \times m) \qquad \boxed{P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}.} \tag{7}$$

Compare with projection onto a line, when A has only one column: $A^{\mathrm{T}}A$ is $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$.

$$\textbf{For } \boldsymbol{n = 1} \qquad \boxed{\widehat{\boldsymbol{x}} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}} \quad \textbf{and} \quad \boxed{\boldsymbol{p} = \boldsymbol{a}\frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}} \quad \textbf{and} \quad \boxed{P = \frac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}.}$$

Those formulas are identical with (5) and (6) and (7). The number $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ becomes the matrix $A^{\mathrm{T}}A$. When it is a number, we divide by it. When it is a matrix, we invert it. The new formulas contain $(A^{\mathrm{T}}A)^{-1}$ instead of $1/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$. The linear independence of the columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ will guarantee that this inverse matrix exists.

The key step was $A^{\mathrm{T}}(\boldsymbol{b} - A\widehat{\boldsymbol{x}}) = \mathbf{0}$. We used geometry ($\boldsymbol{e}$ is orthogonal to each $\boldsymbol{a}$). *Linear algebra gives this "normal equation" too*, in a very quick and beautiful way:

1. Our subspace is the column space of A.

2. The error vector $\boldsymbol{b} - A\widehat{\boldsymbol{x}}$ is perpendicular to that column space.

3. Therefore $\boldsymbol{b} - A\widehat{\boldsymbol{x}}$ is in the nullspace of A^{T}! This means $A^{\mathrm{T}}(\boldsymbol{b} - A\widehat{\boldsymbol{x}}) = \mathbf{0}$.

The left nullspace is important in projections. That nullspace of A^{T} contains the error vector $\boldsymbol{e} = \boldsymbol{b} - A\widehat{\boldsymbol{x}}$. The vector $\boldsymbol{b}$ is being split into the projection $\boldsymbol{p}$ and the error $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$. Projection produces a right triangle with sides $\boldsymbol{p}$, $\boldsymbol{e}$, and $\boldsymbol{b}$.

Example 3 If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ find $\widehat{\boldsymbol{x}}$ and $\boldsymbol{p}$ and P.

Solution Compute the square matrix $A^{\mathrm{T}}A$ and also the vector $A^{\mathrm{T}}\boldsymbol{b}$:

$$A^{\mathrm{T}}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \quad \text{and} \quad A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

Now solve the normal equation $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ to find $\widehat{\boldsymbol{x}}$:

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \quad \text{gives} \quad \widehat{\boldsymbol{x}} = \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{5} \\ \mathbf{-3} \end{bmatrix}. \tag{8}$$

The combination $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$ is the projection of $\boldsymbol{b}$ onto the column space of A:

$$\boldsymbol{p} = \mathbf{5} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mathbf{-3} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}. \quad \text{The error is} \quad \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}. \tag{9}$$

Two checks on the calculation. First, the error $\boldsymbol{e} = (1, -2, 1)$ is perpendicular to both columns $(1, 1, 1)$ and $(0, 1, 2)$. Second, the matrix P times $\boldsymbol{b} = (6, 0, 0)$ correctly gives $\boldsymbol{p} = (5, 2, -1)$. That solves the problem for one particular $\boldsymbol{b}$, as soon as we find P.

The projection matrix is $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. The determinant of $A^{\mathrm{T}}A$ is $15 - 9 = 6$; then $(A^{\mathrm{T}}A)^{-1}$ is easy. Multiply A times $(A^{\mathrm{T}}A)^{-1}$ times A^{T} to reach P:

$$(A^{\mathrm{T}}A)^{-1} = \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \quad \text{and} \quad P = \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}. \tag{10}$$

We must have $P^2 = P$, because a second projection doesn't change the first projection.

Warning The matrix $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is deceptive. You might try to split $(A^{\mathrm{T}}A)^{-1}$ into A^{-1} times $(A^{\mathrm{T}})^{-1}$. If you make that mistake, and substitute it into P, you will find $P = AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}}$. Apparently everything cancels. This looks like $P = I$, the identity matrix. We want to say why this is wrong.

The matrix A is rectangular. It has no inverse matrix. We cannot split $(A^{\mathrm{T}}A)^{-1}$ into A^{-1} times $(A^{\mathrm{T}})^{-1}$ because there is no A^{-1} in the first place.

In our experience, a problem that involves a rectangular matrix almost always leads to $A^{\mathrm{T}}A$. When A has independent columns, $A^{\mathrm{T}}A$ is invertible. This fact is so crucial that we state it clearly and give a proof.

$A^{\mathrm{T}}A$ is invertible if and only if A has linearly independent columns.

Proof $A^{\mathrm{T}}A$ is a square matrix (n by n). For every matrix A, we will now show that $A^{\mathrm{T}}A$ ***has the same nullspace as*** A. When the columns of A are linearly independent, its nullspace contains only the zero vector. Then $A^{\mathrm{T}}A$, with this same nullspace, is invertible.

Let A be any matrix. If $\boldsymbol{x}$ is in its nullspace, then $A\boldsymbol{x} = \mathbf{0}$. Multiplying by A^{T} gives $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$. So $\boldsymbol{x}$ is also in the nullspace of $A^{\mathrm{T}}A$.

Now start with the nullspace of $A^{\mathrm{T}}A$. **From** $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$ **we must prove** $A\boldsymbol{x} = \mathbf{0}$. We can't multiply by $(A^{\mathrm{T}})^{-1}$, which generally doesn't exist. Just multiply by $\boldsymbol{x}^{\mathrm{T}}$:

$$(\boldsymbol{x}^{\mathrm{T}})A^{\mathrm{T}}A\boldsymbol{x} = 0 \quad \text{or} \quad (A\boldsymbol{x})^{\mathrm{T}}(A\boldsymbol{x}) = 0 \quad \text{or} \quad \|A\boldsymbol{x}\|^2 = 0. \tag{11}$$

We have shown: If $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x}$ has length zero. Therefore $A\boldsymbol{x} = \mathbf{0}$. Every vector $\boldsymbol{x}$ in one nullspace is in the other nullspace. If $A^{\mathrm{T}}A$ has dependent columns, so has A. If $A^{\mathrm{T}}A$ has independent columns, so has A. This is the good case: $A^{\mathrm{T}}A$ is invertible.

When A has independent columns, $A^{\mathrm{T}}A$ is square, symmetric, and invertible.

To repeat for emphasis: $A^{\mathrm{T}}A$ is (n by m) times (m by n). Then $A^{\mathrm{T}}A$ is square (n by n). It is symmetric, because its transpose is $(A^{\mathrm{T}}A)^{\mathrm{T}} = A^{\mathrm{T}}(A^{\mathrm{T}})^{\mathrm{T}}$ which equals $A^{\mathrm{T}}A$. We just proved that $A^{\mathrm{T}}A$ is invertible—provided A has independent columns. Watch the difference between dependent and independent columns:

$$\underset{A^{\mathrm{T}}}{\begin{bmatrix}1&1&0\\2&2&0\end{bmatrix}}\underset{\text{dependent}}{\overset{A}{\begin{bmatrix}1&2\\1&2\\0&0\end{bmatrix}}}=\underset{\text{singular}}{\overset{A^{\mathrm{T}}A}{\begin{bmatrix}2&4\\4&8\end{bmatrix}}}\qquad \overset{A^{\mathrm{T}}}{\begin{bmatrix}1&1&0\\2&2&1\end{bmatrix}}\underset{\text{indep.}}{\overset{A}{\begin{bmatrix}1&2\\1&2\\0&1\end{bmatrix}}}=\underset{\text{invertible}}{\overset{A^{\mathrm{T}}A}{\begin{bmatrix}2&4\\4&9\end{bmatrix}}}$$

Very brief summary To find the projection $\boldsymbol{p} = \widehat{x}_1\boldsymbol{a}_1 + \cdots + \widehat{x}_n\boldsymbol{a}_n$, solve $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. This gives $\widehat{\boldsymbol{x}}$. The projection is $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$ and the error is $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \boldsymbol{b} - A\widehat{\boldsymbol{x}}$. The projection matrix $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ gives $\boldsymbol{p} = P\boldsymbol{b}$.

This matrix satisfies $P^2 = P$. *The distance from b to the subspace $C(A)$ is* $||e||$.

■ REVIEW OF THE KEY IDEAS ■

1. The projection of $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ is $\boldsymbol{p} = \boldsymbol{a}\widehat{x} = \boldsymbol{a}(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})$.
2. The rank one projection matrix $P = \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ multiplies $\boldsymbol{b}$ to produce $\boldsymbol{p}$.
3. Projecting $\boldsymbol{b}$ onto a subspace leaves $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ perpendicular to the subspace.
4. When A has full rank n, the equation $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ leads to $\widehat{\boldsymbol{x}}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$.
5. The projection matrix $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ has $P^{\mathrm{T}} = P$ and $P^2 = P$ and $P\boldsymbol{b} = \boldsymbol{p}$.

■ WORKED EXAMPLES ■

4.2 A Project the vector $\boldsymbol{b} = (3, 4, 4)$ onto the line through $\boldsymbol{a} = (2, 2, 1)$ and then onto the plane that also contains $\boldsymbol{a}^* = (1, 0, 0)$. Check that the first error vector $\boldsymbol{b} - \boldsymbol{p}$ is perpendicular to $\boldsymbol{a}$, and the second error vector $\boldsymbol{e}^* = \boldsymbol{b} - \boldsymbol{p}^*$ is also perpendicular to $\boldsymbol{a}^*$.

Find the 3 by 3 projection matrix P onto that plane of $\boldsymbol{a}$ and $\boldsymbol{a}^*$. Find a vector whose *projection onto the plane is the zero vector*. Why is it exactly the error $\boldsymbol{e}^*$?

Solution The projection of $\boldsymbol{b} = (3, 4, 4)$ onto the line through $\boldsymbol{a} = (2, 2, 1)$ is $\boldsymbol{p} = 2\boldsymbol{a}$:

Onto a line $$\boldsymbol{p} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a} = \frac{18}{9}(2, 2, 1) = (4, 4, 2) = 2\boldsymbol{a}.$$

The error vector $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = (-1, 0, 2)$ is perpendicular to $\boldsymbol{a} = (2, 2, 1)$. So $\boldsymbol{p}$ is correct.

The plane of $\boldsymbol{a} = (2, 2, 1)$ and $\boldsymbol{a}^* = (1, 0, 0)$ is the column space of $A = [\boldsymbol{a}\ \boldsymbol{a}^*]$:

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} \quad A^{\mathrm{T}}A = \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix} \quad (A^{\mathrm{T}}A)^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 9 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .8 & .4 \\ 0 & .4 & .2 \end{bmatrix}$$

Now $\boldsymbol{p}^* = P\boldsymbol{b} = (3, 4.8, 2.4)$. The error $\boldsymbol{e}^* = \boldsymbol{b} - \boldsymbol{p}^* = (0, -.8, 1.6)$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{a}^*$. This $\boldsymbol{e}^*$ is in the nullspace of P and *its projection is zero*! Note $P^2 = P = P^{\mathrm{T}}$.

4.2 B Suppose your pulse is measured at $x = 70$ beats per minute, then at $x = 80$, then at $x = 120$. Those three equations $Ax = \boldsymbol{b}$ in one unknown have $A^{\mathrm{T}} = [1\ 1\ 1]$ and $\boldsymbol{b} = (70, 80, 120)$. ***The best $\widehat{x}$ is the ______ of 70, 80, 120.*** Use calculus and projection:

1. Minimize $E = (x - 70)^2 + (x - 80)^2 + (x - 120)^2$ by solving $dE/dx = 0$.
2. Project $\boldsymbol{b} = (70, 80, 120)$ onto $\boldsymbol{a} = (1, 1, 1)$ to find $\widehat{x} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$.

Solution The closest horizontal line to the heights 70, 80, 120 is the *average* $\widehat{x} = 90$:

$$\frac{dE}{dx} = 2(x - 70) + 2(x - 80) + 2(x - 120) = 0 \quad \text{gives} \quad \widehat{x} = \frac{70 + 80 + 120}{3} = 90.$$

Also by projection : $$\widehat{x} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \frac{(1, 1, 1)^{\mathrm{T}}(70, 80, 120)}{(1, 1, 1)^{\mathrm{T}}(1, 1, 1)} = \frac{70 + 80 + 120}{3} = 90.$$

In *recursive* least squares, a fourth measurement 130 changes the average $\widehat{x}_{\text{old}} = 90$ to $\widehat{x}_{\text{new}} = 100$. Verify the *update formula* $\widehat{x}_{\text{new}} = \widehat{x}_{\text{old}} + \frac{1}{4}(130 - \widehat{x}_{\text{old}})$. When a new measurement arrives, we don't have to average all the old measurements again!

Problem Set 4.2

Questions 1–9 ask for projections $\boldsymbol{p}$ onto lines. Also errors $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ and matrices P.

1 Project the vector $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$. Check that $\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$:

$$\text{(a)}\ \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \ \text{and} \ \boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{(b)}\ \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \ \text{and} \ \boldsymbol{a} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}.$$

2 *Draw* the projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ and also compute it from $\boldsymbol{p} = \widehat{x}\boldsymbol{a}$:

$$\text{(a)}\ \boldsymbol{b} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad \text{and} \quad \boldsymbol{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{(b)}\quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

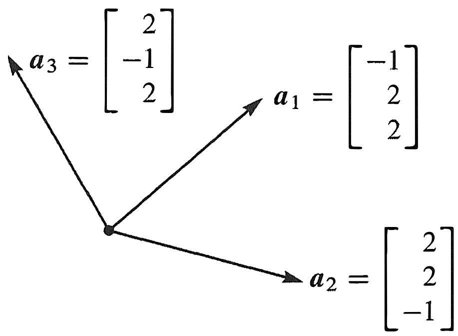
3 In Problem 1, find the projection matrix $P = \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ onto the line through each vector $\boldsymbol{a}$. Verify in both cases that $P^2 = P$. Multiply $P\boldsymbol{b}$ in each case to compute the projection $\boldsymbol{p}$.

4 Construct the projection matrices P_1 and P_2 onto the lines through the $\boldsymbol{a}$'s in Problem 2. Is it true that $(P_1 + P_2)^2 = P_1 + P_2$? This *would* be true if $P_1P_2 = 0$.

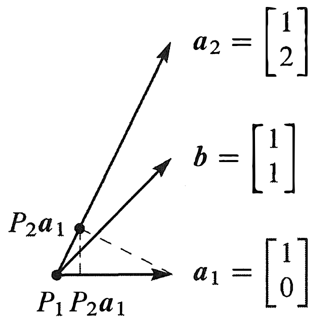
5 Compute the projection matrices $\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ onto the lines through $\boldsymbol{a}_1 = (-1, 2, 2)$ and $\boldsymbol{a}_2 = (2, 2, -1)$. Multiply those projection matrices and explain why their product P_1P_2 is what it is.

6 Project $\boldsymbol{b} = (1, 0, 0)$ onto the lines through $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ in Problem 5 and also onto $\boldsymbol{a}_3 = (2, -1, 2)$. Add up the three projections $\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3$.

7 Continuing Problems 5–6, find the projection matrix P_3 onto $\boldsymbol{a}_3 = (2, -1, 2)$. Verify that $P_1 + P_2 + P_3 = I$. This is because the basis $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ is orthogonal!



Questions 5–6–7: orthogonal



Questions 8–9–10: not orthogonal

8 Project the vector $\boldsymbol{b} = (1, 1)$ onto the lines through $\boldsymbol{a}_1 = (1, 0)$ and $\boldsymbol{a}_2 = (1, 2)$. Draw the projections $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ and add $\boldsymbol{p}_1 + \boldsymbol{p}_2$. The projections do not add to $\boldsymbol{b}$ because the $\boldsymbol{a}$'s are not orthogonal.

9 In Problem 8, the projection of $\boldsymbol{b}$ onto the *plane* of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ will equal $\boldsymbol{b}$. Find $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ for $A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} =$ invertible matrix.

10 Project $\boldsymbol{a}_1 = (1, 0)$ onto $\boldsymbol{a}_2 = (1, 2)$. Then project the result back onto $\boldsymbol{a}_1$. Draw these projections and multiply the projection matrices P_1P_2: Is this a projection?

Questions 11–20 ask for projections, and projection matrices, onto subspaces.

11 Project $\boldsymbol{b}$ onto the column space of A by solving $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$:

$$\text{(a)}\; A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \qquad \text{(b)} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}.$$

Find $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$. It should be perpendicular to the columns of A.

12 Compute the projection matrices P_1 and P_2 onto the column spaces in Problem 11. Verify that $P_1\boldsymbol{b}$ gives the first projection $\boldsymbol{p}_1$. Also verify $P_2^2 = P_2$.

13 (Quick and Recommended) Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\boldsymbol{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

14 Suppose $\boldsymbol{b}$ equals 2 times the first column of A. What is the projection of $\boldsymbol{b}$ onto the column space of A? Is $P = I$ for sure in this case? Compute $\boldsymbol{p}$ and P when $\boldsymbol{b} = (0, 2, 4)$ and the columns of A are $(0, 1, 2)$ and $(1, 2, 0)$.

15 If A is doubled, then $P = 2A(4A^{\mathrm{T}}A)^{-1}2A^{\mathrm{T}}$. This is the same as $A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. The column space of $2A$ is the same as _____. Is $\widehat{\boldsymbol{x}}$ the same for A and $2A$?

16 What linear combination of $(1, 2, -1)$ and $(1, 0, 1)$ is closest to $\boldsymbol{b} = (2, 1, 1)$?

17 (*Important*) If $P^2 = P$ show that $(I - P)^2 = I - P$. When P projects onto the column space of A, $I - P$ projects onto the _____.

18 (a) If P is the 2 by 2 projection matrix onto the line through $(1, 1)$, then $I - P$ is the projection matrix onto _____.

(b) If P is the 3 by 3 projection matrix onto the line through $(1, 1, 1)$, then $I - P$ is the projection matrix onto _____.

19 To find the projection matrix onto the plane $x - y - 2z = 0$, choose two vectors in that plane and make them the columns of A. The plane will be the column space of A! Then compute $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$.

20 To find the projection matrix P onto the same plane $x - y - 2z = 0$, write down a vector $\boldsymbol{e}$ that is perpendicular to that plane. Compute the projection $Q = \boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}/\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}$ and then $P = I - Q$.

Questions 21–26 show that projection matrices satisfy $\boldsymbol{P^2 = P}$ and $\boldsymbol{P^{\mathrm{T}} = P}$.

21 Multiply the matrix $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ by itself. Cancel to prove that $P^2 = P$. Explain why $P(P\boldsymbol{b})$ always equals $P\boldsymbol{b}$: The vector $P\boldsymbol{b}$ is in the column space of A so its projection onto that column space is _____.

22 Prove that $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is symmetric by computing P^{T}. Remember that the inverse of a symmetric matrix is symmetric.

23 If A is square and invertible, the warning against splitting $(A^{\mathrm{T}}A)^{-1}$ does not apply. It is true that $AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}} = I$. *When A is invertible, why is $P = I$?* ***What is the error e?***

24 The nullspace of A^{T} is _____ to the column space $\boldsymbol{C}(A)$. So if $A^{\mathrm{T}}\boldsymbol{b} = \mathbf{0}$, the projection of $\boldsymbol{b}$ onto $\boldsymbol{C}(A)$ should be $\boldsymbol{p} =$ _____. Check that $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ gives this answer.

25 The projection matrix P onto an n-dimensional subspace of $\mathbf{R}^m$ has rank $r = n$. ***Reason:*** The projections $P\boldsymbol{b}$ fill the subspace S. So S is the _____ of P.

26 If an m by m matrix has $A^2 = A$ and its rank is m, prove that $A = I$.

27 The important fact that ends the section is this: ***If*** $\boldsymbol{A^{\mathrm{T}}Ax} = \mathbf{0}$ ***then*** $\boldsymbol{Ax} = \mathbf{0}$. *New Proof*: The vector $A\boldsymbol{x}$ is in the nullspace of _____. $A\boldsymbol{x}$ is always in the column space of _____. To be in both of those perpendicular spaces, $A\boldsymbol{x}$ must be zero.

28 Use $P^{\mathrm{T}} = P$ and $P^2 = P$ to prove that the length squared of column 2 always equals the diagonal entry P_{22}. This number is $\frac{2}{6} = \frac{4}{36} + \frac{4}{36} + \frac{4}{36}$ for

$$P = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}.$$

29 If B has rank m (full row rank, independent rows) show that BB^{T} is invertible.

Challenge Problems

30 (a) Find the projection matrix P_C onto the column space of A (after looking closely at the matrix!)

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}$$

(b) Find the 3 by 3 projection matrix P_R onto the row space of A. Multiply $B = P_C A P_R$. Your answer B should be a little surprising—can you explain it?

31 In $\mathbf{R}^m$, suppose I give you $\boldsymbol{b}$ and also a combination $\boldsymbol{p}$ of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$. How would you test to see if $\boldsymbol{p}$ is the projection of $\boldsymbol{b}$ onto the subspace spanned by the $\boldsymbol{a}$'s?

32 Suppose P_1 is the projection matrix onto the 1-dimensional subspace spanned by the first column of A. Suppose P_2 is the projection matrix onto the 2-dimensional column space of A. After thinking a little, compute the product P_2P_1.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

33 Suppose you know the average $\widehat{x}_{\text{old}}$ of $b_1, b_2, \ldots, b_{999}$. When b_{1000} arrives, check that the new average is a combination of $\widehat{x}_{\text{old}}$ and the mismatch $b_{1000} - \widehat{x}_{\text{old}}$:

$$\widehat{\boldsymbol{x}}_{\mathbf{new}} = \frac{b_1 + \cdots + b_{1000}}{1000} = \frac{b_1 + \cdots + b_{999}}{999} + \frac{1}{1000}\left(b_{1000} - \frac{b_1 + \cdots + b_{999}}{999}\right).$$

This is a **"Kalman filter"** $\widehat{x}_{\text{new}} = \widehat{x}_{\text{old}} + \frac{1}{1000}(b_{1000} - \widehat{x}_{\text{old}})$ with gain matrix $\frac{1}{1000}$. The last page of the book extends the Kalman filter to matrix updates.

34 (2017) Suppose P_1 and P_2 are projection matrices ($P_i^2 = P_i = P_i^{\mathrm{T}}$). Prove this fact:

$$P_1 P_2 \text{ is a projection matrix if and only if } P_1 P_2 = P_2 P_1.$$

4.3 Least Squares Approximations

1 Solving $\boxed{A^{\mathrm{T}} A \widehat{\boldsymbol{x}} = A^{\mathrm{T}} \boldsymbol{b}}$ gives the projection $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$ of $\boldsymbol{b}$ onto the column space of A.

2 When $A\boldsymbol{x} = \boldsymbol{b}$ has no solution, $\widehat{\boldsymbol{x}}$ is the "least-squares solution": $||\boldsymbol{b} - A\widehat{\boldsymbol{x}}||^2 =$ minimum.

3 Setting partial derivatives of $E = ||A\boldsymbol{x} - \boldsymbol{b}||^2$ to zero $\left(\frac{\partial E}{\partial x_i} = 0\right)$ also produces $A^{\mathrm{T}} A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

4 To fit points $(t_1, b_1), \ldots, (t_m, b_m)$ by a straight line, A has columns $(1, \ldots, 1)$ and $(t_1, \ldots, t_m)$.

5 In that case $A^{\mathrm{T}} A$ is the 2 by 2 matrix $\begin{bmatrix} m & \Sigma\, t_i \\ \Sigma\, t_i & \Sigma\, t_i^2 \end{bmatrix}$ and $A^{\mathrm{T}}\boldsymbol{b}$ is the vector $\begin{bmatrix} \Sigma\, b_i \\ \Sigma\, t_i b_i \end{bmatrix}$.

It often happens that $A\boldsymbol{x} = \boldsymbol{b}$ has no solution. The usual reason is: *too many equations*. The matrix A has more rows than columns. There are more equations than unknowns (m is greater than n). The n columns span a small part of m-dimensional space. Unless all measurements are perfect, $\boldsymbol{b}$ is outside that column space of A. Elimination reaches an impossible equation and stops. But we can't stop just because measurements include noise!

To repeat: We cannot always get the error $\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{x}$ down to zero. When $\boldsymbol{e}$ is zero, $\boldsymbol{x}$ is an exact solution to $A\boldsymbol{x} = \boldsymbol{b}$. *When the length of* $\boldsymbol{e}$ *is as small as possible,* $\widehat{\boldsymbol{x}}$ *is a least squares solution.* Our goal in this section is to compute $\widehat{\boldsymbol{x}}$ and use it. These are real problems and they need an answer.

The previous section emphasized $\boldsymbol{p}$ (the projection). This section emphasizes $\widehat{\boldsymbol{x}}$ (the least squares solution). They are connected by $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$. The fundamental equation is still $A^{\mathrm{T}} A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. Here is a short unofficial way to reach this "*normal equation*":

When $A\boldsymbol{x} = \boldsymbol{b}$ has no solution, multiply by A^{T} and solve $A^{\mathrm{T}} A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

Example 1 A crucial application of least squares is fitting a straight line to m points. Start with three points: *Find the closest line to the points* $(0, 6)$, $(1, 0)$, *and* $(2, 0)$.

No straight line $b = C + Dt$ goes through those three points. We are asking for two numbers C and D that satisfy three equations: $n = 2$ and $m = 3$. Here are the three equations at $t = 0, 1, 2$ to match the given values $b = 6, 0, 0$:

$\boldsymbol{t = 0}$	The first point is on the line $b = C + Dt$ if	$C + D \cdot 0 = 6$
$\boldsymbol{t = 1}$	The second point is on the line $b = C + Dt$ if	$C + D \cdot 1 = 0$
$\boldsymbol{t = 2}$	The third point is on the line $b = C + Dt$ if	$C + D \cdot 2 = 0.$

This 3 by 2 system has *no solution*: $\boldsymbol{b} = (6, 0, 0)$ is not a combination of the columns $(1, 1, 1)$ and $(0, 1, 2)$. Read off A, $\boldsymbol{x}$, and $\boldsymbol{b}$ from those equations:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} C \\ D \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \quad A\boldsymbol{x} = \boldsymbol{b} \text{ is } \textit{not} \text{ solvable.}$$

The same numbers were in Example 3 in the last section. We computed $\widehat{\boldsymbol{x}} = (5, -3)$. **Those numbers are the best C and D, so $5 - 3t$ will be the best line for the 3 points.** We must connect projections to least squares, by explaining why $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

In practical problems, there could easily be $m = 100$ points instead of $m = 3$. They don't exactly match any straight line $C + Dt$. Our numbers $6, 0, 0$ exaggerate the error so you can see e_1, e_2, and e_3 in Figure 4.6.

Minimizing the Error

How do we make the error $\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{x}$ as small as possible? This is an important question with a beautiful answer. The best $\boldsymbol{x}$ (called $\widehat{\boldsymbol{x}}$) can be found by geometry (the error $\boldsymbol{e}$ meets the column space of A at $90°$) and by algebra: $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. Calculus gives the same $\widehat{\boldsymbol{x}}$: the derivative of the error $||A\boldsymbol{x} - \boldsymbol{b}||^2$ is zero at $\widehat{\boldsymbol{x}}$.

By geometry Every $A\boldsymbol{x}$ lies in the plane of the columns $(1, 1, 1)$ and $(0, 1, 2)$. In that plane, we look for the point closest to $\boldsymbol{b}$. *The nearest point is the projection $\boldsymbol{p}$.*

The best choice for $A\widehat{\boldsymbol{x}}$ is $\boldsymbol{p}$. The smallest possible error is $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$, perpendicular to the columns. *The three points at heights (p_1, p_2, p_3) do lie on a line*, because $\boldsymbol{p}$ is in the column space of A. In fitting a straight line, $\widehat{\boldsymbol{x}}$ is the best choice for (C, D).

By algebra Every vector $\boldsymbol{b}$ splits into two parts. The part in the column space is $\boldsymbol{p}$. The perpendicular part is $\boldsymbol{e}$. There is an equation we cannot solve ($A\boldsymbol{x} = \boldsymbol{b}$). There is an equation $A\widehat{\boldsymbol{x}} = \boldsymbol{p}$ we can and do solve (by removing $\boldsymbol{e}$ and solving $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$):

$$A\boldsymbol{x} = \boldsymbol{b} = \boldsymbol{p} + \boldsymbol{e} \quad \text{is impossible} \qquad A\widehat{\boldsymbol{x}} = \boldsymbol{p} \quad \text{is solvable} \qquad \widehat{\boldsymbol{x}} \quad \text{is } (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}. \tag{1}$$

The solution to $A\widehat{\boldsymbol{x}} = \boldsymbol{p}$ leaves the least possible error (which is $\boldsymbol{e}$):

$$\textbf{Squared length for any } x \qquad ||A\boldsymbol{x} - \boldsymbol{b}||^2 = ||A\boldsymbol{x} - \boldsymbol{p}||^2 + ||\boldsymbol{e}||^2. \tag{2}$$

This is the law $c^2 = a^2 + b^2$ for a right triangle. The vector $A\boldsymbol{x} - \boldsymbol{p}$ in the column space is perpendicular to $\boldsymbol{e}$ in the left nullspace. We reduce $A\boldsymbol{x} - \boldsymbol{p}$ to **zero** by choosing $\boldsymbol{x} = \widehat{\boldsymbol{x}}$. That leaves the smallest possible error $\boldsymbol{e} = (e_1, e_2, e_3)$ which we can't reduce.

Notice what "smallest" means. The *squared length* of $A\boldsymbol{x} - \boldsymbol{b}$ is minimized:

The least squares solution $\widehat{\boldsymbol{x}}$ makes $E = \|\|A\boldsymbol{x} - \boldsymbol{b}\|\|^2$ as small as possible.

Figure 4.6a shows the closest line. It misses by distances $e_1, e_2, e_3 = 1, -2, 1$. *Those are vertical distances*. The least squares line minimizes $E = e_1^2 + e_2^2 + e_3^2$.

Figure 4.6b shows the same problem in 3-dimensional space ($\boldsymbol{b}\,\boldsymbol{p}\,\boldsymbol{e}$ space). The vector $\boldsymbol{b}$ is not in the column space of A. That is why we could not solve $A\boldsymbol{x} = \boldsymbol{b}$. No line goes through the three points. The smallest possible error is the perpendicular vector $\boldsymbol{e}$. This is $\boldsymbol{e} = \boldsymbol{b} - A\widehat{\boldsymbol{x}}$, the vector of errors $(1, -2, 1)$ in the three equations. Those are the distances from the best line. Behind both figures is the fundamental equation $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

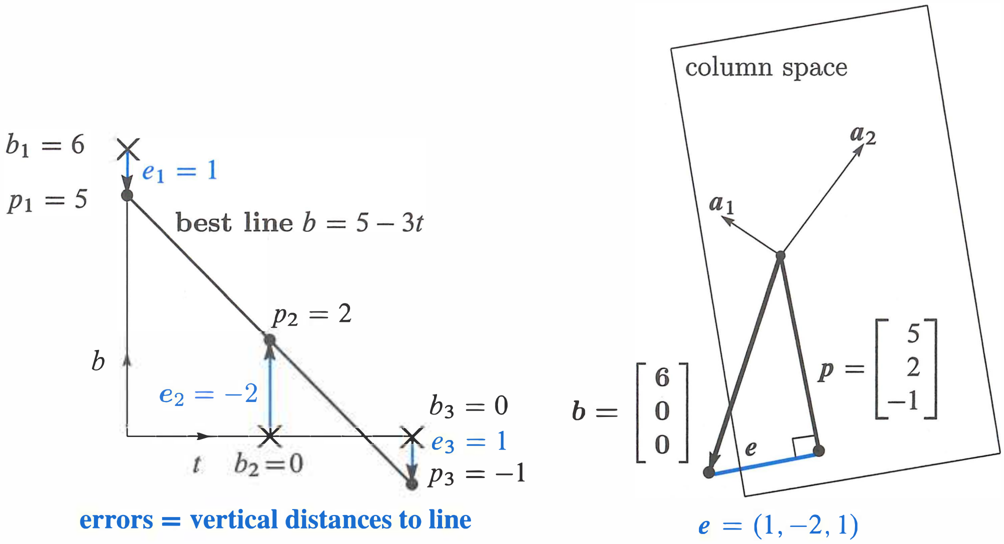


Figure 4.6: **Best line and projection: Two pictures, same problem.** The line has heights $\boldsymbol{p} = (5, 2, -1)$ with errors $\boldsymbol{e} = (1, -2, 1)$. The equations $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ give $\widehat{\boldsymbol{x}} = (5, -3)$. Same answer! The best line is $b = 5 - 3t$ and the closest point is $\boldsymbol{p} = 5\boldsymbol{a}_1 - 3\boldsymbol{a}_2$.

Notice that the errors $1, -2, 1$ add to zero. *Reason*: The error $\boldsymbol{e} = (e_1, e_2, e_3)$ is perpendicular to the first column $(1, 1, 1)$ in A. The dot product gives $e_1 + e_2 + e_3 = 0$.

By calculus Most functions are minimized by calculus! The graph bottoms out and the derivative in every direction is zero. Here the error function E to be minimized is a *sum of squares* $e_1^2 + e_2^2 + e_3^2$ (the square of the error in each equation):

$$E = \|A\boldsymbol{x} - \boldsymbol{b}\|^2 = (C + D \cdot 0 - 6)^2 + (C + D \cdot 1)^2 + (C + D \cdot 2)^2. \qquad (3)$$

The unknowns are C and D. With two unknowns there are *two derivatives*—both zero at the minimum. They are "partial derivatives" because $\partial E/\partial C$ treats D as constant and $\partial E/\partial D$ treats C as constant:

$$\partial E/\partial C = 2(C + D \cdot 0 - 6) \quad + 2(C + D \cdot 1) \quad + 2(C + D \cdot 2) \quad = 0$$

$$\partial E/\partial D = 2(C + D \cdot 0 - 6)(\mathbf{0}) + 2(C + D \cdot 1)(\mathbf{1}) + 2(C + D \cdot 2)(\mathbf{2}) = 0.$$

$\partial E/\partial D$ contains the extra factors $\mathbf{0}, \mathbf{1}, \mathbf{2}$ from the chain rule. (The last derivative from $(C + 2D)^2$ was 2 times $C + 2D$ times that extra 2.) Those factors are just $\mathbf{1}, \mathbf{1}, \mathbf{1}$ in $\partial E/\partial C$.

It is no accident that those factors 1, 1, 1 and 0, 1, 2 in the derivatives of $||Ax - b||^2$ are the columns of A. Now cancel 2 from every term and collect all C's and all D's:

$$\begin{array}{ll}\text{The } C \text{ derivative is zero:} & 3C + 3D = 6 \\ \text{The } D \text{ derivative is zero:} & 3C + 5D = 0\end{array} \qquad \textbf{This matrix} \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \textbf{ is } \boldsymbol{A^{\mathrm{T}}A} \tag{4}$$

These equations are identical with $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. The best C and D are the components of $\widehat{\boldsymbol{x}}$. The equations from calculus are the same as the "normal equations" from linear algebra. These are the key equations of least squares:

The partial derivatives of $\|A\boldsymbol{x} - \boldsymbol{b}\|^2$ ***are zero when*** $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

The solution is $C = 5$ and $D = -3$. Therefore $b = 5 - 3t$ is the best line—it comes closest to the three points. At $t = 0, 1, 2$ this line goes through $\boldsymbol{p} = 5, 2, -1$. It could not go through $\boldsymbol{b} = 6, 0, 0$. The errors are $1, -2, 1$. This is the vector $\boldsymbol{e}$!

The Big Picture for Least Squares

The key figure of this book shows the four subspaces and the true action of a matrix. The vector $\boldsymbol{x}$ on the left side of Figure 4.3 went to $\boldsymbol{b} = A\boldsymbol{x}$ on the right side. In that figure $\boldsymbol{x}$ was split into $\boldsymbol{x}_r + \boldsymbol{x}_n$. There were *many* solutions to $A\boldsymbol{x} = \boldsymbol{b}$.

In this section the situation is just the opposite. There are *no* solutions to $A\boldsymbol{x} = \boldsymbol{b}$. *Instead of splitting up* $\boldsymbol{x}$ *we are splitting up* $\boldsymbol{b} = \boldsymbol{p} + \boldsymbol{e}$. Figure 4.7 shows the big picture for least squares. Instead of $A\boldsymbol{x} = \boldsymbol{b}$ we solve $A\widehat{\boldsymbol{x}} = \boldsymbol{p}$. The error $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ is unavoidable.

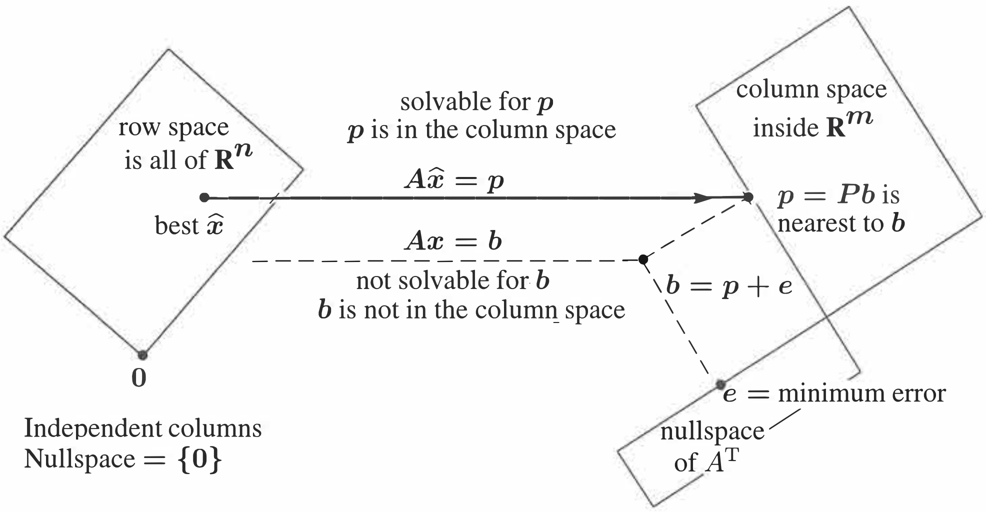


Figure 4.7: The projection $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$ is closest to $\boldsymbol{b}$, so $\widehat{\boldsymbol{x}}$ minimizes $E = \|\boldsymbol{b} - A\boldsymbol{x}\|^2$.

Notice how the nullspace $\boldsymbol{N}(A)$ is very small—just one point. With independent columns, the only solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$. Then $A^{\mathrm{T}}A$ is invertible. The equation $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ fully determines the best vector $\widehat{\boldsymbol{x}}$. The error has $A^{\mathrm{T}}\boldsymbol{e} = \mathbf{0}$.

Chapter 7 will have the complete picture—all four subspaces included. Every $\boldsymbol{x}$ splits into $\boldsymbol{x}_r + \boldsymbol{x}_n$, and every $\boldsymbol{b}$ splits into $\boldsymbol{p} + \boldsymbol{e}$. The best solution is $\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{x}}_r$ in the row space. We can't help $\boldsymbol{e}$ and we don't want $\boldsymbol{x}_n$ from the nullspace—this leaves $A\widehat{\boldsymbol{x}} = \boldsymbol{p}$.

Fitting a Straight Line

Fitting a line is the clearest application of least squares. It starts with $m > 2$ points, hopefully near a straight line. At times $t_1, \ldots, t_m$ those m points are at heights $b_1, \ldots, b_m$. The best line $C + Dt$ misses the points by vertical distances $e_1, \ldots, e_m$. No line is perfect, and the least squares line minimizes $E = e_1^2 + \cdots + e_m^2$.

The first example in this section had three points in Figure 4.6. Now we allow m points (and m can be large). The two components of $\widehat{\boldsymbol{x}}$ are still C and D.

A line goes through the m points when we exactly solve $A\boldsymbol{x} = \boldsymbol{b}$. Generally we can't do it. Two unknowns C and D determine a line, so A has only $n = 2$ columns. To fit the m points, we are trying to solve m equations (and we only have two unknowns!).

$$A\boldsymbol{x} = \boldsymbol{b} \quad \text{is} \quad \begin{array}{c} C + Dt_1 = b_1 \\ C + Dt_2 = b_2 \\ \vdots \\ C + Dt_m = b_m \end{array} \quad \text{with} \quad A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix}. \tag{5}$$

The column space is so thin that almost certainly $\boldsymbol{b}$ is outside of it. When $\boldsymbol{b}$ happens to lie in the column space, the points happen to lie on a line. In that case $\boldsymbol{b} = \boldsymbol{p}$. Then $A\boldsymbol{x} = \boldsymbol{b}$ is solvable and the errors are $\boldsymbol{e} = (0, \ldots, 0)$.

The closest line $C + Dt$ has heights $p_1, \ldots, p_m$ with errors $e_1, \ldots, e_m$.

Solve $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ for $\widehat{\boldsymbol{x}} = (C, D)$. The errors are $e_i = b_i - C - Dt_i$.

Fitting points by a straight line is so important that we give the two equations $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$, once and for all. The two columns of A are independent (unless all times t_i are the same). So we turn to least squares and solve $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

$$\textbf{Dot-product matrix} \quad \boldsymbol{A^{\mathrm{T}}A} = \begin{bmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_m \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} = \begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix}. \tag{6}$$

On the right side of the normal equation is the 2 by 1 vector $A^{\mathrm{T}}\boldsymbol{b}$:

$$A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_m \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix}. \tag{7}$$

In a specific problem, these numbers are given. The best $\widehat{\boldsymbol{x}} = (C, D)$ is $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}$.

The line $C + Dt$ minimizes $e_1^2 + \cdots + e_m^2 = \|A\boldsymbol{x} - \boldsymbol{b}\|^2$ when $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$:

$$A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b} \qquad \begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix}. \tag{8}$$

The vertical errors at the m points on the line are the components of $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$. This error vector (the *residual*) $\boldsymbol{b} - A\widehat{\boldsymbol{x}}$ is perpendicular to the columns of A (geometry). The error is in the nullspace of A^{T} (linear algebra). The best $\widehat{\boldsymbol{x}} = (C, D)$ minimizes the total error E, the sum of squares (calculus):

$$E(\boldsymbol{x}) = \|A\boldsymbol{x} - \boldsymbol{b}\|^2 = (C + Dt_1 - b_1)^2 + \cdots + (C + Dt_m - b_m)^2.$$

Calculus sets the derivatives $\partial E/\partial C$ and $\partial E/\partial D$ to zero, and produces $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

Other least squares problems have more than two unknowns. Fitting by the best parabola has $n = 3$ coefficients C, D, E (see below). In general we are fitting m data points by n parameters $x_1, \ldots, x_n$. The matrix A has n columns and $n < m$. The derivatives of $\|A\boldsymbol{x} - \boldsymbol{b}\|^2$ give the n equations $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. **The derivative of a square is linear.** This is why the method of least squares is so popular.

Example 2 A has *orthogonal columns* when the measurement times t_i add to zero.

Suppose $b = 1, 2, 4$ at times $t = -2, 0, 2$. *Those times add to zero*. The columns of A have *zero dot product*: $(1, 1, 1)$ is orthogonal to $(-2, 0, 2)$:

$$\begin{aligned} C + D(-2) &= 1 \\ C + \quad D(0) &= 2 \\ C + \quad D(2) &= 4 \end{aligned} \qquad \text{or} \qquad A\boldsymbol{x} = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

When the columns of A are orthogonal, $A^{\mathrm{T}}A$ will be a diagonal matrix:

$$A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b} \qquad \text{is} \qquad \begin{bmatrix} 3 & \mathbf{0} \\ \mathbf{0} & 8 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}. \tag{9}$$

Main point: Since $A^{\mathrm{T}}A$ *is diagonal*, we can solve separately for $C = \frac{7}{3}$ and $D = \frac{6}{8}$. The zeros in $A^{\mathrm{T}}A$ are dot products of perpendicular columns in A. The diagonal matrix $A^{\mathrm{T}}A$, with entries $m = 3$ and $t_1^2 + t_2^2 + t_3^2 = 8$, is virtually as good as the identity matrix.

Orthogonal columns are so helpful that it is worth *shifting the times by subtracting the average time* $\widehat{t} = (t_1 + \cdots + t_m)/m$. If the original times were $1, 3, 5$ then their average is $\widehat{t} = 3$. The shifted times $T = t - \widehat{t} = t - 3$ add up to zero!

$$\begin{aligned} T_1 &= 1 - 3 = -2 \\ T_2 &= 3 - 3 = 0 \\ T_3 &= 5 - 3 = 2 \end{aligned} \qquad A_{\text{new}} = \begin{bmatrix} 1 & T_1 \\ 1 & T_2 \\ 1 & T_3 \end{bmatrix} \qquad A_{\text{new}}^{\mathrm{T}}A_{\text{new}} = \begin{bmatrix} 3 & \mathbf{0} \\ \mathbf{0} & 8 \end{bmatrix}.$$

Now C and D come from the easy equation (9). Then the best straight line uses $C + DT$ which is $C + D(t - \widehat{t}\,) = C + D(t - 3)$. Problem 30 even gives a formula for C and D.

That was a perfect example of the "Gram-Schmidt idea" coming in the next section: *Make the columns orthogonal in advance*. Then $A_{\text{new}}^{\mathrm{T}} A_{\text{new}}$ is diagonal and $\widehat{\boldsymbol{x}}_{\text{new}}$ is easy.

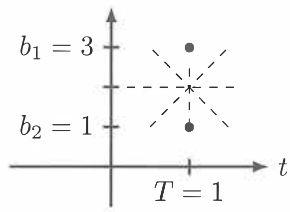
Dependent Columns in A: What is $\widehat{\boldsymbol{x}}$?

From the start, this chapter has assumed independent columns in A. Then $A^{\mathrm{T}}A$ is invertible. Then $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ produces the least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$.

Which $\widehat{\boldsymbol{x}}$ is best if A has *dependent columns*? Here is a specific example.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \boldsymbol{b} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \boldsymbol{p}$$

$$A\boldsymbol{x} = \boldsymbol{b} \qquad\qquad A\widehat{\boldsymbol{x}} = \boldsymbol{p}$$



The measurements $b_1 = 3$ and $b_2 = 1$ are at the same time T! A straight line $C + Dt$ cannot go through both points. I think we are right to project $\boldsymbol{b} = (3, 1)$ to $\boldsymbol{p} = (2, 2)$ in the column space of A. That changes the equation $A\boldsymbol{x} = \boldsymbol{b}$ to the equation $A\widehat{\boldsymbol{x}} = \boldsymbol{p}$. An equation with no solution has become an equation with infinitely many solutions. The problem is that A has dependent columns and $(1, -1)$ is in its nullspace.

Which solution $\widehat{\boldsymbol{x}}$ should we choose? All the dashed lines in the figure have the same two errors 1 and -1 at time T. Those errors $(1, -1) = \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ are as small as possible. But this doesn't tell us which dashed line is best.

My instinct is to go for the horizontal line at height 2. If the equation for the best line is $b = C + Dt$, then my choice will have $\widehat{x}_1 = C = 2$ and $\widehat{x}_2 = D = 0$. But what if the line had been written as $b = ct + d$? This is equally correct (just reversing C and D). Now the horizontal line has $\widehat{x}_1 = c = 0$ and $\widehat{x}_2 = d = 2$. I don't see any way out.

In Section 7.4, the *"pseudoinverse" of* A will choose the **shortest solution to** $A\widehat{\boldsymbol{x}} = \boldsymbol{p}$. Here, that shortest solution will be $\boldsymbol{x}^+ = (1, 1)$. This is the particular solution in the row space of A, and $\boldsymbol{x}^+$ has length $\sqrt{2}$. (Both solutions $\widehat{\boldsymbol{x}} = (2, 0)$ and $(0, 2)$ have length 2.) We are arbitrarily choosing the nullspace component of the solution $\boldsymbol{x}^+$ to be zero.

When A has independent columns, the nullspace only contains the zero vector and the pseudoinverse is our usual left inverse $L = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. When I write it that way, the pseudoinverse sounds like the best way to choose $\boldsymbol{x}$.

Comment MATLAB experiments with singular matrices produced either **Inf** or **NaN** (Not a Number) or $\mathbf{10^{16}}$ (a bad number). There is a warning in every case! I believe that **Inf** and **NaN** and $\mathbf{10^{16}}$ come from the possibilities $\mathbf{0}\boldsymbol{x} = \boldsymbol{b}$ and $\mathbf{0}\boldsymbol{x} = \mathbf{0}$ and $\mathbf{10^{-16}}\,\boldsymbol{x} = \mathbf{1}$.

Those are three small examples of three big difficulties: singular with no solution, singular with many solutions, and very very close to singular.

Fitting by a Parabola

If we throw a ball, it would be crazy to fit the path by a straight line. A parabola $b = C + Dt + Et^2$ allows the ball to go up and come down again (b is the height at time t). The actual path is not a perfect parabola, but the whole theory of projectiles starts with that approximation.

When Galileo dropped a stone from the Leaning Tower of Pisa, it accelerated. The distance contains a quadratic term $\frac{1}{2}gt^2$. (Galileo's point was that the stone's mass is not involved.) Without that t^2 term we could never send a satellite into its orbit. But even with a nonlinear function like t^2, the unknowns C, D, E still appear linearly! Fitting points by the best parabola is still a problem in linear algebra.

Problem Fit heights $b_1, \ldots, b_m$ at times $t_1, \ldots, t_m$ by a parabola $C + Dt + Et^2$.

Solution With $m > 3$ points, the m equations for an exact fit are generally unsolvable:

$$\begin{aligned} C + Dt_1 + Et_1^2 &= b_1 \\ &\vdots \\ C + Dt_m + Et_m^2 &= b_m \end{aligned} \qquad \begin{array}{l}\text{is } A\boldsymbol{x} = \boldsymbol{b} \text{ with} \\ \text{the } m \text{ by 3 matrix}\end{array} \qquad A = \begin{bmatrix} 1 & t_1 & t_1^2 \\ \vdots & \vdots & \vdots \\ 1 & t_m & t_m^2 \end{bmatrix}. \tag{10}$$

Least squares The closest parabola $C + Dt + Et^2$ chooses $\widehat{\boldsymbol{x}} = (C, D, E)$ to satisfy the three normal equations $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

May I ask you to convert this to a problem of projection? The column space of A has dimension ____. The projection of $\boldsymbol{b}$ is $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$, which combines the three columns using the coefficients C, D, E. The error at the first data point is $e_1 = b_1 - C - Dt_1 - Et_1^2$. The total squared error is $e_1^2 +$ ____. If you prefer to minimize by calculus, take the partial derivatives of E with respect to ____, ____, ____. These three derivatives will be zero when $\widehat{\boldsymbol{x}} = (C, D, E)$ solves the 3 by 3 system of equations $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

Section 10.5 has more least squares applications. The big one is Fourier series—approximating functions instead of vectors. The function to be minimized changes from a sum of squared errors $e_1^2 + \cdots + e_m^2$ to an integral of the squared error.

Example 3 For a parabola $b = C + Dt + Et^2$ to go through the three heights $b = 6, 0, 0$ when $t = 0, 1, 2$, the equations for C, D, E are

$$\begin{aligned} C + D \cdot 0 + E \cdot 0^2 &= 6 \\ C + D \cdot 1 + E \cdot 1^2 &= 0 \\ C + D \cdot 2 + E \cdot 2^2 &= 0. \end{aligned} \tag{11}$$

This is $A\boldsymbol{x} = \boldsymbol{b}$. We can solve it exactly. Three data points give three equations and a square matrix. The solution is $\boldsymbol{x} = (C, D, E) = (\mathbf{6}, \mathbf{-9}, \mathbf{3})$. The parabola through the three points in Figure 4.8a is $b = \mathbf{6 - 9}\boldsymbol{t} + \mathbf{3}\boldsymbol{t}^{\mathbf{2}}$.

What does this mean for projection? The matrix has three columns, which span the whole space $\mathbf{R}^3$. The projection matrix is the identity. The projection of $\boldsymbol{b}$ is $\boldsymbol{b}$. The error is zero. We didn't need $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$, because we solved $A\boldsymbol{x} = \boldsymbol{b}$. Of course we could multiply by A^{T}, but there is no reason to do it.

Figure 4.8 also shows a fourth point b_4 at time t_4. If that falls on the parabola, the new $A\boldsymbol{x} = \boldsymbol{b}$ (four equations) is still solvable. When the fourth point is not on the parabola, we turn to $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. Will the least squares parabola stay the same, with all the error at the fourth point? Not likely!

Least squares balances the four errors to get three equations for C, D, E.

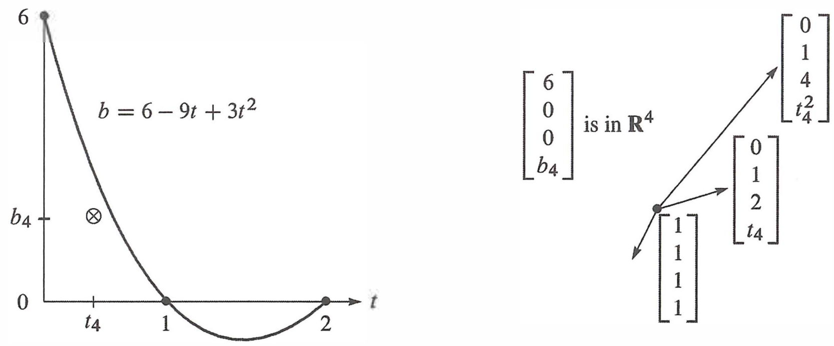


Figure 4.8: An exact fit of the parabola at $t = 0, 1, 2$ means that $\boldsymbol{p} = \boldsymbol{b}$ and $\boldsymbol{e} = \boldsymbol{0}$. The fourth point ⓧ off the parabola makes $m > n$ and we need least squares: project $\boldsymbol{b}$ on $\boldsymbol{C}(\boldsymbol{A})$. The figure on the right shows $\boldsymbol{b}$—not a combination of the three columns of A.

■ REVIEW OF THE KEY IDEAS ■

1. The least squares solution $\widehat{\boldsymbol{x}}$ minimizes $\|A\boldsymbol{x} - \boldsymbol{b}\|^2 = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{x} - 2\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{b} + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b}$. This is E, the sum of squares of the errors in the m equations $(m > n)$.

2. The best $\widehat{\boldsymbol{x}}$ comes from the normal equations $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. E is a minimum.

3. To fit m points by a line $b = C + Dt$, the normal equations give C and D.

4. The heights of the best line are $\boldsymbol{p} = (p_1, \ldots, p_m)$. The vertical distances to the data points are the errors $\boldsymbol{e} = (e_1, \ldots, e_m)$. A key equation is $A^{\mathrm{T}}\boldsymbol{e} = \boldsymbol{0}$.

5. If we try to fit m points by a combination of $n < m$ functions, the m equations $A\boldsymbol{x} = \boldsymbol{b}$ are generally unsolvable. The n equations $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ give the least squares solution—the combination with smallest MSE (mean square error).

■ WORKED EXAMPLES ■

4.3 A Start with nine measurements b_1 to b_9, *all zero*, at times $t = 1, \ldots, 9$. The tenth measurement $b_{10} = 40$ is an outlier. Find the **best horizontal line** $\boldsymbol{y = C}$ to fit the ten points $(1,0), (2,0), \ldots, (9,0), (10,40)$ using three options for the error E:

(1) Least *squares* $E_2 = e_1^2 + \cdots + e_{10}^2$ (then the normal equation for C is linear)

(2) Least *maximum* error $E_\infty = |e_{\max}|$ **(3)** Least *sum* of errors $E_1 = |e_1| + \cdots + |e_{10}|$.

Solution **(1)** The least squares fit to $0, 0, \ldots, 0, 40$ by a horizontal line is $\boldsymbol{C = 4}$:

$$A = \text{column of 1's} \quad A^{\mathrm{T}}A = 10 \quad A^{\mathrm{T}}\boldsymbol{b} = \text{sum of } b_i = 40. \quad \text{So } 10\,C = 40.$$

(2) The least maximum error requires $\boldsymbol{C = 20}$, halfway between 0 and 40.

(3) The least sum requires $\boldsymbol{C = 0}$ (!!). The sum of errors $9|C| + |40 - C|$ would increase if C moves up from zero.

The least sum comes from the *median* measurement (the median of $0, \ldots, 0, 40$ is zero). Many statisticians feel that the least squares solution is too heavily influenced by outliers like $b_{10} = 40$, and they prefer least sum. But the equations become *nonlinear*.

Now find the least squares line $C + Dt$ through those ten points $(1,0)$ to $(10,40)$:

$$A^{\mathrm{T}}A = \begin{bmatrix} 10 & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix} = \begin{bmatrix} 10 & 55 \\ 55 & 385 \end{bmatrix} \qquad A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix} = \begin{bmatrix} 40 \\ 400 \end{bmatrix}$$

Those come from equation (8). Then $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ gives $C = -8$ and $D = 24/11$.

What happens to C and D if you multiply $\boldsymbol{b} = (0, 0, \ldots, 40)$ by 3 and then add 30 to get $\boldsymbol{b}_{\text{new}} = (30, 30, \ldots, 150)$? Linearity allows us to rescale $\boldsymbol{b}$. Multiplying $\boldsymbol{b}$ by 3 will multiply C and D by 3. Adding 30 to all b_i will add 30 to C.

4.3 B Find the parabola $C + Dt + Et^2$ that comes closest (least squares error) to the values $\boldsymbol{b} = (0, 0, 1, 0, 0)$ at the times $t = -2, -1, 0, 1, 2$. First write down the five equations $A\boldsymbol{x} = \boldsymbol{b}$ in three unknowns $\boldsymbol{x} = (C, D, E)$ for a parabola to go through the five points. No solution because no such parabola exists. Solve $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

I would predict $D = 0$. Why should the best parabola be symmetric around $t = 0$? In $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$, equation 2 for D should uncouple from equations 1 and 3.

Solution The five equations $A\boldsymbol{x} = \boldsymbol{b}$ have a rectangular *Vandermonde matrix* A:

$$\begin{array}{l} C + D\,(-2) + E\,(-2)^2 = 0 \\ C + D\,(-1) + E\,(-1)^2 = 0 \\ C + D\quad(0) + E\quad(0)^2 = 1 \\ C + D\quad(1) + E\quad(1)^2 = 0 \\ C + D\quad(2) + E\quad(2)^2 = 0 \end{array} \qquad A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \qquad A^{\mathrm{T}}A = \begin{bmatrix} 5 & \mathbf{0} & 10 \\ \mathbf{0} & 10 & \mathbf{0} \\ 10 & \mathbf{0} & 34 \end{bmatrix}$$

Those zeros in $A^{\mathrm{T}}A$ mean that column 2 of A is orthogonal to columns 1 and 3. We see this directly in A (the times $-2,-1,0,1,2$ are symmetric). The best C, D, E in the parabola $C + Dt + Et^2$ come from $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$, and D is uncoupled from C and E:

$$\begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{leads to} \quad \begin{array}{l} C = 34/70 \\ D = 0 \ \text{ as predicted} \\ E = -10/70 \end{array}$$

Problem Set 4.3

Problems 1–11 use four data points $\boldsymbol{b} = (0, 8, 8, 20)$ to bring out the key ideas.

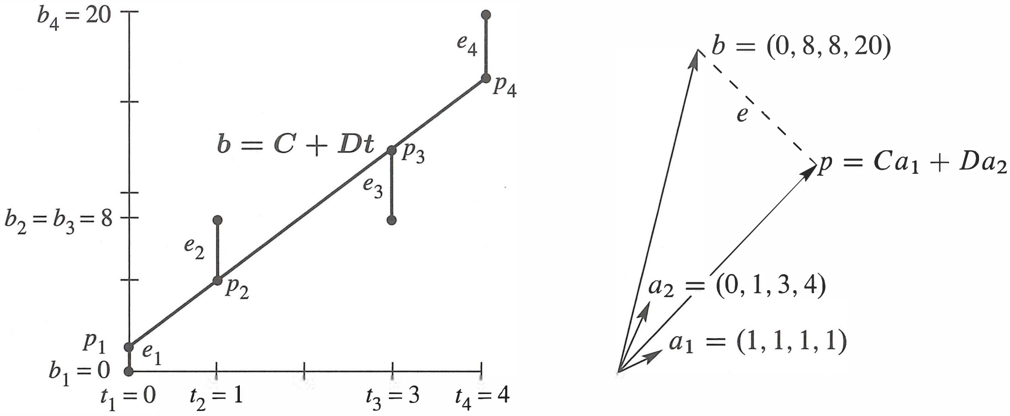


Figure 4.9: **Problems 1–11**: The closest line $C + Dt$ matches $C\boldsymbol{a}_1 + D\boldsymbol{a}_2$ in $\mathbf{R}^4$.

1 With $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$, set up and solve the normal equations $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. For the best straight line in Figure 4.9a, find its four heights p_i and four errors e_i. What is the minimum value $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$?

2 (Line $C + Dt$ does go through p's) With $b = 0, 8, 8, 20$ at times $t = 0, 1, 3, 4$, write down the four equations $A\boldsymbol{x} = \boldsymbol{b}$ (unsolvable). Change the measurements to $p = 1, 5, 13, 17$ and find an exact solution to $A\widehat{\boldsymbol{x}} = \boldsymbol{p}$.

3 Check that $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = (-1, 3, -5, 3)$ is perpendicular to both columns of the same matrix A. What is the shortest distance $\|\boldsymbol{e}\|$ from $\boldsymbol{b}$ to the column space of A?

4 (By calculus) Write down $E = \|A\boldsymbol{x} - \boldsymbol{b}\|^2$ as a sum of four squares—the last one is $(C + 4D - 20)^2$. Find the derivative equations $\partial E/\partial C = 0$ and $\partial E/\partial D = 0$. Divide by 2 to obtain the normal equations $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

5 Find the height C of the best *horizontal line* to fit $\boldsymbol{b} = (0, 8, 8, 20)$. An exact fit would solve the unsolvable equations $C = 0$, $C = 8$, $C = 8$, $C = 20$. Find the 4 by 1 matrix A in these equations and solve $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. Draw the horizontal line at height $\widehat{x} = C$ and the four errors in $\boldsymbol{e}$.

6 Project $\boldsymbol{b} = (0, 8, 8, 20)$ onto the line through $\boldsymbol{a} = (1, 1, 1, 1)$. Find $\widehat{x} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ and the projection $\boldsymbol{p} = \widehat{x}\boldsymbol{a}$. Check that $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ is perpendicular to $\boldsymbol{a}$, and find the shortest distance $\|\boldsymbol{e}\|$ from $\boldsymbol{b}$ to the line through $\boldsymbol{a}$.

7 Find the closest line $b = Dt$, *through the origin*, to the same four points. An exact fit would solve $D \cdot 0 = 0, D \cdot 1 = 8, D \cdot 3 = 8, D \cdot 4 = 20$. Find the 4 by 1 matrix and solve $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}\boldsymbol{b}$. Redraw Figure 4.9a showing the best line $b = Dt$ and the e's.

8 Project $\boldsymbol{b} = (0, 8, 8, 20)$ onto the line through $\boldsymbol{a} = (0, 1, 3, 4)$. Find $\widehat{x} = D$ and $\boldsymbol{p} = \widehat{x}\boldsymbol{a}$. The best C in Problems 5–6 and the best D in Problems 7–8 do *not* agree with the best (C, D) in Problems 1–4. That is because $(1, 1, 1, 1)$ and $(0, 1, 3, 4)$ are _____ perpendicular.

9 For the closest parabola $b = C + Dt + Et^2$ to the same four points, write down the unsolvable equations $A\boldsymbol{x} = \boldsymbol{b}$ in three unknowns $\boldsymbol{x} = (C, D, E)$. Set up the three normal equations $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ (solution not required). In Figure 4.9a you are now fitting a parabola to 4 points—what is happening in Figure 4.9b?

10 For the closest cubic $b = C + Dt + Et^2 + Ft^3$ to the same four points, write down the four equations $A\boldsymbol{x} = \boldsymbol{b}$. Solve them by elimination. In Figure 4.9a this cubic now goes exactly through the points. What are $\boldsymbol{p}$ and $\boldsymbol{e}$?

11 The average of the four times is $\widehat{t} = \frac{1}{4}(0 + 1 + 3 + 4) = 2$. The average of the four b's is $\widehat{b} = \frac{1}{4}(0 + 8 + 8 + 20) = 9$.

(a) Verify that the best line goes through the center point $(\widehat{t}, \widehat{b}) = (2, 9)$.

(b) Explain why $C + D\widehat{t} = \widehat{b}$ comes from the first equation in $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

Questions 12–16 introduce basic ideas of statistics—the foundation for least squares.

12 (Recommended) This problem projects $\boldsymbol{b} = (b_1, \ldots, b_m)$ onto the line through $\boldsymbol{a} = (1, \ldots, 1)$. We solve m equations $\boldsymbol{a}x = \boldsymbol{b}$ in 1 unknown (by least squares).

(a) Solve $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}\widehat{x} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}$ to show that $\widehat{x}$ is the *mean* (the average) of the $\boldsymbol{b}$'s.

(b) Find $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{a}\widehat{x}$ and the *variance* $\|\boldsymbol{e}\|^2$ and the *standard deviation* $\|\boldsymbol{e}\|$.

(c) The horizontal line $\widehat{\boldsymbol{b}} = 3$ is closest to $\boldsymbol{b} = (1, 2, 6)$. Check that $\boldsymbol{p} = (3, 3, 3)$ is perpendicular to $\boldsymbol{e}$ and find the 3 by 3 projection matrix P.

13 First assumption behind least squares: $A\boldsymbol{x} = \boldsymbol{b} -$ ***(noise e with mean zero)***. Multiply the error vectors $\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{x}$ by $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ to get $\widehat{\boldsymbol{x}} - \boldsymbol{x}$ on the right. The estimation errors $\widehat{\boldsymbol{x}} - \boldsymbol{x}$ also average to zero. The estimate $\widehat{\boldsymbol{x}}$ is *unbiased.*

14 Second assumption behind least squares: The m errors e_i are independent with variance σ^2, so the average of $(\boldsymbol{b} - A\boldsymbol{x})(\boldsymbol{b} - A\boldsymbol{x})^{\mathrm{T}}$ is $\sigma^2 I$. Multiply on the left by $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ and on the right by $A(A^{\mathrm{T}}A)^{-1}$ to show that the average matrix $(\widehat{\boldsymbol{x}} - \boldsymbol{x})(\widehat{\boldsymbol{x}} - \boldsymbol{x})^{\mathrm{T}}$ is $\sigma^2(A^{\mathrm{T}}A)^{-1}$. This is the ***covariance matrix*** W in Section 10.2.

15 A doctor takes 4 readings of your heart rate. The best solution to $x = b_1, \ldots, x = b_4$ is the average $\widehat{x}$ of $b_1, \ldots, b_4$. The matrix A is a column of 1's. Problem 14 gives the expected error $(\widehat{x} - x)^2$ as $\sigma^2(A^{\mathrm{T}}A)^{-1} =$ _____. ***By averaging, the variance drops from σ^2 to $\sigma^2/4$.***

16 If you know the average $\widehat{x}_9$ of 9 numbers $b_1, \ldots, b_9$, how can you quickly find the average $\widehat{x}_{10}$ with one more number b_{10}? The idea of *recursive* least squares is to avoid adding 10 numbers. What number multiplies $\widehat{x}_9$ in computing $\widehat{x}_{10}$?

$$\widehat{x}_{10} = \tfrac{1}{10}b_{10} + \underline{\qquad}\,\widehat{x}_9 = \tfrac{1}{10}(b_1 + \cdots + b_{10}) \quad \text{as in Worked Example 4.2 C.}$$

Questions 17–24 give more practice with $\widehat{x}$ and p and e.

17 Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1$, $b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\widehat{\boldsymbol{x}} = (C, D)$ and draw the closest line.

18 Find the projection $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$ in Problem 17. This gives the three heights of the closest line. Show that the error vector is $\boldsymbol{e} = (2, -6, 4)$. Why is $P\boldsymbol{e} = \boldsymbol{0}$?

19 Suppose the measurements at $t = -1, 1, 2$ are the errors $2, -6, 4$ in Problem 18. Compute $\widehat{\boldsymbol{x}}$ and the closest line to these new measurements. Explain the answer: $\boldsymbol{b} = (2, -6, 4)$ is perpendicular to _____ so the projection is $\boldsymbol{p} = \boldsymbol{0}$.

20 Suppose the measurements at $t = -1, 1, 2$ are $\boldsymbol{b} = (5, 13, 17)$. Compute $\widehat{\boldsymbol{x}}$ and the closest line and $\boldsymbol{e}$. The error is $\boldsymbol{e} = \boldsymbol{0}$ because this $\boldsymbol{b}$ is _____.

21 Which of the four subspaces contains the error vector $\boldsymbol{e}$? Which contains $\boldsymbol{p}$? Which contains $\widehat{\boldsymbol{x}}$? What is the nullspace of A?

22 Find the best line $C + Dt$ to fit $b = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

23 Is the error vector $\boldsymbol{e}$ orthogonal to $\boldsymbol{b}$ or $\boldsymbol{p}$ or $\boldsymbol{e}$ or $\widehat{\boldsymbol{x}}$? Show that $\|\boldsymbol{e}\|^2$ equals $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{b}$ which equals $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} - \boldsymbol{p}^{\mathrm{T}}\boldsymbol{b}$. This is the smallest total error E.

24 The partial derivatives of $\|A\boldsymbol{x}\|^2$ with respect to $x_1, \ldots, x_n$ fill the vector $2A^{\mathrm{T}}A\boldsymbol{x}$. The derivatives of $2\boldsymbol{b}^{\mathrm{T}}A\boldsymbol{x}$ fill the vector $2A^{\mathrm{T}}\boldsymbol{b}$. So the derivatives of $\|A\boldsymbol{x} - \boldsymbol{b}\|^2$ are zero when _____.

Challenge Problems

25 *What condition on $(t_1, b_1), (t_2, b_2), (t_3, b_3)$ puts those three points onto a straight line*? A column space answer is: (b_1, b_2, b_3) must be a combination of $(1,\ 1,\ 1)$ and $(t_1,\ t_2,\ t_3)$. Try to reach a specific equation connecting the t's and b's. I should have thought of this question sooner!

26 Find the *plane* that gives the best fit to the 4 values $\boldsymbol{b} = (0, 1, 3, 4)$ at the corners $(1, 0)$ and $(0, 1)$ and $(-1, 0)$ and $(0, -1)$ of a square. The equations $C + Dx + Ey = b$ at those 4 points are $A\boldsymbol{x} = \boldsymbol{b}$ with 3 unknowns $\boldsymbol{x} = (C, D, E)$. What is A? At the center $(0, 0)$ of the square, show that $C + Dx + Ey =$ average of the b's.

27 (Distance between lines) The points $P = (x, x, x)$ and $Q = (y, 3y, -1)$ are on two lines in space that don't meet. Choose x and y to minimize the squared distance $\|P - Q\|^2$. The line connecting the closest P and Q is perpendicular to ____ .

28 Suppose the columns of A are not independent. How could you find a matrix B so that $P = B(B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}$ does give the projection onto the column space of A? (The usual formula will fail when $A^{\mathrm{T}}A$ is not invertible.)

29 Usually there will be exactly one hyperplane in $\mathbf{R}^n$ that contains the n given points $\boldsymbol{x} = \boldsymbol{0}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}$. (Example for $n = 3$: There will be one plane containing $0, \boldsymbol{a}_1, \boldsymbol{a}_2$ unless ____.) What is the test to have exactly one plane in $\mathbf{R}^n$?

30 Example 2 shifted the times t_i to make them add to zero. We subtracted away the average time $\widehat{t} = (t_1 + \cdots + t_m)/m$ to get $T_i = t_i - \widehat{t}$. Those T_i add to zero.

With the columns $(1, \ldots, 1)$ and $(T_1, \ldots, T_m)$ now orthogonal, $A^{\mathrm{T}}A$ is diagonal. Its entries are m and $T_1^2 + \cdots + T_m^2$. Show that the best C and D have direct formulas:

$$\boldsymbol{T} \textbf{ is } \boldsymbol{t} - \widehat{\boldsymbol{t}} \qquad C = \frac{b_1 + \cdots + b_m}{m} \qquad \text{and} \qquad D = \frac{b_1T_1 + \cdots + b_mT_m}{T_1^2 + \cdots + T_m^2}.$$

The best line is $C + DT$ ***or*** $C + D(t - \widehat{t})$. The time shift that makes $A^{\mathrm{T}}A$ diagonal is an example of the Gram-Schmidt process: *orthogonalize the columns of A in advance*.

4.4 Orthonormal Bases and Gram-Schmidt

1 The columns $q_1, \ldots, q_n$ are orthonormal if $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j = \begin{Bmatrix} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \end{Bmatrix}$. Then $\boxed{Q^{\mathrm{T}}Q = I\,.}$

2 If Q is also square, then $QQ^{\mathrm{T}} = I$ and $\boxed{Q^{\mathrm{T}} = Q^{-1}}$. Q is an "orthogonal matrix".

3 The least squares solution to $Q\boldsymbol{x} = \boldsymbol{b}$ is $\widehat{\boldsymbol{x}} = Q^{\mathrm{T}}\boldsymbol{b}$. Projection of $\boldsymbol{b}$: $\boldsymbol{p} = QQ^{\mathrm{T}}\boldsymbol{b} = P\boldsymbol{b}$.

4 The **Gram-Schmidt** process takes independent $\boldsymbol{a}_i$ to orthonormal $\boldsymbol{q}_i$. Start with $\boldsymbol{q}_1 = \boldsymbol{a}_1 / \,||\boldsymbol{a}_1||$.

5 $\boldsymbol{q}_i$ is $(\boldsymbol{a}_i -$ projection $\boldsymbol{p}_i) \,/\, ||\boldsymbol{a}_i - \boldsymbol{p}_i||$; projection $\boldsymbol{p}_i = (\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{q}_1)\boldsymbol{q}_1 + \cdots + (\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{q}_{i-1})\boldsymbol{q}_{i-1}$.

6 Each $\boldsymbol{a}_i$ will be a combination of $\boldsymbol{q}_1$ to $\boldsymbol{q}_i$. Then $\boldsymbol{A} = \boldsymbol{QR}$: orthogonal Q and triangular R.

This section has two goals, **why** and **how**. The first is to see why orthogonality is good. Dot products are zero, so $A^{\mathrm{T}}A$ will be diagonal. It becomes so easy to find $\widehat{\boldsymbol{x}}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$. ***The second goal is to construct orthogonal vectors.*** You will see how Gram-Schmidt chooses combinations of the original basis vectors to produce right angles. Those original vectors are the columns of A, probably *not* orthogonal. ***The orthonormal basis vectors will be the columns of a new matrix Q.***

From Chapter 3, a basis consists of independent vectors that span the space. The basis vectors could meet at any angle (except $0°$ and $180°$). But every time we visualize axes, they are perpendicular. *In our imagination, the coordinate axes are practically always orthogonal.* This simplifies the picture and it greatly simplifies the computations.

The vectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ are ***orthogonal*** when their dot products $\boldsymbol{q}_i \cdot \boldsymbol{q}_j$ are zero. More exactly $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j = 0$ whenever $i \neq j$. With one more step—just *divide each vector by its length*—the vectors become ***orthogonal unit vectors***. Their lengths are all 1 (normal). Then the basis is called ***orthonormal***.

DEFINITION The vectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ are ***orthonormal*** if

$$\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j = \begin{cases} 0 & \text{when } i \neq j \quad (\textbf{\textit{orthogonal}} \text{ vectors}) \\ 1 & \text{when } i = j \quad (\textbf{\textit{unit}} \text{ vectors: } \|\boldsymbol{q}_i\| = 1) \end{cases}$$

A matrix with orthonormal columns is assigned the special letter Q.

The matrix Q is easy to work with because $Q^{\mathrm{T}}Q = I$. This repeats in matrix language that the columns $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ are orthonormal. Q is not required to be square.

A matrix Q with orthonormal columns satisfies $\boxed{Q^{\mathrm{T}}Q = I}$ **:**

$$\boxed{Q^{\mathrm{T}}Q} = \begin{bmatrix} -\,\boldsymbol{q}_1^{\mathrm{T}}\,- \\ -\,\boldsymbol{q}_2^{\mathrm{T}}\,- \\ \\ -\,\boldsymbol{q}_n^{\mathrm{T}}\,- \end{bmatrix} \begin{bmatrix} | & | & & | \\ \boldsymbol{q}_1 & \boldsymbol{q}_2 & & \boldsymbol{q}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \boxed{I}. \quad (1)$$

When row i of Q^{T} multiplies column j of Q, the dot product is $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j$. Off the diagonal $(i \neq j)$ that dot product is zero by orthogonality. On the diagonal $(i = j)$ the unit vectors give $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_i = \|\boldsymbol{q}_i\|^2 = 1$. Often Q is rectangular $(m > n)$. Sometimes $m = n$.

When Q is square, $Q^{\mathrm{T}}Q = I$ means that $Q^{\mathrm{T}} = Q^{-1}$: transpose = inverse.

If the columns are only orthogonal (not unit vectors), dot products still give a diagonal matrix (not the identity matrix). This diagonal matrix is almost as good as I. The important thing is orthogonality—then it is easy to produce unit vectors.

To repeat: $Q^{\mathrm{T}}Q = I$ even when Q is rectangular. In that case Q^{T} is only an inverse from the left. For square matrices we also have $QQ^{\mathrm{T}} = I$, so Q^{T} is the two-sided inverse of Q. The rows of a square Q are orthonormal like the columns. ***The inverse is the transpose***. In this square case we call Q an ***orthogonal matrix***.[1]

Here are three examples of orthogonal matrices—rotation and permutation and reflection. The quickest test is to check $Q^{\mathrm{T}}Q = I$.

Example 1 **(Rotation)** Q rotates every vector in the plane by the angle θ:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad Q^{\mathrm{T}} = Q^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

The columns of Q are orthogonal (take their dot product). They are unit vectors because $\sin^2\theta + \cos^2\theta = 1$. Those columns give an ***orthonormal basis*** for the plane $\mathbf{R}^2$.

The standard basis vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are rotated through θ (see Figure 4.10a). Q^{-1} rotates vectors back through $-\theta$. It agrees with Q^{T}, because the cosine of $-\theta$ equals the cosine of θ, and $\sin(-\theta) = -\sin\theta$. We have $Q^{\mathrm{T}}Q = I$ and $QQ^{\mathrm{T}} = I$.

Example 2 **(Permutation)** These matrices change the order to (y, z, x) and (y, x):

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}.$$

All columns of these Q's are unit vectors (their lengths are obviously 1). They are also orthogonal (the 1's appear in different places). *The inverse of a permutation matrix is its transpose*: $Q^{-1} = Q^{\mathrm{T}}$. The inverse puts the components back into their original order:

[1]"Orthonormal matrix" would have been a better name for Q, but it's not used. Any matrix with orthonormal columns has the letter Q. But we only call it an **orthogonal matrix** when it is square.

$$\textbf{Inverse = transpose:}\quad \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}\boldsymbol{y}\\\boldsymbol{z}\\\boldsymbol{x}\end{bmatrix}=\begin{bmatrix}\boldsymbol{x}\\\boldsymbol{y}\\\boldsymbol{z}\end{bmatrix} \quad\text{and}\quad \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}\boldsymbol{y}\\\boldsymbol{x}\end{bmatrix}=\begin{bmatrix}\boldsymbol{x}\\\boldsymbol{y}\end{bmatrix}.$$

Every permutation matrix is an orthogonal matrix.

Example 3 **(Reflection)** If $\boldsymbol{u}$ is any unit vector, set $Q = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$. Notice that $\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is a matrix while $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}$ is the number $\|\boldsymbol{u}\|^2 = 1$. Then Q^{T} and Q^{-1} both equal Q:

$$Q^{\mathrm{T}} = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} = Q \quad\text{and}\quad Q^{\mathrm{T}}Q = I - 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} + 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} = I. \tag{2}$$

Reflection matrices $I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ are symmetric and also orthogonal. If you square them, you get the identity matrix: $Q^2 = Q^{\mathrm{T}}Q = I$. Reflecting twice through a mirror brings back the original, like $(-1)^2 = 1$. Notice $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$ inside $4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ in equation (2).

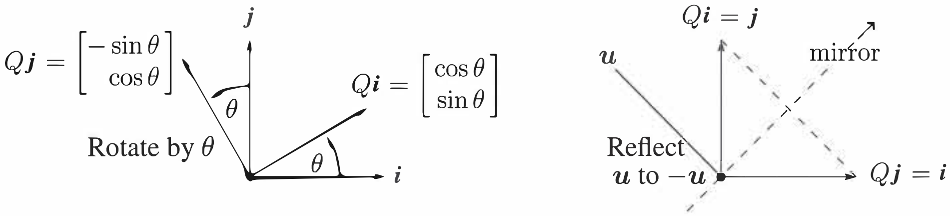


Figure 4.10: Rotation by $Q = \begin{bmatrix}c & -s\\ s & c\end{bmatrix}$ and reflection across 45° by $Q = \begin{bmatrix}0&1\\1&0\end{bmatrix}$.

As example choose the direction $\boldsymbol{u} = (-1/\sqrt{2}, 1/\sqrt{2})$. Compute $2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ (column times row) and subtract from I to get the reflection matrix Q in the direction of $\boldsymbol{u}$:

$$\textbf{Reflection}\quad Q = I - 2\begin{bmatrix}.5 & -.5\\ -.5 & .5\end{bmatrix} = \begin{bmatrix}0&1\\1&0\end{bmatrix} \quad\text{and}\quad \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}y\\x\end{bmatrix}.$$

When (x, y) goes to (y, x), a vector like $(3, 3)$ doesn't move. It is on the mirror line.

Rotations preserve the length of every vector. So do reflections. So do permutations. So does multiplication by any orthogonal matrix Q—***lengths and angles don't change***.

Proof $\|Q\boldsymbol{x}\|^2$ equals $\|\boldsymbol{x}\|^2$ because $(Q\boldsymbol{x})^{\mathrm{T}}(Q\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}Q^{\mathrm{T}}Q\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}I\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$.

If Q has orthonormal columns ($Q^{\mathrm{T}}Q = I$), it leaves lengths unchanged:

Same length for $Q\boldsymbol{x}$ $\quad \|Q\boldsymbol{x}\| = \|\boldsymbol{x}\|$ **for every vector** $\boldsymbol{x}$. (3)

Q also preserves dot products: $(Q\boldsymbol{x})^{\mathrm{T}}(Q\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{T}}Q^{\mathrm{T}}Q\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$. Just use $Q^{\mathrm{T}}Q = I$!

Projections Using Orthonormal Bases: Q Replaces A

Orthogonal matrices are excellent for computations—numbers can never grow too large when lengths of vectors are fixed. Stable computer codes use Q's as much as possible.

For projections onto subspaces, all formulas involve $A^{\mathrm{T}}A$. The entries of $A^{\mathrm{T}}A$ are the dot products $\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{a}_j$ of the basis vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.

Suppose the basis vectors are actually orthonormal. The $\boldsymbol{a}$'s become the $\boldsymbol{q}$'s. Then $A^{\mathrm{T}}A$ *simplifies to* $Q^{\mathrm{T}}Q = I$. Look at the improvements in $\widehat{\boldsymbol{x}}$ and $\boldsymbol{p}$ and P. Instead of $Q^{\mathrm{T}}Q$ we print a blank for the identity matrix:

$$\underline{\qquad}\,\widehat{\boldsymbol{x}} = Q^{\mathrm{T}}\boldsymbol{b} \quad \text{and} \quad \boldsymbol{p} = Q\widehat{\boldsymbol{x}} \quad \text{and} \quad P = Q\,\underline{\qquad}\,Q^{\mathrm{T}}. \tag{4}$$

The least squares solution of $Q\boldsymbol{x} = \boldsymbol{b}$ is $\widehat{\boldsymbol{x}} = Q^{\mathrm{T}}\boldsymbol{b}$. The projection matrix is QQ^{T}.

There are no matrices to invert. This is the point of an orthonormal basis. The best $\widehat{\boldsymbol{x}} = Q^{\mathrm{T}}\boldsymbol{b}$ just has dot products of $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ with $\boldsymbol{b}$. We have 1-dimensional projections! The "coupling matrix" or "correlation matrix" $A^{\mathrm{T}}A$ is now $Q^{\mathrm{T}}Q = I$. There is no coupling. When A is Q, with orthonormal columns, here is $\boldsymbol{p} = Q\widehat{\boldsymbol{x}} = QQ^{\mathrm{T}}\boldsymbol{b}$:

Projection onto q's

$$\boldsymbol{p} = \begin{bmatrix} | & & | \\ \boldsymbol{q}_1 & \cdots & \boldsymbol{q}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b} \\ \vdots \\ \boldsymbol{q}_n^{\mathrm{T}}\boldsymbol{b} \end{bmatrix} = \boldsymbol{q}_1(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b}) + \cdots + \boldsymbol{q}_n(\boldsymbol{q}_n^{\mathrm{T}}\boldsymbol{b}). \tag{5}$$

Important case: When Q is square and $m = n$, the subspace is the whole space. Then $Q^{\mathrm{T}} = Q^{-1}$ and $\widehat{\boldsymbol{x}} = Q^{\mathrm{T}}\boldsymbol{b}$ is the same as $\boldsymbol{x} = Q^{-1}\boldsymbol{b}$. The solution is exact! The projection of $\boldsymbol{b}$ onto the whole space is $\boldsymbol{b}$ itself. In this case $\boldsymbol{p} = \boldsymbol{b}$ and $P = QQ^{\mathrm{T}} = I$.

You may think that projection onto the whole space is not worth mentioning. But when $\boldsymbol{p} = \boldsymbol{b}$, our formula assembles $\boldsymbol{b}$ out of its 1-dimensional projections. If $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ is an orthonormal basis for the whole space, then Q is square. Every $\boldsymbol{b} = QQ^{\mathrm{T}}\boldsymbol{b}$ *is the sum of its components along the* $\boldsymbol{q}$'s:

$$\boldsymbol{b} = \boldsymbol{q}_1(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b}) + \boldsymbol{q}_2(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{b}) + \cdots + \boldsymbol{q}_n(\boldsymbol{q}_n^{\mathrm{T}}\boldsymbol{b}). \tag{6}$$

Transforms $QQ^{\mathrm{T}} = I$ is the foundation of Fourier series and all the great "transforms" of applied mathematics. They break vectors $\boldsymbol{b}$ or functions $f(x)$ into perpendicular pieces. Then by adding the pieces in (6), the inverse transform puts $\boldsymbol{b}$ and $f(x)$ back together.

Example 4 The columns of this orthogonal Q are orthonormal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$:

$$\boldsymbol{m = n = 3} \qquad Q = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \quad \text{has} \quad \boldsymbol{Q^{\mathrm{T}}Q = QQ^{\mathrm{T}} = I}.$$

The separate projections of $\boldsymbol{b} = (0,0,1)$ onto $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$ are $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$:

$$\boldsymbol{q}_1(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b}) = \tfrac{2}{3}\boldsymbol{q}_1 \quad \text{and} \quad \boldsymbol{q}_2(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{b}) = \tfrac{2}{3}\boldsymbol{q}_2 \quad \text{and} \quad \boldsymbol{q}_3(\boldsymbol{q}_3^{\mathrm{T}}\boldsymbol{b}) = -\tfrac{1}{3}\boldsymbol{q}_3.$$

The sum of the first two is the projection of $\boldsymbol{b}$ onto the *plane* of $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. The sum of all three is the projection of $\boldsymbol{b}$ onto the *whole space*—which is $\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 = \boldsymbol{b}$ itself:

Reconstruct $\boldsymbol{b}$
$\boldsymbol{b} = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3$

$$\frac{2}{3}\boldsymbol{q}_1 + \frac{2}{3}\boldsymbol{q}_2 - \frac{1}{3}\boldsymbol{q}_3 = \frac{1}{9}\begin{bmatrix} -2+4-2 \\ 4-2-2 \\ 4+4+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{b}.$$

The Gram-Schmidt Process

The point of this section is that "orthogonal is good". Projections and least squares always involve $A^{\mathrm{T}}A$. When this matrix becomes $Q^{\mathrm{T}}Q = I$, the inverse is no problem. The one-dimensional projections are uncoupled. The best $\widehat{\boldsymbol{x}}$ is $Q^{\mathrm{T}}\boldsymbol{b}$ (just n separate dot products). For this to be true, we had to say "*If* the vectors are orthonormal". ***Now we explain the "Gram-Schmidt way" to create orthonormal vectors***.

Start with three independent vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. We intend to construct three orthogonal vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$. Then (at the end may be easiest) we divide $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ by their lengths. That produces three orthonormal vectors $\boldsymbol{q}_1 = \boldsymbol{A}/\|\boldsymbol{A}\|$, $\boldsymbol{q}_2 = \boldsymbol{B}/\|\boldsymbol{B}\|$, $\boldsymbol{q}_3 = \boldsymbol{C}/\|\boldsymbol{C}\|$.

Gram-Schmidt Begin by choosing $\boldsymbol{A} = \boldsymbol{a}$. This first direction is accepted as it comes. The next direction $\boldsymbol{B}$ must be perpendicular to $\boldsymbol{A}$. ***Start with $\boldsymbol{b}$ and subtract its projection along $\boldsymbol{A}$***. This leaves the perpendicular part, which is the orthogonal vector $\boldsymbol{B}$:

First Gram-Schmidt step

$$\boxed{\boldsymbol{B} = \boldsymbol{b} - \frac{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}}\boldsymbol{A}.} \tag{7}$$

$\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal in Figure 4.11. Multiply equation (7) by $\boldsymbol{A}^{\mathrm{T}}$ to verify that $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b} = 0$. This vector $\boldsymbol{B}$ is what we have called the error vector $\boldsymbol{e}$, perpendicular to $\boldsymbol{A}$. Notice that $\boldsymbol{B}$ in equation (7) is not zero (otherwise $\boldsymbol{a}$ and $\boldsymbol{b}$ would be dependent). The directions $\boldsymbol{A}$ and $\boldsymbol{B}$ are now set.

The third direction starts with $\boldsymbol{c}$. This is not a combination of $\boldsymbol{A}$ and $\boldsymbol{B}$ (because $\boldsymbol{c}$ is not a combination of $\boldsymbol{a}$ and $\boldsymbol{b}$). But most likely $\boldsymbol{c}$ is not perpendicular to $\boldsymbol{A}$ and $\boldsymbol{B}$. So subtract off its components in those two directions to get a perpendicular direction $\boldsymbol{C}$:

Next Gram-Schmidt step

$$\boxed{\boldsymbol{C} = \boldsymbol{c} - \frac{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}}\boldsymbol{A} - \frac{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}}\boldsymbol{B}.} \tag{8}$$

This is the one and only idea of the Gram-Schmidt process. ***Subtract from every new vector its projections in the directions already set.*** That idea is repeated at every step.[2] If we had a fourth vector $\boldsymbol{d}$, we would subtract three projections onto $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ to get $\boldsymbol{D}$.

[2] I think Gram had the idea. I don't really know where Schmidt came in.

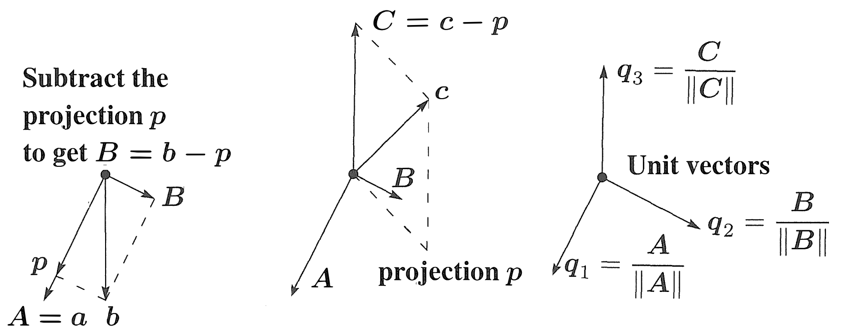


Figure 4.11: First project $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ and find the orthogonal $\boldsymbol{B}$ as $\boldsymbol{b} - \boldsymbol{p}$. Then project $\boldsymbol{c}$ onto the $\boldsymbol{AB}$ plane and find $\boldsymbol{C}$ as $\boldsymbol{c} - \boldsymbol{p}$. Divide by $\|\boldsymbol{A}\|$, $\|\boldsymbol{B}\|$, $\|\boldsymbol{C}\|$.

At the end, *or immediately when each one is found*, divide the orthogonal vectors $\boldsymbol{A}, \boldsymbol{B}$, $\boldsymbol{C}, \boldsymbol{D}$ by their lengths. The resulting vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4$ are orthonormal.

Example of **Gram-Schmidt** Suppose the independent non-orthogonal vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{c} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}.$$

Then $\boldsymbol{A} = \boldsymbol{a}$ has $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = 2$ and $A^{\mathrm{T}}\boldsymbol{b} = 2$. Subtract from $\boldsymbol{b}$ its projection $\boldsymbol{p}$ along $\boldsymbol{A}$:

First step $$\boldsymbol{B} = \boldsymbol{b} - \frac{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}}\boldsymbol{A} = \boldsymbol{b} - \frac{2}{2}\boldsymbol{A} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{-2} \end{bmatrix}.$$

Check: $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B} = 0$ as required. Now subtract the projections of $\boldsymbol{c}$ on $\boldsymbol{A}$ and $\boldsymbol{B}$ to get $\boldsymbol{C}$:

Next step $$\boldsymbol{C} = \boldsymbol{c} - \frac{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}}\boldsymbol{A} - \frac{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}}\boldsymbol{B} = \boldsymbol{c} - \frac{6}{2}\boldsymbol{A} + \frac{6}{6}\boldsymbol{B} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}.$$

Check: $\boldsymbol{C} = (1, 1, 1)$ is perpendicular to both $\boldsymbol{A}$ and $\boldsymbol{B}$. Finally convert $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ to unit vectors (length 1, orthonormal). The lengths of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are $\sqrt{2}$ and $\sqrt{6}$ and $\sqrt{3}$. Divide by those lengths, for an orthonormal basis:

$$\boldsymbol{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{q}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{q}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Usually $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ contain fractions. Almost always $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ contain square roots.

The Factorization $A = QR$

We started with a matrix A, whose columns were $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. We ended with a matrix Q, whose columns are $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$. How are those matrices related? Since the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are combinations of the $\boldsymbol{q}$'s (and vice versa), there must be a third matrix connecting A to Q. This third matrix is the triangular R in $A = QR$.

The first step was $\boldsymbol{q}_1 = \boldsymbol{a}/\|\boldsymbol{a}\|$ (other vectors not involved). The second step was equation (7), where $\boldsymbol{b}$ is a combination of $\boldsymbol{A}$ and $\boldsymbol{B}$. At that stage $\boldsymbol{C}$ and $\boldsymbol{q}_3$ were not involved. This non-involvement of later vectors is the key point of Gram-Schmidt:

- The vectors $\boldsymbol{a}$ and $\boldsymbol{A}$ and $\boldsymbol{q}_1$ are all along a single line.
- The vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{q}_1, \boldsymbol{q}_2$ are all in the same plane.
- The vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ and $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ are in one subspace (dimension 3).

At every step $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k$ are combinations of $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_k$. Later $\boldsymbol{q}$'s are not involved. The connecting matrix R is ***triangular***, and we have $A = QR$:

$$\begin{bmatrix} & & \\ a & b & c \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ q_1 & q_2 & q_3 \\ & & \end{bmatrix} \begin{bmatrix} q_1^{\mathrm{T}} a & q_1^{\mathrm{T}} b & q_1^{\mathrm{T}} c \\ & q_2^{\mathrm{T}} b & q_2^{\mathrm{T}} c \\ & & q_3^{\mathrm{T}} c \end{bmatrix} \quad \text{or} \quad \boldsymbol{A = QR}. \tag{9}$$

$A = QR$ is Gram-Schmidt in a nutshell. Multiply by Q^{T} to recognize $\boldsymbol{R = Q^{\mathrm{T}} A}$ above.

(Gram-Schmidt) From independent vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, Gram-Schmidt constructs orthonormal vectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$. The matrices with these columns satisfy $A = QR$. Then $R = Q^{\mathrm{T}} A$ is ***upper triangular*** because later $\boldsymbol{q}$'s are orthogonal to earlier $\boldsymbol{a}$'s.

Here are the original $\boldsymbol{a}$'s and the final $\boldsymbol{q}$'s from the example. The i, j entry of $R = Q^{\mathrm{T}} A$ is row i of Q^{T} times column j of A. The dot products $\boldsymbol{q}_i^{\mathrm{T}} \boldsymbol{a}_j$ go into R. **Then $\boldsymbol{A = QR}$:**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -3 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{\mathbf{2}} & \sqrt{2} & \sqrt{18} \\ \mathbf{0} & \sqrt{\mathbf{6}} & -\sqrt{6} \\ \mathbf{0} & \mathbf{0} & \sqrt{\mathbf{3}} \end{bmatrix} = QR.$$

Look closely at Q and R. The lengths of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are $\sqrt{2}, \sqrt{6}, \sqrt{3}$ on the diagonal of R. The columns of Q are orthonormal. Because of the square roots, QR might look harder than LU. Both factorizations are absolutely central to calculations in linear algebra.

Any m by n matrix A with independent columns can be factored into $A = QR$. The m by n matrix Q has orthonormal columns, and the square matrix R is upper triangular with positive diagonal. We must not forget why this is useful for least squares: $\boldsymbol{A^{\mathrm{T}} A = (QR)^{\mathrm{T}} QR = R^{\mathrm{T}} Q^{\mathrm{T}} QR = R^{\mathrm{T}} R}$. The least squares equation $A^{\mathrm{T}} A \widehat{\boldsymbol{x}} = A^{\mathrm{T}} \boldsymbol{b}$ simplifies to $R^{\mathrm{T}} R \widehat{\boldsymbol{x}} = R^{\mathrm{T}} Q^{\mathrm{T}} \boldsymbol{b}$. Then finally we reach $R \widehat{\boldsymbol{x}} = Q^{\mathrm{T}} \boldsymbol{b}$: good.

Least squares $\quad R^{\mathrm{T}}R\widehat{x} = R^{\mathrm{T}}Q^{\mathrm{T}}b$ or $R\widehat{x} = Q^{\mathrm{T}}b$ or $\boxed{\widehat{x} = R^{-1}Q^{\mathrm{T}}b}$ (10)

Instead of solving $Ax = b$, which is impossible, we solve $R\widehat{x} = Q^{\mathrm{T}}b$ by back substitution—which is very fast. The real cost is the mn^2 multiplications in the Gram-Schmidt process, which are needed to construct the orthogonal Q and the triangular R with $A = QR$.

Below is an informal code. It executes equations (11) for $j = 1$ then $j = 2$ and eventually $j = n$. The important lines 4-5 subtract from $v = a_j$ its projection onto each $q_i, i < j$. The last line of that code normalizes v (divides by $r_{jj} = ||v||$) to get the unit vector q_j:

$$r_{kj} = \sum_{i=1}^{m} q_{ik}v_{ij} \text{ and } v_{ij} = v_{ij} - q_{ik}r_{kj} \text{ and } r_{jj} = \left(\sum_{i=1}^{m} v_{ij}^2\right)^{1/2} \text{ and } q_{ij} = \frac{v_{ij}}{r_{jj}}. \tag{11}$$

Starting from $a, b, c = a_1, a_2, a_3$ this code will construct q_1, then B, q_2, then C, q_3:

$$q_1 = a_1/\|a_1\| \qquad B = a_2 - (q_1^{\mathrm{T}}a_2)q_1 \qquad q_2 = B/\|B\|$$

$$C^* = a_3 - (q_1^{\mathrm{T}}a_3)q_1 \quad C = C^* - (q_2^{\mathrm{T}}C^*)q_2 \quad q_3 = C/\|C\|$$

Equation (11) subtracts **one projection at a time** as in C^* and C. That change is called ***modified Gram-Schmidt***. This code is numerically more stable than equation (8) which subtracts all projections at once.

```
for j = 1:n                      % modified Gram-Schmidt
  v = A(:,j);                    % v begins as column j of the original A
  for i = 1:j-1                  % columns q_1 to q_{j-1} are already settled in Q
    R(i,j) = Q(:,i)'*v;          % compute R_ij = q_i^T a_j which is q_i^T v
    v = v-R(i,j)*Q(:,i);         % subtract the projection (q_i^T v) q_i
  end                            % v is now perpendicular to all of q_1, ..., q_{j-1}
  R(j,j) = norm(v);              % the diagonal entries R_jj are lengths
  Q(:,j) = v/R(j,j);             % divide v by its length to get the next q_j
end                              % the "for j = 1 : n loop" produces all of the q_j
```

To recover column j of A, undo the last step and the middle steps of the code:

$$R(j,j)q_j = (v \textbf{\textit{ minus its projections}}) = (\text{column } j \text{ of } A) - \sum_{i=1}^{j-1} R(i,j)q_i\,. \tag{12}$$

Moving the sum to the far left, this is column j in the multiplication $QR = A$.

Confession Good software like LAPACK, used in good systems like MATLAB and Julia and Python, will not use this Gram-Schmidt code. There is now a better way. "Householder reflections" act on A to produce the upper triangular R. This happens one column at a time in the same way that elimination produces the upper triangular U in LU.

Those reflection matrices $I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ will be described in Chapter 11 on numerical linear algebra. If A is tridiagonal we can simplify even more to use 2 by 2 rotations. The result is always $A = QR$ and the MATLAB command to orthogonalize A is $[Q, R] = \mathsf{qr}(A)$. I believe that Gram-Schmidt is still the good process to understand, even if the reflections or rotations lead to a more perfect Q.

■ REVIEW OF THE KEY IDEAS ■

1. If the orthonormal vectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ are the columns of Q, then $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j = 0$ and $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_i = 1$ translate into the matrix multiplication $Q^{\mathrm{T}}Q = I$.
2. If Q is square (an ***orthogonal matrix***) then $Q^{\mathrm{T}} = Q^{-1}$: ***transpose*** $=$ ***inverse***.
3. The length of $Q\boldsymbol{x}$ equals the length of $\boldsymbol{x}$: $\|Q\boldsymbol{x}\| = \|\boldsymbol{x}\|$.
4. The projection onto the column space of Q spanned by the $\boldsymbol{q}$'s is $P = QQ^{\mathrm{T}}$.
5. If Q is square then $P = QQ^{\mathrm{T}} = I$ and every $\boldsymbol{b} = \boldsymbol{q}_1(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b}) + \cdots + \boldsymbol{q}_n(\boldsymbol{q}_n^{\mathrm{T}}\boldsymbol{b})$.
6. Gram-Schmidt produces orthonormal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ from independent $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. In matrix form this is the factorization $A = QR =$ (orthogonal Q)(triangular R).

■ WORKED EXAMPLES ■

4.4 A Add two more columns with all entries 1 or -1, so the columns of this 4 by 4 "Hadamard matrix" are orthogonal. How do you turn H_4 into an *orthogonal matrix* Q?

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad H_4 = \begin{bmatrix} 1 & 1 & x & x \\ 1 & -1 & x & x \\ 1 & 1 & x & x \\ 1 & -1 & x & x \end{bmatrix} \qquad \text{and} \qquad Q_4 = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

The block matrix $H_8 = \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix}$ is the next Hadamard matrix with 1's and -1's. What is the product $H_8^{\mathrm{T}} H_8$?

The projection of $\boldsymbol{b} = (6, 0, 0, 2)$ onto the first column of H_4 is $\boldsymbol{p}_1 = (2, 2, 2, 2)$. The projection onto the second column is $\boldsymbol{p}_2 = (1, -1, 1, -1)$. What is the projection $\boldsymbol{p}_{1,2}$ of $\boldsymbol{b}$ onto the 2-dimensional space spanned by the first two columns?

Solution H_4 can be built from H_2 just as H_8 is built from H_4:

$$H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \text{ has orthogonal columns.}$$

Then $Q = H/2$ has orthonormal columns. Dividing by 2 gives unit vectors in Q. A 5 by 5 Hadamard matrix is impossible because the dot product of columns would have five 1's and/or -1's and could not add to zero. H_8 has orthogonal columns of length $\sqrt{8}$.

$$H_8^{\mathrm{T}} H_8 = \begin{bmatrix} H^{\mathrm{T}} & H^{\mathrm{T}} \\ H^{\mathrm{T}} & -H^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} H & H \\ H & -H \end{bmatrix} = \begin{bmatrix} 2H^{\mathrm{T}}H & 0 \\ 0 & 2H^{\mathrm{T}}H \end{bmatrix} = \begin{bmatrix} 8I & 0 \\ 0 & 8I \end{bmatrix}. \; Q_8 = \frac{H_8}{\sqrt{8}}$$

4.4 B What is the key point of orthogonal columns? Answer: $A^{\mathrm{T}}A$ is diagonal and easy to invert. **We can project onto lines and just add**. The axes are orthogonal.

Add p's Projection $\boldsymbol{p}_{1,2}$ onto a plane equals $\boldsymbol{p}_1 + \boldsymbol{p}_2$ onto orthogonal lines.

Problem Set 4.4

Problems 1–12 are about orthogonal vectors and orthogonal matrices.

1 Are these pairs of vectors orthonormal or only orthogonal or only independent?

(a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} .6 \\ .8 \end{bmatrix}$ and $\begin{bmatrix} .4 \\ -.3 \end{bmatrix}$ (c) $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$.

Change the second vector when necessary to produce orthonormal vectors.

2 The vectors $(2, 2, -1)$ and $(-1, 2, 2)$ are orthogonal. Divide them by their lengths to find orthonormal vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. Put those into the columns of Q and multiply $Q^{\mathrm{T}}Q$ and QQ^{T}.

3 (a) If A has three orthogonal columns each of length 4, what is $A^{\mathrm{T}}A$?

(b) If A has three orthogonal columns of lengths $1, 2, 3$, what is $A^{\mathrm{T}}A$?

4 Give an example of each of the following:

(a) A matrix Q that has orthonormal columns but $QQ^{\mathrm{T}} \neq I$.

(b) Two orthogonal vectors that are not linearly independent.

(c) An orthonormal basis for $\mathbf{R}^3$, including the vector $\boldsymbol{q}_1 = (1, 1, 1)/\sqrt{3}$.

5 Find two orthogonal vectors in the plane $x + y + 2z = 0$. Make them orthonormal.

6 If Q_1 and Q_2 are orthogonal matrices, show that their product Q_1Q_2 is also an orthogonal matrix. (Use $Q^{\mathrm{T}}Q = I$.)

7 If Q has orthonormal columns, what is the least squares solution $\widehat{x}$ to $Qx = b$?

8 If q_1 and q_2 are orthonormal vectors in $\mathbf{R}^5$, what combination ____ q_1 + ____ q_2 is closest to a given vector b?

9 (a) Compute $P = QQ^{\mathrm{T}}$ when $q_1 = (.8, .6, 0)$ and $q_2 = (-.6, .8, 0)$. Verify that $P^2 = P$.

(b) Prove that always $(QQ^{\mathrm{T}})^2 = QQ^{\mathrm{T}}$ by using $Q^{\mathrm{T}}Q = I$. Then $P = QQ^{\mathrm{T}}$ is the projection matrix onto the column space of Q.

10 Orthonormal vectors are automatically linearly independent.

(a) Vector proof: When $c_1q_1 + c_2q_2 + c_3q_3 = \mathbf{0}$, what dot product leads to $c_1 = 0$? Similarly $c_2 = 0$ and $c_3 = 0$. Thus the q's are independent.

(b) Matrix proof: Show that $Qx = \mathbf{0}$ leads to $x = \mathbf{0}$. Since Q may be rectangular, you can use Q^{T} but not Q^{-1}.

11 (a) Gram-Schmidt: Find orthonormal vectors q_1 and q_2 in the plane spanned by $a = (1, 3, 4, 5, 7)$ and $b = (-6, 6, 8, 0, 8)$.

(b) Which vector in this plane is closest to $(1, 0, 0, 0, 0)$?

12 If a_1, a_2, a_3 is a basis for $\mathbf{R}^3$, any vector b can be written as

$$b = x_1a_1 + x_2a_2 + x_3a_3 \qquad \text{or} \qquad \begin{bmatrix} & & \\ a_1 & a_2 & a_3 \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b.$$

(a) Suppose the a's are orthonormal. Show that $x_1 = a_1^{\mathrm{T}}b$.

(b) Suppose the a's are orthogonal. Show that $x_1 = a_1^{\mathrm{T}}b/a_1^{\mathrm{T}}a_1$.

(c) If the a's are independent, x_1 is the first component of ____ times b.

Problems 13–25 are about the Gram-Schmidt process and $A = QR$.

13 What multiple of $a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ should be subtracted from $b = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ to make the result B orthogonal to a? Sketch a figure to show a, b, and B.

14 Complete the Gram-Schmidt process in Problem 13 by computing $q_1 = a/\|a\|$ and $q_2 = B/\|B\|$ and factoring into QR:

$$\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \end{bmatrix}\begin{bmatrix} \|a\| & ? \\ 0 & \|B\| \end{bmatrix}.$$

15 (a) Find orthonormal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ such that $\boldsymbol{q}_1, \boldsymbol{q}_2$ span the column space of

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

(b) Which of the four fundamental subspaces contains $\boldsymbol{q}_3$?

(c) Solve $A\boldsymbol{x} = (1, 2, 7)$ by least squares.

16 What multiple of $\boldsymbol{a} = (4, 5, 2, 2)$ is closest to $\boldsymbol{b} = (1, 2, 0, 0)$? Find orthonormal vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ in the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$.

17 Find the projection of $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$:

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \text{and} \quad \boldsymbol{p} = ? \quad \text{and} \quad \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = ?$$

Compute the orthonormal vectors $\boldsymbol{q}_1 = \boldsymbol{a}/\|\boldsymbol{a}\|$ and $\boldsymbol{q}_2 = \boldsymbol{e}/\|\boldsymbol{e}\|$.

18 (Recommended) Find orthogonal vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ by Gram-Schmidt from $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$:

$$\boldsymbol{a} = (1, -1, 0, 0) \qquad \boldsymbol{b} = (0, 1, -1, 0) \qquad \boldsymbol{c} = (0, 0, 1, -1).$$

$\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are bases for the vectors perpendicular to $\boldsymbol{d} = (1, 1, 1, 1)$.

19 If $A = QR$ then $A^{\mathrm{T}}A = R^{\mathrm{T}}R =$ _____ triangular times _____ triangular. *Gram-Schmidt on A corresponds to elimination on $A^{\mathrm{T}}A$.* The pivots for $A^{\mathrm{T}}A$ must be the squares of diagonal entries of R. Find Q and R by Gram-Schmidt for this A:

$$A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad A^{\mathrm{T}}A = \begin{bmatrix} 9 & 9 \\ 9 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & \\ & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

20 True or false (give an example in either case):

(a) Q^{-1} is an orthogonal matrix when Q is an orthogonal matrix.

(b) If Q (3 by 2) has orthonormal columns then $\|Q\boldsymbol{x}\|$ always equals $\|\boldsymbol{x}\|$.

21 Find an orthonormal basis for the column space of A:

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} -4 \\ -3 \\ 3 \\ 0 \end{bmatrix}.$$

Then compute the projection of $\boldsymbol{b}$ onto that column space.

22 Find orthogonal vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ by Gram-Schmidt from

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}.$$

23 Find $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ (orthonormal) as combinations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ (independent columns). Then write A as QR:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}.$$

24 (a) Find a basis for the subspace S in $\mathbf{R}^4$ spanned by all solutions of

$$x_1 + x_2 + x_3 - x_4 = 0.$$

(b) Find a basis for the orthogonal complement $S^{\perp}$.

(c) Find $\boldsymbol{b}_1$ in S and $\boldsymbol{b}_2$ in $S^{\perp}$ so that $\boldsymbol{b}_1 + \boldsymbol{b}_2 = \boldsymbol{b} = (1, 1, 1, 1)$.

25 If $ad - bc > 0$, the entries in $A = QR$ are

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{\begin{bmatrix} a & -c \\ c & a \end{bmatrix}}{\sqrt{a^2 + c^2}} \frac{\begin{bmatrix} a^2 + c^2 & ab + cd \\ 0 & ad - bc \end{bmatrix}}{\sqrt{a^2 + c^2}}.$$

Write $A = QR$ when $a, b, c, d = 2, 1, 1, 1$ and also $1, 1, 1, 1$. Which entry of R becomes zero when the columns are dependent and Gram-Schmidt breaks down?

Problems 26–29 use the QR code in equation (11). It executes Gram-Schmidt.

26 Show why $\boldsymbol{C}$ (found via $\boldsymbol{C}^*$ in the steps after (11)) is equal to $\boldsymbol{C}$ in equation (8).

27 Equation (8) subtracts from $\boldsymbol{c}$ its components along $\boldsymbol{A}$ and $\boldsymbol{B}$. Why not subtract the components along $\boldsymbol{a}$ and along $\boldsymbol{b}$?

28 Where are the mn^2 multiplications in equation (11)?

29 Apply the MATLAB qr code to $\boldsymbol{a} = (2, 2, -1)$, $\boldsymbol{b} = (0, -3, 3)$, $\boldsymbol{c} = (1, 0, 0)$. What are the $\boldsymbol{q}$'s?

Problems 30–35 involve orthogonal matrices that are special.

30 The first four ***wavelets*** are in the columns of this wavelet matrix W:

$$W = \frac{1}{2} \begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}.$$

What is special about the columns? Find the inverse wavelet transform W^{-1}.

31 (a) Choose c so that Q is an orthogonal matrix:

$$Q = c\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

Project $\boldsymbol{b} = (1,1,1,1)$ onto the first column. Then project $\boldsymbol{b}$ onto the plane of the first two columns.

32 If $\boldsymbol{u}$ is a unit vector, then $Q = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is a reflection matrix (Example 3). Find Q_1 from $\boldsymbol{u} = (0,1)$ and Q_2 from $\boldsymbol{u} = (0, \sqrt{2}/2, \sqrt{2}/2)$. Draw the reflections when Q_1 and Q_2 multiply the vectors $(1,2)$ and $(1,1,1)$.

33 Find all matrices that are both orthogonal and lower triangular.

34 $Q = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is a reflection matrix when $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$. Two reflections give $Q^2 = I$.

(a) Show that $Q\boldsymbol{u} = -\boldsymbol{u}$. The mirror is perpendicular to $\boldsymbol{u}$.

(b) Find $Q\boldsymbol{v}$ when $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = 0$. The mirror contains $\boldsymbol{v}$. It reflects to itself.

Challenge Problems

35 (MATLAB) Factor $[Q, R] = \mathbf{qr}(A)$ for $A = \mathbf{eye}(4) - \mathbf{diag}([1\ \ 1\ \ 1], -1)$. You are orthogonalizing the columns $(1,-1,0,0)$ and $(0,1,-1,0)$ and $(0,0,1,-1)$ and $(0,0,0,1)$ of A. Can you scale the orthogonal columns of Q to get nice integer components?

36 If A is m by n with rank n, $\mathbf{qr}(A)$ produces a *square* Q and zeros below R:

The factors from MATLAB are $(m \text{ by } m)(m \text{ by } n)$ $\qquad A = [Q_1 \quad Q_2]\begin{bmatrix} R \\ 0 \end{bmatrix}.$

The n columns of Q_1 are an orthonormal basis for which fundamental subspace? The $m-n$ columns of Q_2 are an orthonormal basis for which fundamental subspace?

37 We know that $P = QQ^{\mathrm{T}}$ is the projection onto the column space of $Q(m \text{ by } n)$. Now add another column $\boldsymbol{a}$ to produce $A = [Q \quad \boldsymbol{a}]$. Gram-Schmidt replaces $\boldsymbol{a}$ by what vector $\boldsymbol{q}$? Start with $\boldsymbol{a}$, subtract _____, divide by _____ to find $\boldsymbol{q}$.

Chapter 5

Determinants

1 The **determinant** of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\boldsymbol{ad - bc}$. Singular matrix $A = \begin{bmatrix} a & xa \\ c & xc \end{bmatrix}$ has $\mathbf{det = 0}$.

2 **Row exchange reverses signs** $PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ has $\det PA = bc - ad = \mathbf{-\det \boldsymbol{A}}$.

3 The determinant of $\begin{bmatrix} xa + yA & xb + yB \\ c & d \end{bmatrix}$ is $x(ad - bc) + y(Ad - Bc)$. **Det is linear in row 1 by itself.**

4 **Elimination** $EA = \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{bmatrix}$ $\quad \det \boldsymbol{EA} = a\left(d - \frac{c}{a} b\right) =$ **product of pivots** $= \det \boldsymbol{A}$.

5 If A is n by n then $\mathbf{1, 2, 3, 4}$ remain true: $\mathbf{det = 0}$ when A is singular, **det reverses sign** when rows are exchanged, **det is linear in row 1 by itself**, det $=$ **product of the pivots**. Always $\det \boldsymbol{BA} = (\det \boldsymbol{B})(\det \boldsymbol{A})$ and $\det \boldsymbol{A}^{\mathrm{T}} = \det \boldsymbol{A}$. This is an amazing number.

5.1 The Properties of Determinants

The determinant of a square matrix is a single number. That number contains an amazing amount of information about the matrix. It tells immediately whether the matrix is invertible. ***The determinant is zero when the matrix has no inverse***. When A is invertible, the determinant of A^{-1} is $1/(\det A)$. If $\det A = 2$ then $\det A^{-1} = \frac{1}{2}$. In fact the determinant leads to a formula for every entry in A^{-1}.

This is one use for determinants—to find formulas for inverse matrices and pivots and solutions $A^{-1}\boldsymbol{b}$. For a large matrix we seldom use those formulas, because elimination is faster. For a 2 by 2 matrix with entries a, b, c, d, its determinant $ad - bc$ shows how A^{-1} changes as A changes. Notice the division by the determinant!

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has inverse} \quad A^{-1} = \frac{\mathbf{1}}{\boldsymbol{ad - bc}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{1}$$

Multiply those matrices to get I. When the determinant is $ad - bc = 0$, we are asked to divide by zero and we can't—then A has no inverse. (The rows are parallel when $a/c = b/d$. This gives $ad = bc$ and $\det A = 0$.) Dependent rows always lead to $\det A = 0$.

The determinant is also connected to the pivots. For a 2 by 2 matrix the pivots are a and $d - (c/a)b$. ***The product of the pivots is the determinant***:

$$\textbf{Product of pivots} \qquad a\left(d - \frac{c}{a}b\right) = ad - bc \quad \textbf{which is} \quad \det A.$$

After a row exchange the pivots change to c and $b - (a/c)d$. Those new pivots multiply to give $bc - ad$. The row exchange to $\left[\begin{smallmatrix} \mathbf{c} & \mathbf{d} \\ \mathbf{a} & \mathbf{b} \end{smallmatrix}\right]$ reversed the sign of the determinant.

Looking ahead The determinant of an n by n matrix can be found in three ways:

1 Multiply the n pivots (times 1 or -1)	This is the **pivot formula**.
2 Add up $n!$ terms (times 1 or -1)	This is the **"big" formula**.
3 Combine n smaller determinants (times 1 or -1)	This is the **cofactor formula**.

You see that *plus or minus signs*—the decisions between 1 and -1—play a big part in determinants. That comes from the following rule for n by n matrices:

The determinant changes sign when two rows (**or two columns*****) are exchanged.***

The identity matrix has determinant $+1$. Exchange two rows and $\det P = -1$. Exchange two more rows and the new permutation has $\det P = +1$. Half of all permutations are *even* ($\det P = 1$) and half are *odd* ($\det P = -1$). Starting from I, half of the P's involve an even number of exchanges and half require an odd number. In the 2 by 2 case, ad has a plus sign and bc has minus—coming from the row exchange:

$$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \quad \text{and} \quad \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1.$$

The other essential rule is linearity—but a warning comes first. Linearity does not mean that $\det(A + B) = \det A + \det B$. ***This is absolutely false***. That kind of linearity is not even true when $A = I$ and $B = I$. The false rule would say that $\det(I + I) = 1 + 1 = 2$. The true rule is $\det 2I = 2^n$. Determinants are multiplied by 2^n (not just by 2) when matrices are multiplied by 2.

We don't intend to define the determinant by its formulas. It is better to start with its properties—*sign reversal and linearity*. The properties are simple (Section 5.1). They prepare for the formulas (Section 5.2). Then come the applications, including these three:

(1) Determinants give A^{-1} and $A^{-1}\boldsymbol{b}$ (this formula is called **Cramer's Rule**).

(2) When the edges of a box are the rows of A, the **volume** is $|\det A|$.

(3) For n special numbers λ, called **eigenvalues**, the determinant of $A - \lambda I$ is zero. This is a truly important application and it fills Chapter 6.

The Properties of the Determinant

Determinants have three basic properties (rules 1, 2, 3). By using those rules we can compute the determinant of any square matrix A. ***This number is written in two ways,* det A *and* $|A|$.** Notice: Brackets for the matrix, straight bars for its determinant. When A is a 2 by 2 matrix, the rules 1, 2, 3 lead to the answer we expect:

$$\text{The determinant of} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

From rules 1–3 we will reach rules 4–10. The last two are $\det(AB) = (\det A)(\det B)$ and $\det A^{\mathrm{T}} = \det A$. We will check all rules with the 2 by 2 formula, but do not forget: The rules apply to any n by n matrix A.

Rule 1 (the easiest) matches $\det I = 1$ with volume $= 1$ for a unit cube.

1 ***The determinant of the n by n identity matrix is* 1.**

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \text{and} \qquad \begin{vmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{vmatrix} = 1.$$

2 ***The determinant changes sign when two rows are exchanged*** (sign reversal):

$$\text{Check:} \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \text{(both sides equal } bc - ad\text{).}$$

Because of this rule, we can find $\det P$ for any permutation matrix. Just exchange rows of I until you reach P. Then $\det P = +1$ for an ***even*** number of row exchanges and $\det P = -1$ for an ***odd*** number.

The third rule has to make the big jump to the determinants of all matrices.

3 ***The determinant is a linear function of each row separately*** (all other rows stay fixed). If the first row is multiplied by t, the determinant is multiplied by t. If first rows are added, determinants are added. This rule only applies when the other rows do not change! Notice how c and d stay the same:

multiply row 1 by any number t **det is multiplied by t**	$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
add row 1 of A to row 1 of A': **then determinants add**	$\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$

In the first case, both sides are $tad - tbc$. Then t factors out. In the second case, both sides are $ad + a'd - bc - b'c$. These rules still apply when A is n by n, and **one row changes.**

$$A = \begin{vmatrix} 4 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 4 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}.$$

By itself, rule 3 does not say what those determinants are ($\det A$ is 4).

Combining multiplication and addition, we get *any linear combination in one row*. Rule 2 for row exchanges can put that row into the first row and back again.

This rule does not mean that $\det 2I = 2 \det I$. To obtain $2I$ we have to multiply *both* rows by 2, and the factor 2 comes out both times:

$$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2^2 = 4 \qquad \text{and} \qquad \begin{vmatrix} t & 0 \\ 0 & t \end{vmatrix} = t^2.$$

This is just like area and volume. Expand a rectangle by 2 and its area increases by 4. Expand an n-dimensional box by t and its volume increases by t^n. The connection is no accident—we will see how *determinants equal volumes*.

Pay special attention to rules 1–3. They completely determine the number $\det A$. We could stop here to find a formula for n by n determinants (a little complicated). We prefer to go gradually, because rules $4 - 10$ make determinants much easier to work with.

4 ***If two rows of A are equal, then*** $\det A = 0$.

Equal rows Check 2 by 2 : $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0.$

Rule 4 follows from rule 2. (Remember we must use the rules and not the 2 by 2 formula.) *Exchange the two equal rows*. The determinant D is supposed to change sign. But also D has to stay the same, because the matrix is not changed. The only number with $-D = D$ is $D = 0$—this must be the determinant. (Note: In Boolean algebra the reasoning fails, because $-1 = 1$. Then D is defined by rules 1, 3, 4.)

A matrix with two equal rows has no inverse. Rule 4 makes $\det A = 0$. But matrices can be singular and determinants can be zero without having equal rows! Rule 5 will be the key. We can do row operations (like elimination) without changing $\det A$.

5 ***Subtracting a multiple of one row from another row leaves*** $\det A$ ***unchanged.***

ℓ times row 1 from row 2 $\begin{vmatrix} a & b \\ c - \ell a & d - \ell b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$

Rule 3 (linearity) splits the left side into the right side plus another term $-\ell \begin{vmatrix} a & b \\ a & b \end{vmatrix}$. This extra term is zero by rule 4: equal rows. Therefore rule 5 is correct (not just 2 by 2).

Conclusion *The determinant is not changed by the usual elimination steps from A to U.* Thus $\det A$ *equals* $\det U$. If we can find determinants of triangular matrices U, we can find determinants of all matrices A. Every row exchange reverses the sign, so always $\det A = \pm \det U$. Rule 5 has narrowed the problem to triangular matrices.

6 ***A matrix with a row of zeros has*** $\det A = 0$.

Row of zeros $\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0.$

For an easy proof, add some other row to the zero row. The determinant is not changed (rule 5). But the matrix now has two equal rows. So $\det A = 0$ by rule 4.

7 ***If A is triangular then*** $\det A = a_{11}a_{22}\cdots a_{nn}$ ***= product of diagonal entries.***

Triangular $\quad \begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad \quad$ and also $\quad \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = ad.$

Suppose all diagonal entries are nonzero. Remove the off-diagonal entries by elimination! (If A is lower triangular, subtract multiples of each row from lower rows. If A is upper triangular, subtract from higher rows.) By rule 5 the determinant is not changed—and now the matrix is diagonal:

Diagonal matrix $$\det \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix} = (a_{11})(a_{22})\cdots(a_{nn}).$$

Factor a_{11} from the first row by rule 3. Then factor a_{22} from the second row. Eventually factor a_{nn} from the last row. The determinant is a_{11} times a_{22} times $\cdots$ times a_{nn} times $\det I$. Then rule 1 (used at last!) is $\det I = 1$.

What if a diagonal entry a_{ii} is zero? Then the triangular A is singular. Elimination produces a *zero row*. By rule 5 the determinant is unchanged, and by rule 6 a zero row means $\det A = 0$. We reach the great test for **singular or invertible** matrices.

8 ***If A is singular then*** $\det A = 0$. ***If A is invertible then*** $\det A \neq 0$.

Singular $\quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad$ is singular if and only if $\quad ad - bc = 0.$

Proof Elimination goes from A to U. If A is singular then U has a zero row. The rules give $\det A = \det U = 0$. If A is invertible then U has the pivots along its diagonal. The product of nonzero pivots (using rule 7) gives a nonzero determinant:

Multiply pivots $\quad$ $\det A = \pm \det U = \pm$ **(product of the pivots).** $\quad$ (2)

The pivots of a 2 by 2 matrix (if $a \neq 0$) are a and $d - (c/a)b$:

The determinant is $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d-(c/a)b \end{vmatrix} = ad - bc.$$

This is the first formula for the determinant. MATLAB multiplies the pivots to find $\det A$. The sign in $\pm \det U$ depends on whether the number of row exchanges is even or odd: $+1$ or -1 is the determinant of the permutation P that exchanges rows.

With no row exchanges, $P = I$ and $\det A = \det U =$ *product of pivots*. And $\det L = 1$:

$$\text{If} \quad PA = LU \quad \text{then} \quad \det P \, \det A = \det L \, \det U \quad \text{and} \quad \det A = \pm \det U. \qquad (3)$$

9 ***The determinant of AB is* det A *times* det B: $|AB| = |A|\,|B|$.**

Product rule $$\begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{vmatrix} p & q \\ r & s \end{vmatrix} = \begin{vmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{vmatrix}.$$

When the matrix B is A^{-1}, *this rule says that the determinant of* A^{-1} is $\mathbf{1/\det A}$:

A times A^{-1} $\quad \boldsymbol{AA^{-1} = I}$ **so** $\boxed{(\det \boldsymbol{A})(\det \boldsymbol{A^{-1}}) = \det \boldsymbol{I} = \mathbf{1}.}$

This product rule is the most intricate so far. Even the 2 by 2 case needs some algebra:

$$|A|\,|B| = (ad-bc)(ps-qr) = (ap+br)(cq+ds) - (aq+bs)(cp+dr) = |AB|.$$

For the n by n case, here is a snappy proof that $|AB| = |A|\,|B|$. When $|B|$ is not zero, consider the ratio $\boldsymbol{D(A) = |AB|/|B|}$. *Check that this ratio $D(A)$ has properties* 1,2,3. Then $D(A)$ has to be the determinant and we have $|AB|/|B| = |A|$. Good.

Property 1 (*Determinant of I*) If $A = I$ then the ratio $D(A)$ becomes $|B|/|B| = 1$.

Property 2 (*Sign reversal*) When two rows of A are exchanged, so are the same two rows of AB. Therefore $|AB|$ changes sign and so does the ratio $|AB|/|B|$.

Property 3 (*Linearity*) When row 1 of A is multiplied by t, so is row 1 of AB. This multiplies the determinant $|AB|$ by t. So the ratio $|AB|/|B|$ is multiplied by t.

Add row 1 of A to row 1 of A'. Then row 1 of AB adds to row 1 of $A'B$. By rule 3, determinants add. After dividing by $|B|$, the ratios add—as desired.

Conclusion This ratio $|AB|/|B|$ has the same three properties that define $|A|$. Therefore it equals $|A|$. This proves the product rule $|AB| = |A|\,|B|$. The case $|B| = 0$ is separate and easy, because AB is singular when B is singular. Then $|AB| = |A|\,|B|$ is $0 = 0$.

10 ***The transpose A^{T} has the same determinant as A.***

Transpose $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} \quad \text{since both sides equal} \quad ad-bc.$$

The equation $|A^{\mathrm{T}}| = |A|$ becomes $0 = 0$ when A is singular (we know that A^{T} is also singular). Otherwise A has the usual factorization $PA = LU$. Transposing both sides gives $A^{\mathrm{T}}P^{\mathrm{T}} = U^{\mathrm{T}}L^{\mathrm{T}}$. The proof of $|A| = |A^{\mathrm{T}}|$ comes by using rule 9 for products:

$$\text{Compare} \quad \det P \det A = \det L \det U \quad \text{with} \quad \det A^{\mathrm{T}} \det P^{\mathrm{T}} = \det U^{\mathrm{T}} \det L^{\mathrm{T}}.$$

First, $\det L = \det L^{\mathrm{T}} = 1$ (both have 1's on the diagonal). Second, $\det U = \det U^{\mathrm{T}}$ (those triangular matrices have the same diagonal). Third, $\det P = \det P^{\mathrm{T}}$ (permutations have $P^{\mathrm{T}}P = I$, so $|P^{\mathrm{T}}|\,|P| = 1$ by rule 9; thus $|P|$ and $|P^{\mathrm{T}}|$ both equal 1 or both equal -1). So L, U, P have the same determinants as $L^{\mathrm{T}}, U^{\mathrm{T}}, P^{\mathrm{T}}$ and this leaves $\det A = \det A^{\mathrm{T}}$.

Important comment on columns Every rule for the rows can apply to the columns (just by transposing, since $|A| = |A^{\mathrm{T}}|$). The determinant changes sign when two columns are exchanged. *A zero column or two equal columns will make the determinant zero.* If a column is multiplied by t, so is the determinant. The determinant is a linear function of each column separately.

It is time to stop. The list of properties is long enough. Next we find and use an explicit formula for the determinant.

■ REVIEW OF THE KEY IDEAS ■

1. The determinant is defined by $\det I = 1$, sign reversal, and linearity in each row.

2. After elimination $\det A$ is $\pm$ (product of the pivots).

3. The determinant is zero exactly when A is not invertible.

4. Two remarkable properties are $\det AB = (\det A)(\det B)$ and $\det A^{\mathrm{T}} = \det A$.

■ WORKED EXAMPLES ■

5.1 A Apply these operations to A and find the determinants of M_1, M_2, M_3, M_4:
In M_1, multiplying each a_{ij} by $(-1)^{i+j}$ gives a checkerboard sign pattern.
In M_2, rows $1, 2, 3$ of A are *subtracted* from rows $2, 3, 1$.
In M_3, rows $1, 2, 3$ of A are *added* to rows $2, 3, 1$.
How are the determinants of M_1, M_2, M_3 related to the determinant of A?

$$\begin{bmatrix} a_{11} & -a_{12} & a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ a_{31} & -a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} \text{row } 1 - \text{row } 3 \\ \text{row } 2 - \text{row } 1 \\ \text{row } 3 - \text{row } 2 \end{bmatrix} \qquad \begin{bmatrix} \text{row } 1 + \text{row } 3 \\ \text{row } 2 + \text{row } 1 \\ \text{row } 3 + \text{row } 2 \end{bmatrix}$$

Solution The three determinants are $\det A$, 0, and $2 \det A$. Here are reasons:

$$M_1 = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \quad \text{so } \det M_1 = (-1)(\det A)(-1).$$

M_2 is singular because its rows add to the zero row. Its determinant is zero.
M_3 can be split into *eight matrices* by Rule 3 (linearity in each row separately):

$$\begin{vmatrix} \text{row } 1 + \text{row } 3 \\ \text{row } 2 + \text{row } 1 \\ \text{row } 3 + \text{row } 3 \end{vmatrix} = \begin{vmatrix} \text{row } 1 \\ \text{row } 2 \\ \text{row } 3 \end{vmatrix} + \begin{vmatrix} \text{row } 3 \\ \text{row } 2 \\ \text{row } 3 \end{vmatrix} + \begin{vmatrix} \text{row } 1 \\ \text{row } 1 \\ \text{row } 3 \end{vmatrix} + \cdots + \begin{vmatrix} \text{row } 3 \\ \text{row } 1 \\ \text{row } 2 \end{vmatrix}.$$

All but the first and last have repeated rows and zero determinant. The first is A and the last has *two* row exchanges. So $\det M_3 = \det A + \det A$. (Try $A = I$.)

5.1 B Explain how to reach this determinant by row operations:

$$\det \begin{bmatrix} 1-a & 1 & 1 \\ 1 & 1-a & 1 \\ 1 & 1 & 1-a \end{bmatrix} = a^2(3-a). \qquad (4)$$

Solution Subtract row 3 from row 1 and then from row 2. This leaves

$$\det \begin{bmatrix} -a & 0 & a \\ 0 & -a & a \\ 1 & 1 & 1-a \end{bmatrix}.$$

Now add column 1 to column 3, and also column 2 to column 3. This leaves a lower triangular matrix with $-a, -a, 3-a$ on the diagonal: $\det = (-a)(-a)(3-a)$.

The determinant is zero if $a = 0$ or $a = 3$. For $a = 0$ we have the *all-ones matrix*—certainly singular. For $a = 3$, each row adds to zero—again singular. Those numbers 0 and 3 are the **eigenvalues** of the all-ones matrix. This example is revealing and important, leading toward Chapter 6.

Problem Set 5.1

Questions 1–12 are about the rules for determinants.

1 If a 4 by 4 matrix has $\det A = \frac{1}{2}$, find $\det(2A)$ and $\det(-A)$ and $\det(A^2)$ and $\det(A^{-1})$.

2 If a 3 by 3 matrix has $\det A = -1$, find $\det\left(\frac{1}{2}A\right)$ and $\det(-A)$ and $\det(A^2)$ and $\det(A^{-1})$.

3 True or false, with a reason if true or a counterexample if false:

(a) The determinant of $I + A$ is $1 + \det A$.

(b) The determinant of ABC is $|A|\,|B|\,|C|$.

(c) The determinant of $4A$ is $4|A|$.

(d) The determinant of $AB - BA$ is zero. Try an example with $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

4 Which row exchanges show that these "reverse identity matrices" J_3 and J_4 have $|J_3| = -1$ but $|J_4| = +1$?

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = -1 \qquad \text{but} \qquad \det \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = +1.$$

5 For $n = 5, 6, 7$, count the row exchanges to permute the reverse identity J_n to the identity matrix I_n. Propose a rule for every size n and predict whether J_{101} has determinant $+1$ or -1.

6 Show how Rule 6 (determinant $= 0$ if a row is all zero) comes from Rule 3.

7 Find the determinants of rotations and reflections:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1-2\cos^2\theta & -2\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 1-2\sin^2\theta \end{bmatrix}.$$

8 Prove that every orthogonal matrix ($Q^{\mathrm{T}}Q = I$) has determinant 1 or -1.

(a) Use the product rule $|AB| = |A|\,|B|$ and the transpose rule $|Q| = |Q^{\mathrm{T}}|$.

(b) Use only the product rule. If $|\det Q| > 1$ then $\det Q^n = (\det Q)^n$ blows up. How do you know this can't happen to Q^n?

9 Do these matrices have determinant $0, 1, 2$, or 3?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

10 If the entries in every row of A add to zero, solve $Ax = \mathbf{0}$ to prove $\det A = 0$. If those entries add to one, show that $\det(A - I) = 0$. Does this mean $\det A = 1$?

11 Suppose that $CD = -DC$ and find the flaw in this reasoning: Taking determinants gives $|C|\,|D| = -|D|\,|C|$. Therefore $|C| = 0$ or $|D| = 0$. One or both of the matrices must be singular. (That is not true.)

12 The inverse of a 2 by 2 matrix seems to have determinant $= 1$:

$$\det A^{-1} = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{ad-bc}{ad-bc} = 1.$$

What is wrong with this calculation? What is the correct $\det A^{-1}$?

Questions 13–27 use the rules to compute specific determinants.

13 Reduce A to U and find $\det A =$ product of the pivots:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}.$$

14 By applying row operations to produce an upper triangular U, compute

$$\det \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} \quad \text{and} \quad \det \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

15 Use row operations to simplify and compute these determinants:

$$\det\begin{bmatrix}101 & 201 & 301\\ 102 & 202 & 302\\ 103 & 203 & 303\end{bmatrix} \quad \text{and} \quad \det\begin{bmatrix}1 & t & t^2\\ t & 1 & t\\ t^2 & t & 1\end{bmatrix}.$$

16 Find the determinants of a rank one matrix and a skew-symmetric matrix :

$$A=\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & -4 & 5\end{bmatrix} \quad \text{and} \quad A=\begin{bmatrix}0 & 1 & 3\\ -1 & 0 & 4\\ -3 & -4 & 0\end{bmatrix}.$$

17 A skew-symmetric matrix has $A^{\mathrm{T}}=-A$. Insert a, b, c for $1, 3, 4$ in Question 16 and show that $|A|=0$. Write down a 4 by 4 example with $|A|=1$.

18 Use row operations to show that the 3 by 3 "Vandermonde determinant" is

$$\det\begin{bmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{bmatrix}=(b-a)(c-a)(c-b).$$

19 Find the determinants of U and U^{-1} and U^2:

$$U=\begin{bmatrix}1 & 4 & 6\\ 0 & 2 & 5\\ 0 & 0 & 3\end{bmatrix} \quad \text{and} \quad U=\begin{bmatrix}a & b\\ 0 & d\end{bmatrix}.$$

20 Suppose you do two row operations at once, going from

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad \text{to} \quad \begin{bmatrix}a-Lc & b-Ld\\ c-la & d-lb\end{bmatrix}.$$

Find the second determinant. Does it equal $ad-bc$?

21 *Row exchange*: Add row 1 of A to row 2, then subtract row 2 from row 1. Then add row 1 to row 2 and multiply row 1 by -1 to reach B. Which rules show

$$\det B=\begin{vmatrix}c & d\\ a & b\end{vmatrix} \quad \text{equals} \quad -\det A=-\begin{vmatrix}a & b\\ c & d\end{vmatrix}?$$

Those rules could replace Rule 2 in the definition of the determinant.

22 From $ad-bc$, find the determinants of A and A^{-1} and $A-\lambda I$:

$$A=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} \quad \text{and} \quad A^{-1}=\frac{1}{3}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix} \quad \text{and} \quad A-\lambda I=\begin{bmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{bmatrix}.$$

Which two numbers λ lead to $\det(A-\lambda I)=0$? Write down the matrix $A-\lambda I$ for each of those numbers λ—it should not be invertible.

23 From $A = \left[\begin{smallmatrix} 4 & 1 \\ 2 & 3 \end{smallmatrix}\right]$ find A^2 and A^{-1} and $A - \lambda I$ and their determinants. Which two numbers λ lead to $\det(A - \lambda I) = 0$?

24 Elimination reduces A to U. Then $A = LU$:

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = LU.$$

Find the determinants of L, U, A, $U^{-1}L^{-1}$, and $U^{-1}L^{-1}A$.

25 If the i, j entry of A is i times j, show that $\det A = 0$. (Exception when $A = [\,1\,]$.)

26 If the i, j entry of A is $i + j$, show that $\det A = 0$. (Exception when $n = 1$ or 2.)

27 Compute the determinants of these matrices by row operations:

$$A = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}.$$

28 True or false (give a reason if true or a 2 by 2 example if false):

(a) If A is not invertible then AB is not invertible.
(b) The determinant of A is always the product of its pivots.
(c) The determinant of $A - B$ equals $\det A - \det B$.
(d) AB and BA have the same determinant.

29 What is wrong with this proof that projection matrices have $\det P = 1$?

$$P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} \qquad \text{so} \qquad |P| = |A| \frac{1}{|A^{\mathrm{T}}||A|} |A^{\mathrm{T}}| = 1.$$

30 (Calculus question) Show that the partial derivatives of $\ln(\det A)$ give A^{-1}!

$$f(a, b, c, d) = \ln(ad - bc) \qquad \text{leads to} \qquad \begin{bmatrix} \partial f/\partial a & \partial f/\partial c \\ \partial f/\partial b & \partial f/\partial d \end{bmatrix} = A^{-1}.$$

31 (MATLAB) The Hilbert matrix **hilb**(n) has i, j entry equal to $1/(i + j - 1)$. Print the determinants of **hilb**(1), **hilb**(2), . . ., **hilb**(10). Hilbert matrices are hard to work with! What are the pivots of **hilb** (5)?

32 (MATLAB) What is a typical determinant (experimentally) of **rand**(n) and **randn**(n) for $n = 50, 100, 200, 400$? (And what does “Inf” mean in MATLAB?)

33 (MATLAB) Find the largest determinant of a 6 by 6 matrix of 1’s and -1’s.

34 If you know that $\det A = 6$, what is the determinant of B?

$$\text{From } \det A = \begin{vmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{vmatrix} = 6 \text{ find } \det B = \begin{vmatrix} \text{row 3 + row 2 + row 1} \\ \text{row 2 + row 1} \\ \text{row 1} \end{vmatrix}.$$

5.2 Permutations and Cofactors

1 **2 by 2**: $ad - bc$ has $2\,!$ terms with $\pm$ signs. **n by n: det A adds $n!$ terms with $\pm$ signs.**

2 For $n = 3$, det A adds $3! = 6$ terms. Two terms are $+a_{12}a_{23}a_{31}$ and $-a_{13}a_{22}a_{31}$. **Rows 1, 2, 3 and columns 1, 2, 3 appear once in each term.**

3 That minus sign came because the column order $3, 2, 1$ needs one exchange to recover $1, 2, 3$.

4 The six terms include $+a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} = a_{11}(\boldsymbol{a_{22}a_{33} - a_{23}a_{32}}) = a_{11}(\textbf{cofactor } \boldsymbol{C_{11}})$.

5 Always $\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$. Cofactors are determinants of size $n-1$.

A computer finds the determinant from the pivots. This section explains two other ways to do it. There is a "big formula" using all $n!$ permutations. There is a "cofactor formula" using determinants of size $n-1$. The best example is my favorite 4 by 4 matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad \text{has} \quad \det A = 5.$$

We can find this determinant in all three ways: ***pivots, big formula, cofactors***.

1. The product of the pivots is $2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4}$. Cancellation produces 5.

2. The "big formula" in equation (8) has $4! = 24$ terms. Only five terms are nonzero:

$$\det A = 16 - 4 - 4 - 4 + 1 = 5.$$

   The 16 comes from $2 \cdot 2 \cdot 2 \cdot 2$ on the diagonal of A. Where do -4 and $+1$ come from? When you can find those five terms, you have understood formula (8).

3. The numbers $2, -1, 0, 0$ in the first row multiply their cofactors $4, 3, 2, 1$ from the other rows. That gives $2 \cdot 4 - 1 \cdot 3 = 5$. Those cofactors are 3 by 3 determinants. Cofactors use the rows and columns that are *not* used by the entry in the first row. ***Every term in a determinant uses each row and column once!***

The Pivot Formula

When elimination leads to $A = LU$, the pivots $d_1, \ldots, d_n$ are on the diagonal of the upper triangular U. If no row exchanges are involved, ***multiply those pivots*** to find the determinant:

$$\det A = (\det L)(\det U) = (1)(d_1 d_2 \cdots d_n). \tag{1}$$

This formula for $\det A$ appeared in Section 5.1, with the further possibility of row exchanges. Then a permutation enters $PA = LU$. The determinant of P is -1 or $+1$.

$$\boxed{(\det \boldsymbol{P})(\det \boldsymbol{A}) = (\det \boldsymbol{L})(\det \boldsymbol{U}) \quad \text{gives} \quad \det A = \pm(d_1 d_2 \cdots d_n).} \tag{2}$$

Example 1 A row exchange produces pivots 4, 2, 1 and that important minus sign:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad PA = \begin{bmatrix} 4 & 5 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \det A = -(4)(2)(1) = -8.$$

The odd number of row exchanges (namely one exchange) means that $\det P = -1$.

The next example has no row exchanges. It may be the first matrix we factored into LU (when it was 3 by 3). What is remarkable is that we can go directly to n by n. Pivots give the determinant. We will also see how determinants give the pivots.

Example 2 The first pivots of this tridiagonal matrix A are $2, \frac{3}{2}, \frac{4}{3}$. The next are $\frac{5}{4}$ and $\frac{6}{5}$ and eventually $\frac{n+1}{n}$. Factoring this n by n matrix reveals its determinant:

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \cdot & \\ & & \cdot & \cdot & -1 \\ & & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ -\frac{1}{2} & 1 & & & \\ & -\frac{2}{3} & 1 & & \\ & & \cdot & \cdot & \\ & & & -\frac{n-1}{n} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{2} & -1 & & & \\ & \mathbf{\frac{3}{2}} & -1 & & \\ & & \mathbf{\frac{4}{3}} & -1 & \\ & & & \cdot & \cdot \\ & & & & \mathbf{\frac{n+1}{n}} \end{bmatrix}$$

The pivots are on the diagonal of U (the last matrix). When 2 and $\frac{3}{2}$ and $\frac{4}{3}$ and $\frac{5}{4}$ are multiplied, the fractions cancel. The determinant of the 4 by 4 matrix is 5. The 3 by 3 determinant is 4. *The n by n determinant is* $n+1$:

$$\mathbf{-1, 2, -1 \ matrix} \qquad \det A = (\mathbf{2})\left(\tfrac{\mathbf{3}}{\mathbf{2}}\right)\left(\tfrac{\mathbf{4}}{\mathbf{3}}\right)\cdots\left(\tfrac{\mathbf{n+1}}{\mathbf{n}}\right) = \boldsymbol{n+1}.$$

Important point: The first pivots depend only on the *upper left corner* of the original matrix A. This is a rule for all matrices without row exchanges.

> The first k pivots come from the k by k matrix A_k in the top left corner of A.
> ***The determinant of that corner submatrix*** A_k ***is*** $d_1 d_2 \cdots d_k$ ***(first k pivots)***.

The 1 by 1 matrix A_1 contains the very first pivot d_1. This is $\det A_1$. The 2 by 2 matrix in the corner has $\det A_2 = d_1 d_2$. Eventually the n by n determinant multiplies all n pivots.

Elimination deals with the matrix A_k in the upper left corner while starting on the whole matrix. We assume no row exchanges—then $A = LU$ and $A_k = L_k U_k$. Dividing one determinant by the previous determinant ($\det A_k$ divided by $\det A_{k-1}$) cancels everything but the latest pivot d_k. ***Each pivot is a ratio of determinants***:

$$\textbf{Pivots from determinants} \qquad \textbf{The } k\textbf{th pivot is } \boldsymbol{d_k} = \frac{d_1 d_2 \cdots d_k}{d_1 d_2 \cdots d_{k-1}} = \boxed{\frac{\det A_k}{\det A_{k-1}}}. \tag{3}$$

We don't need row exchanges when all the upper left submatrices have $\det A_k \neq 0$.

The Big Formula for Determinants

Pivots are good for computing. They concentrate a lot of information—enough to find the determinant. But it is hard to connect them to the original a_{ij}. That part will be clearer if we go back to rules 1-2-3, linearity and sign reversal and $\det I = 1$. We want to derive a single explicit formula for the determinant, directly from the entries a_{ij}.

The formula has $n!$ terms. Its size grows fast because $n! = 1, 2, 6, 24, 120, \ldots$. For $n = 11$ there are about forty million terms. For $n = 2$, the two terms are ad and bc. Half the terms have minus signs (as in $-bc$). The other half have plus signs (as in ad). For $n = 3$ there are $3! = (3)(2)(1)$ terms. Here are those six terms:

$$\text{3 by 3 determinant} \quad \begin{vmatrix} a_{11} & a_{12} & \mathbf{a_{13}} \\ \mathbf{a_{21}} & a_{22} & a_{23} \\ a_{31} & \mathbf{a_{32}} & a_{33} \end{vmatrix} = \begin{array}{l} +a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + \mathbf{a_{13}a_{21}a_{32}} \\ -a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \end{array} \tag{4}$$

Notice the pattern. Each product like $a_{11}a_{23}a_{32}$ has ***one entry from each row***. It also has ***one entry from each column***. The column order 1, 3, 2 means that this particular term comes with a minus sign. The column order 3, 1, 2 in $a_{13}a_{21}a_{32}$ has a plus sign (boldface). It will be "permutations" that tell us the sign.

The next step ($n = 4$) brings $4! = 24$ terms. There are 24 ways to choose one entry from each row and column. Down the main diagonal, $a_{11}a_{22}a_{33}a_{44}$ with column order $1, 2, 3, 4$ always has a plus sign. That is the "identity permutation".

To derive the big formula I start with $n = 2$. The goal is to reach $ad - bc$ in a systematic way. Break each row into two simpler rows:

$$\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} c & 0 \end{bmatrix} + \begin{bmatrix} 0 & d \end{bmatrix}.$$

Now apply linearity, first in row 1 (with row 2 fixed) and then in row 2 (with row 1 fixed):

$$\begin{aligned} \begin{vmatrix} a & b \\ c & d \end{vmatrix} &= \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} && \textit{(break up row 1)} \\ &= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} && \textit{(break up row 2).} \end{aligned} \tag{5}$$

The last line has $2^2 = 4$ determinants. The first and fourth are zero because one row is a multiple of the other row. We are left with $2! = 2$ determinants to compute:

$$\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = ad - bc.$$

The splitting led to permutation matrices. Their determinants give a plus or minus sign. The permutation tells the column sequence. In this case the column order is $(1, 2)$ or $(2, 1)$.

Now try $n = 3$. Each row splits into 3 simpler rows like $\begin{bmatrix} a_{11} & 0 & 0 \end{bmatrix}$. Using linearity in each row, $\det A$ splits into $3^3 = 27$ simple determinants. If a column choice is repeated—for example if we also choose the row $\begin{bmatrix} a_{21} & 0 & 0 \end{bmatrix}$—then the simple determinant is zero.

We pay attention only when ***the entries a_{ij} come from different columns***, like $(\mathbf{3},\mathbf{1},\mathbf{2})$:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & \boldsymbol{a_{13}} \\ \boldsymbol{a_{21}} & & \\ & \boldsymbol{a_{32}} & \end{vmatrix}$$

Six terms

$$+ \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}.$$

There are $3! = 6$ ***ways to order the columns, so six determinants***. The six permutations of $(1,2,3)$ include the identity permutation $(1,2,3)$ from $P = I$.

$$\textbf{Column numbers} = (1,2,3), (2,3,1), (\mathbf{3},\mathbf{1},\mathbf{2}), (1,3,2), (2,1,3), (3,2,1). \tag{6}$$

The last three are *odd permutations* (one exchange). The first three are *even permutations* (0 or 2 exchanges). When the column sequence is $(\mathbf{3},\mathbf{1},\mathbf{2})$, we have chosen the entries $a_{\mathbf{13}}a_{\mathbf{21}}a_{\mathbf{32}}$—that particular column sequence comes with a plus sign (2 exchanges). The determinant of A is now split into six simple terms. Factor out the a_{ij}:

$$\begin{aligned} \det A = {} & a_{11}a_{22}a_{33} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31} \begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{\mathbf{13}}a_{\mathbf{21}}a_{\mathbf{32}} \begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix} \\ & + a_{11}a_{23}a_{32} \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12}a_{21}a_{33} \begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + a_{13}a_{22}a_{31} \begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}. \end{aligned} \tag{7}$$

The first three (even) permutations have $\det P = +1$, the last three (odd) permutations have $\det P = -1$. We have proved the 3 by 3 formula in a systematic way.

Now you can see the n by n formula. There are $n!$ orderings of the columns. The columns $(1,2,\ldots,n)$ go in each possible order $(\alpha,\beta,\ldots,\omega)$. Taking $a_{1\alpha}$ from row 1 and $a_{2\beta}$ from row 2 and eventually $a_{n\omega}$ from row n, the determinant contains the product $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ times $+1$ or -1. Half the column orderings have sign -1.

The determinant of A is the sum of these $n!$ simple determinants, times 1 or -1. The simple determinants $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ choose ***one entry from every row and column***. For 5 by 5, the term $a_{15}a_{22}a_{33}a_{44}a_{51}$ would have $\det P = -1$ from exchanging 5 and 1.

$$\begin{aligned} \det A &= \text{sum over all } \mathbf{n}! \text{ column permutations } P = (\alpha,\beta,\ldots,\omega) \\ &= \boxed{\sum (\det P)\, a_{1\alpha}a_{2\beta}\cdots a_{n\omega}} = \textbf{BIG FORMULA}. \end{aligned} \tag{8}$$

The 2 by 2 case is $+a_{11}a_{22} - a_{12}a_{21}$ (which is $ad - bc$). Here P is $(1,2)$ or $(2,1)$.

The 3 by 3 case has three products "down to the right" (see Problem 28) and three products "down to the left". Warning: Many people believe they should follow this pattern in the 4 by 4 case. They only take 8 products—but we need 24.

Example 3 (Determinant of U) When U is upper triangular, only one of the $n!$ products can be nonzero. This one term comes from the diagonal: $\det U = +u_{11}u_{22}\cdots u_{nn}$. All other column orderings pick at least one entry below the diagonal, where U has zeros. As soon as we pick a number like $u_{21} = 0$, that term in equation (8) is sure to be zero.

Of course $\det I = 1$. The only nonzero term is $+(1)(1)\cdots(1)$ from the diagonal.

Example 4 Suppose Z is the identity matrix except for column 3. Then

$$\text{The determinant of } Z = \begin{vmatrix} 1 & 0 & \boldsymbol{a} & 0 \\ 0 & 1 & \boldsymbol{b} & 0 \\ 0 & 0 & \boldsymbol{c} & 0 \\ 0 & 0 & \boldsymbol{d} & 1 \end{vmatrix} \text{ is } \boldsymbol{c}. \tag{9}$$

The term $(1)(1)(c)(1)$ comes from the main diagonal with a plus sign. There are $4! = 24$ products (choosing one factor from each row and column) but the other 23 products are zero. Reason: If we pick a, b, or d from column 3, that column is used up. Then the only available choice from row 3 is zero.

Here is a different reason for the same answer. If $c = 0$, then Z has a row of zeros and $\det Z = c = 0$ is correct. If c is not zero, *use elimination*. Subtract multiples of row 3 from the other rows, to knock out a, b, d. That leaves a diagonal matrix and $\det Z = c$.

This example will soon be used for "Cramer's Rule". If we move a, b, c, d into the first column of Z, the determinant is $\det Z = a$. (*Why*?) Changing one column of I leaves Z with an easy determinant, coming from its main diagonal only.

Example 5 Suppose A has 1's just above and below the main diagonal. Here $n = 4$:

$$A = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 \end{bmatrix} \quad \text{have } \textbf{determinant 1}.$$

The only nonzero choice in the first row is column 2. The only nonzero choice in row 4 is column 3. Then rows 2 and 3 *must* choose columns 1 and 4. In other words $\det P = \det A$. The determinant of P is $+1$ (two exchanges to reach $2, 1, 4, 3$). Therefore $\det A = +1$.

Determinant by Cofactors

Formula (8) is a direct definition of the determinant. It gives you everything at once—but you have to digest it. Somehow this sum of $n!$ terms must satisfy rules 1-2-3 (then all the other properties 4-10 will follow). The easiest is $\det I = 1$, already checked.

When you separate out the factor a_{11} or a_{12} or $a_{1\alpha}$ that comes from the first row, you see linearity. For 3 by 3, separate the usual 6 terms of the determinant into 3 pairs:

$$\det A = \boldsymbol{a_{11}}\,(a_{22}a_{33} - a_{23}a_{32}) + \boldsymbol{a_{12}}\,(a_{23}a_{31} - a_{21}a_{33}) + \boldsymbol{a_{13}}\,(a_{21}a_{32} - a_{22}a_{31}). \tag{10}$$

Those three quantities in parentheses are called ***"cofactors"***. They are **2 by 2 determinants**, from rows 2 and 3. The first row contributes the factors a_{11}, a_{12}, a_{13}. *The lower rows contribute the cofactors* C_{11}, C_{12}, C_{13}. Certainly the determinant $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ depends linearly on a_{11}, a_{12}, a_{13}—this is Rule 3.

The cofactor of a_{11} is $C_{11} = a_{22}a_{33} - a_{23}a_{32}$. You can see it in this splitting:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}.$$

We are still choosing ***one entry from each row and column***. Since a_{11} uses up row 1 and column 1, that leaves a 2 by 2 determinant as its cofactor.

As always, we have to watch signs. The 2 by 2 determinant that goes with a_{12} looks like $a_{21}a_{33} - a_{23}a_{31}$. But in the cofactor C_{12}, *its sign is reversed.* Then $a_{12}C_{12}$ is the correct 3 by 3 determinant. The sign pattern for cofactors along the first row is *plus-minus-plus-minus*. ***You cross out row* 1 *and column* j *to get a submatrix* M_{1j} *of size* $n-1$.** Multiply its determinant by the sign $(-1)^{1+j}$ to get the cofactor:

$$\text{The cofactors along row 1 are } C_{1j} = (-1)^{1+j} \det M_{1j}.$$
$$\textbf{The cofactor expansion is } \det \boldsymbol{A} = \boldsymbol{a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}}. \tag{11}$$

In the big formula (8), the terms that multiply a_{11} combine to give $C_{11} = \det M_{11}$. The sign is $(-1)^{1+1}$, meaning *plus*. Equation (11) is another form of equation (8) and also equation (10), with factors from row 1 multiplying cofactors that use only the other rows.

Note Whatever is possible for row 1 is possible for row i. The entries a_{ij} in that row also have cofactors C_{ij}. Those are determinants of order $n-1$, multiplied by $(-1)^{i+j}$. Since a_{ij} accounts for row i and column j, ***the submatrix* M_{ij} *throws out row* i *and column* j.** The display shows a_{43} and M_{43} (with row 4 and column 3 removed). The sign $(-1)^{4+3}$ multiplies the determinant of M_{43} to give C_{43}. The sign matrix shows the $\pm$ pattern:

$$A = \begin{bmatrix} \bullet & \bullet & & \bullet \\ \bullet & \bullet & & \bullet \\ \bullet & \bullet & & \bullet \\ & & a_{43} & \end{bmatrix} \qquad \textbf{signs } (-1)^{i+j} = \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}.$$

The determinant is the dot product of any row i of A with its cofactors using other rows:

COFACTOR FORMULA $$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}. \tag{12}$$

Each cofactor C_{ij} (order $n-1$, without row i and column j) includes its correct sign:

Cofactor $$C_{ij} = (-1)^{i+j} \det M_{ij}.$$

A determinant of order n is a combination of determinants of order $n-1$. A recursive person would keep going. Each subdeterminant breaks into determinants of order $n-2$. *We could define all determinants via equation* (12). This rule goes from order n to $n-1$ to $n-2$ and eventually to order 1. Define the 1 by 1 determinant $|a|$ to be the number a. Then the cofactor method is complete.

We preferred to construct $\det A$ from its properties (linearity, sign reversal, $\det I = 1$). The big formula (8) and the cofactor formulas (10)–(12) follow from those rules. One last formula comes from the rule that $\det A = \det A^{\mathrm{T}}$. We can expand in cofactors, *down a column* instead of across a row. Down column j the entries are a_{1j} to a_{nj}. The cofactors are C_{1j} to C_{nj}. The determinant is the dot product:

Cofactors down column j $$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}. \tag{13}$$

Cofactors are useful when matrices have many zeros—as in the next examples.

Example 6 The $-1, 2, -1$ matrix has only two nonzeros in its first row. So only two cofactors C_{11} and C_{12} are involved in the determinant. I will highlight C_{12}:

$$\begin{vmatrix} 2 & -1 & & \\ \mathbf{-1} & 2 & \mathbf{-1} & \\ & -1 & \mathbf{2} & \mathbf{-1} \\ & & \mathbf{-1} & \mathbf{2} \end{vmatrix} = 2\begin{vmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} \mathbf{-1} & \mathbf{-1} & \\ & \mathbf{2} & \mathbf{-1} \\ & \mathbf{-1} & \mathbf{2} \end{vmatrix}. \tag{14}$$

You see 2 times C_{11} first on the right, from crossing out row 1 and column 1. This cofactor C_{11} has exactly the same $-1, 2, -1$ pattern as the original A—but one size smaller.

To compute the boldface C_{12}, *use cofactors down its first column*. The only nonzero is at the top. That contributes another -1 (so we are back to minus). Its cofactor is the $-1, 2, -1$ determinant which is 2 by 2, *two sizes smaller* than the original A.

Summary ***Each determinant*** D_n ***of order*** n ***comes from*** D_{n-1} ***and*** D_{n-2}:

$$D_4 = 2D_3 - D_2 \qquad \text{and generally} \qquad D_n = 2D_{n-1} - D_{n-2}. \tag{15}$$

Direct calculation gives $D_2 = 3$ and $D_3 = 4$. Equation (14) has $D_4 = 2(4) - 3 = 5$. These determinants 3, 4, 5 fit the formula $\boldsymbol{D_n = n+1}$. Then D_n equals $2n - (n-1)$. That "special tridiagonal answer" also came from the product of pivots in Example 2.

Example 7 This is the same matrix, except the first entry (upper left) is now 1:

$$B_4 = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}.$$

All pivots of this matrix turn out to be 1. So its determinant is 1. How does that come from cofactors? Expanding on row 1, the cofactors all agree with Example 6. Just change $a_{11} = 2$ to $b_{11} = 1$:

$$\det B_4 = D_3 - D_2 \qquad \text{instead of} \qquad \det A_4 = 2D_3 - D_2.$$

The determinant of B_4 is $4 - 3 = 1$. The determinant of every B_n is $n - (n-1) = 1$.
If you also change the last 2 into 1, why is $\det = 0$?

■ REVIEW OF THE KEY IDEAS ■

1. With no row exchanges, $\det A$ = (*product of pivots*). In the upper left corner of A, $\det A_k$ = (*product of the first k pivots*).

2. Every term in the big formula (8) uses each row and column once. Half of the $n!$ terms have plus signs (when $\det P = +1$) and half have minus signs.

3. The cofactor C_{ij} is $(-1)^{i+j}$ times the smaller determinant that omits row i and column j (because a_{ij} uses that row and column).

4. The determinant is the dot product of any row of A with its row of cofactors. When a row of A has a lot of zeros, we only need a few cofactors.

■ WORKED EXAMPLES ■

5.2 A A *Hessenberg matrix* is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule $|H_4| = |H_3| + |H_2|$. The same rule will continue for all sizes, $|H_n| = |H_{n-1}| + |H_{n-2}|$. Which Fibonacci number is $|H_n|$?

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad H_3 = \begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad H_4 = \begin{bmatrix} 2 & 1 & & \\ \mathbf{1} & 2 & \mathbf{1} & \\ \mathbf{1} & 1 & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

Solution The cofactor C_{11} for H_4 is the determinant $|H_3|$. We also need C_{12} (in boldface):

$$C_{12} = -\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \end{vmatrix} = -\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

Rows 2 and 3 stayed the same and we used linearity in row 1. The two determinants on the right are $-|H_3|$ and $+|H_2|$. Then the 4 by 4 determinant is

$$|H_4| = 2C_{11} + 1C_{12} = 2|H_3| - |H_3| + |H_2| = |H_3| + |H_2|.$$

The actual numbers are $|H_2| = 3$ and $|H_3| = 5$ (and of course $|H_1| = 2$). Since $|H_n| = 2, 3, 5, 8, \ldots$ follows Fibonacci's rule $|H_{n-1}| + |H_{n-2}|$, it must be $|H_n| = F_{n+2}$.

5.2 B These questions use the $\pm$ signs (even and odd P's) in the big formula for $\det A$:

1. If A is the 10 by 10 all-ones matrix, how does the big formula give $\det A = 0$?
2. If you multiply all $n!$ permutations together into a single P, is P odd or even?
3. If you multiply each a_{ij} by the fraction i/j, why is $\det A$ unchanged?

Solution In Question **1**, with all $a_{ij} = 1$, all the products in the big formula (8) will be 1. Half of them come with a plus sign, and half with minus. So they cancel to leave $\det A = 0$. (Of course the all-ones matrix is singular. I am assuming $n > 1$.)

In Question **2**, multiplying $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ gives an odd permutation. Also for 3 by 3, the three odd permutations multiply (in any order) to give *odd*. But for $n > 3$ the product of all permutations will be *even*. There are $n!/2$ odd permutations and that is an even number as soon as $n!$ includes the factor 4.

In Question **3**, each a_{ij} is multiplied by i/j. So each product $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ in the big formula is multiplied by all the row numbers $i = 1, 2, \ldots, n$ and divided by all the column numbers $j = 1, 2, \ldots, n$. (The columns come in some permuted order!) Then each product is unchanged and $\det A$ stays the same.

Another approach to Question **3**: We are multiplying the matrix A by the diagonal matrix $D = \textbf{diag}(1 : n)$ when row i is multiplied by i. And we are postmultiplying by D^{-1} when column j is divided by j. The determinant of DAD^{-1} is the same as $\det A$ by the product rule.

Problem Set 5.2

Problems 1–10 use the big formula with $n!$ terms: $|A| = \sum \pm a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$. Every term uses each row and each column once.

1 Compute the determinants of A, B, C from six terms. Are their rows independent?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 6 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

2 Compute the determinants of A, B, C, D. Are their columns independent?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad C = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \quad D = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

3 Show that $\det A = 0$, regardless of the five nonzeros marked by x's:

$$A = \begin{bmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}.$$

What are the cofactors of row 1?
What is the rank of A?
What are the 6 terms in $\det A$?

4 Find two ways to choose nonzeros from four different rows and columns:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$ (B has the same zeros as A).

Is $\det A$ equal to $1 + 1$ or $1 - 1$ or $-1 - 1$? What is $\det B$?

5 Place the smallest number of zeros in a 4 by 4 matrix that will guarantee $\det A = 0$. Place as many zeros as possible while still allowing $\det A \neq 0$.

6 (a) If $a_{11} = a_{22} = a_{33} = 0$, how many of the six terms in $\det A$ will be zero?

(b) If $a_{11} = a_{22} = a_{33} = a_{44} = 0$, how many of the 24 products $a_{1j}a_{2k}a_{3l}a_{4m}$ are sure to be zero?

7 How many 5 by 5 permutation matrices have $\det P = +1$? Those are even permutations. Find one that needs four exchanges to reach the identity matrix.

8 If $\det A$ is not zero, at least one of the $n!$ terms in formula (8) is not zero. Deduce from the big formula that some ordering of the rows of A leaves no zeros on the diagonal. (Don't use P from elimination; that PA can have zeros on the diagonal.)

9 Show that 4 is the largest determinant for a 3 by 3 matrix of 1's and -1's.

10 How many permutations of $(1, 2, 3, 4)$ are even and what are they? Extra credit: What are all the possible 4 by 4 determinants of $I + P_{\text{even}}$?

Problems 11–22 use cofactors $C_{ij} = (-1)^{i+j} \det M_{ij}$. Remove row i and column j.

11 Find all cofactors and put them into cofactor matrices C, D. Find AC and $\det B$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}.$$

12 Find the cofactor matrix C and multiply A times C^{T}. Compare AC^{T} with A^{-1}:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

13 The n by n determinant C_n has 1's above and below the main diagonal:

$$C_1 = |0| \quad C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \quad C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

(a) What are these determinants C_1, C_2, C_3, C_4?

(b) By cofactors find the relation between C_n and C_{n-1} and C_{n-2}. Find C_{10}.

14 The matrices in Problem 13 have 1's just above and below the main diagonal. Going down the matrix, which order of columns (if any) gives all 1's? Explain why that permutation is *even* for $n = 4, 8, 12, \ldots$ and *odd* for $n = 2, 6, 10, \ldots$. Then

$$C_n = 0 \text{ (odd } n) \qquad C_n = 1 \ (n = 4, 8, \cdots) \qquad C_n = -1 \ (n = 2, 6, \cdots).$$

15 The tridiagonal $1, 1, 1$ matrix of order n has determinant E_n:

$$E_1 = |1| \quad E_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \quad E_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} \quad E_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}.$$

(a) By cofactors show that $E_n = E_{n-1} - E_{n-2}$.

(b) Starting from $E_1 = 1$ and $E_2 = 0$ find $E_3, E_4, \ldots, E_8$.

(c) By noticing how these numbers eventually repeat, find E_{100}.

16 F_n is the determinant of the $1, 1, -1$ tridiagonal matrix of order n:

$$F_2 = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \quad F_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 3 \quad F_4 = \begin{vmatrix} 1 & -1 & & \\ 1 & 1 & -1 & \\ & 1 & 1 & -1 \\ & & 1 & 1 \end{vmatrix} \neq 4.$$

Expand in cofactors to show that $F_n = F_{n-1} + F_{n-2}$. These determinants are *Fibonacci numbers* $1, 2, 3, 5, 8, 13, \ldots$. The sequence usually starts $1, 1, 2, 3$ (with two 1's) so our F_n is the usual F_{n+1}.

17 The matrix B_n is the $-1, 2, -1$ matrix A_n except that $b_{11} = 1$ instead of $a_{11} = 2$. Using cofactors of the *last* row of B_4 show that $|B_4| = 2|B_3| - |B_2| = 1$.

$$B_4 = \begin{bmatrix} \mathbf{1} & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \quad B_3 = \begin{bmatrix} \mathbf{1} & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} \quad B_2 = \begin{bmatrix} \mathbf{1} & -1 \\ -1 & 2 \end{bmatrix}.$$

The recursion $|B_n| = 2|B_{n-1}| - |B_{n-2}|$ is satisfied when every $|B_n| = 1$. This recursion is the same as for the A's in Example 6. The difference is in the starting values $1, 1, 1$ for the determinants of sizes $n = 1, 2, 3$.

18 Go back to B_n in Problem 17. It is the same as A_n except for $b_{11} = 1$. So use linearity in the first row, where $[\,1 \ \ -1 \ \ 0\,]$ equals $[\,2 \ \ -1 \ \ 0\,]$ minus $[\,1 \ \ 0 \ \ 0\,]$:

$$|B_n| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & & \\ & A_{n-1} & \\ 0 & & \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & & \\ & A_{n-1} & \\ 0 & & \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 \\ -1 & & \\ & A_{n-1} & \\ 0 & & \end{vmatrix}.$$

Linearity gives $|B_n| = |A_n| - |A_{n-1}| =$ ____.

19 Explain why the 4 by 4 Vandermonde determinant contains x^3 but not x^4 or x^5:

$$V_4 = \det \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & x & x^2 & x^3 \end{bmatrix}.$$

The determinant is zero at $x =$ ____, ____, and ____. The cofactor of x^3 is $V_3 = (b-a)(c-a)(c-b)$. Then $V_4 = (b-a)(c-a)(c-b)(x-a)(x-b)(x-c)$.

20 Find G_2 and G_3 and then by row operations G_4. Can you predict G_n?

$$G_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad G_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \quad G_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}.$$

21 Compute S_1, S_2, S_3 for these $1, 3, 1$ matrices. By Fibonacci guess and check S_4.

$$S_1 = |3| \quad S_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \quad S_3 = \begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix}$$

22 Change 3 to 2 in the upper left corner of the matrices in Problem 21. Why does that subtract S_{n-1} from the determinant S_n? Show that the determinants of the new matrices become the Fibonacci numbers $2, 5, 13$ (always F_{2n+1}).

Problems 23–26 are about block matrices and block determinants.

23 With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A|\,|D| \qquad \text{but} \qquad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A|\,|D| - |C|\,|B|.$$

(a) Why is the first statement true? Somehow B doesn't enter.

(b) Show by example that equality fails (as shown) when C enters.

(c) Show by example that the answer $\det(AD - CB)$ is also wrong.

24 With block multiplication, $A = LU$ has $A_k = L_k U_k$ in the top left corner:

$$A = \begin{bmatrix} A_k & * \\ * & * \end{bmatrix} = \begin{bmatrix} L_k & 0 \\ * & * \end{bmatrix} \begin{bmatrix} U_k & * \\ 0 & * \end{bmatrix}.$$

(a) Suppose the first three pivots of A are $2, 3, -1$. What are the determinants of L_1, L_2, L_3 (with diagonal 1's) and U_1, U_2, U_3 and A_1, A_2, A_3?

(b) If A_1, A_2, A_3 have determinants $5, 6, 7$ find the three pivots from equation (3).

25 Block elimination subtracts CA^{-1} times the first row $[\,A \;\; B\,]$ from the second row $[C \;\; D]$. This leaves the *Schur complement* $D - CA^{-1}B$ in the corner:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}.$$

Take determinants of these block matrices to prove correct rules if A^{-1} exists:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\,|D - CA^{-1}B| = |AD - CB| \quad \text{provided } AC = CA.$$

26 If A is m by n and B is n by m, block multiplication gives $\det M = \det AB$:

$$M = \begin{bmatrix} 0 & A \\ -B & I \end{bmatrix} = \begin{bmatrix} AB & A \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix}.$$

If A is a single row and B is a single column what is $\det M$? If A is a column and B is a row what is $\det M$? Do a 3 by 3 example of each.

27 (A calculus question) Show that the derivative of $\det A$ with respect to a_{11} is the cofactor C_{11}. The other entries are fixed—we are only changing a_{11}.

28 A 3 by 3 determinant has three products "down to the right" and three "down to the left" with minus signs. Compute the six terms like $(1)(5)(9) = 45$ to find D.

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \begin{matrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{matrix}$$

$$- \quad - \quad - \qquad + \quad + \quad +$$

Explain without determinants why this particular matrix is or is not invertible.

29 For E_4 in Problem 15, five of the $4! = 24$ terms in the big formula (8) are nonzero. Find those five terms to show that $E_4 = -1$.

30 For the 4 by 4 tridiagonal second difference matrix (entries -1, 2, -1) find the five terms in the big formula that give $\det A = 16 - 4 - 4 - 4 + 1$.

31 Find the determinant of this cyclic P by cofactors of row 1 and then the "big formula". How many exchanges reorder $4, 1, 2, 3$ into $1, 2, 3, 4$? Is $|P^2| = 1$ or -1?

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Challenge Problems

32 Cofactors of the $1, 3, 1$ matrices in Problem 21 give a recursion $S_n = 3S_{n-1} - S_{n-2}$. Amazingly that recursion produces every second Fibonacci number. Here is the challenge.

Show that S_n is the Fibonacci number F_{2n+2} by proving $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using Fibonacci's rule $F_k = F_{k-1} + F_{k-2}$ starting with $k = 2n + 2$.

33 The symmetric Pascal matrices have determinant 1. If I subtract 1 from the n, n entry, why does the determinant become zero? (Use rule 3 or cofactors.)

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = 1 \text{ (known)} \qquad \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & \mathbf{19} \end{bmatrix} = \mathbf{0} \text{ (to explain).}$$

34 This problem shows in two ways that $\det A = 0$ (the x's are any numbers):

$$A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}.$$

(a) How do you know that the rows are linearly dependent?

(b) Explain why all 120 terms are zero in the big formula for $\det A$.

35 If $|\det(A)| > 1$, prove that the powers A^n cannot stay bounded. But if $|\det(A)| \leq 1$, show that some entries of A^n might still grow large. Eigenvalues will give the right test for stability, determinants tell us only one number.

5.3 Cramer's Rule, Inverses, and Volumes

1 $\boldsymbol{A^{-1}}$ **equals** $\boldsymbol{C^{\mathrm{T}}}$ **/ det** $\boldsymbol{A}$. Then $(A^{-1})_{ij} =$ cofactor $\boldsymbol{C_{ji}}$ divided by the determinant of A.

2 **Cramer's Rule** computes $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ from $x_j = \det(A$ with column j changed to $\boldsymbol{b}) / \det A$.

3 **Area of parallelogram** $= |ad-bc|$ if the four corners are $(0,0)$, (a,b), (c,d), and $(a+c, b+d)$.

4 **Volume of box** $= |\mathbf{det}\ \boldsymbol{A}|$ if the rows of A (or the columns of A) give the sides of the box.

5 The **cross product** $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$ is det $\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$. Notice $\boldsymbol{v} \times \boldsymbol{u} = -(\boldsymbol{u} \times \boldsymbol{v})$. w_1, w_2, w_3 are cofactors of row 1. Notice $\boldsymbol{w}^{\mathrm{T}}\boldsymbol{u} = 0$ and $\boldsymbol{w}^{\mathrm{T}}\boldsymbol{v} = 0$.

This section solves $A\boldsymbol{x} = \boldsymbol{b}$ and also finds A^{-1}—by algebra and not by elimination. In all formulas you will see a division by det A. Each entry in A^{-1} and $A^{-1}\boldsymbol{b}$ is a determinant divided by the determinant of A. Let me start with Cramer's Rule.

Cramer's Rule solves $A\boldsymbol{x} = \boldsymbol{b}$. A neat idea gives the first component x_1. Replacing the first column of I by $\boldsymbol{x}$ gives a matrix with determinant x_1. When you multiply it by A, *the first column becomes* $A\boldsymbol{x}$ *which is* $\boldsymbol{b}$. The other columns of B_1 are copied from A:

$$\textbf{Key idea} \qquad \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \boldsymbol{x_1} & 0 & 0 \\ \boldsymbol{x_2} & 1 & 0 \\ \boldsymbol{x_3} & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b_1} & a_{12} & a_{13} \\ \boldsymbol{b_2} & a_{22} & a_{23} \\ \boldsymbol{b_3} & a_{32} & a_{33} \end{bmatrix} = B_1. \qquad (1)$$

We multiplied a column at a time. *Take determinants of the three matrices to find* x_1 :

$$\boxed{\textbf{Product rule} \qquad (\det A)(x_1) = \det B_1 \qquad \text{or} \qquad x_1 = \frac{\det B_1}{\det A}.} \qquad (2)$$

This is the first component of $\boldsymbol{x}$ in Cramer's Rule! Changing a column of A gave B_1. To find x_2 and B_2, put the vectors $\boldsymbol{x}$ and $\boldsymbol{b}$ into the *second* columns of I and A:

$$\textbf{Same idea} \qquad \begin{bmatrix} & & \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ & & \end{bmatrix} \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} = \begin{bmatrix} & & \\ \boldsymbol{a}_1 & \boldsymbol{b} & \boldsymbol{a}_3 \\ & & \end{bmatrix} = B_2. \qquad (3)$$

Take determinants to find $(\det A)(x_2) = \det B_2$. This gives $\boldsymbol{x_2} = (\mathbf{det}\ \boldsymbol{B_2})/(\mathbf{det}\ \boldsymbol{A})$.

Example 1 Solving $3x_1 + 4x_2 = \mathbf{2}$ and $5x_1 + 6x_2 = \mathbf{4}$ needs three determinants:

$$\det A = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} \qquad \det B_1 = \begin{vmatrix} \mathbf{2} & 4 \\ \mathbf{4} & 6 \end{vmatrix} \qquad \det B_2 = \begin{vmatrix} 3 & \mathbf{2} \\ 5 & \mathbf{4} \end{vmatrix}$$

Those determinants of A, B_1, B_2 are -2 and -4 and 2. All ratios divide by $\det A = -2$:

$$\textbf{Find } \boldsymbol{x} = \boldsymbol{A^{-1}b} \qquad x_1 = \frac{-4}{-2} = 2 \qquad x_2 = \frac{2}{-2} = -1 \qquad \textbf{Check} \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

CRAMER's RULE If $\det A$ is not zero, $A\boldsymbol{x} = \boldsymbol{b}$ is solved by determinants:

$$x_1 = \frac{\det B_1}{\det A} \qquad x_2 = \frac{\det B_2}{\det A} \qquad \cdots \qquad x_n = \frac{\det B_n}{\det A} \tag{4}$$

The matrix B_j has the jth column of A replaced by the vector b.

To solve an n by n system, Cramer's Rule evaluates $n+1$ determinants (of A and the n different B's). When each one is the sum of $n!$ terms—applying the "big formula" with all permutations—this makes a total of $(n+1)!$ terms. *It would be crazy to solve equations that way*. But we do finally have an explicit formula for the solution $\boldsymbol{x}$.

Example 2 Cramer's Rule is inefficient for numbers but it is well suited to letters. For $n = 2$, find the columns of $A^{-1} = [\boldsymbol{x}\ \ \boldsymbol{y}]$ by solving $AA^{-1} = I$:

$$\begin{matrix}\textbf{Columns of } \boldsymbol{A^{-1}} \\ \textbf{are } \boldsymbol{x} \textbf{ and } \boldsymbol{y}\end{matrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$$

Those share the same matrix A. We need $|A|$ and four determinants for $x_1,\ x_2,\ y_1,\ y_2$:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ and } \begin{vmatrix} \mathbf{1} & b \\ \mathbf{0} & d \end{vmatrix} \quad \begin{vmatrix} a & \mathbf{1} \\ c & \mathbf{0} \end{vmatrix} \quad \begin{vmatrix} \mathbf{0} & b \\ \mathbf{1} & d \end{vmatrix} \quad \begin{vmatrix} a & \mathbf{0} \\ c & \mathbf{1} \end{vmatrix}$$

The last four determinants are d, $-c$, $-b$, and a. (They are the cofactors!) Here is A^{-1}:

$$x_1 = \frac{d}{|A|},\ x_2 = \frac{-c}{|A|},\ y_1 = \frac{-b}{|A|},\ y_2 = \frac{a}{|A|} \text{ and then } A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

I chose 2 by 2 so that the main points could come through clearly. The new idea is: $\boldsymbol{A^{-1}}$ **involves the cofactors**. When the right side is a column of the identity matrix I, as in $AA^{-1} = I$, **the determinant of each $\boldsymbol{B_j}$ in Cramer's Rule is a cofactor of $\boldsymbol{A}$.**

You can see those cofactors for $n = 3$. Solve $A\boldsymbol{x} = (1, 0, 0)$ to find column 1 of A^{-1}:

$$\begin{matrix}\textbf{Determinants of } \boldsymbol{B}\textbf{'s} \\ = \textbf{Cofactors of } \boldsymbol{A}\end{matrix} \qquad \begin{vmatrix} \mathbf{1} & a_{12} & a_{13} \\ \mathbf{0} & a_{22} & a_{23} \\ \mathbf{0} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & \mathbf{1} & a_{13} \\ a_{21} & \mathbf{0} & a_{23} \\ a_{31} & \mathbf{0} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & a_{12} & \mathbf{1} \\ a_{21} & a_{22} & \mathbf{0} \\ a_{31} & a_{32} & \mathbf{0} \end{vmatrix} \tag{5}$$

That first determinant $|B_1|$ is the cofactor $C_{11} = a_{22}a_{33} - a_{23}a_{32}$. Then $|B_2|$ is the cofactor C_{12}. Notice that the correct minus sign appears in $-(a_{21}a_{33} - a_{23}a_{31})$. This cofactor C_{12} goes into column 1 of A^{-1}. When we divide by $\det A$, we have the inverse matrix!

The i, j ***entry of*** A^{-1} ***is the cofactor*** C_{ji} ***(not*** C_{ij}***) divided by*** $\det A$:

FORMULA FOR A^{-1}
$$(A^{-1})_{ij} = \frac{C_{ji}}{\det A} \quad \text{and} \quad A^{-1} = \frac{C^{\mathrm{T}}}{\det A}. \tag{6}$$

The cofactors C_{ij} go into the *"cofactor matrix"* C. ***The transpose of C leads to*** A^{-1}. To compute the i, j entry of A^{-1}, cross out row j and column i of A. Multiply the determinant by $(-1)^{i+j}$ to get the cofactor C_{ji}, and divide by $\det A$.

Check this rule for the $3, 1$ entry of A^{-1}. For column 1 we solve $Ax = (1, 0, 0)$. The third component x_3 needs the third determinant in equation (5), divided by $\det A$. That determinant is exactly the cofactor $C_{13} = a_{21}a_{32} - a_{22}a_{31}$. So $(A^{-1})_{31} = C_{13}/\det A$.

Summary In solving $AA^{-1} = I$, each column of I leads to a column of A^{-1}. Every entry of A^{-1} is a ratio: determinant of size $n-1$ / determinant of size n.

Direct proof of the formula $A^{-1} = C^{\mathrm{T}}/\det A$ This means $AC^{\mathrm{T}} = (\det A)I$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix}. \tag{7}$$

(Row 1 of A) times (column 1 of C^{T}) yields the first $\det A$ on the right:

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \det A \quad \text{This is exactly the cofactor rule!}$$

Similarly row 2 of A times column 2 of C^{T} (*notice the transpose*) also yields $\det A$. The entries a_{2j} are multiplying cofactors C_{2j} as they should, to give the determinant.

How to explain the zeros off the main diagonal in equation (7)? The rows of A are multiplying cofactors from *different* rows. Why is the answer zero?

Row 2 of A
Row 1 of C
$$a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13} = 0. \tag{8}$$

Answer: This is the cofactor rule for a new matrix, when the second row of A is copied into its first row. The new matrix A^* has two equal rows, so $\det A^* = 0$ in equation (8). Notice that A^* has the same cofactors C_{11}, C_{12}, C_{13} as A—because all rows agree after the first row. Thus the remarkable multiplication (7) is correct:

$$AC^{\mathrm{T}} = (\det A)I \qquad \text{or} \qquad A^{-1} = \frac{C^{\mathrm{T}}}{\det A}.$$

Example 3 The "sum matrix" A has determinant 1. Then A^{-1} contains cofactors:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{has inverse} \quad A^{-1} = \frac{C^{\mathrm{T}}}{1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Cross out row 1 and column 1 of A to see the 3 by 3 cofactor $C_{11} = 1$. Now cross out row 1 and column 2 for C_{12}. The 3 by 3 submatrix is still triangular with determinant 1. But the cofactor C_{12} is -1 because of the sign $(-1)^{1+2}$. This number -1 goes into the $(2, 1)$ entry of A^{-1}—don't forget to transpose C.

The inverse of a triangular matrix is triangular. Cofactors give a reason why.

Example 4 If all cofactors are nonzero, is A sure to be invertible? *No way.*

Area of a Triangle

Everybody knows the area of a rectangle—base times height. The area of a triangle is *half* the base times the height. But here is a question that those formulas don't answer. ***If we know the corners*** (x_1, y_1) ***and*** (x_2, y_2) ***and*** (x_3, y_3) ***of a triangle, what is the area?*** Using the corners to find the base and height is not a good way to compute area.

Determinants are the best way to find area. ***The area of a triangle is half of a 3 by 3 determinant.*** The square roots in the base and height cancel out in the good formula. If one corner is at the origin, say $(x_3, y_3) = (0, 0)$, the determinant is only 2 by 2.

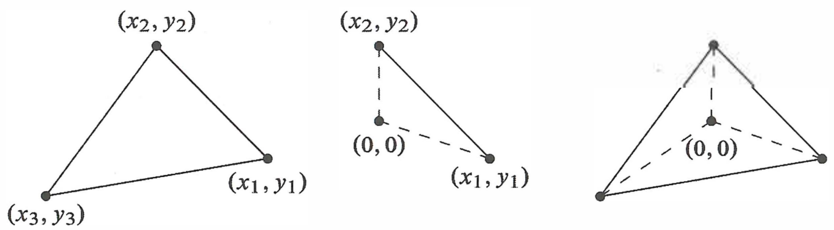


Figure 5.1: General triangle; special triangle from $(0, 0)$; general from three specials.

The triangle with corners (x_1, y_1) and (x_2, y_2) and (x_3, y_3) has **area** $= \dfrac{\textbf{determinant}}{\textbf{2}}$:

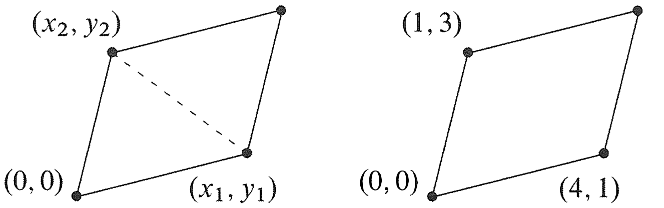
Area of triangle $\dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ Area $= \dfrac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ when $(x_3, y_3) = (0, 0)$.

When you set $x_3 = y_3 = 0$ in the 3 by 3 determinant, you get the 2 by 2 determinant. These formulas have no square roots—they are reasonable to memorize. The 3 by 3 determinant breaks into a sum of three 2 by 2's (cofactors), just as the third triangle in Figure 5.1 breaks into three special triangles from $(0,0)$:

$$\textbf{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{matrix} +\frac{1}{2}(x_1y_2 - x_2y_1) \\ +\frac{1}{2}(x_2y_3 - x_3y_2) \\ +\frac{1}{2}(x_3y_1 - x_1y_3). \end{matrix} \tag{9}$$

If $(0,0)$ is outside the triangle, two of the special areas can be negative—but the sum is still correct. The real problem is to explain the area of a triangle with corner $(0,0)$.

Why is $\frac{1}{2}|x_1y_2 - x_2y_1|$ the area of this triangle? We can remove the factor $\frac{1}{2}$ for a parallelogram (twice as big, because the parallelogram contains two equal triangles). We now prove that the parallelogram area is the determinant $x_1y_2 - x_2y_1$. This area in Figure 5.2 is 11, and therefore the triangle has area $\frac{11}{2}$.



Parallelogram

$$\text{Area} = \begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix} = 11$$

$$\text{Triangle: Area} = \frac{\mathbf{11}}{\mathbf{2}}$$

Figure 5.2: A triangle is half of a parallelogram. Area is half of a determinant.

Proof that a parallelogram starting from $(0,0)$ has area $=$ 2 by 2 determinant.

There are many proofs but this one fits with the book. We show that the area has the same properties 1-2-3 as the determinant. Then area = determinant! Remember that those three rules defined the determinant and led to all its other properties.

1 When $A = I$, the parallelogram becomes the unit square. Its area is $\det I = 1$.

2 When rows are exchanged, the determinant reverses sign. The absolute value (positive area) stays the same—it is the same parallelogram.

3 If row 1 is multiplied by t, Figure 5.3a shows that the area is also multiplied by t. Suppose a new row (x_1', y_1') is added to (x_1, y_1) (keeping row 2 fixed). Figure 5.3b shows that the solid parallelogram areas add to the dotted parallelogram area (because the two triangles completed by dotted lines are the same).

That is an exotic proof, when we could use plane geometry. But the proof has a major attraction—it applies in n dimensions. The n edges going out from the origin are given by the *rows of an n by n matrix*. The box is completed by more edges, like the parallelogram.

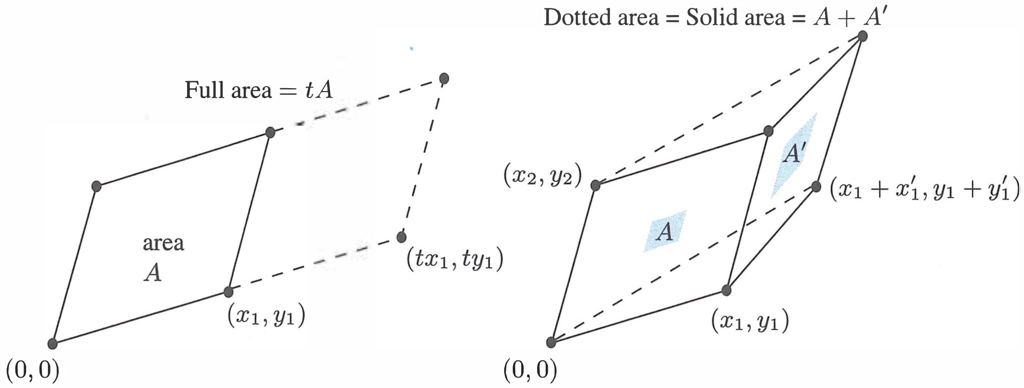


Figure 5.3: Areas obey the rule of linearity in side 1 (keeping the side (x_2, y_2) constant).

Figure 5.4 shows a three-dimensional box—whose edges are not at right angles. ***The volume equals the absolute value of* det *A*.** Our proof checks again that rules 1–3 for determinants are also obeyed by volumes. When an edge is stretched by a factor t, the volume is multiplied by t. When edge 1 is added to edge $1'$, the volume is the sum of the two original volumes. This is Figure 5.3b lifted into three dimensions or n dimensions. I would draw the boxes but this paper is only two-dimensional.

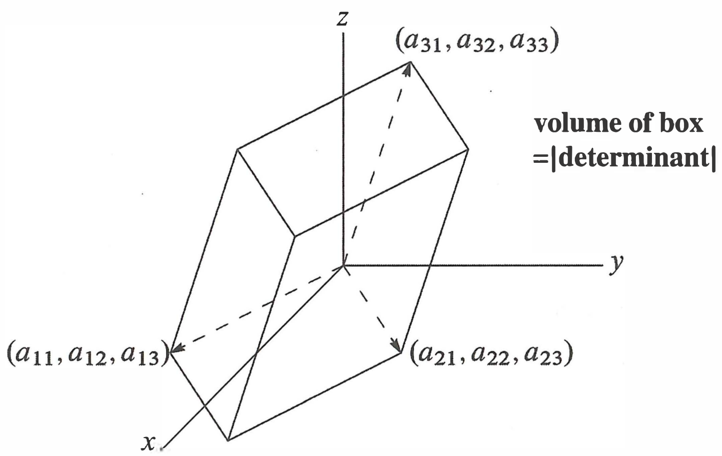


Figure 5.4: Three-dimensional box formed from the three rows of A.

The unit cube has volume $= 1$, which is $\det I$. Row exchanges or edge exchanges leave the same box and the same absolute volume. The determinant changes sign, to indicate whether the edges are a *right-handed triple* $(\det A > 0)$ or a *left-handed triple* $(\det A < 0)$. The box volume follows the rules for determinants, so volume of $\det A =$ absolute value.

Example 5 Suppose a rectangular box ($90°$ angles) has side lengths r, s, and t. Its volume is r times s times t. The diagonal matrix A with entries r, s, and t produces those three sides. Then $\det A$ also equals the volume $r\,s\,t$.

Example 6 In calculus, the box is infinitesimally small! To integrate over a circle, we might change x and y to r and θ. Those are polar coordinates: $x = r\cos\theta$ and $y = r\sin\theta$. The area of a "polar box" is a determinant J times $dr\,d\theta$:

$$\textbf{Area } \boldsymbol{r\,dr\,d\theta} \textbf{ in calculus} \qquad J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \boldsymbol{r}.$$

This determinant is the r in the small area $dA = r\,dr\,d\theta$. The stretching factor J goes into double integrals just as dx/du goes into an ordinary integral $\int dx = \int (dx/du)\,du$. For triple integrals the Jacobian matrix J with nine derivatives will be 3 by 3.

The Cross Product

The *cross product* is an extra (and optional) application, special for three dimensions. Start with vectors $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$. Unlike the dot product, which is a number, **the cross product is a vector**—also in three dimensions. It is written $\boldsymbol{u} \times \boldsymbol{v}$ and pronounced "$\boldsymbol{u}$ cross $\boldsymbol{v}$." *The components of this cross product are* 2 *by* 2 *cofactors*. We will explain the properties that make $\boldsymbol{u} \times \boldsymbol{v}$ useful in geometry and physics.

This time we bite the bullet, and write down the formula before the properties.

DEFINITION The ***cross product*** of $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$ is a vector

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\boldsymbol{i} + (u_3v_1 - u_1v_3)\boldsymbol{j} + (u_1v_2 - u_2v_1)\boldsymbol{k}. \tag{10}$$

This vector $\boldsymbol{u} \times \boldsymbol{v}$ ***is perpendicular to*** $\boldsymbol{u}$ ***and*** $\boldsymbol{v}$. The cross product $\boldsymbol{v} \times \boldsymbol{u}$ is $-(\boldsymbol{u} \times \boldsymbol{v})$.

Comment The 3 by 3 determinant is the easiest way to remember $\boldsymbol{u} \times \boldsymbol{v}$. It is not especially legal, because the first row contains vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and the other rows contain numbers. In the determinant, the vector $\boldsymbol{i} = (1, 0, 0)$ multiplies u_2v_3 and $-u_3v_2$. The result is $(u_2v_3 - u_3v_2, 0, 0)$, which displays the first component of the cross product.

Notice the cyclic pattern of the subscripts: 2 and 3 give component 1 of $\boldsymbol{u} \times \boldsymbol{v}$, then 3 and 1 give component 2, then 1 and 2 give component 3. This completes the definition of $\boldsymbol{u} \times \boldsymbol{v}$. Now we list the properties of the cross product:

Property 1 $\boldsymbol{v} \times \boldsymbol{u}$ reverses rows 2 and 3 in the determinant so it equals $-(\boldsymbol{u} \times \boldsymbol{v})$.

Property 2 The cross product $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ (and also to $\boldsymbol{v}$). The direct proof is to watch terms cancel, producing a zero dot product:

$$\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0. \tag{11}$$

The determinant for $\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v})$ has rows $\boldsymbol{u}, \boldsymbol{u}$ and $\boldsymbol{v}$ (2 equal rows) so it is zero.

Property 3 The cross product of any vector with itself (two equal rows) is $\boldsymbol{u} \times \boldsymbol{u} = \mathbf{0}$.

When $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel, the cross product is zero. When $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular, the dot product is zero. One involves $\sin\theta$ and the other involves $\cos\theta$:

$$\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\,|\sin\theta| \qquad \text{and} \qquad |\boldsymbol{u} \cdot \boldsymbol{v}| = \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\,|\cos\theta|. \tag{12}$$

Example 7 $\boldsymbol{u} = (3, 2, 0)$ and $\boldsymbol{v} = (1, 4, 0)$ are in the xy plane, $\boldsymbol{u} \times \boldsymbol{v}$ goes up the z axis:

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & 0 \\ 1 & 4 & 0 \end{vmatrix} = 10\boldsymbol{k}. \quad \text{The cross product is } \boldsymbol{u} \times \boldsymbol{v} = (\mathbf{0}, \mathbf{0}, \mathbf{10}).$$

The length of $\boldsymbol{u} \times \boldsymbol{v}$ ***equals the area of the parallelogram with sides*** $\boldsymbol{u}$ ***and*** $\boldsymbol{v}$***.*** This will be important: In this example the area is 10.

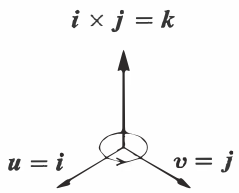
Example 8 The cross product of $\boldsymbol{u} = (1, 1, 1)$ and $\boldsymbol{v} = (1, 1, 2)$ is $(1, -1, 0)$:

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \boldsymbol{i} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = \boldsymbol{i} - \boldsymbol{j}.$$

This vector $(1, -1, 0)$ is perpendicular to $(1, 1, 1)$ and $(1, 1, 2)$ as predicted. Area $= \sqrt{2}$.

Example 9 The cross product of $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{j} = (0, 1, 0)$ obeys the *right hand rule*. That cross product $\boldsymbol{k} = \boldsymbol{i} \times \boldsymbol{j}$ goes up not down:

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \boldsymbol{k}$$



Rule $\boldsymbol{u} \times \boldsymbol{v}$ points along your right thumb when the fingers curl from $\boldsymbol{u}$ to $\boldsymbol{v}$.

Thus $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$. The right hand rule also gives $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$ and $\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$. Note the cyclic order. In the opposite order (anti-cyclic) the thumb is reversed and the cross product goes the other way: $\boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i}$ and $\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$ and $\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}$. You see the three plus signs and three minus signs from a 3 by 3 determinant.

The definition of $\boldsymbol{u} \times \boldsymbol{v}$ can be based on vectors instead of their components:

DEFINITION The ***cross product*** is a vector with length $\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\,|\sin\theta|$. Its direction is perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$. It points "up" or "down" by the right hand rule.

This definition appeals to physicists, who hate to choose axes and coordinates. They see (u_1, u_2, u_3) as the position of a mass and (F_x, F_y, F_z) as a force acting on it. If $\boldsymbol{F}$ is parallel to $\boldsymbol{u}$, then $\boldsymbol{u} \times \boldsymbol{F} = \mathbf{0}$—there is no turning. The cross product $\boldsymbol{u} \times \boldsymbol{F}$ is the turning force or *torque*. It points along the turning axis (perpendicular to $\boldsymbol{u}$ and $\boldsymbol{F}$). Its length $\|\boldsymbol{u}\|\,\|\boldsymbol{F}\|\sin\theta$ measures the "moment" that produces turning.

Triple Product = Determinant = Volume

Since $\boldsymbol{u} \times \boldsymbol{v}$ is a vector, we can take its dot product with a third vector $\boldsymbol{w}$. That produces the ***triple product*** $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$. It is called a "scalar" triple product, because it is a number. In fact it is a determinant—it gives the volume of the $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ box:

Triple product
$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \qquad (13)$$

We can put $\boldsymbol{w}$ in the top or bottom row. The two determinants are the same because _____ row exchanges go from one to the other. Notice when this determinant is zero:

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = 0 \quad \text{exactly when the vectors } \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \text{ lie in the } \textit{same plane}.$$

First reason $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to that plane so its dot product with $\boldsymbol{w}$ is zero.

Second reason Three vectors in a plane are dependent. The matrix is singular ($\det = 0$).

Third reason Zero volume when the $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ box is squashed onto a plane.

It is remarkable that $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$ equals the volume of the box with sides $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. This 3 by 3 determinant carries tremendous information. Like $ad - bc$ for a 2 by 2 matrix, it separates invertible from singular. Chapter 6 will be looking for singular.

■ REVIEW OF THE KEY IDEAS ■

1. Cramer's Rule solves $A\boldsymbol{x} = \boldsymbol{b}$ by ratios like $x_1 = |B_1|/|A| = |\boldsymbol{b}\,\boldsymbol{a}_2 \cdots \boldsymbol{a}_n|/|A|$.

2. When C is the cofactor matrix for A, the inverse is $A^{-1} = C^{\mathrm{T}} / \det A$.

3. The volume of a box is $|\det A|$, when the box edges are the rows of A.

4. Area and volume are needed to change variables in double and triple integrals.

5. In $\mathbf{R}^3$, the cross product $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$. Notice $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$.

WORKED EXAMPLES

5.3 A If A is singular, the equation $AC^{\mathrm{T}} = (\det A)I$ becomes $\boldsymbol{AC^{\mathrm{T}}} =$ **zero matrix**. *Then each column of C^{T} is in the nullspace of A.* Those columns contain cofactors along rows of A. So the cofactors quickly find the nullspace for a 3 by 3 matrix of rank 2. My apologies that this comes so late!

Solve $A\boldsymbol{x} = \mathbf{0}$ by $\boldsymbol{x} =$ cofactors along a row, for these singular matrices of rank 2:

Cofactors give nullspace $\quad A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 3 & 9 \\ 2 & 2 & 8 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Solution The first matrix has these cofactors along its top row (note each minus sign):

$$\begin{vmatrix} 3 & 9 \\ 2 & 8 \end{vmatrix} = 6 \qquad -\begin{vmatrix} 2 & 9 \\ 2 & 8 \end{vmatrix} = 2 \qquad \begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} = -2$$

Then $\boldsymbol{x} = (6, 2, -2)$ solves $A\boldsymbol{x} = \mathbf{0}$. The cofactors along the second row are $(-18, -6, 6)$ which is just $-3\boldsymbol{x}$. This is also in the one-dimensional nullspace of A.

The second matrix has *zero cofactors* along its first row. The nullvector $\boldsymbol{x} = (0, 0, 0)$ is not interesting. The cofactors of row 2 give $\boldsymbol{x} = (1, -1, 0)$ which solves $A\boldsymbol{x} = \mathbf{0}$.

Every n by n matrix of rank $n-1$ has at least one nonzero cofactor by Problem 3.3.12. But for rank $n-2$, all cofactors are zero and we only find $\boldsymbol{x} = \mathbf{0}$.

5.3 B Use Cramer's Rule with ratios $\det B_j / \det A$ to solve $A\boldsymbol{x} = \boldsymbol{b}$. Also find the inverse matrix $A^{-1} = C^{\mathrm{T}} / \det A$. For this $\boldsymbol{b} = (0, 0, 1)$ the solution $\boldsymbol{x}$ is column 3 of A^{-1}! Which cofactors are involved in computing that column $\boldsymbol{x} = (x, y, z)$?

Column 3 of $\boldsymbol{A^{-1}}$ $\quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$

Find the volumes of two boxes: edges are *columns* of A and edges are rows of A^{-1}.

Solution The determinants of the B_j (with right side $\boldsymbol{b}$ placed in column j) are

$$|B_1| = \begin{vmatrix} \mathbf{0} & 6 & 2 \\ \mathbf{0} & 4 & 2 \\ \mathbf{1} & 9 & 0 \end{vmatrix} = 4 \qquad |B_2| = \begin{vmatrix} 2 & \mathbf{0} & 2 \\ 1 & \mathbf{0} & 2 \\ 5 & \mathbf{1} & 0 \end{vmatrix} = -2 \qquad |B_3| = \begin{vmatrix} 2 & 6 & \mathbf{0} \\ 1 & 4 & \mathbf{0} \\ 5 & 9 & \mathbf{1} \end{vmatrix} = 2.$$

Those are cofactors C_{31}, C_{32}, C_{33} of row 3. Their dot product with row 3 is $\det A = 2$:

$$\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = (5, 9, 0) \cdot (4, -2, 2) = 2.$$

The three ratios $\det B_j / \det A$ give the three components of $\boldsymbol{x} = (2, -1, 1)$. This $\boldsymbol{x}$ is the third column of A^{-1} because $\boldsymbol{b} = (0, 0, 1)$ is the third column of I.

The cofactors along the other *rows* of A, divided by $\det A$, give the other *columns* of A^{-1}:

$$A^{-1} = \frac{C^{\mathrm{T}}}{\det A} = \frac{1}{2}\begin{bmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{bmatrix}. \quad \text{Multiply to check} \quad AA^{-1} = I$$

The box from the columns of A has volume $= \det A = 2$. The box from the rows also has volume 2, since $|A^{\mathrm{T}}| = |A|$. The box from the rows of A^{-1} has volume $1/|A| = \frac{1}{2}$.

Problem Set 5.3

Problems 1–5 are about Cramer's Rule for $x = A^{-1}b$.

1 Solve these linear equations by Cramer's Rule $x_j = \det B_j / \det A$:

(a) $\begin{aligned} 2x_1 + 5x_2 &= 1 \\ x_1 + 4x_2 &= 2 \end{aligned}$ (b) $\begin{aligned} 2x_1 + x_2 \phantom{{}+x_3} &= 1 \\ x_1 + 2x_2 + x_3 &= 0 \\ x_2 + 2x_3 &= 0. \end{aligned}$

2 Use Cramer's Rule to solve for y (only). Call the 3 by 3 determinant D:

(a) $\begin{aligned} ax + by &= 1 \\ cx + dy &= 0 \end{aligned}$ (b) $\begin{aligned} ax + by + cz &= 1 \\ dx + ey + fz &= 0 \\ gx + hy + iz &= 0. \end{aligned}$

3 Cramer's Rule breaks down when $\det A = 0$. Example (a) has no solution while (b) has infinitely many. What are the ratios $x_j = \det B_j / \det A$ in these two cases?

(a) $\begin{aligned} 2x_1 + 3x_2 &= 1 \\ 4x_1 + 6x_2 &= 1 \end{aligned}$ (parallel lines) (b) $\begin{aligned} 2x_1 + 3x_2 &= 1 \\ 4x_1 + 6x_2 &= 2 \end{aligned}$ (same line)

4 *Quick proof of Cramer's rule.* The determinant is a linear function of column 1. It is zero if two columns are equal. When $\boldsymbol{b} = A\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3$ goes into the first column of A, the determinant of this matrix B_1 is

$$|\boldsymbol{b} \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3| = |x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3| = x_1|\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3| = x_1 \det A.$$

(a) What formula for x_1 comes from left side = right side?

(b) What steps lead to the middle equation?

5 If the right side $\boldsymbol{b}$ is the first column of A, solve the 3 by 3 system $A\boldsymbol{x} = \boldsymbol{b}$. How does each determinant in Cramer's Rule lead to this solution $\boldsymbol{x}$?

Problems 6–15 are about $A^{-1} = C^{\mathrm{T}} / \det A$. Remember to transpose C.

6 Find A^{-1} from the cofactor formula $C^{\mathrm{T}} / \det A$. Use symmetry in part (b).

$$\text{(a)}\ A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 7 & 1 \end{bmatrix} \qquad \text{(b)}\quad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

7 If all the cofactors are zero, how do you know that A has no inverse? If none of the cofactors are zero, is A sure to be invertible?

8 Find the cofactors of A and multiply AC^{T} to find $\det A$:

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 6 & -3 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad \text{and } AC^{\mathrm{T}} = \underline{\qquad}.$$

If you change that 4 to 100, why is $\det A$ unchanged?

9 Suppose $\det A = 1$ and you know all the cofactors in C. How can you find A?

10 From the formula $AC^{\mathrm{T}} = (\det A)I$ show that $\det C = (\det A)^{n-1}$.

11 If all entries of A are integers, and $\det A = 1$ or -1, prove that all entries of A^{-1} are integers. Give a 2 by 2 example with no zero entries.

12 If all entries of A and A^{-1} are integers, prove that $\det A = 1$ or -1. Hint: What is $\det A$ times $\det A^{-1}$?

13 Complete the calculation of A^{-1} by cofactors that was started in Example 5.

14 L is lower triangular and S is symmetric. Assume they are invertible:

To invert
triangular L
symmetric S

$$L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \qquad S = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}.$$

(a) Which three cofactors of L are zero? Then L^{-1} is also lower triangular.

(b) Which three pairs of cofactors of S are equal? Then S^{-1} is also symmetric.

(c) The cofactor matrix C of an orthogonal Q will be _____. *Why*?

15 For $n = 5$ the matrix C contains _____ cofactors. Each 4 by 4 cofactor contains _____ terms and each term needs _____ multiplications. Compare with $5^3 = 125$ for the Gauss-Jordan computation of A^{-1} in Section 2.4.

Problems 16–26 are about area and volume by determinants.

16 (a) Find the area of the parallelogram with edges $\boldsymbol{v} = (3, 2)$ and $\boldsymbol{w} = (1, 4)$.

(b) Find the area of the triangle with sides $\boldsymbol{v}$, $\boldsymbol{w}$, and $\boldsymbol{v} + \boldsymbol{w}$. Draw it.

(c) Find the area of the triangle with sides $\boldsymbol{v}$, $\boldsymbol{w}$, and $\boldsymbol{w} - \boldsymbol{v}$. Draw it.

17 A box has edges from $(0,0,0)$ to $(3,1,1)$ and $(1,3,1)$ and $(1,1,3)$. Find its volume. Also find the area of each parallelogram face using $\|\boldsymbol{u} \times \boldsymbol{v}\|$.

18 (a) The corners of a triangle are $(2,1)$ and $(3,4)$ and $(0,5)$. What is the area?

(b) Add a corner at $(-1,0)$ to make a lopsided region (four sides). Find the area.

19 The parallelogram with sides $(2,1)$ and $(2,3)$ has the same area as the parallelogram with sides $(2,2)$ and $(1,3)$. Find those areas from 2 by 2 determinants and say why they must be equal. (I can't see why from a picture. Please write to me if you do.)

20 The Hadamard matrix H has orthogonal rows. The box is a hypercube!

$$\text{What is} \quad |H| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} = \text{volume of a hypercube in } \mathbf{R}^4?$$

21 If the columns of a 4 by 4 matrix have lengths L_1, L_2, L_3, L_4, what is the largest possible value for the determinant (based on volume)? If all entries of the matrix are 1 or -1, what are those lengths and the maximum determinant?

22 Show by a picture how a rectangle with area x_1y_2 minus a rectangle with area x_2y_1 produces the same area as our parallelogram.

23 When the edge vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are perpendicular, the volume of the box is $\|\boldsymbol{a}\|$ times $\|\boldsymbol{b}\|$ times $\|\boldsymbol{c}\|$. The matrix $A^{\mathrm{T}}A$ is _____. Find $\det A^{\mathrm{T}}A$ and $\det A$.

24 The box with edges $\boldsymbol{i}$ and $\boldsymbol{j}$ and $\boldsymbol{w} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$ has height _____. What is the volume? What is the matrix with this determinant? What is $\boldsymbol{i} \times \boldsymbol{j}$ and what is its dot product with $\boldsymbol{w}$?

25 An n-dimensional cube has how many corners? How many edges? How many $(n-1)$-dimensional faces? The cube in $\mathbf{R}^n$ whose edges are the rows of $2I$ has volume _____. A hypercube computer has parallel processors at the corners with connections along the edges.

26 The triangle with corners $(0,0), (1,0), (0,1)$ has area $\frac{1}{2}$. The pyramid in $\mathbf{R}^3$ with four corners $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$ has volume _____. What is the volume of a pyramid in $\mathbf{R}^4$ with five corners at $(0,0,0,0)$ and the rows of I?

Problems 27–30 are about areas dA and volumes dV in calculus.

27 Polar coordinates satisfy $x = r\cos\theta$ and $y = r\sin\theta$. Polar area is $J\,dr\,d\theta$:

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}.$$

The two columns are orthogonal. Their lengths are _____. Thus $J =$ _____.

28 Spherical coordinates ρ, ϕ, θ satisfy $x = \rho \sin\phi \cos\theta$ and $y = \rho \sin\phi \sin\theta$ and $z = \rho\cos\phi$. Find the 3 by 3 matrix of partial derivatives: $\partial x/\partial\rho, \partial x/\partial\phi, \partial x/\partial\theta$ in row 1. Simplify its determinant to $J = \rho^2 \sin\phi$. Then dV in spherical coordinates is $\rho^2 \sin\,\phi\, d\rho\, d\phi d\theta$, the volume of an infinitesimal "coordinate box".

29 The matrix that connects r, θ to x, y is in Problem 27. Invert that 2 by 2 matrix:

$$J^{-1} = \begin{vmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{vmatrix} = \begin{vmatrix} \cos\theta & ? \\ ? & ? \end{vmatrix} = ?$$

It is surprising that $\partial r/\partial x = \partial x/\partial r$ (***Calculus***, Gilbert Strang, p. 501). Multiplying the matrices J and J^{-1} gives the chain rule $\frac{\partial x}{\partial x} = \frac{\partial x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial x}{\partial \theta}\frac{\partial \theta}{\partial x} = 1$.

30 The triangle with corners $(0,0)$, $(6,0)$, and $(1,4)$ has area ______. When you rotate it by $\theta = 60^\circ$ the area is ______. The determinant of the rotation matrix is

$$J = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & ? \\ ? & ? \end{vmatrix} = ?$$

Problems 31–38 are about the triple product $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$ in three dimensions.

31 A box has base area $\|\boldsymbol{u} \times \boldsymbol{v}\|$. Its perpendicular height is $\|\boldsymbol{w}\| \cos\theta$. Base area times height = volume = $\|\boldsymbol{u} \times \boldsymbol{v}\|\,\|\boldsymbol{w}\| \cos\theta$ which is $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$. Compute base area, height, and volume for $\boldsymbol{u} = (2,4,0)$, $\boldsymbol{v} = (-1,3,0)$, $\boldsymbol{w} = (1,2,2)$.

32 The volume of the same box is given more directly by a 3 by 3 determinant. Evaluate that determinant.

33 Expand the 3 by 3 determinant in equation (13) in cofactors of its row u_1, u_2, u_3. This expansion is the dot product of $\boldsymbol{u}$ with the vector _____.

34 Which of the triple products $(\boldsymbol{u} \times \boldsymbol{w}) \cdot \boldsymbol{v}$ and $(\boldsymbol{w} \times \boldsymbol{u}) \cdot \boldsymbol{v}$ and $(\boldsymbol{v} \times \boldsymbol{w}) \cdot \boldsymbol{u}$ are the same as $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$? Which orders of the rows $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ give the correct determinant?

35 Let $P = (1,0,-1)$ and $Q = (1,1,1)$ and $R = (2,2,1)$. Choose S so that $PQRS$ is a parallelogram and compute its area. Choose T, U, V so that $OPQRSTUV$ is a tilted box and compute its volume.

36 Suppose (x,y,z) and $(1,1,0)$ and $(1,2,1)$ lie on a plane through the origin. What determinant is zero? What equation does this give for the plane?

37 Suppose (x,y,z) is a linear combination of $(2,3,1)$ and $(1,2,3)$. What determinant is zero? What equation does this give for the plane of all combinations?

38 (a) Explain from volumes why $\det 2A = 2^n \det A$ for n by n matrices.

(b) For what size matrix is the false statement $\det A + \det A = \det(A + A)$ true?

Challenge Problems

39 If you know all 16 cofactors of a 4 by 4 invertible matrix A, how would you find A?

40 Suppose A is a 5 by 5 matrix. Its entries in row 1 multiply determinants (cofactors) in rows 2–5 to give the determinant. Can you guess a "Jacobi formula" for $\det A$ using 2 by 2 determinants from rows 1–2 *times* 3 by 3 determinants from rows 3–5?

Test your formula on the $-1, 2, -1$ tridiagonal matrix that has determinant $= 6$.

41 The 2 by 2 matrix AB =(2 by 3)(3 by 2) has a "Cauchy-Binet formula" for $\det AB$:

$$\det AB = \text{sum of (2 by 2 determinants in } A) \text{ (2 by 2 determinants in } B)$$

(a) Guess which 2 by 2 determinants to use from A and B.

(b) Test your formula when the rows of A are $1, 2, 3$ and $1, 4, 7$ with $B = A^{\mathrm{T}}$.

42 The big formula has $n!$ terms. But if an entry of A is zero, $(n-1)!$ terms disappear. If A has only *three diagonals*, how many terms are left?

For $n = 1, 2, 3, 4$ the tridiagonal determinant has $1, 2, 3, 5$ terms. Those are Fibonacci numbers in Section 6.2! Show why a tridiagonal 5 by 5 determinant has $5 + 3 = 8$ nonzero terms (Fibonacci again). Use the cofactors of a_{11} and a_{12}.

Chapter 6

Eigenvalues and Eigenvectors

6.1 Introduction to Eigenvalues

1 An **eigenvector** $\boldsymbol{x}$ lies along the same line as $A\boldsymbol{x}$: $\boxed{A\boldsymbol{x} = \lambda\boldsymbol{x}.}$ The **eigenvalue** is λ.

2 If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $A^2\boldsymbol{x} = \lambda^2\boldsymbol{x}$ and $A^{-1}\boldsymbol{x} = \lambda^{-1}\boldsymbol{x}$ and $(A + cI)\boldsymbol{x} = (\lambda + c)\boldsymbol{x}$: the same $\boldsymbol{x}$.

3 If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $(A–\lambda I)\boldsymbol{x} = \mathbf{0}$ and $A–\lambda I$ is singular and $\boxed{\det(\boldsymbol{A}-\boldsymbol{\lambda I}) = \mathbf{0}.}$ n eigenvalues.

4 Check λ's by $\det A = (\lambda_1)(\lambda_2)\cdots(\lambda_n)$ and diagonal sum $a_{11} + a_{22} + \cdots + a_{nn} =$ sum of λ's.

5 Projections have $\lambda = 1$ and 0. Reflections have 1 and -1. Rotations have $e^{i\theta}$ and $e^{-i\theta}$: *complex*!

This chapter enters a new part of linear algebra. The first part was about $A\boldsymbol{x} = \boldsymbol{b}$: balance and equilibrium and steady state. Now the second part is about **change**. Time enters the picture—continuous time in a differential equation $d\boldsymbol{u}/dt = A\boldsymbol{u}$ or time steps in a difference equation $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$. Those equations are NOT solved by elimination.

The key idea is to avoid all the complications presented by the matrix A. Suppose the solution vector $\boldsymbol{u}(t)$ stays in the direction of a fixed vector $\boldsymbol{x}$. Then we only need to find the number (changing with time) that multiplies $\boldsymbol{x}$. A number is easier than a vector. **We want "eigenvectors" $\boldsymbol{x}$ that don't change direction when you multiply by A.**

A good model comes from the powers $A, A^2, A^3, \ldots$ of a matrix. Suppose you need the hundredth power A^{100}. Its columns are very close to the *eigenvector* $(.6,\ .4)$:

$$A, A^2, A^3 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \quad \begin{bmatrix} .650 & .525 \\ .350 & .475 \end{bmatrix} \qquad \boldsymbol{A^{100}} \approx \begin{bmatrix} \mathbf{.6000} & \mathbf{.6000} \\ \mathbf{.4000} & \mathbf{.4000} \end{bmatrix}$$

A^{100} was found by using the *eigenvalues* of A, not by multiplying 100 matrices. Those eigenvalues (here they are $\lambda = 1$ and $1/2$) are a new way to see into the heart of a matrix.

To explain eigenvalues, we first explain eigenvectors. Almost all vectors change direction, when they are multiplied by A. ***Certain exceptional vectors x are in the same direction as Ax. Those are the "eigenvectors"***. Multiply an eigenvector by A, and the vector Ax is a number λ times the original x.

The basic equation is $Ax = \lambda x$. The number λ is an eigenvalue of A.

The eigenvalue λ tells whether the special vector x is stretched or shrunk or reversed or left unchanged—when it is multiplied by A. We may find $\lambda = 2$ or $\frac{1}{2}$ or -1 or 1. The eigenvalue λ could be zero! Then $Ax = 0x$ means that this eigenvector x is in the nullspace.

If A is the identity matrix, every vector has $Ax = x$. All vectors are eigenvectors of I. All eigenvalues "lambda" are $\lambda = 1$. This is unusual to say the least. Most 2 by 2 matrices have *two* eigenvector directions and *two* eigenvalues. We will show that $\det(A - \lambda I) = 0$.

This section will explain how to compute the x's and λ's. It can come early in the course because we only need the determinant of a 2 by 2 matrix. Let me use $\det(A - \lambda I) = 0$ to find the eigenvalues for this first example, and then derive it properly in equation (3).

Example 1 The matrix A has two eigenvalues $\lambda = 1$ and $\lambda = 1/2$. Look at $\det(A - \lambda I)$:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \det \begin{bmatrix} .8-\lambda & .3 \\ .2 & .7-\lambda \end{bmatrix} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda - 1)\left(\lambda - \frac{1}{2}\right).$$

I factored the quadratic into $\lambda - 1$ times $\lambda - \frac{1}{2}$, to see the two eigenvalues $\boldsymbol{\lambda} = \mathbf{1}$ and $\boldsymbol{\lambda} = \frac{\mathbf{1}}{\mathbf{2}}$. For those numbers, the matrix $A - \lambda I$ becomes *singular* (zero determinant). The eigenvectors x_1 and x_2 are in the nullspaces of $A - I$ and $A - \frac{1}{2}I$.

$(A - I)x_1 = 0$ is $Ax_1 = x_1$ and the first eigenvector is $(\mathbf{.6}, \mathbf{.4})$.

$(A - \frac{1}{2}I)x_2 = 0$ is $Ax_2 = \frac{1}{2}x_2$ and the second eigenvector is $(\mathbf{1}, \mathbf{-1})$:

$$x_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \quad \text{and} \quad Ax_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} .6 \\ .4 \end{bmatrix} = x_1 \quad (Ax = x \text{ means that } \lambda_1 = 1)$$

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad Ax_2 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} .5 \\ -.5 \end{bmatrix} \quad (\text{this is } \tfrac{1}{2}\, x_2 \text{ so } \lambda_2 = \tfrac{1}{2}).$$

If x_1 is multiplied again by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2}x_2$, and if we multiply again we get $(\frac{1}{2})^2$ times x_2.

When A is squared, the eigenvectors stay the same. The eigenvalues are squared.

This pattern keeps going, because the eigenvectors stay in their own directions (Figure 6.1) and never get mixed. The eigenvectors of A^{100} are the same x_1 and x_2. The eigenvalues of A^{100} are $1^{100} = 1$ and $(\frac{1}{2})^{100}$ = very small number.

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $x_1 + (.2)x_2$:

Separate into eigenvectors
Then multiply by A

$$\begin{bmatrix} .8 \\ .2 \end{bmatrix} = x_1 + (.2)x_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .2 \\ -.2 \end{bmatrix}. \qquad (1)$$

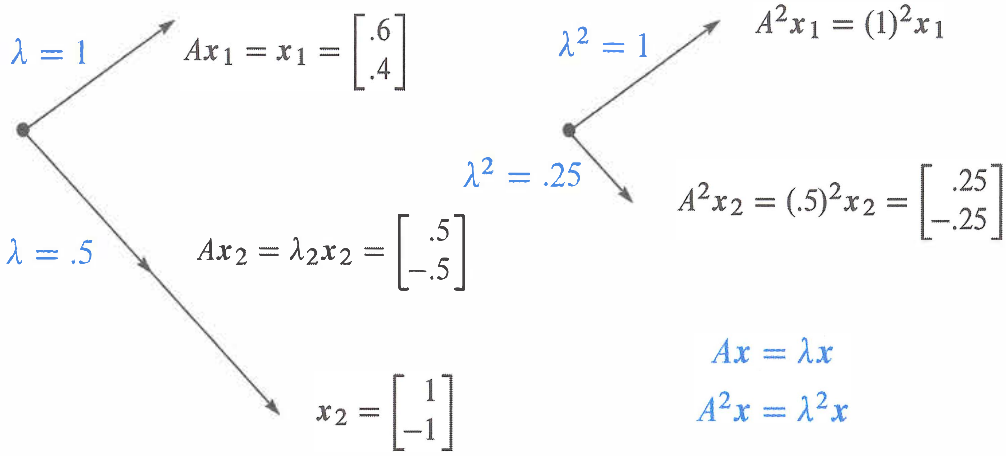


Figure 6.1: The eigenvectors keep their directions. $A^2\boldsymbol{x} = \lambda^2\boldsymbol{x}$ with $\lambda^2 = 1^2$ and $(.5)^2$.

When we multiply separately for $\boldsymbol{x}_1$ and $(.2)\boldsymbol{x}_2$, A multiplies $\boldsymbol{x}_2$ by its eigenvalue $\frac{1}{2}$:

$$\textbf{Multiply each } \boldsymbol{x}_i \textbf{ by } \boldsymbol{\lambda}_i \qquad A\begin{bmatrix} .8 \\ .2 \end{bmatrix} \quad \text{is} \quad \boldsymbol{x}_1 + \frac{1}{2}(.2)\boldsymbol{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .1 \\ -.1 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix}.$$

Each eigenvector is multiplied by its eigenvalue, when we multiply by A. At every step $\boldsymbol{x}_1$ is unchanged and $\boldsymbol{x}_2$ is multiplied by $\left(\frac{1}{2}\right)$, so 99 steps give the small number $\left(\frac{1}{2}\right)^{99}$:

$$\boxed{A^{99}\begin{bmatrix} .8 \\ .2 \end{bmatrix} \quad \text{is really} \quad \boldsymbol{x}_1 + (.2)\left(\tfrac{1}{2}\right)^{99}\boldsymbol{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} \text{very} \\ \text{small} \\ \text{vector} \end{bmatrix}.}$$

This is the first column of A^{100}. The number we originally wrote as .6000 was not exact. We left out $(.2)(\frac{1}{2})^{99}$ which wouldn't show up for 30 decimal places.

The eigenvector $\boldsymbol{x}_1$ is a "steady state" that doesn't change (because $\lambda_1 = 1$). The eigenvector $\boldsymbol{x}_2$ is a "decaying mode" that virtually disappears (because $\lambda_2 = .5$). The higher the power of A, the more closely its columns approach the steady state.

This particular A is a ***Markov matrix***. Its largest eigenvalue is $\lambda = 1$. Its eigenvector $\boldsymbol{x}_1 = (.6, .4)$ is the *steady state*—which all columns of A^k will approach. Section 10.3 shows how Markov matrices appear when you search with Google.

For projection matrices P, we can see when Px is parallel to x. The eigenvectors for $\lambda = 1$ and $\lambda = 0$ fill the column space and nullspace. The column space doesn't move ($P\boldsymbol{x} = \boldsymbol{x}$). The nullspace goes to zero ($P\boldsymbol{x} = 0\,\boldsymbol{x}$).

Example 2 **The projection matrix** $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ has eigenvalues $\lambda = 1$ and $\lambda = 0$.

Its eigenvectors are $\boldsymbol{x}_1 = (1, 1)$ and $\boldsymbol{x}_2 = (1, -1)$. For those vectors, $P\boldsymbol{x}_1 = \boldsymbol{x}_1$ (steady state) and $P\boldsymbol{x}_2 = \mathbf{0}$ (nullspace). This example illustrates Markov matrices and singular matrices and (most important) symmetric matrices. All have special λ's and $\boldsymbol{x}$'s:

1. **Markov matrix** : Each column of P adds to 1, so $\lambda = 1$ is an eigenvalue.
2. P is **singular**, so $\lambda = 0$ is an eigenvalue.
3. P is **symmetric**, so its eigenvectors $(1, 1)$ and $(1, -1)$ are perpendicular.

The only eigenvalues of a projection matrix are 0 and 1. The eigenvectors for $\lambda = 0$ (which means $P\boldsymbol{x} = 0\boldsymbol{x}$) fill up the nullspace. The eigenvectors for $\lambda = 1$ (which means $P\boldsymbol{x} = \boldsymbol{x}$) fill up the column space. The nullspace is projected to zero. The column space projects onto itself. The projection keeps the column space and destroys the nullspace:

$$\textbf{Project each part} \qquad \boldsymbol{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad \textbf{projects onto} \qquad P\boldsymbol{v} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \end{bmatrix}.$$

Projections have $\lambda = 0$ and 1. Permutations have all $|\lambda| = 1$. The next matrix R is a reflection and at the same time a permutation. R also has special eigenvalues.

Example 3 **The reflection matrix** $R = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ **has eigenvalues 1 and** $-\mathbf{1}$.

The eigenvector $(1, 1)$ is unchanged by R. The second eigenvector is $(1, -1)$—its signs are reversed by R. A matrix with no negative entries can still have a negative eigenvalue! The eigenvectors for R are the same as for P, because *reflection* $= 2$(*projection*) $- I$:

$$\boldsymbol{R = 2P - I} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2}$$

***When a matrix is shifted by $\boldsymbol{I}$, each $\boldsymbol{\lambda}$ is shifted by* 1.** No change in eigenvectors.

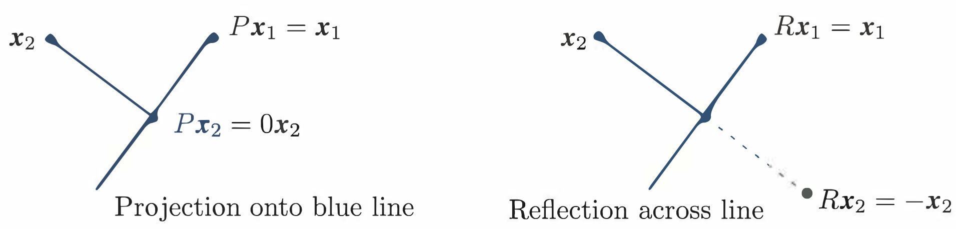


Figure 6.2: Projections P have eigenvalues 1 and 0. Reflections R have $\lambda = 1$ and -1. A typical $\boldsymbol{x}$ changes direction, but an eigenvector stays along the same line.

The Equation for the Eigenvalues

For projection matrices we found λ's and $\boldsymbol{x}$'s by geometry: $P\boldsymbol{x} = \boldsymbol{x}$ and $P\boldsymbol{x} = \mathbf{0}$. For other matrices we use determinants and linear algebra. *This is the key calculation in the chapter*—almost every application starts by solving $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

First move $\lambda\boldsymbol{x}$ to the left side. Write the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ as $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$. The matrix $A - \lambda I$ times the eigenvector $\boldsymbol{x}$ is the zero vector. ***The eigenvectors make up the nullspace of $A - \lambda I$***. When we know an eigenvalue λ, we find an eigenvector by solving $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$.

Eigenvalues first. If $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ has a nonzero solution, $A - \lambda I$ is not invertible. ***The determinant of $A - \lambda I$ must be zero.*** This is how to recognize an eigenvalue λ:

Eigenvalues The number λ is an eigenvalue of A if and only if $A - \lambda I$ is singular.

Equation for the eigenvalues $\det(A - \lambda I) = 0.$ (3)

This "*characteristic polynomial*" $\det(A - \lambda I)$ involves only λ, not $\boldsymbol{x}$. When A is n by n, equation (3) has degree n. Then A has n eigenvalues (repeats possible!) Each λ leads to $\boldsymbol{x}$:

For each eigenvalue λ solve $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ or $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to find an eigenvector $\boldsymbol{x}$.

Example 4 $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{4} \end{bmatrix}$ is already singular (zero determinant). Find its λ's and $\boldsymbol{x}$'s.

When A is singular, $\lambda = 0$ is one of the eigenvalues. The equation $A\boldsymbol{x} = 0\boldsymbol{x}$ has solutions. They are the eigenvectors for $\lambda = 0$. But $\det(A - \lambda I) = 0$ is the way to find *all* λ's and $\boldsymbol{x}$'s. Always subtract λI from A:

$$\textit{\textbf{Subtract } } \lambda \textit{\textbf{ from the diagonal to find}} \quad A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}. \tag{4}$$

Take the determinant "$ad - bc$" of this 2 by 2 matrix. From $1 - \lambda$ times $4 - \lambda$, the "ad" part is $\lambda^2 - 5\lambda + 4$. The "bc" part, not containing λ, is 2 times 2.

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - (2)(2) = \lambda^2 - 5\lambda. \tag{5}$$

Set this determinant $\lambda^2 - 5\lambda$ to zero. One solution is $\lambda = 0$ (as expected, since A is singular). Factoring into λ times $\lambda - 5$, the other root is $\lambda = 5$:

$\det(A - \lambda I) = \lambda^2 - 5\lambda = 0$ yields the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 5$.

Now find the eigenvectors. Solve $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ separately for $\lambda_1 = 0$ and $\lambda_2 = 5$:

$$(A - 0I)\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \boxed{\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}} \text{ for } \lambda_1 = 0$$

$$(A - 5I)\boldsymbol{x} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \boxed{\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}} \text{ for } \lambda_2 = 5.$$

The matrices $A - 0I$ and $A - 5I$ are singular (because 0 and 5 are eigenvalues). The eigenvectors $(2, -1)$ and $(1, 2)$ are in the nullspaces: $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ is $A\boldsymbol{x} = \lambda \boldsymbol{x}$.

We need to emphasize: *There is nothing exceptional about* $\lambda = 0$. Like every other number, zero might be an eigenvalue and it might not. If A is singular, the eigenvectors for $\lambda = 0$ fill the nullspace: $A\boldsymbol{x} = 0\boldsymbol{x} = \mathbf{0}$. If A is invertible, zero is not an eigenvalue. We shift A by a multiple of I to *make it singular.*

In the example, the shifted matrix $A - 5I$ is singular and 5 is the other eigenvalue.

Summary To solve the eigenvalue problem for an n by n matrix, follow these steps:

1. ***Compute the determinant of*** $A - \lambda I$. With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n.

2. ***Find the roots of this polynomial***, by solving $\det(A - \lambda I) = 0$. The n roots are the n eigenvalues of A. They make $A - \lambda I$ singular.

3. For each eigenvalue λ, ***solve*** $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ ***to find an eigenvector*** $\boldsymbol{x}$.

A note on the eigenvectors of 2 by 2 matrices. When $A - \lambda I$ is singular, both rows are multiples of a vector (a, b). *The eigenvector is any multiple of* $(b, -a)$. The example had

$\lambda = 0$: rows of $A - 0I$ in the direction $(1, 2)$; eigenvector in the direction $(2, -1)$

$\lambda = 5$: rows of $A - 5I$ in the direction $(-4, 2)$; eigenvector in the direction $(2, 4)$.

Previously we wrote that last eigenvector as $(1, 2)$. Both $(1, 2)$ and $(2, 4)$ are correct. There is a whole *line of eigenvectors*—any nonzero multiple of $\boldsymbol{x}$ is as good as $\boldsymbol{x}$. MATLAB's **eig**(A) divides by the length, to make the eigenvector into a unit vector.

We must add a warning. Some 2 by 2 matrices have only *one* line of eigenvectors. This can only happen when two eigenvalues are equal. (On the other hand $A = I$ has equal eigenvalues and plenty of eigenvectors.) Without a full set of eigenvectors, we don't have a basis. We can't write every $\boldsymbol{v}$ as a combination of eigenvectors. In the language of the next section, *we can't diagonalize a matrix without* n *independent eigenvectors*.

Determinant and Trace

Bad news first: If you add a row of A to another row, or exchange rows, the eigenvalues usually change. *Elimination does not preserve the* λ's. The triangular U has *its* eigenvalues sitting along the diagonal—they are the pivots. But they are not the eigenvalues of A! Eigenvalues are changed when row 1 is added to row 2:

$$U = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \quad \text{has } \lambda = 0 \text{ and } \lambda = 1; \quad A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{has } \lambda = 0 \text{ and } \lambda = 7.$$

Good news second: The *product* λ_1 *times* λ_2 *and the sum* $\lambda_1 + \lambda_2$ *can be found quickly from the matrix*. For this A, the product is 0 times 7. That agrees with the determinant (which is 0). The sum of eigenvalues is $0 + 7$. That agrees with the sum down the main diagonal (the **trace** is $1 + 6$). These quick checks always work:

> ***The product of the n eigenvalues** equals **the determinant.***
> ***The sum of the n eigenvalues** equals **the sum of the n diagonal entries.***

The sum of the entries along the main diagonal is called the ***trace*** of A:

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \textbf{\textit{trace}} = a_{11} + a_{22} + \cdots + a_{nn}. \tag{6}$$

Those checks are very useful. They are proved in Problems 16–17 and again in the next section. They don't remove the pain of computing λ's. But when the computation is wrong, they generally tell us so. To compute the correct λ's, go back to $\det(A - \lambda I) = 0$.

The trace and determinant *do* tell everything when the matrix is 2 by 2. We never want to get those wrong! Here trace $= \mathbf{3}$ and det $= \mathbf{2}$, so the eigenvalues are $\lambda = \mathbf{1}$ and $\mathbf{2}$:

$$A = \begin{bmatrix} 1 & 9 \\ 0 & 2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 7 & -3 \\ 10 & -4 \end{bmatrix}. \tag{7}$$

And here is a question about the best matrices for finding eigenvalues: *triangular*.

Why do the eigenvalues of a triangular matrix lie along its diagonal?

Imaginary Eigenvalues

One more bit of news (not too terrible). The eigenvalues might not be real numbers.

> **Example 5** *The* $90°$ *rotation* $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ***has no real eigenvectors. Its eigenvalues are*** $\boldsymbol{\lambda_1 = i}$ ***and*** $\boldsymbol{\lambda_2 = -i}$. ***Then*** $\boldsymbol{\lambda_1 + \lambda_2 =}$ **trace** $= 0$ ***and*** $\boldsymbol{\lambda_1\lambda_2} =$ ***determinant*** $= 1$.

After a rotation, *no real vector $Q\boldsymbol{x}$ stays in the same direction as $\boldsymbol{x}$* ($\boldsymbol{x} = \mathbf{0}$ is useless). There cannot be an eigenvector, unless we go to ***imaginary numbers***. Which we do.

To see how $i = \sqrt{-1}$ can help, look at Q^2 which is $-I$. If Q is rotation through $90°$, then Q^2 is rotation through $180°$. Its eigenvalues are -1 and -1. (Certainly $-I\boldsymbol{x} = -1\boldsymbol{x}$.) Squaring Q will square each λ, so we must have $\lambda^2 = -1$. *The eigenvalues of the* $90°$ *rotation matrix* Q *are* $+i$ *and* $-i$, because $i^2 = -1$.

Those λ's come as usual from $\det(Q - \lambda I) = 0$. This equation gives $\lambda^2 + 1 = 0$. Its roots are i and $-i$. We meet the imaginary number i also in the eigenvectors:

Complex eigenvectors $\qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = i \begin{bmatrix} i \\ 1 \end{bmatrix}$.

Somehow these complex vectors $\boldsymbol{x}_1 = (1, i)$ and $\boldsymbol{x}_2 = (i, 1)$ keep their direction as they are rotated. Don't ask me how. This example makes the all-important point that real matrices can easily have complex eigenvalues and eigenvectors. The particular eigenvalues i and $-i$ also illustrate two special properties of Q:

1. Q is an orthogonal matrix so the absolute value of each λ is $|\lambda| = 1$.

2. Q is a skew-symmetric matrix so each λ is pure imaginary.

A symmetric matrix ($S^{\mathrm{T}} = S$) can be compared to a real number. A skew-symmetric matrix ($A^{\mathrm{T}} = -A$) can be compared to an imaginary number. An orthogonal matrix ($Q^{\mathrm{T}}Q = I$) corresponds to a complex number with $|\lambda| = 1$. For the eigenvalues of S and A and Q, those are more than analogies—they are facts to be proved in Section 6.4.

The eigenvectors for all these special matrices are perpendicular. Somehow $(i, 1)$ and $(1, i)$ are perpendicular (Chapter 9 explains the dot product of complex vectors).

Eigenvalues of AB and $A+B$

The first guess about the eigenvalues of AB is not true. An eigenvalue λ of A times an eigenvalue β of B usually does *not* give an eigenvalue of AB:

False proof $\qquad AB\boldsymbol{x} = A\beta\boldsymbol{x} = \beta A\boldsymbol{x} = \beta\lambda\boldsymbol{x}. \qquad (8)$

It seems that β times λ is an eigenvalue. When $\boldsymbol{x}$ is an eigenvector for A and B, this proof is correct. ***The mistake is to expect that A and B automatically share the same eigenvector*** $\boldsymbol{x}$. Usually they don't. Eigenvectors of A are not generally eigenvectors of B. A and B could have all zero eigenvalues while 1 is an eigenvalue of AB:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad \text{then} \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A+B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For the same reason, the eigenvalues of $A + B$ are generally not $\lambda + \beta$. Here $\lambda + \beta = 0$ while $A + B$ has eigenvalues 1 and -1. (At least they add to zero.)

The false proof suggests what is true. Suppose x really is an eigenvector for both A and B. Then we do have $ABx = \lambda\beta x$ and $BAx = \lambda\beta x$. When all n eigenvectors are shared, we *can* multiply eigenvalues. The test $AB = BA$ for shared eigenvectors is important in quantum mechanics—time out to mention this application of linear algebra:

A and B share the same n *independent* eigenvectors if and only if $\boldsymbol{AB = BA}$.

Heisenberg's uncertainty principle In quantum mechanics, the position matrix P and the momentum matrix Q do not commute. In fact $QP - PQ = I$ (these are infinite matrices). To have $Px = \mathbf{0}$ at the same time as $Qx = \mathbf{0}$ would require $x = Ix = \mathbf{0}$. If we knew the position exactly, we could not also know the momentum exactly. Problem 36 derives Heisenberg's uncertainty principle $\|Px\|\,\|Qx\| \geq \frac{1}{2}\|x\|^2$.

■ REVIEW OF THE KEY IDEAS ■

1. $Ax = \lambda x$ says that eigenvectors x keep the same direction when multiplied by A.
2. $Ax = \lambda x$ also says that $\det(A - \lambda I) = 0$. This determines n eigenvalues.
3. The eigenvalues of A^2 and A^{-1} are λ^2 and λ^{-1}, with the same eigenvectors.
4. The sum of the λ's equals the sum down the main diagonal of A (*the trace*). The product of the λ's equals the determinant of A.
5. Projections P, reflections R, 90° rotations Q have special eigenvalues $1, 0, -1, i, -i$. Singular matrices have $\lambda = 0$. Triangular matrices have λ's on their diagonal.
6. *Special properties of a matrix lead to special eigenvalues and eigenvectors.* That is a major theme of this chapter (it is captured in a table at the very end).

■ WORKED EXAMPLES ■

6.1 A Find the eigenvalues and eigenvectors of A and A^2 and A^{-1} and $A + 4I$:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

Check the trace $\lambda_1 + \lambda_2 = 4$ and the determinant $\lambda_1\lambda_2 = 3$.

Solution The eigenvalues of A come from $\det(A - \lambda I) = 0$:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} \mathbf{2-\lambda} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{2-\lambda} \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0.$$

This factors into $(\lambda - 1)(\lambda - 3) = 0$ so the eigenvalues of A are $\lambda_1 = \mathbf{1}$ and $\lambda_2 = \mathbf{3}$. For the trace, the sum $2+2$ agrees with $1+3$. The determinant 3 agrees with the product $\lambda_1\lambda_2$.

The eigenvectors come separately by solving $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ which is $A\boldsymbol{x} = \lambda\boldsymbol{x}$:

$$\boldsymbol{\lambda = 1:} \quad (A - I)\boldsymbol{x} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{\lambda = 3:} \quad (A - 3I)\boldsymbol{x} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A^2 and A^{-1} and $A + 4I$ keep the *same eigenvectors as* A. Their eigenvalues are λ^2 and λ^{-1} and $\lambda + 4$:

$$A^2 \text{ has eigenvalues } 1^2 = 1 \text{ and } 3^2 = 9 \qquad A^{-1} \text{ has } \frac{1}{1} \text{ and } \frac{1}{3} \qquad A + 4I \text{ has } \begin{matrix} 1 + 4 = 5 \\ 3 + 4 = 7 \end{matrix}$$

Notes for later sections: A has *orthogonal eigenvectors* (Section 6.4 on symmetric matrices). A can be *diagonalized* since $\lambda_1 \neq \lambda_2$ (Section 6.2). A is *similar* to any 2 by 2 matrix with eigenvalues 1 and 3 (Section 6.2). A is a *positive definite matrix* (Section 6.5) since $A = A^{\mathrm{T}}$ and the λ's are positive.

6.1 B **How can you estimate the eigenvalues of any A?** Gershgorin gave this answer.

Every eigenvalue of A must be "near" at least one of the entries a_{ii} on the main diagonal. For λ to be "near a_{ii}" means that $|a_{ii} - \lambda|$ is no more than **the sum R_i of all other $|a_{ij}|$ in that row i of the matrix**. Then $R_i = \Sigma_{j \neq i} |a_{ij}|$ is the radius of a circle centered at a_{ii}.

Every λ is in the circle around one or more diagonal entries a_{ii} : $|a_{ii} - \lambda| \leq R_i$.

Here is the reasoning. If λ is an eigenvalue, then $A - \lambda I$ is not invertible. Then $A - \lambda I$ cannot be diagonally dominant (see Section 2.5). So at least one diagonal entry $a_{ii} - \lambda$ is *not larger* than the sum R_i of all other entries $|a_{ij}|$ (we take absolute values!) in row i.

Example 1. Every eigenvalue λ of this A falls into one or both of the **Gershgorin circles**: The centers are a and d, the radii are $R_1 = |b|$ and $R_2 = |c|$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{matrix} \text{First circle:} & |\lambda - a| \leq |b| \\ \text{Second circle:} & |\lambda - d| \leq |c| \end{matrix}$$

Those are circles in the complex plane, since λ could certainly be complex.

Example 2. All eigenvalues of this A lie in a circle of radius $R = 3$ around *one or more* of the diagonal entries d_1, d_2, d_3:

$$A = \begin{bmatrix} d_1 & 1 & 2 \\ 2 & d_2 & 1 \\ -1 & 2 & d_3 \end{bmatrix} \qquad \begin{matrix} |\lambda - d_1| \leq 1 + 2 = R_1 \\ |\lambda - d_2| \leq 2 + 1 = R_2 \\ |\lambda - d_3| \leq 1 + 2 = R_3 \end{matrix}$$

You see that "near" means not more than 3 away from d_1 or d_2 or d_3, for this example.

6.1 C Find the eigenvalues and eigenvectors of this symmetric 3 by 3 matrix S:

Symmetric matrix
Singular matrix
Trace $1+2+1=4$

$$S = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution Since all rows of S add to zero, the vector $\boldsymbol{x} = (1,1,1)$ gives $S\boldsymbol{x} = \mathbf{0}$. This is an eigenvector for $\lambda = 0$. To find λ_2 and λ_3 I will compute the 3 by 3 determinant:

$$\det(S - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} \begin{aligned} &= (1-\lambda)(2-\lambda)(1-\lambda) - 2(1-\lambda) \\ &= (1-\lambda)[(2-\lambda)(1-\lambda) - 2] \\ &= (\mathbf{1}-\boldsymbol{\lambda})(-\boldsymbol{\lambda})(\mathbf{3}-\boldsymbol{\lambda}). \end{aligned}$$

Those three factors give $\lambda = 0, 1, 3$. Each eigenvalue corresponds to an eigenvector (or a line of eigenvectors):

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad S\boldsymbol{x}_1 = \mathbf{0}\boldsymbol{x}_1 \qquad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad S\boldsymbol{x}_2 = \mathbf{1}\boldsymbol{x}_2 \qquad \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad S\boldsymbol{x}_3 = \mathbf{3}\boldsymbol{x}_3\,.$$

I notice again that eigenvectors are perpendicular when S is symmetric. We were lucky to find $\lambda = 0, 1, 3$. For a larger matrix I would use **eig**(A), and never touch determinants.

The full command $[\boldsymbol{X}, \boldsymbol{E}] = \mathbf{eig}(A)$ will produce unit eigenvectors in the columns of X.

Problem Set 6.1

1 The example at the start of the chapter has powers of this matrix A:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

Find the eigenvalues of these matrices. All powers have the same eigenvectors.

(a) Show from A how a row exchange can produce different eigenvalues.

(b) Why is a zero eigenvalue *not* changed by the steps of elimination?

2 Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$ has the _____ eigenvectors as A. Its eigenvalues are _____ by 1.

3 Compute the eigenvalues and eigenvectors of A and A^{-1}. Check the trace !

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

A^{-1} has the _____ eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues _____ .

4 Compute the eigenvalues and eigenvectors of A and A^2:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

A^2 has the same _____ as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues _____. In this example, why is $\lambda_1^2 + \lambda_2^2 = 13$?

5 Find the eigenvalues of A and B (easy for triangular matrices) and $A + B$:

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A + B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Eigenvalues of $A + B$ (*are equal to*)(*are not equal to*) eigenvalues of A plus eigenvalues of B.

6 Find the eigenvalues of A and B and AB and BA:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

(a) Are the eigenvalues of AB equal to eigenvalues of A times eigenvalues of B?

(b) Are the eigenvalues of AB equal to the eigenvalues of BA?

7 Elimination produces $A = LU$. The eigenvalues of U are on its diagonal; they are the _____. The eigenvalues of L are on its diagonal; they are all _____. The eigenvalues of A are not the same as _____.

8 (a) If you know that $\boldsymbol{x}$ is an eigenvector, the way to find λ is to _____.

(b) If you know that λ is an eigenvalue, the way to find $\boldsymbol{x}$ is to _____.

9 What do you do to the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$, in order to prove (a), (b), and (c)?

(a) λ^2 is an eigenvalue of A^2, as in Problem 4.

(b) λ^{-1} is an eigenvalue of A^{-1}, as in Problem 3.

(c) $\lambda + 1$ is an eigenvalue of $A + I$, as in Problem 2.

10 Find the eigenvalues and eigenvectors for both of these Markov matrices A and A^∞. Explain from those answers why A^{100} is close to A^∞:

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}.$$

11 Here is a strange fact about 2 by 2 matrices with eigenvalues $\lambda_1 \neq \lambda_2$: The columns of $A - \lambda_1 I$ are multiples of the eigenvector $\boldsymbol{x}_2$. Any idea why this should be?

12 Find three eigenvectors for this matrix P (projection matrices have $\lambda=1$ and 0):

$$\textbf{Projection matrix} \qquad P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If two eigenvectors share the same λ, so do all their linear combinations. Find an eigenvector of P with no zero components.

13 From the unit vector $\boldsymbol{u} = \left(\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6}\right)$ construct the rank one projection matrix $P = \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$. This matrix has $P^2 = P$ because $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$.

(a) $P\boldsymbol{u} = \boldsymbol{u}$ comes from $(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u} = \boldsymbol{u}(\_\_\_\_)$. Then $\boldsymbol{u}$ is an eigenvector with $\lambda=1$.

(b) If $\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ show that $P\boldsymbol{v} = \mathbf{0}$. Then $\lambda = 0$.

(c) Find three independent eigenvectors of P all with eigenvalue $\lambda = 0$.

14 Solve $\det(Q - \lambda I) = 0$ by the quadratic formula to reach $\lambda = \cos\theta \pm i\sin\theta$:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{rotates the } xy \text{ plane by the angle } \theta. \text{ No real } \lambda\text{'s.}$$

Find the eigenvectors of Q by solving $(Q - \lambda I)\boldsymbol{x} = \mathbf{0}$. Use $i^2 = -1$.

15 Every permutation matrix leaves $\boldsymbol{x} = (1, 1, \ldots, 1)$ unchanged. Then $\lambda = 1$. Find two more λ's (possibly complex) for these permutations, from $\det(P - \lambda I) = 0$:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

16 **The determinant of A equals the product $\lambda_1\lambda_2\cdots\lambda_n$.** Start with the polynomial $\det(A - \lambda I)$ separated into its n factors (always possible). Then set $\lambda = 0$:

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda) \quad \text{so} \quad \det A = \_\_\_\_.$$

Check this rule in Example 1 where the Markov matrix has $\lambda = 1$ and $\frac{1}{2}$.

17 The sum of the diagonal entries (the *trace*) equals the sum of the eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has} \quad \det(A - \lambda I) = \lambda^2 - (a+d)\lambda + ad - bc = 0.$$

The quadratic formula gives the eigenvalues $\lambda = (a+d+\sqrt{})/2$ and $\lambda = \_\_\_\_$. Their sum is $\_\_\_\_$. If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A - \lambda I) = \_\_\_\_$.

18 If A has $\lambda_1 = 4$ and $\lambda_2 = 5$ then $\det(A - \lambda I) = (\lambda - 4)(\lambda - 5) = \lambda^2 - 9\lambda + 20$. Find three matrices that have trace $a + d = 9$ and determinant 20 and $\lambda = 4, 5$.

19 A 3 by 3 matrix B is known to have eigenvalues $0, 1, 2$. This information is enough to find three of these (give the answers where possible):

(a) the rank of B

(b) the determinant of $B^{\mathrm{T}}B$

(c) the eigenvalues of $B^{\mathrm{T}}B$

(d) the eigenvalues of $(B^2 + I)^{-1}$.

20 Choose the last rows of A and C to give eigenvalues $4, 7$ and $1, 2, 3$:

Companion matrices $$A = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix}.$$

21 ***The eigenvalues of A equal the eigenvalues of*** A^{T}. This is because $\det(A - \lambda I)$ equals $\det(A^{\mathrm{T}} - \lambda I)$. That is true because _____. Show by an example that the eigenvectors of A and A^{T} are *not* the same.

22 Construct any 3 by 3 Markov matrix M: positive entries down each column add to 1. Show that $M^{\mathrm{T}}(1,1,1) = (1,1,1)$. By Problem 21, $\lambda = 1$ is also an eigenvalue of M. Challenge: A 3 by 3 singular Markov matrix with trace $\frac{1}{2}$ has what λ's ?

23 Find three 2 by 2 matrices that have $\lambda_1 = \lambda_2 = 0$. The trace is zero and the determinant is zero. A might not be the zero matrix but check that $A^2 = 0$.

24 This matrix is singular with rank one. Find three λ's and three eigenvectors:

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$

25 Suppose A and B have the same eigenvalues $\lambda_1, \ldots, \lambda_n$ with the same independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. Then $A = B$. *Reason*: Any vector $\boldsymbol{x}$ is a combination $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$. What is $A\boldsymbol{x}$? What is $B\boldsymbol{x}$?

26 The block B has eigenvalues $1, 2$ and C has eigenvalues $3, 4$ and D has eigenvalues $5, 7$. Find the eigenvalues of the 4 by 4 matrix A:

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$

27 Find the rank and the four eigenvalues of A and C:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

28 Subtract I from the previous A. Find the λ's and then the determinants of

$$B = A - I = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C = I - A = \begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

29 (Review) Find the eigenvalues of A, B, and C:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

30 When $a + b = c + d$ show that $(1, 1)$ is an eigenvector and find both eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

31 If we exchange rows 1 and 2 *and* columns 1 and 2, the eigenvalues don't change. Find eigenvectors of A and B for $\lambda = 11$. *Rank one gives* $\lambda_2 = \lambda_3 = 0$.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix} \quad \text{and} \quad B = PAP^{\mathrm{T}} = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}.$$

32 Suppose A has eigenvalues $0, 3, 5$ with independent eigenvectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

(a) Give a basis for the nullspace and a basis for the column space.

(b) Find a particular solution to $A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$. Find all solutions.

(c) $A\boldsymbol{x} = \boldsymbol{u}$ has no solution. If it did then _____ would be in the column space.

Challenge Problems

33 Show that $\boldsymbol{u}$ is an eigenvector of the rank one 2×2 matrix $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$. Find both eigenvalues of A. Check that $\lambda_1 + \lambda_2$ agrees with the trace $u_1v_1 + u_2v_2$.

34 Find the eigenvalues of this permutation matrix P from $\det(P - \lambda I) = 0$. Which vectors are not changed by the permutation? They are eigenvectors for $\lambda = 1$. Can you find three more eigenvectors?

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

35 There are six 3 by 3 permutation matrices P. What numbers can be the *determinants* of P? What numbers can be *pivots*? What numbers can be the *trace* of P? What *four numbers* can be eigenvalues of P, as in Problem 15?

36 (**Heisenberg's Uncertainty Principle**) $AB - BA = I$ can happen for infinite matrices with $A = A^{\mathrm{T}}$ and $B = -B^{\mathrm{T}}$. Then

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}AB\boldsymbol{x} - \boldsymbol{x}^{\mathrm{T}}BA\boldsymbol{x} \leq 2\|A\boldsymbol{x}\|\,\|B\boldsymbol{x}\|.$$

Explain that last step by using the Schwarz inequality $|\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v}| \leq ||\boldsymbol{u}||\,||\boldsymbol{v}||$. Then Heisenberg's inequality says that $\|A\boldsymbol{x}\|/\|\boldsymbol{x}\|$ times $\|B\boldsymbol{x}\|/\|\boldsymbol{x}\|$ is at least $\frac{1}{2}$. It is impossible to get the position error and momentum error both very small.

37 *Find a* 2 *by* 2 *rotation matrix* (other than I) *with* $A^3 = I$. Its eigenvalues must satisfy $\lambda^3 = 1$. They can be $e^{2\pi i/3}$ and $e^{-2\pi i/3}$. What are the trace and determinant?

38 (a) Find the eigenvalues and eigenvectors of A. They depend on c:

$$A = \begin{bmatrix} .4 & 1-c \\ .6 & c \end{bmatrix}.$$

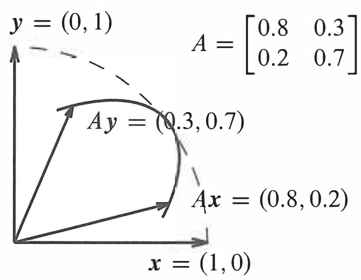
(b) Show that A has just one line of eigenvectors when $c = 1.6$.

(c) This is a Markov matrix when $c = .8$. Then A^n will approach what matrix A^∞?

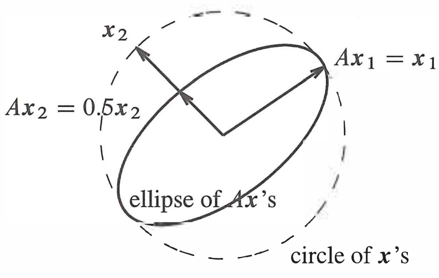
Eigshow in MATLAB

There is a MATLAB demo (just type **eigshow**), displaying the eigenvalue problem for a 2 by 2 matrix. It starts with the unit vector $\boldsymbol{x} = (1, 0)$. *The mouse makes this vector move around the unit circle.* At the same time the screen shows $A\boldsymbol{x}$, in color and also moving. Possibly $A\boldsymbol{x}$ is ahead of $\boldsymbol{x}$. Possibly $A\boldsymbol{x}$ is behind $\boldsymbol{x}$. *Sometimes $A\boldsymbol{x}$ is parallel to $\boldsymbol{x}$.*

At that parallel moment, $A\boldsymbol{x} = \lambda\boldsymbol{x}$ (at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in the second figure).



These are not eigenvectors



$A\boldsymbol{x}$ lines up with $\boldsymbol{x}$ at eigenvectors

The eigenvalue λ is the length of $A\boldsymbol{x}$, when the unit eigenvector $\boldsymbol{x}$ lines up. The built-in choices for A illustrate three possibilities: $0, 1$, or 2 real vectors where $A\boldsymbol{x}$ crosses $\boldsymbol{x}$.

The axes of the ellipse are **singular vectors** in 7.4—and eigenvectors if $A^{\mathrm{T}} = A$.

6.2 Diagonalizing a Matrix

1 The columns of $AX = X\Lambda$ are $A\boldsymbol{x}_k = \lambda_k \boldsymbol{x}_k$. The eigenvalue matrix Λ is diagonal.

2 $\boldsymbol{n}$ independent eigenvectors in X diagonalize A $\boxed{\boldsymbol{A = X\Lambda X^{-1}} \text{ and } \boldsymbol{\Lambda = X^{-1}AX}}$

3 The eigenvector matrix X also diagonalizes all powers A^k: $\boxed{\boldsymbol{A^k = X\Lambda^k X^{-1}}}$

4 Solve $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$ by $\boldsymbol{u}_k = A^k\boldsymbol{u}_0 = X\Lambda^k X^{-1}\boldsymbol{u}_0 = \boxed{c_1(\lambda_1)^k\boldsymbol{x}_1 + \cdots + c_n(\lambda_n)^k\boldsymbol{x}_n}$

5 **No equal eigenvalues** $\Rightarrow$ X is invertible and A can be diagonalized.
Equal eigenvalues $\Rightarrow$ A *might* have too few independent eigenvectors. Then X^{-1} fails.

6 Every matrix $C = B^{-1}AB$ has the **same eigenvalues** as A. These C's are "**similar**" to A.

When $\boldsymbol{x}$ is an eigenvector, multiplication by A is just multiplication by a number λ: $A\boldsymbol{x} = \lambda\boldsymbol{x}$. All the difficulties of matrices are swept away. Instead of an interconnected system, we can follow the eigenvectors separately. It is like having a *diagonal matrix*, with no off-diagonal interconnections. The 100th power of a diagonal matrix is easy.

The point of this section is very direct. ***The matrix A turns into a diagonal matrix Λ when we use the eigenvectors properly***. This is the matrix form of our key idea. We start right off with that one essential computation. The next page explains why $\boldsymbol{AX = X\Lambda}$.

Diagonalization Suppose the n by n matrix A has n linearly independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. Put them into the columns of an ***eigenvector matrix X***. Then $X^{-1}AX$ is the ***eigenvalue matrix Λ***:

Eigenvector matrix X
Eigenvalue matrix Λ

$$\boldsymbol{X^{-1}AX = \Lambda} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}. \qquad (1)$$

The matrix A is "diagonalized." We use capital lambda for the eigenvalue matrix, because the small λ's (the eigenvalues) are on its diagonal.

Example 1 This A is triangular so its eigenvalues are on the diagonal: $\lambda = 1$ and $\lambda = 6$.

Eigenvectors go into X $\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$

$$\underset{X^{-1}}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} \mathbf{1} & \mathbf{5} \\ 0 & \mathbf{6} \end{bmatrix}} \underset{X}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} \underset{=}{=} \underset{\Lambda}{\begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{6} \end{bmatrix}}$$

In other words $A = X\Lambda X^{-1}$. Then watch $A^2 = X\Lambda X^{-1}X\Lambda X^{-1}$. So $\boldsymbol{A^2}$ **is** $\boldsymbol{X\Lambda^2 X^{-1}}$.

A^2 has the same eigenvectors in X and squared eigenvalues in Λ^2.

Why is $AX = X\Lambda$? A multiplies its eigenvectors, which are the columns of X. The first column of AX is $A\boldsymbol{x}_1$. That is $\lambda_1\boldsymbol{x}_1$. Each column of X is multiplied by its eigenvalue:

A times X
$$AX = A\begin{bmatrix} \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \\ \\ \end{bmatrix}.$$

The trick is to split this matrix AX into X times Λ:

X times Λ
$$\begin{bmatrix} \\ \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ \\ \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = X\Lambda.$$

Keep those matrices in the right order! Then λ_1 multiplies the first column $\boldsymbol{x}_1$, as shown. The diagonalization is complete, and we can write $AX = X\Lambda$ in two good ways:

$$AX = X\Lambda \quad \text{is} \quad \boxed{\boldsymbol{X^{-1}AX = \Lambda}} \quad \text{or} \quad \boxed{\boldsymbol{A = X\Lambda X^{-1}.}} \tag{2}$$

The matrix X has an inverse, because its columns (the eigenvectors of A) were assumed to be linearly independent. *Without n independent eigenvectors, we can't diagonalize.*

A and Λ have the same eigenvalues $\lambda_1, \ldots, \lambda_n$. The eigenvectors are different. The job of the original eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ was to diagonalize A. Those eigenvectors in X produce $A = X\Lambda X^{-1}$. You will soon see their simplicity and importance and meaning. The kth power will be $A^k = X\Lambda^k X^{-1}$ which is easy to compute:

$$\boldsymbol{A^k = (X\Lambda X^{-1})(X\Lambda X^{-1})\ldots(X\Lambda X^{-1}) = X\Lambda^k X^{-1}}.$$

Powers of A
Example 1
$$\begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}^k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ & 6^k \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{6^k - 1} \\ 0 & \mathbf{6^k} \end{bmatrix} = A^k.$$

With $k = 1$ we get A. With $k = 0$ we get $A^0 = I$ (and $\lambda^0 = 1$). *With $k = -1$ we get A^{-1}.* You can see how $A^2 = [1\ \ 35;\ \ 0\ \ 36]$ fits that formula when $k = 2$.

Here are four small remarks before we use Λ again in Example 2.

Remark 1 Suppose the eigenvalues $\lambda_1, \ldots, \lambda_n$ are all different. Then it is automatic that the eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are independent. The eigenvector matrix X will be *invertible*. ***Any matrix that has no repeated eigenvalues can be diagonalized.***

Remark 2 ***We can multiply eigenvectors by any nonzero constants.*** $A(c\boldsymbol{x}) = \lambda(c\boldsymbol{x})$ is still true. In Example 1, we can divide $\boldsymbol{x} = (1, 1)$ by $\sqrt{2}$ to produce a unit vector.

MATLAB and virtually all other codes produce eigenvectors of length $||\boldsymbol{x}|| = 1$.

Remark 3 The eigenvectors in X come in the same order as the eigenvalues in Λ. To reverse the order in Λ, put the eigenvector $(1,1)$ before $(1,0)$ in X:

$$\textbf{New order } 6,1 \qquad \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} = \Lambda_{\textbf{new}}$$

To diagonalize A we *must* use an eigenvector matrix. From $X^{-1}AX = \Lambda$ we know that $AX = X\Lambda$. Suppose the first column of X is $\boldsymbol{x}$. Then the first columns of AX and $X\Lambda$ are $A\boldsymbol{x}$ and $\lambda_1 \boldsymbol{x}$. For those to be equal, $\boldsymbol{x}$ must be an eigenvector.

Remark 4 (repeated warning for repeated eigenvalues) Some matrices have too few eigenvectors. *Those matrices cannot be diagonalized.* Here are two examples:

$$\textbf{Not diagonalizable} \qquad A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad \textbf{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Their eigenvalues happen to be 0 and 0. Nothing is special about $\lambda = 0$, the problem is the repetition of λ. All eigenvectors of the first matrix are multiples of $(1,1)$:

$$\begin{array}{l}\textbf{Only one line}\\ \textbf{of eigenvectors}\end{array} \quad A\boldsymbol{x} = 0\boldsymbol{x} \quad \text{means} \quad \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

There is no second eigenvector, so this unusual matrix A cannot be diagonalized.

Those matrices are the best examples to test any statement about eigenvectors. In many true-false questions, non-diagonalizable matrices lead to *false*.

Remember that there is no connection between invertibility and diagonalizability:

- ***Invertibility*** is concerned with the ***eigenvalues*** ($\lambda = 0$ or $\lambda \neq 0$).
- ***Diagonalizability*** is concerned with the ***eigenvectors*** (too few or enough for X).

Each eigenvalue has at least one eigenvector! $A - \lambda I$ is singular. If $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ leads you to $\boldsymbol{x} = \boldsymbol{0}$, λ is *not* an eigenvalue. Look for a mistake in solving $\det(A - \lambda I) = 0$.

Eigenvectors for n different λ's are independent. Then we can diagonalize A.

Independent $\boldsymbol{x}$ from different λ Eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_j$ that correspond to distinct (all different) eigenvalues are linearly independent. An n by n matrix that has n different eigenvalues (no repeated λ's) must be diagonalizable.

Proof Suppose $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = \boldsymbol{0}$. Multiply by A to find $c_1\lambda_1\boldsymbol{x}_1 + c_2\lambda_2\boldsymbol{x}_2 = \boldsymbol{0}$. Multiply by λ_2 to find $c_1\lambda_2\boldsymbol{x}_1 + c_2\lambda_2\boldsymbol{x}_2 = \boldsymbol{0}$. Now subtract one from the other:

$$\text{Subtraction leaves} \quad (\lambda_1 - \lambda_2)c_1\boldsymbol{x}_1 = \boldsymbol{0}. \quad \text{Therefore } c_1 = 0.$$

Since the λ's are different and $\boldsymbol{x}_1 \neq \boldsymbol{0}$, we are forced to the conclusion that $c_1 = 0$. Similarly $c_2 = 0$. Only the combination with $c_1 = c_2 = 0$ gives $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = \boldsymbol{0}$. So the eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ must be independent.

This proof extends directly to j eigenvectors. Suppose that $c_1\boldsymbol{x}_1 + \cdots + c_j\boldsymbol{x}_j = \mathbf{0}$. Multiply by A, multiply by λ_j, and subtract. This multiplies $\boldsymbol{x}_j$ by $\lambda_j - \lambda_j = 0$, and $\boldsymbol{x}_j$ is gone. Now multiply by A and by λ_{j-1} and subtract. This removes $\boldsymbol{x}_{j-1}$. Eventually only $\boldsymbol{x}_1$ is left:

$$\text{We reach} \quad (\lambda_1 - \lambda_2)\cdots(\lambda_1 - \lambda_j)c_1\boldsymbol{x}_1 = \mathbf{0} \quad \text{which forces} \quad c_1 = 0. \tag{3}$$

Similarly every $c_i = 0$. When the λ's are all different, the eigenvectors are independent. A full set of eigenvectors can go into the columns of the eigenvector matrix X.

Example 2 **Powers of A** The Markov matrix $A = \left[\begin{smallmatrix} .8 & .3 \\ .2 & .7 \end{smallmatrix}\right]$ in the last section had $\lambda_1 = 1$ and $\lambda_2 = .5$. Here is $A = X\Lambda X^{-1}$ with those eigenvalues in the diagonal Λ:

$$\textbf{Markov example} \qquad \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = X\Lambda X^{-1}.$$

The eigenvectors $(.6, .4)$ and $(1, -1)$ are in the columns of X. They are also the eigenvectors of A^2. Watch how A^2 has the same X, and ***the eigenvalue matrix of A^2 is Λ^2***:

$$\textbf{Same } \boldsymbol{X} \textbf{ for } \boldsymbol{A^2} \qquad \boxed{A^2} = X\Lambda X^{-1}X\Lambda X^{-1} = \boxed{X\Lambda^2 X^{-1}.} \tag{4}$$

Just keep going, and you see why the high powers A^k approach a "steady state":

$$\textbf{Powers of } \boldsymbol{A} \qquad A^k = X\Lambda^k X^{-1} = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (.5)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix}.$$

As k gets larger, $(.5)^k$ gets smaller. In the limit it disappears completely. That limit is A^∞:

$$\textbf{Limit } \boldsymbol{k \to \infty} \qquad A^\infty = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

The limit has the eigenvector $\boldsymbol{x}_1$ in both columns. We saw this A^∞ on the very first page of Chapter 6. Now we see it coming from powers like $A^{100} = X\Lambda^{100}X^{-1}$.

Question ***When does $A^k \to$ zero matrix?*** **Answer** ***All $|\lambda| < 1$.***

Similar Matrices: Same Eigenvalues

Suppose the eigenvalue matrix Λ is fixed. As we change the eigenvector matrix X, we get a whole family of different matrices $A = X\Lambda X^{-1}$—*all with the same eigenvalues in* Λ. All those matrices A (with the same Λ) are called **similar**.

This idea extends to matrices that can't be diagonalized. Again we choose one constant matrix C (not necessarily Λ). And we look at the whole family of matrices $\boldsymbol{A = BCB^{-1}}$, allowing all invertible matrices B. Again those matrices A and C are called **similar**.

We are using C instead of Λ because C might not be diagonal. We are using B instead of X because the columns of B might not be eigenvectors. We only require that B is invertible—its columns can contain any basis for $\mathbf{R}^n$. The key fact about similar matrices stays true. **Similar matrices A and C have the same eigenvalues.**

All the matrices $A = BCB^{-1}$ are "similar." They all share the eigenvalues of C.

Proof Suppose $C\boldsymbol{x} = \lambda\boldsymbol{x}$. Then BCB^{-1} has the same eigenvalue λ with the new eigenvector $B\boldsymbol{x}$:

$$\textbf{\textit{Same }}\boldsymbol{\lambda} \qquad (BCB^{-1})(B\boldsymbol{x}) = BC\boldsymbol{x} = B\lambda\boldsymbol{x} = \lambda(B\boldsymbol{x}). \tag{5}$$

A fixed matrix C produces a family of similar matrices BCB^{-1}, allowing all B. When C is the identity matrix, the "family" is very small. The only member is $BIB^{-1} = I$. The identity matrix is the only diagonalizable matrix with all eigenvalues $\lambda = 1$.

The family is larger when $\lambda = 1$ and 1 *with only one eigenvector* (not diagonalizable). The simplest C is the *Jordan form*—to be developed in Section 8.3. All the similar A's have two parameters r and s, not both zero: always determinant $= 1$ and trace $= 2$.

$$C = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} \end{bmatrix} = \text{Jordan form gives } A = BCB^{-1} = \begin{bmatrix} 1 - rs & r^2 \\ -s^2 & 1 + rs \end{bmatrix}. \tag{6}$$

For an important example I will take eigenvalues $\lambda = 1$ and 0 (not repeated!). Now the whole family is diagonalizable with the same eigenvalue matrix Λ. We get every 2 by 2 matrix that has eigenvalues 1 and 0. The trace is 1 and the determinant is zero:

$$\begin{array}{c}\textbf{All}\\ \textbf{similar}\end{array} \quad \Lambda = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{0} \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{or } A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \quad \text{or any } A = \frac{\boldsymbol{x}\boldsymbol{y}^{\mathrm{T}}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}}.$$

The family contains all matrices with $A^2 = A$, including $A = \Lambda$ when $B = I$. When A is symmetric these are also projection matrices. Eigenvalues 1 and 0 make life easy.

Fibonacci Numbers

We present a famous example, where eigenvalues tell how fast the Fibonacci numbers grow. ***Every new Fibonacci number is the sum of the two previous F's***:

The sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ ***comes from*** $F_{k+2} = F_{k+1} + F_k$.

These numbers turn up in a fantastic variety of applications. Plants and trees grow in a spiral pattern, and a pear tree has 8 growths for every 3 turns. For a willow those numbers can be 13 and 5. The champion is a sunflower of Daniel O'Connell, which had 233 seeds in 144 loops. Those are the Fibonacci numbers F_{13} and F_{12}. Our problem is more basic.

Problem: Find the Fibonacci number F_{100} The slow way is to apply the rule $F_{k+2} = F_{k+1} + F_k$ one step at a time. By adding $F_6 = 8$ to $F_7 = 13$ we reach $F_8 = 21$. Eventually we come to F_{100}. Linear algebra gives a better way.

The key is to begin with a matrix equation $\mathbf{u}_{k+1} = A\boldsymbol{u}_k$. That is a *one-step* rule for vectors, while Fibonacci gave a two-step rule for scalars. We match those rules by putting two Fibonacci numbers into a vector. Then you will see the matrix A.

$$\text{Let } \boldsymbol{u}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}. \text{ The rule } \begin{matrix} F_{k+2} = F_{k+1} + F_k \\ F_{k+1} = F_{k+1} \end{matrix} \text{ is } \boldsymbol{u}_{k+1} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \boldsymbol{u}_k. \tag{7}$$

Every step multiplies by $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. After 100 steps we reach $\boldsymbol{u}_{100} = A^{100}\boldsymbol{u}_0$:

$$\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \ldots, \quad \boldsymbol{u}_{100} = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}.$$

This problem is just right for eigenvalues. Subtract λ from the diagonal of A:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \quad \text{leads to} \quad \det(A - \lambda I) = \lambda^2 - \lambda - 1.$$

The equation $\lambda^2 - \lambda - 1 = 0$ is solved by the quadratic formula $\left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$:

$$\textbf{Eigenvalues} \qquad \lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -.618.$$

These eigenvalues lead to eigenvectors $\boldsymbol{x}_1 = (\lambda_1, 1)$ and $\boldsymbol{x}_2 = (\lambda_2, 1)$. Step 2 finds the combination of those eigenvectors that gives $\boldsymbol{u}_0 = (1, 0)$:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2}\left(\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}\right) \quad \text{or} \quad \boldsymbol{u}_0 = \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{\lambda_1 - \lambda_2}. \tag{8}$$

Step 3 multiplies $\boldsymbol{u}_0$ by A^{100} to find $\boldsymbol{u}_{100}$. The eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ stay separate! They are multiplied by $(\lambda_1)^{100}$ and $(\lambda_2)^{100}$:

$$\textbf{100 steps from } \boldsymbol{u}_0 \qquad \boldsymbol{u}_{100} = \frac{(\lambda_1)^{100}\boldsymbol{x}_1 - (\lambda_2)^{100}\boldsymbol{x}_2}{\lambda_1 - \lambda_2}. \tag{9}$$

We want F_{100} = second component of $\boldsymbol{u}_{100}$. The second components of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are 1. The difference between $\lambda_1 = (1+\sqrt{5})/2$ and $\lambda_2 = (1-\sqrt{5})/2$ is $\sqrt{5}$. And $\lambda_2^{100} \approx 0$.

$$\text{100th Fibonacci number} = \frac{\lambda_1^{100} - \lambda_2^{100}}{\lambda_1 - \lambda_2} = \text{nearest integer to } \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{100}. \tag{10}$$

Every F_k is a whole number. The ratio F_{101}/F_{100} must be very close to the limiting ratio $\left(1+\sqrt{5}\right)/2$. The Greeks called this number the *"golden mean"*. For some reason a rectangle with sides 1.618 and 1 looks especially graceful.

Matrix Powers A^k

Fibonacci's example is a typical difference equation $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$. ***Each step multiplies by A.*** The solution is $\boldsymbol{u}_k = A^k\boldsymbol{u}_0$. We want to make clear how diagonalizing the matrix gives a quick way to compute A^k and find $\boldsymbol{u}_k$ in three steps.

The eigenvector matrix X produces $A = X\Lambda X^{-1}$. This is a factorization of the matrix, like $A = LU$ or $A = QR$. The new factorization is perfectly suited to computing powers, because ***every time*** $\boldsymbol{X^{-1}}$ ***multiplies*** $\boldsymbol{X}$ ***we get*** $\boldsymbol{I}$:

$$\textbf{Powers of } A \qquad A^k\boldsymbol{u}_0 = (X\Lambda X^{-1})\cdots(X\Lambda X^{-1})\boldsymbol{u}_0 = X\Lambda^k X^{-1}\boldsymbol{u}_0$$

I will split $X\Lambda^k X^{-1}\boldsymbol{u}_0$ into three steps that show how eigenvalues work:

1. Write $\boldsymbol{u}_0$ as a combination $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$ of the eigenvectors. Then $\boldsymbol{c} = X^{-1}\boldsymbol{u}_0$.

2. Multiply each eigenvector $\boldsymbol{x}_i$ by $(\lambda_i)^k$. Now we have $\Lambda^k X^{-1}\boldsymbol{u}_0$.

3. Add up the pieces $c_i(\lambda_i)^k\boldsymbol{x}_i$ to find the solution $\boldsymbol{u}_k = A^k\boldsymbol{u}_0$. This is $X\Lambda^k X^{-1}\boldsymbol{u}_0$.

$$\textbf{Solution for } \boldsymbol{u}_{k+1} = A\boldsymbol{u}_k \qquad \boldsymbol{u}_k = A^k\boldsymbol{u}_0 = c_1(\lambda_1)^k\boldsymbol{x}_1 + \cdots + c_n(\lambda_n)^k\boldsymbol{x}_n. \tag{11}$$

In matrix language A^k equals $(X\Lambda X^{-1})^k$ which is X times Λ^k times X^{-1}. In Step 1, the eigenvectors in X lead to the c's in the combination $\boldsymbol{u}_0 = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$:

$$\textbf{Step 1} \qquad \boldsymbol{u}_0 = \begin{bmatrix} & & \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ & & \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}. \quad \text{This says that} \quad \boldsymbol{u}_0 = X\boldsymbol{c}. \tag{12}$$

The coefficients in Step 1 are $\boldsymbol{c} = X^{-1}\boldsymbol{u}_0$. Then Step 2 multiplies by Λ^k. The final result $\boldsymbol{u}_k = \sum c_i(\lambda_i)^k\boldsymbol{x}_i$ in Step 3 is the product of X and Λ^k and $X^{-1}\boldsymbol{u}_0$:

$$A^k\boldsymbol{u}_0 = X\Lambda^k X^{-1}\boldsymbol{u}_0 = X\Lambda^k\boldsymbol{c} = \begin{bmatrix} & & \\ \boldsymbol{x}_1 & \ldots & \boldsymbol{x}_n \\ & & \end{bmatrix}\begin{bmatrix} (\lambda_1)^k & & \\ & \ddots & \\ & & (\lambda_n)^k \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}. \tag{13}$$

This result is exactly $\boldsymbol{u}_k = c_1(\lambda_1)^k\boldsymbol{x}_1 + \cdots + c_n(\lambda_n)^k\boldsymbol{x}_n$. It solves $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$.

Example 3 Start from $\boldsymbol{u}_0 = (1, 0)$. Compute $A^k\boldsymbol{u}_0$ for this faster Fibonacci:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \quad \text{has} \quad \lambda_1 = \mathbf{2} \quad \text{and} \quad \boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \lambda_2 = \mathbf{-1} \quad \text{and} \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This matrix is like Fibonacci except the rule is changed to $F_{k+2} = F_{k+1} + \mathbf{2}F_k$. The new numbers start with $0, 1, 1, 3$. They grow faster because of $\lambda = 2$.

Find $\boldsymbol{u}_k = A^k \boldsymbol{u}_0$ in 3 steps $\quad \boldsymbol{u}_0 = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ and $\boldsymbol{u}_k = c_1(\lambda_1)^k\boldsymbol{x}_1 + c_2(\lambda_2)^k\boldsymbol{x}_2$

Step 1 $\quad \boldsymbol{u}_0 = \begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1\\-1\end{bmatrix}$ so $c_1 = c_2 = \frac{1}{3}$

Step 2 $\quad$ Multiply the two parts by $(\lambda_1)^k = 2^k$ and $(\lambda_2)^k = (-1)^k$

Step 3 $\quad$ Combine eigenvectors $c_1(\lambda_1)^k\boldsymbol{x}_1$ and $c_2(\lambda_2)^k\boldsymbol{x}_2$ into $\boldsymbol{u}_k$:

$$\boldsymbol{u}_k = \boldsymbol{A}^k\boldsymbol{u}_0 \qquad \boldsymbol{u}_k = \frac{1}{3}2^k\begin{bmatrix}2\\1\end{bmatrix} + \frac{1}{3}(-1)^k\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}F_{k+1}\\F_k\end{bmatrix}.$$

The new number is $F_k = \mathbf{(2^k - (-1)^k)/3}$. After $0, 1, 1, 3$ comes $F_4 = 15/3 = 5$.

Behind these numerical examples lies a fundamental idea: ***Follow the eigenvectors***. In Section 6.3 this is the crucial link from linear algebra to differential equations (λ^k will become $e^{\lambda t}$). Chapter 8 sees the same idea as "transforming to an eigenvector basis." The best example of all is a ***Fourier series***, built from the eigenvectors e^{ikx} of d/dx.

Nondiagonalizable Matrices (Optional)

Suppose λ is an eigenvalue of A. We discover that fact in two ways:

1. **Eigenvectors** (geometric) $\quad$ There are nonzero solutions to $A\boldsymbol{x} = \lambda\boldsymbol{x}$.
2. **Eigenvalues** (algebraic) $\quad$ The determinant of $A - \lambda I$ is zero.

The number λ may be a simple eigenvalue or a multiple eigenvalue, and we want to know its ***multiplicity***. Most eigenvalues have multiplicity $M = 1$ (simple eigenvalues). Then there is a single line of eigenvectors, and $\det(A - \lambda I)$ does not have a double factor.

For exceptional matrices, an eigenvalue can be ***repeated***. Then there are two different ways to count its multiplicity. Always GM $\leq$ AM for each λ:

1. (Geometric Multiplicity = GM) $\quad$ Count the **independent eigenvectors** for λ. Then GM is the dimension of the nullspace of $A - \lambda I$.
2. (Algebraic Multiplicity = AM) $\quad$ AM counts the **repetitions of $\boldsymbol{\lambda}$** among the eigenvalues. Look at the n roots of $\det(A - \lambda I) = 0$.

If A has $\lambda = 4, 4, 4$, then that eigenvalue has AM $= 3$ and GM $= 1, 2$, or 3.

The following matrix A is the standard example of trouble. Its eigenvalue $\lambda = 0$ is repeated. It is a double eigenvalue (AM $= 2$) with only one eigenvector (GM $= 1$).

AM = 2
GM = 1 $\quad A = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ has $\det(A - \lambda I) = \begin{vmatrix}-\lambda & 1\\0 & -\lambda\end{vmatrix} = \lambda^2.$ $\quad$ **$\boldsymbol{\lambda = 0, 0}$ but 1 eigenvector**

There "should" be two eigenvectors, because $\lambda^2 = 0$ has a double root. The double factor λ^2 makes AM $= 2$. But there is only one eigenvector $\boldsymbol{x} = (1, 0)$ and GM $= 1$. ***This shortage of eigenvectors when* GM *is below* AM *means that A is not diagonalizable*.**

These three matrices all have the same shortage of eigenvectors. Their repeated eigenvalue is $\lambda = 5$. Traces are 10 and determinants are 25:

$$A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 7 & 2 \\ -2 & 3 \end{bmatrix}.$$

Those all have $\det(A - \lambda I) = (\lambda - 5)^2$. The algebraic multiplicity is AM $= 2$. But each $A - 5I$ has rank $r = 1$. The geometric multiplicity is GM $= 1$. There is only one line of eigenvectors for $\lambda = 5$, and these matrices are not diagonalizable.

■ REVIEW OF THE KEY IDEAS ■

1. If A has n independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, they go into the columns of X.

   A is diagonalized by X $\qquad X^{-1}AX = \Lambda \quad \text{and} \quad A = X\Lambda X^{-1}$.

2. The powers of A are $A^k = X\Lambda^k X^{-1}$. The eigenvectors in X are unchanged.

3. The eigenvalues of A^k are $(\lambda_1)^k, \ldots, (\lambda_n)^k$ in the matrix Λ^k.

4. The solution to $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$ starting from $\boldsymbol{u}_0$ is $\boldsymbol{u}_k = A^k\boldsymbol{u}_0 = X\Lambda^k X^{-1}\boldsymbol{u}_0$:

   $$\boxed{\boldsymbol{u}_k = c_1(\lambda_1)^k\boldsymbol{x}_1 + \cdots + c_n(\lambda_n)^k\boldsymbol{x}_n \quad \text{provided} \quad \boldsymbol{u}_0 = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n.}$$

   That shows Steps $1, 2, 3$ (c's from $X^{-1}\boldsymbol{u}_0$, λ^k from Λ^k, and $\boldsymbol{x}$'s from X)

5. A is diagonalizable if every eigenvalue has enough eigenvectors (GM = AM).

■ WORKED EXAMPLES ■

6.2 A The **Lucas numbers** are like the Fibonacci numbers except they start with $L_1 = 1$ and $L_2 = 3$. Using the same rule $L_{k+2} = L_{k+1} + L_k$, the next Lucas numbers are $4, 7, 11, 18$. Show that the Lucas number L_{100} is $\lambda_1^{100} + \lambda_2^{100}$.

Solution $\boldsymbol{u}_{k+1} = \left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]\boldsymbol{u}_k$ is the same as for Fibonacci, because $L_{k+2} = L_{k+1} + L_k$ is the same rule (with different starting values). The equation becomes a 2 by 2 system:

$$\text{Let} \quad \boldsymbol{u}_k = \begin{bmatrix} L_{k+1} \\ L_k \end{bmatrix}. \qquad \text{The rule} \quad \begin{matrix} L_{k+2} = L_{k+1} + L_k \\ L_{k+1} = L_{k+1} \end{matrix} \quad \text{is} \quad \boldsymbol{u}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{u}_k.$$

The eigenvalues and eigenvectors of $A = \left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]$ still come from $\lambda^2 = \lambda + 1$:

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \boldsymbol{x}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2} \quad \text{and} \quad \boldsymbol{x}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}.$$

Now solve $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = \boldsymbol{u}_1 = (3, 1)$. The solution is $c_1 = \lambda_1$ and $c_2 = \lambda_2$. Check:

$$\lambda_1\boldsymbol{x}_1 + \lambda_2\boldsymbol{x}_2 = \begin{bmatrix} \lambda_1^2 + \lambda_2^2 \\ \lambda_1 + \lambda_2 \end{bmatrix} = \begin{bmatrix} \text{trace of } A^2 \\ \text{trace of } A \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \boldsymbol{u}_1$$

$\boldsymbol{u}_{100} = A^{99}\boldsymbol{u}_1$ tells us the Lucas numbers (L_{101}, L_{100}). The second components of the eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are 1, so the second component of $\boldsymbol{u}_{100}$ is the answer we want:

$$\textbf{Lucas number} \qquad \boldsymbol{L_{100}} = c_1\lambda_1^{99} + c_2\lambda_2^{99} = \boldsymbol{\lambda_1^{100}} + \boldsymbol{\lambda_2^{100}}.$$

Lucas starts faster than Fibonacci, and ends up larger by a factor near $\sqrt{5}$.

6.2 B Find the inverse and the eigenvalues and the determinant of this matrix A:

$$A = 5 * \textbf{eye}(4) - \textbf{ones}(4) = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}.$$

Describe an eigenvector matrix X that gives $X^{-1}AX = \Lambda$.

Solution What are the eigenvalues of the all-ones matrix? Its rank is certainly 1, so three eigenvalues are $\lambda = 0, 0, 0$. Its trace is 4, so the other eigenvalue is $\lambda = 4$. Subtract this all-ones matrix from $5I$ to get our matrix A:

Subtract the eigenvalues $4, 0, 0, 0$ from $5, 5, 5, 5$. The eigenvalues of A are $1, 5, 5, 5$.

The determinant of A is 125, the product of those four eigenvalues. The eigenvector for $\lambda = 1$ is $\boldsymbol{x} = (1, 1, 1, 1)$ or (c, c, c, c). The other eigenvectors are perpendicular to $\boldsymbol{x}$ (since A is symmetric). The nicest eigenvector matrix X is the symmetric orthogonal **Hadamard matrix $\boldsymbol{H}$** The factor $\frac{1}{2}$ produces unit column vectors.

$$\textbf{Orthonormal eigenvectors} \qquad X = H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = H^{\mathrm{T}} = H^{-1}.$$

The eigenvalues of A^{-1} are $1, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}$. The eigenvectors are not changed so $A^{-1} = H\Lambda^{-1}H^{-1}$. The inverse matrix is surprisingly neat:

$$A^{-1} = \frac{1}{5} * (\textbf{eye}(4) + \textbf{ones}(4)) = \frac{1}{5}\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

A is a rank-one change from $5I$. So A^{-1} is a rank-one change from $I/5$.

In a graph with 5 nodes, the determinant 125 counts the "spanning trees" (trees that touch all nodes). *Trees have no loops* (graphs and trees are in Section 10.1).

With 6 nodes, the matrix $6 * \textbf{eye}(5) - \textbf{ones}(5)$ has the five eigenvalues $1, 6, 6, 6, 6$.

Problem Set 6.2

Questions 1–7 are about the eigenvalue and eigenvector matrices Λ and X.

1 (a) Factor these two matrices into $A = X\Lambda X^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}.$$

(b) If $A = X\Lambda X^{-1}$ then $A^3 = (\quad)(\quad)(\quad)$ and $A^{-1} = (\quad)(\quad)(\quad)$.

2 If A has $\lambda_1 = 2$ with eigenvector $\boldsymbol{x}_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\lambda_2 = 5$ with $\boldsymbol{x}_2 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, use $X\Lambda X^{-1}$ to find A. No other matrix has the same λ's and $\boldsymbol{x}$'s.

3 Suppose $A = X\Lambda X^{-1}$. What is the eigenvalue matrix for $A + 2I$? What is the eigenvector matrix? Check that $A + 2I = (\quad)(\quad)(\quad)^{-1}$.

4 True or false: If the columns of X (eigenvectors of A) are linearly independent, then

(a) A is invertible (b) A is diagonalizable

(c) X is invertible (d) X is diagonalizable.

5 If the eigenvectors of A are the columns of I, then A is a _____ matrix. If the eigenvector matrix X is triangular, then X^{-1} is triangular. Prove that A is also triangular.

6 Describe all matrices X that diagonalize this matrix A (find all eigenvectors):

$$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}.$$

Then describe all matrices that diagonalize A^{-1}.

7 Write down the most general matrix that has eigenvectors $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$.

Questions 8–10 are about Fibonacci and Gibonacci numbers.

8 Diagonalize the Fibonacci matrix by completing X^{-1}:

$$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}\lambda_1 & \lambda_2\\1 & 1\end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}.$$

Do the multiplication $X\Lambda^kX^{-1}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ to find its second component. This is the kth Fibonacci number $F_k = \left(\lambda_1^k - \lambda_2^k\right)/\left(\lambda_1 - \lambda_2\right)$.

9 Suppose G_{k+2} is the *average* of the two previous numbers G_{k+1} and G_k:

$$\begin{aligned}G_{k+2} &= \tfrac{1}{2}G_{k+1} + \tfrac{1}{2}G_k\\ G_{k+1} &= G_{k+1}\end{aligned} \qquad \text{is} \qquad \begin{bmatrix}G_{k+2}\\G_{k+1}\end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix}G_{k+1}\\G_k\end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Find the limit as $n \to \infty$ of the matrices $A^n = X\Lambda^nX^{-1}$.

(c) If $G_0 = 0$ and $G_1 = 1$ show that the Gibonacci numbers approach $\frac{2}{3}$.

10 Prove that every third Fibonacci number in $0, 1, 1, 2, 3, \ldots$ is even.

Questions 11–14 are about diagonalizability.

11 True or false: If the eigenvalues of A are $2, 2, 5$ then the matrix is certainly

(a) invertible (b) diagonalizable (c) not diagonalizable.

12 True or false: If the only eigenvectors of A are multiples of $(1, 4)$ then A has

(a) no inverse (b) a repeated eigenvalue (c) no diagonalization $X\Lambda X^{-1}$.

13 Complete these matrices so that $\det A = 25$. Then check that $\lambda = 5$ is repeated—the trace is 10 so the determinant of $A - \lambda I$ is $(\lambda - 5)^2$. Find an eigenvector with $A\boldsymbol{x} = 5\boldsymbol{x}$. These matrices will not be diagonalizable because there is no second line of eigenvectors.

$$A = \begin{bmatrix}8 & \\ & 2\end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix}9 & 4\\ & 1\end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix}10 & 5\\-5 & \end{bmatrix}$$

14 The matrix $A = \left[\begin{smallmatrix}3 & 1\\0 & 3\end{smallmatrix}\right]$ is not diagonalizable because the rank of $A - 3I$ is _____. Change one entry to make A diagonalizable. Which entries could you change?

Questions 15–19 are about powers of matrices.

15 $A^k = X\Lambda^kX^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than _____. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix}.6 & .9\\.4 & .1\end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix}.6 & .9\\.1 & .6\end{bmatrix}.$$

16 (Recommended) Find Λ and X to diagonalize A_1 in Problem 15. What is the limit of Λ^k as $k \to \infty$? What is the limit of $X\Lambda^k X^{-1}$? In the columns of this limiting matrix you see the _____.

17 Find Λ and X to diagonalize A_2 in Problem 15. What is $(A_2)^{10}\boldsymbol{u}_0$ for these $\boldsymbol{u}_0$?

$$\boldsymbol{u}_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}_0 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

18 Diagonalize A and compute $X\Lambda^k X^{-1}$ to prove this formula for A^k:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{has} \quad A^k = \frac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}.$$

19 Diagonalize B and compute $X\Lambda^k X^{-1}$ to prove this formula for B^k:

$$B = \begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix} \quad \text{has} \quad B^k = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}.$$

20 Suppose $A = X\Lambda X^{-1}$. Take determinants to prove $\det A = \det \Lambda = \lambda_1\lambda_2\cdots\lambda_n$. This quick proof only works when A can be _____.

21 Show that trace XY = trace YX, by adding the diagonal entries of XY and YX:

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} q & r \\ s & t \end{bmatrix}.$$

Now choose Y to be ΛX^{-1}. Then $X\Lambda X^{-1}$ has the same trace as $\Lambda X^{-1}X = \Lambda$. This proves that *the trace of A equals the trace of Λ = sum of the eigenvalues.*

22 $AB - BA = I$ is impossible since the left side has trace = _____. But find an elimination matrix so that $A = E$ and $B = E^{\mathrm{T}}$ give

$$AB - BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{which has trace zero.}$$

23 If $A = X\Lambda X^{-1}$, diagonalize the block matrix $B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix}$. Find its eigenvalue and eigenvector (block) matrices.

24 Consider all 4 by 4 matrices A that are diagonalized by the same fixed eigenvector matrix X. Show that the A's form a subspace (cA and $A_1 + A_2$ have this same X). What is this subspace when $X = I$? What is its dimension?

25 Suppose $A^2 = A$. On the left side A multiplies each column of A. Which of our four subspaces contains eigenvectors with $\lambda = 1$? Which subspace contains eigenvectors with $\lambda = 0$? From the dimensions of those subspaces, A has a full set of independent eigenvectors. So a matrix with $A^2 = A$ can be diagonalized.

26 (Recommended) Suppose $A\boldsymbol{x} = \lambda\boldsymbol{x}$. If $\lambda = 0$ then $\boldsymbol{x}$ is in the nullspace. If $\lambda \neq 0$ then $\boldsymbol{x}$ is in the column space. Those spaces have dimensions $(n - r) + r = n$. So why doesn't every square matrix have n linearly independent eigenvectors?

27 The eigenvalues of A are 1 and 9, and the eigenvalues of B are -1 and 9:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}.$$

Find a matrix square root of A from $R = X\sqrt{\Lambda}\,X^{-1}$. Why is there no real matrix square root of B?

28 If A and B have the same λ's with the same independent eigenvectors, their factorizations into _____ are the same. So $A = B$.

29 Suppose the same X diagonalizes both A and B. They have the *same eigenvectors* in $A = X\Lambda_1 X^{-1}$ and $B = X\Lambda_2 X^{-1}$. Prove that $AB = BA$.

30 (a) If $A = \left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]$ then the determinant of $A - \lambda I$ is $(\lambda - a)(\lambda - d)$. Check the "Cayley-Hamilton Theorem" that $(A - aI)(A - dI) =$ *zero matrix.*

(b) Test the Cayley-Hamilton Theorem on Fibonacci's $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. The theorem predicts that $A^2 - A - I = 0$, since the polynomial $\det(A - \lambda I)$ is $\lambda^2 - \lambda - 1$.

31 Substitute $A = X\Lambda X^{-1}$ into the product $(A - \lambda_1 I)(A - \lambda_2 I)\cdots(A - \lambda_n I)$ and explain why this produces the zero matrix. We are substituting the matrix A for the number λ in the polynomial $p(\lambda) = \det(A - \lambda I)$. The ***Cayley-Hamilton Theorem*** says that this product is always $p(A) =$ *zero matrix*, even if A is not diagonalizable.

32 If $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$ and $AB = BA$, show that $B = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is also a diagonal matrix. B has the same eigen_____ as A but different eigen_____ . These diagonal matrices B form a two-dimensional subspace of matrix space. $AB - BA = 0$ gives four equations for the unknowns $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$—find the rank of the 4 by 4 matrix.

33 The powers A^k approach zero if all $|\lambda_i| < 1$ and they blow up if any $|\lambda_i| > 1$. Peter Lax gives these striking examples in his book *Linear Algebra*:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} 5 & 7 \\ -3 & -4 \end{bmatrix} \qquad D = \begin{bmatrix} 5 & 6.9 \\ -3 & -4 \end{bmatrix}$$

$$\|A^{1024}\| > 10^{700} \qquad B^{1024} = I \qquad C^{1024} = -C \qquad \|D^{1024}\| < 10^{-78}$$

Find the eigenvalues $\lambda = e^{i\theta}$ of B and C to show $B^4 = I$ and $C^3 = -I$.

Challenge Problems

34 The nth power of rotation through θ is rotation through $n\theta$:

$$A^n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Prove that neat formula by diagonalizing $A = X\Lambda X^{-1}$. The eigenvectors (columns of X) are $(1, i)$ and $(i, 1)$. You need to know Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

35 The transpose of $A = X\Lambda X^{-1}$ is $A^{\mathrm{T}} = (X^{-1})^{\mathrm{T}}\Lambda X^{\mathrm{T}}$. The eigenvectors in $A^{\mathrm{T}}\boldsymbol{y} = \lambda\boldsymbol{y}$ are the columns of that matrix $(X^{-1})^{\mathrm{T}}$. They are often called ***left eigenvectors of A***, because $\boldsymbol{y}^{\mathrm{T}}A = \lambda\boldsymbol{y}^{\mathrm{T}}$. How do you multiply matrices to find this formula for A?

$$\boxed{\textbf{Sum of rank-1 matrices} \quad A = X\Lambda X^{-1} = \lambda_1\boldsymbol{x}_1\boldsymbol{y}_1^{\mathrm{T}} + \cdots + \lambda_n\boldsymbol{x}_n\boldsymbol{y}_n^{\mathrm{T}}.}$$

36 The inverse of $A = \textbf{eye}(n) + \textbf{ones}(n)$ is $A^{-1} = \textbf{eye}(n) + C * \textbf{ones}(n)$. Multiply AA^{-1} to find that number C (depending on n).

37 Suppose A_1 and A_2 are n by n invertible matrices. What matrix B shows that $A_2A_1 = B(A_1A_2)B^{-1}$? Then A_2A_1 is similar to A_1A_2: *same eigenvalues*.

38 **When is a matrix A similar to its eigenvalue matrix Λ?**

A and Λ always have the same eigenvalues. But similarity requires a matrix B with $A = B\Lambda B^{-1}$. Then B is the ____ matrix and A must have n independent ____.

39 (Pavel Grinfeld) Without writing down any calculations, can you find the eigenvalues of this matrix? Can you find the 2017th power A^{2017}?

$$A = \begin{bmatrix} 110 & 55 & -164 \\ 42 & 21 & -62 \\ 88 & 44 & -131 \end{bmatrix}.$$

If A is m by n and B is n by m, then AB and BA have same nonzero eigenvalues.

Proof. Start with this identity between square matrices (easily checked). The first and third matrices are inverses. The "size matrix" shows the shapes of all blocks.

$$\begin{bmatrix} I & -A \\ 0 & I \end{bmatrix}\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}\begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix} \qquad \begin{bmatrix} m\times m & m\times n \\ n\times m & n\times n \end{bmatrix}$$

This equation $D^{-1}ED = F$ says $\boldsymbol{F}$ **is similar to** $\boldsymbol{E}$—they have the *same* $m+n$ eigenvalues.

$$E = \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \text{ has the } m \text{ eigenvalues of } AB\text{, plus } n \text{ zeros}$$

$$F = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix} \text{ has the } n \text{ eigenvalues of } BA\text{, plus } m \text{ zeros}$$

So AB and BA have the **same eigenvalues** except for $|n - m|$ zeros. Wow.

If $A = [1\ \ 1]$ and $B = A^{\mathrm{T}}$ then $A^{\mathrm{T}}A = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}$ (notice $\lambda = 2$ and 0) and $AA^{\mathrm{T}} = [\ 2\ ]$.

6.3 Systems of Differential Equations

1 If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $\boldsymbol{u}(t) = e^{\lambda t}\boldsymbol{x}$ will solve $\dfrac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$. Each λ and $\boldsymbol{x}$ give a solution $e^{\lambda t}\boldsymbol{x}$.

2 If $A = X\Lambda X^{-1}$ then $\boxed{\boldsymbol{u}(t) = e^{At}\boldsymbol{u}(0) = Xe^{\Lambda t}X^{-1}\boldsymbol{u}(0) = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + \cdots + c_n e^{\lambda_n t}\boldsymbol{x}_n.}$

3 A is **stable** and $\boldsymbol{u}(t) \to \boldsymbol{0}$ and $e^{At} \to 0$ when all eigenvalues of A have real part < 0.

4 **Matrix exponential** $e^{At} = I + At + \cdots + (At)^n/n! + \cdots = Xe^{\Lambda t}X^{-1}$ if A is diagonalizable.

5 **Second order equation** / **First order system** $\quad u'' + Bu' + Cu = 0$ is equivalent to $\begin{bmatrix} u \\ u' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix}\begin{bmatrix} u \\ u' \end{bmatrix}$.

Eigenvalues and eigenvectors and $A = X\Lambda X^{-1}$ are perfect for matrix powers A^k. They are also perfect for differential equations $d\boldsymbol{u}/dt = A\boldsymbol{u}$. This section is mostly linear algebra, but to read it you need one fact from calculus: *The derivative of* $e^{\lambda t}$ *is* $\lambda e^{\lambda t}$. The whole point of the section is this: **To convert constant-coefficient differential equations into linear algebra.**

The ordinary equations $\dfrac{du}{dt} = u$ and $\dfrac{du}{dt} = \lambda u$ are solved by exponentials:

$$\frac{du}{dt} = u \;\text{ produces }\; u(t) = \boldsymbol{Ce^t} \qquad \frac{du}{dt} = \lambda u \;\text{ produces }\; u(t) = \boldsymbol{Ce^{\lambda t}} \tag{1}$$

At time $t = 0$ those solutions include $e^0 = 1$. So they both reduce to $u(0) = C$. This "initial value" tells us the right choice for C. **The solutions that start from the number $\boldsymbol{u(0)}$ at time $\boldsymbol{t = 0}$ are $\boldsymbol{u(t) = u(0)e^t}$ and $\boldsymbol{u(t) = u(0)e^{\lambda t}}$.**

We just solved a 1 by 1 problem. Linear algebra moves to n by n. The unknown is a vector $\boldsymbol{u}$ (now boldface). It starts from the initial vector $\boldsymbol{u}(0)$, which is given. The n equations contain a square matrix A. We expect n exponents $e^{\lambda t}$ in $\boldsymbol{u}(t)$, from n λ's:

System of $\boldsymbol{n}$ equations $\quad \dfrac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$ $\quad$ starting from the vector $\boldsymbol{u}(0) = \begin{bmatrix} u_1(0) \\ \cdots \\ u_n(0) \end{bmatrix}$ at $t = 0$. $\quad$ (2)

These differential equations are *linear*. If $\boldsymbol{u}(t)$ and $\boldsymbol{v}(t)$ are solutions, so is $C\boldsymbol{u}(t) + D\boldsymbol{v}(t)$. We will need n constants like C and D to match the n components of $\boldsymbol{u}(0)$. Our first job is to find n "pure exponential solutions" $\boldsymbol{u} = e^{\lambda t}\boldsymbol{x}$ by using $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

Notice that A is a *constant* matrix. In other linear equations, A changes as t changes. In nonlinear equations, A changes as $\boldsymbol{u}$ changes. We don't have those difficulties, $d\boldsymbol{u}/dt = A\boldsymbol{u}$ is "linear with constant coefficients". Those and only those are the differential equations that we will convert directly to linear algebra. Here is the key:

Solve linear constant coefficient equations by exponentials $e^{\lambda t}\boldsymbol{x}$, when $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

Solution of $du/dt = Au$

Our pure exponential solution will be $e^{\lambda t}$ times a fixed vector $\boldsymbol{x}$. You may guess that λ is an eigenvalue of A, and $\boldsymbol{x}$ ***is the eigenvector***. Substitute $\boldsymbol{u}(t) = e^{\lambda t}\boldsymbol{x}$ into the equation $d\boldsymbol{u}/dt = A\boldsymbol{u}$ to prove you are right. The factor $e^{\lambda t}$ will cancel to leave $\lambda\boldsymbol{x} = A\boldsymbol{x}$:

$$\textbf{Choose } \boldsymbol{u} = e^{\lambda t}\boldsymbol{x} \textbf{ when } A\boldsymbol{x} = \lambda\boldsymbol{x} \qquad \frac{d\boldsymbol{u}}{dt} = \lambda e^{\lambda t}\boldsymbol{x} \quad \text{agrees with} \quad A\boldsymbol{u} = Ae^{\lambda t}\boldsymbol{x} \tag{3}$$

All components of this special solution $\boldsymbol{u} = e^{\lambda t}\boldsymbol{x}$ share the same $e^{\lambda t}$. The solution grows when $\lambda > 0$. It decays when $\lambda < 0$. If λ is a complex number, its real part decides growth or decay. The imaginary part ω gives oscillation $e^{i\omega t}$ like a sine wave.

Example 1 Solve $\dfrac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\boldsymbol{u}$ starting from $\boldsymbol{u}(0) = \begin{bmatrix} \mathbf{4} \\ \mathbf{2} \end{bmatrix}$.

This is a vector equation for $\boldsymbol{u}$. It contains two scalar equations for the components y and z. They are "coupled together" because the matrix A is not diagonal:

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} \qquad \frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} \quad \text{means that} \quad \frac{\boldsymbol{dy}}{\boldsymbol{dt}} = \boldsymbol{z} \quad \text{and} \quad \frac{\boldsymbol{dz}}{\boldsymbol{dt}} = \boldsymbol{y}.$$

The idea of eigenvectors is to combine those equations in a way that gets back to 1 by 1 problems. The combinations $y + z$ and $y - z$ will do it. Add and subtract equations:

$$\frac{d}{dt}(y+z) = z + y \qquad \text{and} \qquad \frac{d}{dt}(y-z) = -(y-z).$$

The combination $y + z$ grows like e^t, because it has $\lambda = 1$. The combination $y - z$ decays like e^{-t}, because it has $\lambda = -1$. Here is the point: We don't have to juggle the original equations $d\boldsymbol{u}/dt = A\boldsymbol{u}$, looking for these special combinations. The eigenvectors and eigenvalues of A will do it for us.

This matrix A has eigenvalues 1 and -1. The eigenvectors $\boldsymbol{x}$ are $(1, 1)$ and $(1, -1)$. The pure exponential solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ take the form $e^{\lambda t}\boldsymbol{x}$ with $\lambda_1 = 1$ and $\lambda_2 = -1$:

$$\boldsymbol{u}_1(t) = e^{\lambda_1 t}\boldsymbol{x}_1 = e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}_2(t) = e^{\lambda_2 t}\boldsymbol{x}_2 = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{4}$$

Notice: These $\boldsymbol{u}$'s satisfy $A\boldsymbol{u}_1 = \boldsymbol{u}_1$ and $A\boldsymbol{u}_2 = -\boldsymbol{u}_2$, just like $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. The factors e^t and e^{-t} change with time. Those factors give $d\boldsymbol{u}_1/dt = \boldsymbol{u}_1 = A\boldsymbol{u}_1$ and $d\boldsymbol{u}_2/dt = -\boldsymbol{u}_2 = A\boldsymbol{u}_2$. **We have two solutions to** $d\boldsymbol{u}/dt = A\boldsymbol{u}$. To find all other solutions, **multiply those special solutions by any numbers C and D and add:**

$$\textbf{Complete solution} \qquad \boldsymbol{u}(t) = \boldsymbol{C}e^{\boldsymbol{t}}\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} + \boldsymbol{D}e^{-\boldsymbol{t}}\begin{bmatrix} \mathbf{1} \\ -\mathbf{1} \end{bmatrix} = \begin{bmatrix} Ce^t + De^{-t} \\ Ce^t - De^{-t} \end{bmatrix}. \tag{5}$$

With these two constants C and D, we can match any starting vector $\boldsymbol{u}(0) = (u_1(0), u_2(0))$. Set $t = 0$ and $e^0 = 1$. Example 1 asked for the initial value to be $\boldsymbol{u}(0) = (4, 2)$:

$$\boldsymbol{u}(0) \textbf{ decides } \boldsymbol{C, D} \qquad C\begin{bmatrix}1\\1\end{bmatrix} + D\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix} \quad \text{yields} \quad \boldsymbol{C = 3} \quad \text{and} \quad \boldsymbol{D = 1}.$$

With $C = 3$ and $D = 1$ in the solution (5), the initial value problem is completely solved.

The same three steps that solved $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$ now solve $d\boldsymbol{u}/dt = A\boldsymbol{u}$:

1. Write $\boldsymbol{u}(0)$ as a **combination** $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$ **of the eigenvectors of** $\boldsymbol{A}$.
2. Multiply each eigenvector $\boldsymbol{x}_i$ by **its growth factor** $e^{\lambda_i t}$.
3. The solution is the same combination of those pure solutions $e^{\lambda t}\boldsymbol{x}$:

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} \qquad \boldsymbol{u}(t) = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + \cdots + c_n e^{\lambda_n t}\boldsymbol{x}_n. \tag{6}$$

Not included: If two λ's are equal, with only one eigenvector, another solution is needed. (It will be $te^{\lambda t}\boldsymbol{x}$.) Step 1 needs to diagonalize $A = X\Lambda X^{-1}$: a basis of n eigenvectors.

Example 2 Solve $d\boldsymbol{u}/dt = A\boldsymbol{u}$ knowing the eigenvalues $\lambda = 1, 2, 3$ of A:

Typical example
Equation for $\boldsymbol{u}$
Initial condition $\boldsymbol{u}(0)$

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix}1&1&1\\0&2&1\\0&0&3\end{bmatrix}\boldsymbol{u} \quad \text{starting from} \quad \boldsymbol{u}(0) = \begin{bmatrix}9\\7\\4\end{bmatrix}.$$

The eigenvectors are $\boldsymbol{x}_1 = (1,0,0)$ and $\boldsymbol{x}_2 = (1,1,0)$ and $\boldsymbol{x}_3 = (1,1,1)$.

Step 1 The vector $\boldsymbol{u}(0) = (9,7,4)$ is $\mathbf{2}\boldsymbol{x}_1 + \mathbf{3}\boldsymbol{x}_2 + \mathbf{4}\boldsymbol{x}_3$. Thus $(c_1, c_2, c_3) = (\mathbf{2}, \mathbf{3}, \mathbf{4})$.

Step 2 The factors $e^{\lambda t}$ give exponential solutions $e^t\boldsymbol{x}_1$ and $e^{2t}\boldsymbol{x}_2$ and $e^{3t}\boldsymbol{x}_3$.

Step 3 The combination that starts from $\boldsymbol{u}(0)$ is $\boldsymbol{u}(t) = \mathbf{2}e^t\boldsymbol{x}_1 + \mathbf{3}e^{2t}\boldsymbol{x}_2 + \mathbf{4}e^{3t}\boldsymbol{x}_3$.

The coefficients 2, 3, 4 came from solving the linear equation $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + c_3\boldsymbol{x}_3 = \boldsymbol{u}(0)$:

$$\begin{bmatrix} & & \\ \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \\ & & \end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}9\\7\\4\end{bmatrix} \quad \text{which is} \quad X\boldsymbol{c} = \boldsymbol{u}(0). \tag{7}$$

You now have the basic idea—how to solve $d\boldsymbol{u}/dt = A\boldsymbol{u}$. The rest of this section goes further. We solve equations that contain *second* derivatives, because they arise so often in applications. We also decide whether $\boldsymbol{u}(t)$ approaches zero or blows up or just oscillates.

At the end comes the ***matrix exponential*** e^{At}. The short formula $e^{At}\boldsymbol{u}(0)$ solves the equation $d\boldsymbol{u}/dt = A\boldsymbol{u}$ in the same way that $A^k\boldsymbol{u}_0$ solves the equation $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$. Example 3 will show how "difference equations" help to solve differential equations.

All these steps use the λ's and the $\boldsymbol{x}$'s. This section solves the constant coefficient problems that turn into linear algebra. It clarifies these simplest but most important differential equations—whose solution is completely based on growth factors $e^{\lambda t}$.

Second Order Equations

The most important equation in mechanics is $\boldsymbol{my'' + by' + ky = 0}$. The first term is the mass m times the acceleration $a = y''$. This term ma balances the force F (that is *Newton's Law*). The force includes the damping $-by'$ and the elastic force $-ky$, proportional to distance moved. This is a second-order equation because it contains the second derivative $y'' = d^2y/dt^2$. It is still linear with constant coefficients m, b, k.

In a differential equations course, the method of solution is to substitute $y = e^{\lambda t}$. Each derivative of y brings down a factor λ. We want $y = e^{\lambda t}$ to solve the equation:

$$\boldsymbol{m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0} \quad \text{becomes} \quad \boldsymbol{(m\lambda^2 + b\lambda + k)}\,e^{\lambda t} = 0. \tag{8}$$

Everything depends on $m\lambda^2 + b\lambda + k = 0$. This equation for λ has two roots λ_1 and λ_2. Then the equation for y has two pure solutions $y_1 = e^{\lambda_1 t}$ and $y_2 = e^{\lambda_2 t}$. Their combinations $c_1y_1 + c_2y_2$ give the complete solution unless $\lambda_1 = \lambda_2$.

In a linear algebra course we expect matrices and eigenvalues. Therefore we turn the scalar equation (with y'') into a *vector equation for y and y'*: first derivative only. Suppose the mass is $m = 1$. Two equations for $\boldsymbol{u} = (y,\ y')$ give $d\boldsymbol{u}/dt = A\boldsymbol{u}$:

$$\begin{aligned} \boldsymbol{dy/dt} &= \boldsymbol{y'} \\ \boldsymbol{dy'/dt} &= \boldsymbol{-ky - by'} \end{aligned} \quad \text{converts to} \quad \frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{1} \\ \boldsymbol{-k} & \boldsymbol{-b} \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}. \tag{9}$$

The first equation $dy/dt = y'$ is trivial (but true). The second is equation (8) connecting y'' to y' and y. Together they connect $\boldsymbol{u}'$ to $\boldsymbol{u}$. So we solve $\boldsymbol{u}' = A\boldsymbol{u}$ by eigenvalues of A:

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -k & -b-\lambda \end{bmatrix} \quad \text{has determinant} \quad \lambda^2 + b\lambda + k = 0.$$

The equation for the λ's is the same as (8)! It is still $\boldsymbol{\lambda^2 + b\lambda + k = 0}$, since $\boldsymbol{m = 1}$. The roots λ_1 and λ_2 are now *eigenvalues of A*. The eigenvectors and the solution are

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \qquad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \qquad \boldsymbol{u}(t) = c_1e^{\lambda_1 t}\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2e^{\lambda_2 t}\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

The first component of $\boldsymbol{u}(t)$ has $y = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}$—the same solution as before. It can't be anything else. In the second component of $\boldsymbol{u}(t)$ you see the velocity dy/dt. The vector problem is completely consistent with the scalar problem. The 2 by 2 matrix A is called a *companion matrix*—a companion to the second order equation with y''.

Example 3 *Motion around a circle with $y'' + y = 0$ and $y = \cos t$*

This is our master equation with mass $m = 1$ and stiffness $k = 1$ and $d = 0$: no damping. Substitute $y = e^{\lambda t}$ into $y'' + y = 0$ to reach $\boldsymbol{\lambda^2 + 1 = 0}$. *The roots are $\boldsymbol{\lambda = i}$ and $\boldsymbol{\lambda = -i}$*. Then half of $e^{it} + e^{-it}$ gives the solution $y = \cos t$.

As a first-order system, the initial values $y(0) = 1$, $y'(0) = 0$ go into $\boldsymbol{u}(0) = (1, 0)$:

$$\textbf{Use } \boldsymbol{y'' = -y} \qquad \frac{d\boldsymbol{u}}{dt} = \frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}. \tag{10}$$

The eigenvalues of A are again the same $\lambda = i$ and $\lambda = -i$ (no surprise). A is antisymmetric with eigenvectors $\boldsymbol{x}_1 = (1, i)$ and $\boldsymbol{x}_2 = (1, -i)$. The combination that matches $\boldsymbol{u}(0) = (1, 0)$ is $\frac{1}{2}(\boldsymbol{x}_1 + \boldsymbol{x}_2)$. Step 2 multiplies the x's by e^{it} and e^{-it}. Step 3 combines the pure oscillations into $\boldsymbol{u}(t)$ to find $y = \cos t$ as expected:

$$\boldsymbol{u}(t) = \frac{1}{2}e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \cos \boldsymbol{t} \\ -\sin \boldsymbol{t} \end{bmatrix}. \qquad \text{This is } \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}.$$

All good. The vector $\boldsymbol{u} = (\cos t, -\sin t)$ goes around a circle (Figure 6.3). The radius is 1 because $\cos^2 t + \sin^2 t = 1$.

Difference Equations (optional)

To display a circle on a screen, replace $y'' = -y$ by a ***difference equation***. Here are three choices using $\boldsymbol{Y}(t + \Delta t) - 2\boldsymbol{Y}(t) + \boldsymbol{Y}(t - \Delta t)$. Divide by $(\Delta t)^2$ to approximate y''.

F	**Forward from $\boldsymbol{n - 1}$**	$\dfrac{Y_{n+1} - 2Y_n + Y_{n-1}}{(\Delta t)^2} = \boldsymbol{-Y_{n-1}}$	(11 F)
C	**Centered at time $\boldsymbol{n}$**	$\dfrac{Y_{n+1} - 2Y_n + Y_{n-1}}{(\Delta t)^2} = \boldsymbol{-Y_n}$	(11 C)
B	**Backward from $\boldsymbol{n + 1}$**	$\dfrac{Y_{n+1} - 2Y_n + Y_{n-1}}{(\Delta t)^2} = \boldsymbol{-Y_{n+1}}$	(11 B)

Figure 6.3 shows the exact $y(t) = \cos t$ completing a circle at $t = 2\pi$. The three difference methods *don't* complete a perfect circle in 32 time steps of length $\Delta t = 2\pi/32$. Those pictures will be explained by eigenvalues:

Forward $|\lambda| > 1$ **(spiral out)** **Centered** $|\lambda| = 1$ **(best)** **Backward** $|\lambda| < 1$ **(spiral in)**

The 2-step equations (11) reduce to 1-step systems $\boldsymbol{U}_{n+1} = A\,\boldsymbol{U}_n$. Instead of $\boldsymbol{u} = (y, y')$ the discrete unknown is $\boldsymbol{U}_n = (Y_n, Z_n)$. We take n time steps Δt starting from $\boldsymbol{U}_0$:

$$\begin{array}{l} \textbf{Forward} \\ \textbf{(11F)} \end{array} \quad \begin{array}{l} Y_{n+1} = Y_n + \Delta t\, Z_n \\ Z_{n+1} = Z_n - \Delta t\, Y_n \end{array} \quad \text{becomes} \quad \boldsymbol{U}_{n+1} = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{bmatrix}\begin{bmatrix} Y_n \\ Z_n \end{bmatrix} = A\boldsymbol{U}_n. \tag{12}$$

Those are like $Y' = Z$ and $Z' = -Y$. They are first order equations involving times n and $n + 1$. Eliminating Z would bring back the "forward" second order equation (11 **F**).

My question is simple. *Do the points* (Y_n, Z_n) *stay on the circle* $Y^2 + Z^2 = 1$? No, they are growing to infinity in Figure 6.3. ***We are taking powers*** $\boldsymbol{A^n}$ ***and not*** $\boldsymbol{e^{At}}$***, so we test the magnitude*** $|\boldsymbol{\lambda}|$ ***and not the real parts of the eigenvalues.***

Eigenvalues of A $\quad \boldsymbol{\lambda = 1 \pm i\Delta t} \quad$ **Then** $|\boldsymbol{\lambda}| > 1$ **and** $(\boldsymbol{Y_n, Z_n})$ **spirals out**

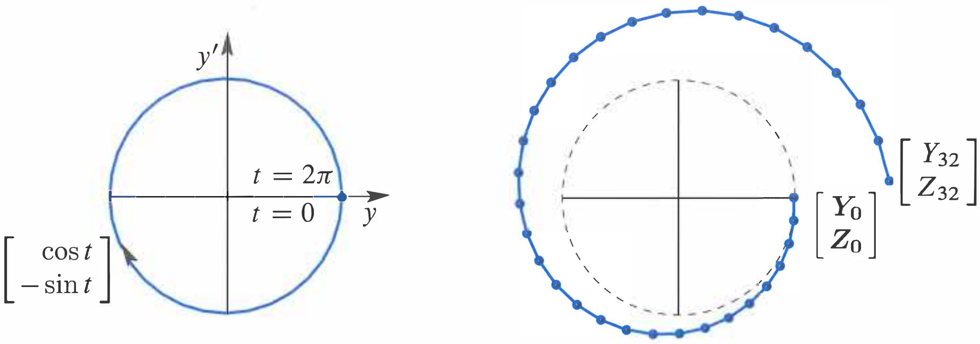


Figure 6.3: Exact $\boldsymbol{u} = (\cos t, -\sin t)$ on a circle. **Forward Euler spirals out** (32 steps).

The backward choice in (11 **B**) will do the opposite in Figure 6.4. Notice the new A:

$$\textbf{Backward} \qquad \begin{array}{l} Y_{n+1} = Y_n + \Delta t\, Z_{n+1} \\ Z_{n+1} = Z_n - \Delta t\, \mathbf{Y_{n+1}} \end{array} \quad \text{is} \quad \begin{bmatrix} 1 & -\Delta t \\ \Delta t & 1 \end{bmatrix} \begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix} = \begin{bmatrix} Y_n \\ Z_n \end{bmatrix} = \boldsymbol{U}_n. \quad (13)$$

That matrix has eigenvalues $1 \pm i\Delta t$. But we *invert* it to reach $\boldsymbol{U}_{n+1}$ from $\boldsymbol{U}_n$. Then $|\lambda| < 1$ explains why *the solution spirals in* to $(0, 0)$ for backward differences.

On the right side of Figure 6.4 you see 32 steps with the *centered* choice. The solution stays close to the circle (Problem 28) if $\Delta t < 2$. This is the **leapfrog method**, constantly used. The second difference $Y_{n+1} - 2Y_n + Y_{n-1}$ "leaps over" the center value Y_n in (11).

This is the way a chemist follows the motion of molecules (molecular dynamics leads to giant computations). Computational science is lively because one differential equation can be replaced by many difference equations—some unstable, some stable, some neutral. Problem 30 has a fourth (very good) method that stays right on the circle.

Real engineering and real physics deal with systems (not just a single mass at one point). The unknown $\boldsymbol{y}$ is a vector. The coefficient of $\boldsymbol{y}''$ is a *mass matrix* M, with n masses. The coefficient of $\boldsymbol{y}$ is a *stiffness matrix* K, not a number k. The coefficient of $\boldsymbol{y}'$ is a damping matrix which might be zero.

The vector equation $My'' + Ky = \boldsymbol{f}$ is a major part of computational mechanics. It is controlled by the eigenvalues of $M^{-1}K$ in $K\boldsymbol{x} = \lambda M\boldsymbol{x}$.

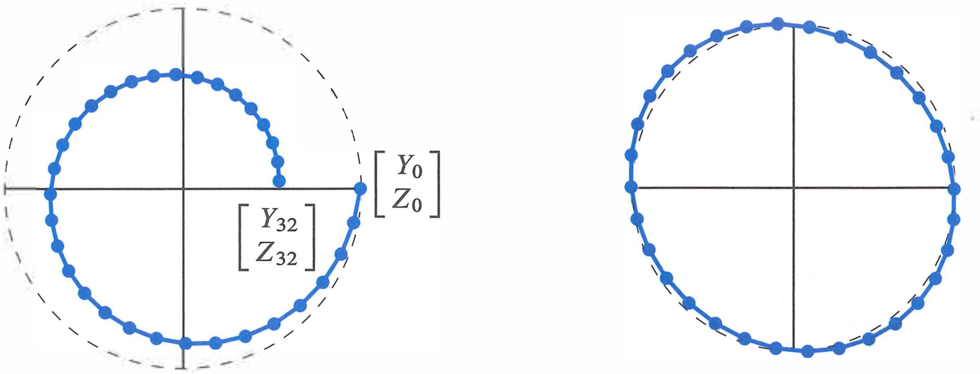


Figure 6.4: Backward differences spiral in. Leapfrog stays near the correct circle.

Stability of 2 by 2 Matrices

For the solution of $d\boldsymbol{u}/dt = A\boldsymbol{u}$, there is a fundamental question. *Does the solution approach* $\boldsymbol{u} = \mathbf{0}$ *as* $t \to \infty$? Is the problem *stable*, by dissipating energy? A solution that includes e^t is unstable. Stability depends on the eigenvalues of A.

The complete solution $\boldsymbol{u}(t)$ is built from pure solutions $e^{\lambda t}\boldsymbol{x}$. If the eigenvalue λ is real, we know exactly when $e^{\lambda t}$ will approach zero: *The number* λ *must be negative.* If the eigenvalue is a complex number $\lambda = r + is$, *the real part* r *must be negative.* When $e^{\lambda t}$ splits into $e^{rt}e^{ist}$, the factor e^{ist} has absolute value fixed at 1:

$$e^{ist} = \cos st + i \sin st \quad \text{has} \quad |e^{ist}|^2 = \cos^2 st + \sin^2 st = 1.$$

The real part of λ controls the growth ($r > 0$) or the decay ($r < 0$).

The question is: ***Which matrices have negative eigenvalues?*** More accurately, when are the ***real parts of the*** λ***'s all negative***? 2 by 2 matrices allow a clear answer.

Stability A is ***stable*** and $\boldsymbol{u}(t) \to \mathbf{0}$ when all eigenvalues λ have ***negative real parts***.
The 2 by 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ must pass two tests:

$\boldsymbol{\lambda_1 + \lambda_2 < 0}$	The trace $T = a + d$ must be negative.
$\boldsymbol{\lambda_1 \lambda_2 > 0}$	The determinant $D = ad - bc$ must be positive.

Reason If the λ's are real and negative, their sum is negative. This is the trace T. Their product is positive. This is the determinant D. The argument also goes in the reverse direction. If $D = \lambda_1\lambda_2$ is positive, then λ_1 and λ_2 have the same sign. If $T = \lambda_1 + \lambda_2$ is negative, that sign will be negative. We can test T and D.

If the λ's are complex numbers, they must have the form $r + is$ and $r - is$. Otherwise T and D will not be real. The determinant D is automatically positive, since $(r + is)(r - is) = r^2 + s^2$. The trace T is $r + is + r - is = 2r$. So a negative trace T means that the real part r is negative and the matrix is stable. Q.E.D.

Figure 6.5 shows the parabola $T^2 = 4D$ separating real λ's from complex λ's. Solving $\lambda^2 - T\lambda + D = 0$ involves the square root $\sqrt{T^2 - 4D}$. This is real below the parabola and imaginary above it. The stable region is the *upper left quarter* of the figure—where the trace T is negative and the determinant D is positive.

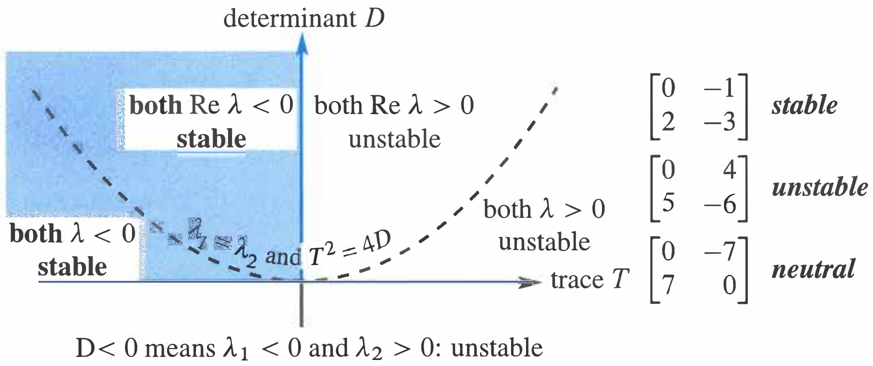


Figure 6.5: A 2 by 2 matrix is stable ($\boldsymbol{u}(t) \to \mathbf{0}$) when **trace** $< \mathbf{0}$ and **det** $> \mathbf{0}$.

The Exponential of a Matrix

We want to write the solution $\boldsymbol{u}(t)$ in a new form $e^{At}\boldsymbol{u}(0)$. First we have to say what e^{At} means, with a matrix in the exponent. To define e^{At} for matrices, we copy e^x for numbers.

The direct definition of e^x is by the infinite series $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$. When you change x to a square matrix At, this series defines the matrix exponential e^{At}:

Matrix exponential e^{At}	$e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots$	(14)
Its t derivative is Ae^{At}	$A + A^2t + \frac{1}{2}A^3t^2 + \cdots = Ae^{At}$	
Its eigenvalues are $e^{\lambda t}$	$(I + At + \frac{1}{2}(At)^2 + \cdots)\boldsymbol{x} = (1 + \lambda t + \frac{1}{2}(\lambda t)^2 + \cdots)\,\boldsymbol{x}$	

The number that divides $(At)^n$ is "n factorial". This is $n! = (1)(2)\cdots(n-1)(n)$. The factorials after $1, 2, 6$ are $4! = 24$ and $5! = 120$. They grow quickly. The series always converges and its derivative is always Ae^{At}. Therefore $e^{At}\boldsymbol{u}(0)$ solves the differential equation with one quick formula—*even if there is a shortage of eigenvectors.*

I will use this series in Example 4, to see it work with a missing eigenvector. **It will produce $te^{\lambda t}$**. First let me reach $Xe^{\Lambda t}X^{-1}$ in the good (diagonalizable) case.

This chapter emphasizes how to find $\boldsymbol{u}(t) = e^{At}\boldsymbol{u}(0)$ by diagonalization. Assume A does have n independent eigenvectors, so it is diagonalizable. Substitute $A = X\Lambda X^{-1}$ into the series for e^{At}. Whenever $X\Lambda X^{-1}X\Lambda X^{-1}$ appears, cancel $X^{-1}X$ in the middle:

$$\begin{aligned}
&\textbf{Use the series} && e^{\boldsymbol{A}t} = I + X\Lambda X^{-1}t + \tfrac{1}{2}(X\Lambda X^{-1}t)(X\Lambda X^{-1}t) + \cdots \\
&\textbf{Factor out } X \textbf{ and } X^{-1} && \phantom{e^{At}} = X\,[I + \Lambda t + \tfrac{1}{2}(\Lambda t)^2 + \cdots]\,X^{-1} \qquad (15)\\
&e^{At} \textbf{ is diagonalized!} && \boxed{\boldsymbol{e^{At} = X\,e^{\Lambda t}\,X^{-1}}.}
\end{aligned}$$

e^{At} has the same eigenvector matrix X as A. Then Λ is a diagonal matrix and so is $e^{\Lambda t}$. The numbers $e^{\lambda_i t}$ are on the diagonal. Multiply $Xe^{\Lambda t}X^{-1}\boldsymbol{u}(0)$ to recognize $\boldsymbol{u}(t)$:

$$e^{At}\boldsymbol{u}(0) = Xe^{\Lambda t}X^{-1}\boldsymbol{u}(0) = \begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}. \qquad (16)$$

This solution $e^{At}\boldsymbol{u}(0)$ is the same answer that came in equation (6) from three steps:

1. $\boldsymbol{u}(0) = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n = X\boldsymbol{c}$. Here we need n independent eigenvectors.
2. Multiply each $\boldsymbol{x}_i$ by its growth factor $e^{\lambda_i t}$ to follow it forward in time.
3. The best form of $e^{At}\boldsymbol{u}(0)$ is $\quad \boldsymbol{u}(t) = \boldsymbol{c_1 e^{\lambda_1 t} x_1 + \cdots + c_n e^{\lambda_n t} x_n}$. (17)

Example 4 When you substitute $y = e^{\lambda t}$ into $y'' - 2y' + y = 0$, you get an equation with **repeated roots**: $\lambda^2 - 2\lambda + 1 = 0$ is $(\lambda - 1)^2 = 0$ with $\boldsymbol{\lambda = 1, 1}$. A differential equations course would propose e^t and te^t as two independent solutions. Here we discover why.

Linear algebra reduces $y'' - 2y' + y = 0$ to a vector equation for $\boldsymbol{u} = (y, y')$:

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ 2y' - y \end{bmatrix} \quad \text{is} \quad \frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\boldsymbol{u}. \qquad (18)$$

A has a **repeated eigenvalue** $\boldsymbol{\lambda = 1, 1}$ (with trace $= 2$ and det $A = 1$). The only eigenvectors are multiples of $\boldsymbol{x} = (1, 1)$. *Diagonalization is not possible*, A has only one line of eigenvectors. So we compute e^{At} from its definition as a series:

$$\textbf{Short series} \qquad e^{\boldsymbol{A}t} = e^{It}\,e^{(A-I)t} = e^t\,[I + (A - I)t]. \qquad (19)$$

That "infinite" series for $e^{(A-I)t}$ ended quickly because $(A - I)^2$ is the zero matrix! You can see $\boldsymbol{te^t}$ in equation (19). The first component of $\boldsymbol{e^{At}}\,\boldsymbol{u}(0)$ is our answer $y(t)$:

$$\begin{bmatrix} y \\ y' \end{bmatrix} = e^t\left[I + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} t\right]\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} \qquad y(t) = e^t y(0) - \boldsymbol{te^t}y(0) + \boldsymbol{te^t}y'(0).$$

Example 5 Use the infinite series to find e^{At} for $A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Notice that $A^4 = I$:

$$A = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \quad A^2 = \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \quad A^3 = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \quad A^4 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}.$$

A^5, A^6, A^7, A^8 will be a repeat of A, A^2, A^3, A^4. The top right corner has $1, 0, -1, 0$ repeating over and over in powers of A. Then $t - \frac{1}{6}t^3$ starts the infinite series for e^{At} in that top right corner, and $1 - \frac{1}{2}t^2$ starts the top left corner:

$$e^{At} = I + At + \tfrac{1}{2}(At)^2 + \tfrac{1}{6}(At)^3 + \cdots = \begin{bmatrix} 1 - \frac{1}{2}t^2 + \cdots & t - \frac{1}{6}t^3 + \cdots \\ -t + \frac{1}{6}t^3 - \cdots & 1 - \frac{1}{2}t^2 + \cdots \end{bmatrix}.$$

The top row of that matrix e^{At} shows the infinite series for the cosine and sine!

$$\boldsymbol{A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}}. \tag{20}$$

A is an antisymmetric matrix ($A^{\mathrm{T}} = -A$). Its exponential e^{At} is an orthogonal matrix. The eigenvalues of A are i and $-i$. The eigenvalues of e^{At} are e^{it} and e^{-it}. Three rules:

1 ***e^{At} always has the inverse e^{-At}.***

2 ***The eigenvalues of e^{At} are always $e^{\lambda t}$.***

3 ***When A is antisymmetric, e^{At} is orthogonal. Inverse = transpose = e^{-At}.***

Antisymmetric is the same as "skew-symmetric". Those matrices have pure imaginary eigenvalues like i and $-i$. Then e^{At} has eigenvalues like e^{it} and e^{-it}. Their absolute value is 1: neutral stability, pure oscillation, energy is conserved. So $||\boldsymbol{u}(t)|| = ||\boldsymbol{u}(0)||$.

Our final example has a triangular matrix A. Then the eigenvector matrix X is triangular. So are X^{-1} and e^{At}. You will see the two forms of the solution: a combination of eigenvectors and the short form $e^{At}\boldsymbol{u}(0)$.

Example 6 Solve $\dfrac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \boldsymbol{u}$ starting from $\boldsymbol{u}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ at $t = 0$.

Solution The eigenvalues 1 and 2 are on the diagonal of A (since A is triangular). The eigenvectors are $(1, 0)$ and $(1, 1)$. The starting $\boldsymbol{u}(0)$ is $\boldsymbol{x}_1 + \boldsymbol{x}_2$ so $c_1 = c_2 = 1$. Then $\boldsymbol{u}(t)$ is the same combination of pure exponentials (*no $te^{\lambda t}$ when* $\lambda = 1$ and 2):

$$\textbf{Solution to } \boldsymbol{u}' = \boldsymbol{A}\boldsymbol{u} \qquad \boldsymbol{u}(t) = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

That is the clearest form. But the matrix form with e^{At} produces $\boldsymbol{u}(t)$ for every $\boldsymbol{u}(0)$:

$$\boldsymbol{u}(t) = Xe^{\Lambda t}X^{-1}\boldsymbol{u}(0) \text{ is } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & \\ & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \boldsymbol{u}(0) = \begin{bmatrix} e^t & e^{2t} + e^t \\ 0 & e^{2t} \end{bmatrix} \boldsymbol{u}(0).$$

That last matrix is e^{At}. It is nice because A is triangular. The situation is the same as for $A\boldsymbol{x} = \boldsymbol{b}$ and inverses. We don't need A^{-1} to find $\boldsymbol{x}$, and we don't need e^{At} to solve $d\boldsymbol{u}/dt = A\boldsymbol{u}$. But as quick formulas for the answers, $A^{-1}\boldsymbol{b}$ and $e^{At}\boldsymbol{u}(0)$ are unbeatable.

■ REVIEW OF THE KEY IDEAS ■

1. The equation $\boldsymbol{u}' = A\boldsymbol{u}$ is linear with constant coefficients in A. Start from $\boldsymbol{u}(0)$.
2. Its solution is usually a combination of exponentials, involving every λ and $\boldsymbol{x}$:

$$\textbf{Independent eigenvectors} \qquad \boldsymbol{u}(t) = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + \cdots + c_n e^{\lambda_n t}\boldsymbol{x}_n.$$

3. The constants $c_1, \ldots, c_n$ are determined by $\boldsymbol{u}(0) = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n = X\boldsymbol{c}$.
4. $\boldsymbol{u}(t)$ approaches zero (**stability**) if every λ has negative real part: All $e^{\lambda t} \to 0$.
5. Solutions have the short form $\boldsymbol{u}(t) = e^{At}\boldsymbol{u}(0)$, with the matrix exponential e^{At}.
6. Equations with y'' reduce to $\boldsymbol{u}' = A\boldsymbol{u}$ by combining y and y' into the vector $\boldsymbol{u}$.

■ WORKED EXAMPLES ■

6.3 A Solve $y'' + 4y' + 3y = 0$ by substituting $e^{\lambda t}$ and also by linear algebra.

Solution Substituting $y = e^{\lambda t}$ yields $(\lambda^2 + 4\lambda + 3)e^{\lambda t} = 0$. That quadratic factors into $\lambda^2+4\lambda+3 = (\lambda+1)(\lambda+3) = 0$. Therefore $\boldsymbol{\lambda_1 = -1}$ and $\boldsymbol{\lambda_2 = -3}$. The pure solutions are $y_1 = e^{-t}$ and $y_2 = e^{-3t}$. The complete solution $\boldsymbol{y} = c_1y_1 + c_2y_2$ approaches zero.

To use linear algebra we set $\boldsymbol{u} = (y, y')$. Then the vector equation is $\boldsymbol{u}' = A\boldsymbol{u}$:

$$\begin{aligned} dy/dt &= y' \\ dy'/dt &= -3y - 4y' \end{aligned} \qquad \text{converts to} \qquad \frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \boldsymbol{u}.$$

This A is a "companion matrix" and its eigenvalues are again -1 and -3:

$$\textbf{Same quadratic} \qquad \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0.$$

The eigenvectors of A are $(1, \lambda_1)$ and $(1, \lambda_2)$. Either way, the decay in $y(t)$ comes from e^{-t} and e^{-3t}. With constant coefficients, calculus leads to linear algebra $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

Note In linear algebra the serious danger is a shortage of eigenvectors. Our eigenvectors $(1, \lambda_1)$ and $(1, \lambda_2)$ are the same if $\lambda_1 = \lambda_2$. Then we can't diagonalize A. In this case we don't yet have two independent solutions to $d\boldsymbol{u}/dt = A\boldsymbol{u}$.

In differential equations the danger is also a repeated λ. After $y = e^{\lambda t}$, a second solution has to be found. It turns out to be $y = \boldsymbol{te^{\lambda t}}$. This "impure" solution (with an extra $\boldsymbol{t}$) appears in the matrix exponential e^{At}. Example 4 showed how.

6.3 B Find the eigenvalues and eigenvectors of A. Then write $\boldsymbol{u}(0) = (0, 2\sqrt{2}, 0)$ as a combination of the eigenvectors. Solve both equations $\boldsymbol{u}' = A\boldsymbol{u}$ and $\boldsymbol{u}'' = A\boldsymbol{u}$:

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \boldsymbol{u} \quad \text{and} \quad \frac{d^2\boldsymbol{u}}{dt^2} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \boldsymbol{u} \quad \text{with} \quad \frac{d\boldsymbol{u}}{dt}(0) = \boldsymbol{0}.$$

$\boldsymbol{u}' = A\boldsymbol{u}$ is like the heat equation $\partial u/\partial t = \partial^2 u/\partial x^2$.
Its solution $u(t)$ will decay (A has negative eigenvalues).
$\boldsymbol{u}'' = A\boldsymbol{u}$ is like the wave equation $\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2$.
Its solution will oscillate (the square roots of λ are imaginary).

Solution The eigenvalues and eigenvectors come from $\det(A - \lambda I) = 0$:

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix} = (-2-\lambda)[(-2-\lambda)^2 - 2] = 0.$$

One eigenvalue is $\lambda = -2$, when $-2 - \lambda$ is zero. The other factor is $\lambda^2 + 4\lambda + 2$, so the other eigenvalues (also real and negative) are $\lambda = -2 \pm \sqrt{2}$. Find the eigenvectors:

$$\boldsymbol{\lambda = -2} \qquad (A + 2I)\boldsymbol{x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for } \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\boldsymbol{\lambda = -2 - \sqrt{2}} \quad (A - \lambda I)\boldsymbol{x} = \begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for } \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

$$\boldsymbol{\lambda = -2 + \sqrt{2}} \quad (A - \lambda I)\boldsymbol{x} = \begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for } \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

The eigenvectors are *orthogonal* (proved in Section 6.4 for all symmetric matrices). All three λ_i are negative. This A is *negative definite* and e^{At} decays to zero (stability).

The starting $\boldsymbol{u}(0) = (0, 2\sqrt{2}, 0)$ is $\boldsymbol{x}_3 - \boldsymbol{x}_2$. The solution is $\boldsymbol{u}(t) = \boldsymbol{e^{\lambda_3 t} x_3 - e^{\lambda_2 t} x_2}$.

Heat equation In Figure 6.6a, the temperature at the center starts at $2\sqrt{2}$. Heat diffuses into the neighboring boxes and then to the outside boxes (frozen at $0°$). The rate of heat flow between boxes is the temperature difference. From box 2, heat flows left and right at the rate $u_1 - u_2$ and $u_3 - u_2$. So the flow out is $u_1 - 2u_2 + u_3$ in the second row of $A\boldsymbol{u}$.

Wave equation $d^2\boldsymbol{u}/dt^2 = A\boldsymbol{u}$ has the same eigenvectors $\boldsymbol{x}$. But now the eigenvalues λ lead to **oscillations** $e^{i\omega t}\boldsymbol{x}$ and $e^{-i\omega t}\boldsymbol{x}$. The frequencies come from $\omega^2 = -\lambda$:

$$\frac{d^2}{dt^2}(e^{i\omega t}\boldsymbol{x}) = A(e^{i\omega t}\boldsymbol{x}) \qquad \text{becomes} \qquad (i\omega)^2 e^{i\omega t}\boldsymbol{x} = \lambda e^{i\omega t}\boldsymbol{x} \quad \text{and} \quad \boldsymbol{\omega^2 = -\lambda}.$$

There are two square roots of $-\lambda$, so we have $e^{i\omega t}\boldsymbol{x}$ and $e^{-i\omega t}\boldsymbol{x}$. With three eigenvectors this makes *six* solutions to $\boldsymbol{u}'' = A\boldsymbol{u}$. A combination will match the six components of $\boldsymbol{u}(0)$ and $\boldsymbol{u}'(0)$. Since $\boldsymbol{u}' = \boldsymbol{0}$ in this problem, $e^{i\omega t}\boldsymbol{x}$ and $e^{-i\omega t}\boldsymbol{x}$ produce $2\cos\omega t\,\boldsymbol{x}$.

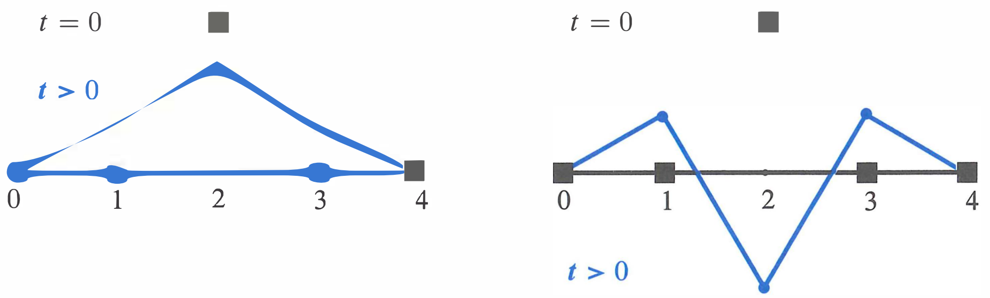


Figure 6.6: Heat diffuses away from box 2 (left). Wave travels from box 2 (right).

6.3 C Solve the four equations $da/dt = 0, db/dt = a, dc/dt = 2b, dz/dt = 3c$ in that order starting from $\boldsymbol{u}(0) = (a(0), b(0), c(0), z(0))$. Solve the same equations by the matrix exponential in $\boldsymbol{u}(t) = e^{At}\boldsymbol{u}(0)$.

Four equations
$\boldsymbol{\lambda = 0, 0, 0, 0}$
Eigenvalues on the diagonal

$$\frac{d}{dt}\begin{bmatrix} a \\ b \\ c \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{2} & 0 & 0 \\ 0 & 0 & \mathbf{3} & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ z \end{bmatrix} \quad \text{is} \quad \frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}.$$

First find A^2, A^3, A^4 and $e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3$. Why does the series stop? Why is it true that $(e^A)(e^A) = (e^{2A})$? ***Always*** $\boldsymbol{e^{As}}$ ***times*** $\boldsymbol{e^{At}}$ ***is*** $\boldsymbol{e^{A(s+t)}}$.

Solution 1 Integrate $da/dt = 0$, then $db/dt = a$, then $dc/dt = 2b$ and $dz/dt = 3c$:

$$\begin{aligned} a(t) &= a(0) \\ b(t) &= ta(0) + b(0) \\ c(t) &= t^2a(0) + 2tb(0) + c(0) \\ z(t) &= t^3a(0) + 3t^2b(0) + 3tc(0) + z(0) \end{aligned}$$

The 4 by 4 matrix which is multiplying $a(0), b(0), c(0), d(0)$ to produce $a(t), b(t), c(t), d(t)$ must be the same e^{At} as below

Solution 2 *The powers of A (strictly triangular) are all zero after A^3.*

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{2} & 0 & 0 \\ 0 & 0 & \mathbf{3} & 0 \end{bmatrix} \quad \boldsymbol{A^2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{2} & 0 & 0 & 0 \\ 0 & \mathbf{6} & 0 & 0 \end{bmatrix} \quad \boldsymbol{A^3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{6} & 0 & 0 & 0 \end{bmatrix} \quad \boldsymbol{A^4 = 0}$$

The diagonals move down at each step. So the series for e^{At} stops after four terms:

Same $\boldsymbol{e^{At}}$ as in Solution 1

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} = \begin{bmatrix} 1 & & & \\ t & 1 & & \\ t^2 & 2t & 1 & \\ t^3 & 3t^2 & 3t & 1 \end{bmatrix}$$

The square of $\boldsymbol{e^A}$ is $\boldsymbol{e^{2A}}$. But $\boldsymbol{e^Ae^B}$ and $\boldsymbol{e^Be^A}$ and $\boldsymbol{e^{A+B}}$ can be all different.

Problem Set 6.3

1 Find two λ's and $\boldsymbol{x}$'s so that $\boldsymbol{u} = e^{\lambda t}\boldsymbol{x}$ solves

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \boldsymbol{u}.$$

What combination $\boldsymbol{u} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2$ starts from $\boldsymbol{u}(0) = (5, -2)$?

2 Solve Problem 1 for $\boldsymbol{u} = (y, z)$ by back substitution, z before y:

$$\text{Solve } \frac{dz}{dt} = z \text{ from } z(0) = -2. \qquad \text{Then solve } \frac{dy}{dt} = 4y + 3z \text{ from } y(0) = 5.$$

The solution for y will be a combination of e^{4t} and e^{t}. The λ's are 4 and 1.

3 (a) If every column of A adds to zero, why is $\lambda = 0$ an eigenvalue?

(b) With negative diagonal and positive off-diagonal adding to zero, $\boldsymbol{u}' = A\boldsymbol{u}$ will be a "continuous" Markov equation. Find the eigenvalues and eigenvectors, and the *steady state* as $t \to \infty$

$$\text{Solve } \frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \boldsymbol{u} \quad \text{with} \quad \boldsymbol{u}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}. \quad \text{What is } \boldsymbol{u}(\infty)?$$

4 A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \qquad \text{and} \qquad \frac{dw}{dt} = v - w.$$

Show that the total $v + w$ is constant (40 people). Find the matrix in $d\boldsymbol{u}/dt = A\boldsymbol{u}$ and its eigenvalues and eigenvectors. What are v and w at $t = 1$ and $t = \infty$?

5 Reverse the diffusion of people in Problem 4 to $d\boldsymbol{u}/dt = -A\boldsymbol{u}$:

$$\frac{dv}{dt} = v - w \qquad \text{and} \qquad \frac{dw}{dt} = w - v.$$

The total $v+w$ still remains constant. How are the λ's changed now that A is changed to $-A$? But show that $v(t)$ grows to infinity from $v(0) = 30$.

6 A has real eigenvalues but B has complex eigenvalues:

$$A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix} \qquad B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix} \qquad (a \text{ and } b \text{ are real})$$

Find the conditions on a and b so that all solutions of $d\boldsymbol{u}/dt = A\boldsymbol{u}$ and $d\boldsymbol{v}/dt = B\boldsymbol{v}$ approach zero as $t \to \infty$: $\mathbf{Re}\,\lambda < 0$ for all eigenvalues.

7 Suppose P is the projection matrix onto the $45°$ line $y = x$ in $\mathbf{R}^2$. What are its eigenvalues? If $d\boldsymbol{u}/dt = -P\boldsymbol{u}$ (notice minus sign) can you find the limit of $\boldsymbol{u}(t)$ at $t = \infty$ starting from $\boldsymbol{u}(0) = (3, 1)$?

8 The rabbit population shows fast growth (from $6r$) but loss to wolves (from $-2w$). The wolf population always grows in this model ($-w^2$ would control wolves):

$$\frac{dr}{dt} = 6r - 2w \qquad \text{and} \qquad \frac{dw}{dt} = 2r + w.$$

Find the eigenvalues and eigenvectors. If $r(0) = w(0) = 30$ what are the populations at time t? After a long time, what is the ratio of rabbits to wolves?

9 (a) Write $(4, 0)$ as a combination $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ of these two eigenvectors of A:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = i\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = -i\begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

(b) The solution to $d\boldsymbol{u}/dt = A\boldsymbol{u}$ starting from $(4, 0)$ is $c_1e^{it}\boldsymbol{x}_1 + c_2e^{-it}\boldsymbol{x}_2$. Substitute $e^{it} = \cos t + i \sin t$ and $e^{-it} = \cos t - i \sin t$ to find $\boldsymbol{u}(t)$.

Questions 10–13 reduce second-order equations to first-order systems for (y, y').

10 Find A to change the scalar equation $y'' = 5y' + 4y$ into a vector equation for $\boldsymbol{u} = (y, y')$:

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}.$$

What are the eigenvalues of A? Find them also by substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$.

11 The solution to $y'' = 0$ is a straight line $y = C + Dt$. Convert to a matrix equation:

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} \text{ has the solution } \begin{bmatrix} y \\ y' \end{bmatrix} = e^{At}\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}.$$

This matrix A has $\lambda = 0, 0$ and it cannot be diagonalized. Find A^2 and compute $e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots$. Multiply your e^{At} times $\big(y(0), y'(0)\big)$ to check the straight line $y(t) = y(0) + y'(0)t$.

12 Substitute $y = e^{\lambda t}$ into $y'' = 6y' - 9y$ to show that $\lambda = 3$ is a repeated root. This is trouble; we need a second solution after e^{3t}. The matrix equation is

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}.$$

Show that this matrix has $\lambda = 3, 3$ and only one line of eigenvectors. *Trouble here too*. Show that the second solution to $y'' = 6y' - 9y$ is $y = te^{3t}$.

13 (a) Write down two familiar functions that solve the equation $d^2y/dt^2 = -9y$. Which one starts with $y(0) = 3$ and $y'(0) = 0$?

(b) This second-order equation $y'' = -9y$ produces a vector equation $\boldsymbol{u}' = A\boldsymbol{u}$:

$$\boldsymbol{u} = \begin{bmatrix} y \\ y' \end{bmatrix} \qquad \frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}.$$

Find $\boldsymbol{u}(t)$ by using the eigenvalues and eigenvectors of A: $\boldsymbol{u}(0) = (3, 0)$.

14 The matrix in this question is skew-symmetric ($A^{\mathrm{T}} = -A$):

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \boldsymbol{u} \qquad \text{or} \qquad \begin{aligned} u_1' &= cu_2 - bu_3 \\ u_2' &= au_3 - cu_1 \\ u_3' &= bu_1 - au_2. \end{aligned}$$

(a) The derivative of $\|\boldsymbol{u}(t)\|^2 = u_1^2 + u_2^2 + u_3^2$ is $2u_1u_1' + 2u_2u_2' + 2u_3u_3'$. Substitute u_1', u_2', u_3' to get *zero*. Then $\|\boldsymbol{u}(t)\|^2$ stays equal to $\|\boldsymbol{u}(0)\|^2$.

(b) *When A is skew-symmetric, $Q = e^{At}$ is orthogonal.* Prove $Q^{\mathrm{T}} = e^{-At}$ from the series for $Q = e^{At}$. Then $Q^{\mathrm{T}}Q = I$.

15 A particular solution to $d\boldsymbol{u}/dt = A\boldsymbol{u} - \boldsymbol{b}$ is $\boldsymbol{u}_p = A^{-1}\boldsymbol{b}$, if A is invertible. The usual solutions to $d\boldsymbol{u}/dt = A\boldsymbol{u}$ give $\boldsymbol{u}_n$. Find the complete solution $\boldsymbol{u} = \boldsymbol{u}_p + \boldsymbol{u}_n$:

(a) $\dfrac{du}{dt} = u - 4$ (b) $\dfrac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \boldsymbol{u} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.

16 If c is not an eigenvalue of A, substitute $\boldsymbol{u} = e^{ct}\boldsymbol{v}$ and find a particular solution to $d\boldsymbol{u}/dt = A\boldsymbol{u} - e^{ct}\boldsymbol{b}$. How does it break down when c is an eigenvalue of A? The "nullspace" of $d\boldsymbol{u}/dt = A\boldsymbol{u}$ contains the usual solutions $e^{\lambda_i t}\boldsymbol{x}_i$.

17 Find a matrix A to illustrate each of the unstable regions in Figure 6.5:

(a) $\lambda_1 < 0$ and $\lambda_2 > 0$ (b) $\lambda_1 > 0$ and $\lambda_2 > 0$ (c) $\lambda = a \pm ib$ with $a > 0$.

Questions 18–27 are about the matrix exponential e^{At}.

18 Write five terms of the infinite series for e^{At}. Take the t derivative of each term. Show that you have four terms of Ae^{At}. Conclusion: $e^{At}\boldsymbol{u}_0$ solves $\boldsymbol{u}' = A\boldsymbol{u}$.

19 The matrix $B = \left[\begin{smallmatrix} 0 & -4 \\ 0 & 0 \end{smallmatrix}\right]$ has $B^2 = 0$. Find e^{Bt} from a (short) infinite series. Check that the derivative of e^{Bt} is Be^{Bt}.

20 Starting from $\boldsymbol{u}(0)$ the solution at time T is $e^{AT}\boldsymbol{u}(0)$. Go an additional time t to reach $e^{At}\ e^{AT}\boldsymbol{u}(0)$. This solution at time $t + T$ can also be written as _____. Conclusion: e^{At} times e^{AT} equals _____.

21 Write $A = \left[\begin{smallmatrix} 1 & 4 \\ 0 & 0 \end{smallmatrix}\right]$ in the form $X\Lambda X^{-1}$. Find e^{At} from $Xe^{\Lambda t}X^{-1}$.

22 If $A^2 = A$ show that the infinite series produces $e^{At} = I + (e^t - 1)A$. For $A = \left[\begin{smallmatrix} 1 & 4 \\ 0 & 0 \end{smallmatrix}\right]$ in Problem 21 this gives $e^{At} =$ ____.

23 Generally $e^A e^B$ is different from $e^B e^A$. They are both different from e^{A+B}. Check this using Problems 21–22 and 19. (If $AB = BA$, all three are the same.)

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix} \qquad A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

24 Write $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 3 \end{smallmatrix}\right]$ as $X\Lambda X^{-1}$. Multiply $Xe^{\Lambda t}X^{-1}$ to find the matrix exponential e^{At}. Check e^{At} and the derivative of e^{At} when $t = 0$.

25 Put $A = \left[\begin{smallmatrix} 1 & 3 \\ 0 & 0 \end{smallmatrix}\right]$ into the infinite series to find e^{At}. First compute A^2 and A^n:

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & 3t \\ 0 & 0 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} + \cdots = \begin{bmatrix} e^t & \quad \\ 0 & \quad \end{bmatrix}.$$

26 (Recommended) Give two reasons why the matrix exponential e^{At} is never singular:

(a) Write down its inverse.

(b) Why are these eigenvalues nonzero? If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $e^{At}\boldsymbol{x} =$ ____ $\boldsymbol{x}$.

27 Find a solution $x(t), y(t)$ that gets large as $t \to \infty$. To avoid this instability a scientist exchanged the two equations:

$$\begin{aligned} dx/dt &= 0x - 4y \\ dy/dt &= -2x + 2y \end{aligned} \qquad \text{becomes} \qquad \begin{aligned} dy/dt &= -2x + 2y \\ dx/dt &= 0x - 4y. \end{aligned}$$

Now the matrix $\left[\begin{smallmatrix} -2 & 2 \\ 0 & -4 \end{smallmatrix}\right]$ is stable. It has negative eigenvalues. How can this be?

Challenge Problems

28 Centering $y'' = -y$ in Example 3 will produce $Y_{n+1} - 2Y_n + Y_{n-1} = -(\Delta t)^2 Y_n$. This can be written as a one-step difference equation for $\boldsymbol{U} = (Y, Z)$:

$$\begin{aligned} Y_{n+1} &= Y_n + \Delta t\, Z_n \\ Z_{n+1} &= Z_n - \Delta t\, Y_{n+1} \end{aligned} \qquad \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix}\begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} Y_n \\ Z_n \end{bmatrix}$$

Invert the matrix on the left side to write this as $\boldsymbol{U}_{n+1} = A\boldsymbol{U}_n$. Show that $\det A = 1$. Choose the large time step $\Delta t = 1$ and find the eigenvalues λ_1 and $\lambda_2 = \overline{\lambda}_1$ of A:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \textbf{ has } |\lambda_1| = |\lambda_2| = 1. \textbf{ Show that } \boldsymbol{A}^6 \textbf{ is exactly } \boldsymbol{I}.$$

After 6 steps to $t = 6$, $\boldsymbol{U}_6$ equals $\boldsymbol{U}_0$. The exact $y = \cos t$ returns to 1 at $t = 2\pi$.

29 That centered choice (*leapfrog method*) in Problem 28 is very successful for small time steps Δt. But find the eigenvalues of A for $\Delta t = \sqrt{2}$ and 2:

$$A = \begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}.$$

Both matrices have $|\lambda| = 1$. Compute A^4 in both cases and find the eigenvectors of A. That second value $\Delta t = 2$ is at the border of instability. Any time step $\Delta t > 2$ will lead to $|\lambda| > 1$, and the powers in $\boldsymbol{U}_n = A^n \boldsymbol{U}_0$ will explode.

Note You might say that nobody would compute with $\Delta t > 2$. But if an atom vibrates with $y'' = -1000000y$, then $\Delta t > .0002$ will give instability. Leapfrog has a very strict stability limit. $Y_{n+1} = Y_n + 3Z_n$ and $Z_{n+1} = Z_n - 3Y_{n+1}$ will explode because $\Delta t = 3$ is too large. The matrix has $|\lambda| > 1$.

30 Another good idea for $y'' = -y$ is the trapezoidal method (half forward/half back). ***This may be the best way to keep (Y_n, Z_n) exactly on a circle.***

$$\textbf{Trapezoidal} \quad \begin{bmatrix} 1 & -\Delta t/2 \\ \Delta t/2 & 1 \end{bmatrix} \begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t/2 \\ -\Delta t/2 & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Z_n \end{bmatrix}.$$

(a) Invert the left matrix to write this equation as $\boldsymbol{U}_{n+1} = A\boldsymbol{U}_n$. *Show that A is an orthogonal matrix*: $A^{\mathrm{T}}A = I$. **These points $\boldsymbol{U}_n$ never leave the circle.** $A = (I - B)^{-1}(I + B)$ is always an orthogonal matrix if $B^{\mathrm{T}} = -B$.

(b) (Optional MATLAB) Take 32 steps from $\boldsymbol{U}_0 = (1, 0)$ to $\boldsymbol{U}_{32}$ with $\Delta t = 2\pi/32$. Is $\boldsymbol{U}_{32} = \boldsymbol{U}_0$? I think there is a small error.

31 The ***cosine of a matrix*** is defined like e^A, by copying the series for $\cos t$:

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \cdots \qquad \cos A = I - \frac{1}{2!}A^2 + \frac{1}{4!}A^4 - \cdots$$

(a) If $A\boldsymbol{x} = \lambda \boldsymbol{x}$, multiply each term times $\boldsymbol{x}$ to find the eigenvalue of $\cos A$.

(b) Find the eigenvalues of $A = \begin{bmatrix} \pi & \pi \\ \pi & \pi \end{bmatrix}$ with eigenvectors $(1, 1)$ and $(1, -1)$. From the eigenvalues and eigenvectors of $\cos A$, find that matrix $C = \cos A$.

(c) The second derivative of $\cos(At)$ is $-A^2 \cos(At)$.

$$\boldsymbol{u}(t) = \mathbf{cos}(\boldsymbol{At})\, \boldsymbol{u}(0) \quad \text{solves} \quad \frac{d^2\boldsymbol{u}}{dt^2} = -A^2\boldsymbol{u} \quad \text{starting from} \quad \boldsymbol{u}'(0) = 0.$$

Construct $\boldsymbol{u}(t) = \cos(At)\,\boldsymbol{u}(0)$ by the usual three steps for that specific A:

1. Expand $\boldsymbol{u}(0) = (4, 2) = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ in the eigenvectors.

2. Multiply those eigenvectors by _____ and _____ (instead of $e^{\lambda t}$).

3. Add up the solution $\boldsymbol{u}(t) = c_1$ _____ $\boldsymbol{x}_1 + c_2$ _____ $\boldsymbol{x}_2$.

32 Explain one of these three proofs that the square of e^A is e^{2A}.

1. Solving with e^A from $t = 0$ to 1 and then 1 to 2 agrees with e^{2A} from 0 to 2.

2. The squared series $(I + A + \frac{A^2}{2} + \cdots)^2$ matches $I + 2A + \frac{(2A)^2}{2} + \cdots = e^{2A}$.

3. If A can be diagonalized then $(Xe^{\Lambda}X^{-1})(Xe^{\Lambda}X^{-1}) = Xe^{2\Lambda}X^{-1}$.

Notes on a Differential Equations Course

Certainly constant-coefficient linear equations are the simplest to solve. This section 6.3 of the book shows you part of a differential equations course, but there is more:

1. The second order equation $mu'' + bu' + ku = 0$ has major importance in applications. The exponents λ in the solutions $u = e^{\lambda t}$ solve $m\lambda^2 + b\lambda + k = 0$. The damping coefficient b is crucial:

   Underdamping $b^2 < 4mk$ **Critical damping** $b^2 = 4mk$ **Overdamping** $b^2 > 4mk$

   This decides whether λ_1 and λ_2 are real roots or repeated roots or complex roots. With complex λ's the solution $u(t)$ oscillates as it decays.

2. Our equations had no forcing term $f(t)$. We were finding the "nullspace solution". To $\boldsymbol{u}_n(t)$ we need to add a particular solution $u_p(t)$ that balances the force $f(t)$:

   Input $f(s)$ at time s
   Growth factor $e^{A(t-s)}$
   Add up outputs at time t

   $$\boldsymbol{u}_{\text{particular}} = \int_0^t e^{A(t-s)} f(s)\, ds.$$

   This solution can also be discovered and studied by *Laplace transform*—that is the established way to convert linear differential equations to linear algebra.

In real applications, nonlinear differential equations are solved numerically. A standard method with good accuracy is "Runge-Kutta"—named after its discoverers. Analysis can find the constant solutions to $du/dt = f(u)$. Those are solutions $u(t) = Y$ with $f(Y) = 0$ and $du/dt = 0$: *no movement*. We can also understand stability or instability near $u = Y$. Far from Y, the computer takes over.

This basic course is the subject of my textbook (companion to this one) on *Differential Equations and Linear Algebra*: **math.mit.edu/dela**.

6.4 Symmetric Matrices

1 A symmetric matrix S has n **real eigenvalues** λ_i and n **orthonormal eigenvectors** $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$.

2 Every real symmetric S can be diagonalized: $\boxed{S = Q\Lambda Q^{-1} = \boldsymbol{Q\Lambda Q}^{\mathrm{T}}}$

3 The number of positive eigenvalues of S equals the number of positive pivots.

4 Antisymmetric matrices $A = -A^{\mathrm{T}}$ have *imaginary* λ's and *orthonormal* (*complex*) $\boldsymbol{q}$'s.

5 Section 9.2 explains why the test $S = S^{\mathrm{T}}$ becomes $\boldsymbol{S = \overline{S}^{\mathrm{T}}}$ for *complex matrices*.

$S = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = \overline{S}^{\mathrm{T}}$ has real $\lambda = 1, -1$. $\qquad A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = -\overline{A}^{\mathrm{T}}$ has $\lambda = i, -i$.

It is no exaggeration to say that symmetric matrices S are the most important matrices the world will ever see—in the theory of linear algebra and also in the applications. We come immediately to the key question about symmetry. Not only the question, but also the two-part answer.

What is special about $\boldsymbol{Sx = \lambda x}$ ***when S is symmetric?***

We look for special properties of the eigenvalues λ and eigenvectors $\boldsymbol{x}$ when $S = S^{\mathrm{T}}$.

The diagonalization $S = X\Lambda X^{-1}$ will reflect the symmetry of S. We get some hint by transposing to $S^{\mathrm{T}} = (X^{-1})^{\mathrm{T}}\Lambda X^{\mathrm{T}}$. Those are the same since $S = S^{\mathrm{T}}$. Possibly X^{-1} in the first form equals X^{T} in the second form? Then $X^{\mathrm{T}}X = I$. That makes each eigenvector in X orthogonal to the other eigenvectors when $S = S^{\mathrm{T}}$. Here are the key facts:

1. **A symmetric matrix has only *real eigenvalues*.**
2. **The *eigenvectors* can be chosen *orthonormal*.**

Those n orthonormal eigenvectors go into the columns of X. Every symmetric matrix can be diagonalized. ***Its eigenvector matrix X becomes an orthogonal matrix Q***. Orthogonal matrices have $Q^{-1} = Q^{\mathrm{T}}$—what we suspected about the eigenvector matrix is true. To remember it we write Q instead of X, when we choose orthonormal eigenvectors.

Why do we use the word "choose"? Because the eigenvectors do not *have* to be unit vectors. Their lengths are at our disposal. We will choose unit vectors—eigenvectors of length one, which are orthonormal and not just orthogonal. Then $A = X\Lambda X^{-1}$ is in its special and particular form $S = Q\Lambda Q^{\mathrm{T}}$ for symmetric matrices.

(Spectral Theorem) Every symmetric matrix has the factorization $S = Q\Lambda Q^{\mathrm{T}}$ with real eigenvalues in Λ and orthonormal eigenvectors in the columns of Q:

Symmetric diagonalization $$S = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}} \quad \text{with} \quad Q^{-1} = Q^{\mathrm{T}}. \qquad (1)$$

It is easy to see that $Q\Lambda Q^{\mathrm{T}}$ is symmetric. Take its transpose. You get $(Q^{\mathrm{T}})^{\mathrm{T}}\Lambda^{\mathrm{T}}Q^{\mathrm{T}}$, which is $Q\Lambda Q^{\mathrm{T}}$ again. The harder part is to prove that every symmetric matrix has real λ's and orthonormal $\boldsymbol{x}$'s. This is the *"spectral theorem"* in mathematics and the *"principal axis theorem"* in geometry and physics. We have to prove it! No choice. I will approach the proof in three steps:

1. By an example, showing real λ's in Λ and orthonormal $\boldsymbol{x}$'s in Q.
2. By a proof of those facts when no eigenvalues are repeated.
3. By a proof that allows repeated eigenvalues (at the end of this section).

Example 1 Find the λ's and $\boldsymbol{x}$'s when $S = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $S - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$.

Solution The determinant of $S - \lambda I$ is $\lambda^2 - 5\lambda$. The eigenvalues are 0 and 5 (*both real*). We can see them directly: $\lambda = 0$ is an eigenvalue because S is singular, and $\lambda = 5$ matches the *trace* down the diagonal of S: $0 + 5$ agrees with $1 + 4$.

Two eigenvectors are $(\mathbf{2}, -\mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$—orthogonal but not yet orthonormal. The eigenvector for $\lambda = 0$ is in the *nullspace* of A. The eigenvector for $\lambda = 5$ is in the *column space*. We ask ourselves, why are the nullspace and column space perpendicular? The Fundamental Theorem says that the nullspace is perpendicular to the ***row space***—not the column space. But our matrix is *symmetric*! Its row and column spaces are the same. Its eigenvectors $(2, -1)$ and $(1, 2)$ must be (and are) perpendicular.

These eigenvectors have length $\sqrt{5}$. Divide them by $\sqrt{5}$ to get unit vectors. Put those unit eigenvectors into the columns of Q. Then $Q^{-1}SQ$ is Λ and $Q^{-1} = Q^{\mathrm{T}}$:

$$Q^{-1}SQ = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = \Lambda.$$

Now comes the n by n case. The λ's are real when $S = S^{\mathrm{T}}$ and $S\boldsymbol{x} = \lambda\boldsymbol{x}$.

Real Eigenvalues All the eigenvalues of a real symmetric matrix are real.

Proof Suppose that $S\boldsymbol{x} = \lambda\boldsymbol{x}$. Until we know otherwise, λ might be a complex number $a + ib$ (a and b real). *Its complex conjugate is* $\overline{\lambda} = a - ib$. Similarly the components of $\boldsymbol{x}$ may be complex numbers, and switching the signs of their imaginary parts gives $\overline{\boldsymbol{x}}$.

The good thing is that $\overline{\lambda}$ times $\overline{\boldsymbol{x}}$ is always the conjugate of λ times $\boldsymbol{x}$. So we can take conjugates of $S\boldsymbol{x} = \lambda\boldsymbol{x}$, remembering that S is real:

$$S\boldsymbol{x} = \lambda\boldsymbol{x} \quad \text{leads to} \quad S\overline{\boldsymbol{x}} = \overline{\lambda}\,\overline{\boldsymbol{x}}. \qquad \text{Transpose to} \quad \overline{\boldsymbol{x}}^{\mathrm{T}} S = \overline{\boldsymbol{x}}^{\mathrm{T}}\,\overline{\lambda}.$$

Now take the dot product of the first equation with $\overline{\boldsymbol{x}}$ and the last equation with $\boldsymbol{x}$:

$$\overline{\boldsymbol{x}}^{\mathrm{T}} S\boldsymbol{x} = \overline{\boldsymbol{x}}^{\mathrm{T}} \lambda\boldsymbol{x} \qquad \text{and also} \qquad \overline{\boldsymbol{x}}^{\mathrm{T}} S\boldsymbol{x} = \overline{\boldsymbol{x}}^{\mathrm{T}}\,\overline{\lambda}\boldsymbol{x}. \tag{2}$$

The left sides are the same so the right sides are equal. One equation has λ, the other has $\overline{\lambda}$. They multiply $\overline{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{x} = |x_1|^2 + |x_2|^2 + \cdots =$ length squared which is not zero. *Therefore λ must equal $\overline{\lambda}$*, and $a + ib$ equals $a - ib$. So $b = 0$ and $\lambda = a = \textit{real}$. Q.E.D.

The eigenvectors come from solving the real equation $(S - \lambda I)\boldsymbol{x} = \boldsymbol{0}$. So the $\boldsymbol{x}$'s are also real. The important fact is that they are perpendicular.

Orthogonal Eigenvectors Eigenvectors of a real symmetric matrix (when they correspond to different λ's) are always perpendicular.

Proof Suppose $S\boldsymbol{x} = \lambda_1\boldsymbol{x}$ and $S\boldsymbol{y} = \lambda_2\boldsymbol{y}$. We are assuming here that $\lambda_1 \neq \lambda_2$. Take dot products of the first equation with $\boldsymbol{y}$ and the second with $\boldsymbol{x}$:

$$\textbf{Use } S^{\mathrm{T}} = S \qquad (\lambda_1\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = (S\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}S^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}S\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}\lambda_2\boldsymbol{y}. \tag{3}$$

The left side is $\boldsymbol{x}^{\mathrm{T}}\lambda_1\boldsymbol{y}$, the right side is $\boldsymbol{x}^{\mathrm{T}}\lambda_2\boldsymbol{y}$. Since $\lambda_1 \neq \lambda_2$, this proves that $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = 0$. The eigenvector $\boldsymbol{x}$ (for λ_1) is perpendicular to the eigenvector $\boldsymbol{y}$ (for λ_2).

Example 2 The eigenvectors of a 2 by 2 symmetric matrix have a special form:

$$\textbf{Not widely known} \quad S = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad \text{has} \quad \boldsymbol{x}_1 = \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}_2 = \begin{bmatrix} \lambda_2 - c \\ b \end{bmatrix}. \tag{4}$$

This is in the Problem Set. The point here is that $\boldsymbol{x}_1$ is perpendicular to $\boldsymbol{x}_2$:

$$\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{x}_2 = b(\lambda_2 - c) + (\lambda_1 - a)b = b(\lambda_1 + \lambda_2 - a - c) = 0.$$

This is zero because $\lambda_1 + \lambda_2$ equals the trace $a + c$. Thus $\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{x}_2 = 0$. Eagle eyes might notice the special case $S = I$, when b and $\lambda_1 - a$ and $\lambda_2 - c$ and $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are all zero. Then $\lambda_1 = \lambda_2 = 1$ is repeated. But of course $S = I$ has perpendicular eigenvectors.

Symmetric matrices S have orthogonal eigenvector matrices Q. Look at this again:

$$\textbf{Symmetry} \qquad S = X\Lambda X^{-1} \quad \text{becomes} \quad S = Q\Lambda Q^{\mathrm{T}} \quad \text{with} \quad Q^{\mathrm{T}}Q = I.$$

This says that every 2 by 2 symmetric matrix is (**rotation**)(**stretch**)(**rotate back**)

$$S = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1^{\mathrm{T}} \\ \boldsymbol{q}_2^{\mathrm{T}} \end{bmatrix}. \tag{5}$$

Columns $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ multiply rows $\lambda_1\boldsymbol{q}_1^{\mathrm{T}}$ and $\lambda_2\boldsymbol{q}_2^{\mathrm{T}}$ to produce $S = \lambda_1\boldsymbol{q}_1\boldsymbol{q}_1^{\mathrm{T}} + \lambda_2\boldsymbol{q}_2\boldsymbol{q}_2^{\mathrm{T}}$.

Every symmetric matrix $\quad S = Q\Lambda Q^{\mathrm{T}} = \lambda_1 \boldsymbol{q}_1 \boldsymbol{q}_1^{\mathrm{T}} + \cdots + \lambda_n \boldsymbol{q}_n \boldsymbol{q}_n^{\mathrm{T}}.$ (6)

Remember the steps to this great result (the spectral theorem).

Section 6.2	Write $A\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i$ in matrix form	$AX = X\Lambda$ or $A = X\Lambda X^{-1}$
Section 6.4	Orthonormal $\boldsymbol{x}_i = \boldsymbol{q}_i$ gives $X = Q$	$S = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$

$Q\Lambda Q^{\mathrm{T}}$ in equation (6) has columns of $Q\Lambda$ times rows of Q^{T}. Here is a direct proof.

S has correct eigenvectors
Those q's are orthonormal $\quad S\boldsymbol{q}_i = (\lambda_1 \boldsymbol{q}_1 \boldsymbol{q}_1^{\mathrm{T}} + \cdots + \lambda_n \boldsymbol{q}_n \boldsymbol{q}_n^{\mathrm{T}})\, \boldsymbol{q}_i = \lambda_i \boldsymbol{q}_i\,.$ (7)

Complex Eigenvalues of Real Matrices

For any real matrix, $S\,x = \lambda\,\boldsymbol{x}$ gives $S\,\overline{\boldsymbol{x}} = \overline{\lambda}\,\overline{\boldsymbol{x}}$. For a symmetric matrix, λ and $\boldsymbol{x}$ turn out to be real. Those two equations become the same. But a *non*symmetric matrix can easily produce λ and $\boldsymbol{x}$ that are complex. Then $A\,\overline{\boldsymbol{x}} = \overline{\lambda}\,\overline{\boldsymbol{x}}$ is true but different from $A\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$. We get another complex eigenvalue (which is $\overline{\lambda}$) and a new eigenvector (which is $\overline{\boldsymbol{x}}$):

For real matrices, complex λ's and x's come in "conjugate pairs."

$\lambda = a + ib$
$\overline{\lambda} = a - ib$

If $\quad A\,\boldsymbol{x} = \lambda\,\boldsymbol{x} \quad$ ***then*** $\quad A\,\overline{\boldsymbol{x}} = \overline{\lambda}\,\overline{\boldsymbol{x}}.$ (8)

Example 3 $\quad A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has $\lambda_1 = \cos\theta + i\sin\theta$ and $\lambda_2 = \cos\theta - i\sin\theta$.

Those eigenvalues are conjugate to each other. They are λ and $\overline{\lambda}$. The eigenvectors must be $\boldsymbol{x}$ and $\overline{\boldsymbol{x}}$, because A is real:

$$\textbf{This is } \lambda\,\boldsymbol{x} \qquad A\boldsymbol{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = (\cos\theta + i\sin\theta) \begin{bmatrix} 1 \\ -i \end{bmatrix}$$
$$\textbf{This is } \overline{\lambda}\,\overline{\boldsymbol{x}} \qquad A\overline{\boldsymbol{x}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = (\cos\theta - i\sin\theta) \begin{bmatrix} 1 \\ i \end{bmatrix}. \tag{9}$$

Those eigenvectors $(1, -i)$ and $(1, i)$ are complex conjugates because A is real.

For this rotation matrix the absolute value is $|\lambda| = 1$, because $\cos^2\theta + \sin^2\theta = 1$. ***This fact*** $|\lambda| = 1$ ***holds for the eigenvalues of every orthogonal matrix Q***.

We apologize that a touch of complex numbers slipped in. They are unavoidable even when the matrix is real. Chapter 9 goes beyond complex numbers λ and complex eigenvectors $\boldsymbol{x}$ to complex matrices A. Then you have the whole picture.

We end with two optional discussions.

Eigenvalues versus Pivots

The eigenvalues of A are very different from the pivots. For eigenvalues, we solve $\det(A - \lambda I) = 0$. For pivots, we use elimination. The only connection so far is this:

$$\textbf{\textit{product of pivots}} = \textbf{\textit{determinant}} = \textbf{\textit{product of eigenvalues.}}$$

We are assuming a full set of pivots $d_1, \ldots, d_n$. There are n real eigenvalues $\lambda_1, \ldots, \lambda_n$. The d's and λ's are not the same, but they come from the same symmetric matrix. Those d's and λ's have a hidden relation. ***For symmetric matrices the pivots and the eigenvalues have the same signs***:

> ***The number of positive eigenvalues of $S = S^{\mathrm{T}}$ equals the number of positive pivots.***
> Special case: S has all $\lambda_i > 0$ if and only if all pivots are positive.

That special case is an all-important fact for **positive definite matrices** in Section 6.5.

Example 4 This symmetric matrix has one positive eigenvalue and one positive pivot:

$$\textbf{Matching signs} \qquad S = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \qquad \begin{matrix} \text{has pivots } 1 \text{ and } -8 \\ \text{eigenvalues } 4 \text{ and } -2. \end{matrix}$$

The signs of the pivots match the signs of the eigenvalues, one plus and one minus. This could be false when the matrix is not symmetric:

$$\textbf{Opposite signs} \quad B = \begin{bmatrix} 1 & 6 \\ -1 & -4 \end{bmatrix} \qquad \begin{matrix} \text{has pivots } 1 \text{ and } 2 \\ \text{eigenvalues } -1 \text{ and } -2. \end{matrix}$$

The diagonal entries are a third set of numbers and we say nothing about them.

> Here is a proof that the pivots and eigenvalues have matching signs, when $S = S^{\mathrm{T}}$.

You see it best when the pivots are divided out of the rows of U. Then S is LDL^{T}. The diagonal pivot matrix D goes between triangular matrices L and L^{T}:

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathbf{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -8 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{3} \\ 0 & 1 \end{bmatrix} \qquad \textbf{This is } S = LDL^{\mathrm{T}}\textbf{. It is symmetric.}$$

Watch the eigenvalues of LDL^{T} when L moves to I. S changes to D.

The eigenvalues of LDL^{T} are 4 and -2. The eigenvalues of IDI^{T} are 1 and -8 (the pivots!). The eigenvalues are changing, as the "3" in L moves to zero. But to change *sign*, a real eigenvalue would have to cross zero. The matrix would at that moment be singular. Our changing matrix always has pivots 1 and -8, so it is *never* singular. The signs cannot change, as the λ's move to the d's.

We repeat the proof for any $S = LDL^{\mathrm{T}}$. Move L toward I, by moving the off-diagonal entries to zero. The pivots are not changing and not zero. The eigenvalues λ of LDL^{T} change to the eigenvalues d of IDI^{T}. Since these eigenvalues cannot cross zero as they move into the pivots, their signs cannot change. **Same signs for the $\boldsymbol{\lambda}$'s and $\boldsymbol{d}$'s.**

This connects the two halves of applied linear algebra—pivots and eigenvalues.

All Symmetric Matrices are Diagonalizable

When no eigenvalues of A are repeated, the eigenvectors are sure to be independent. Then A can be diagonalized. But a repeated eigenvalue can produce a shortage of eigenvectors. This *sometimes* happens for nonsymmetric matrices. It *never* happens for symmetric matrices. ***There are always enough eigenvectors to diagonalize*** $S = S^{\mathrm{T}}$.

Here is one idea for a proof. Change S slightly by a diagonal matrix **diag**$(c, 2c, \ldots, nc)$. If c is very small, the new symmetric matrix will have no repeated eigenvalues. Then we know it has a full set of orthonormal eigenvectors. As $c \to 0$ we obtain n orthonormal eigenvectors of the original S—even if some eigenvalues of that S are repeated.

Every mathematician knows that this argument is incomplete. How do we guarantee that the small diagonal matrix will separate the eigenvalues? (I am sure this is true.)

A different proof comes from a useful new factorization that applies to *all square matrices* A, symmetric or not. This new factorization quickly produces $S = Q\Lambda Q^{\mathrm{T}}$ with a full set of real orthonormal eigenvectors when S is any real symmetric matrix.

Every square A ***factors into*** QTQ^{-1} ***where*** T ***is upper triangular and*** $\overline{Q}^{\mathrm{T}} = Q^{-1}$.
If A ***has real eigenvalues then*** Q ***and*** T ***can be chosen real:*** $Q^{\mathrm{T}}Q = I$.

This is Schur's Theorem. Its proof will go onto the website **math.mit.edu/linearalgebra**. Here I will show how T is diagonal ($T = \Lambda$) when S is symmetric. Then S is $Q\Lambda Q^{\mathrm{T}}$.

We know that every symmetric S has real eigenvalues, and Schur allows repeated λ's:

Schur's $S = QTQ^{-1}$ means that $T = Q^{\mathrm{T}}SQ$. The transpose is again $Q^{\mathrm{T}}SQ$.
The triangular T *is symmetric when* $S^{\mathrm{T}} = S$. Then T must be diagonal and $T = \Lambda$.

This proves that $S = Q\Lambda Q^{-1}$. The symmetric S has n orthonormal eigenvectors in Q.

Note. I have added another proof in Section 7.2 of this book. That proof shows how the eigenvalues λ can be described *one at a time*. The largest λ_1 is the maximum of $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$. Then λ_2 (second largest) is again the same maximum, if we only allow vectors $\boldsymbol{x}$ that are perpendicular to the first eigenvector. The third eigenvalue λ_3 comes by requiring $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{q}_1 = 0$ and $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{q}_2 = 0 \ldots$

This proof is placed in Chapter 7 because the same one-at-a-time idea succeeds for the *singular values of any matrix* A. **Singular values come from** $\boldsymbol{A^{\mathrm{T}}A}$ **and** $\boldsymbol{AA^{\mathrm{T}}}$.

■ REVIEW OF THE KEY IDEAS ■

1. Every symmetric matrix S has *real eigenvalues* and *perpendicular eigenvectors*.
2. Diagonalization becomes $\boldsymbol{S = Q\Lambda Q^{\mathrm{T}}}$ with an orthogonal eigenvector matrix Q.
3. All symmetric matrices are diagonalizable, even with repeated eigenvalues.
4. The signs of the eigenvalues match the signs of the pivots, when $S = S^{\mathrm{T}}$.
5. Every square matrix can be "triangularized" by $A = QTQ^{-1}$. If $A = S$ then $T = \Lambda$.

WORKED EXAMPLES

6.4 A What matrix A has eigenvalues $\lambda = 1, -1$ and eigenvectors $\boldsymbol{x}_1 = (\cos\theta, \sin\theta)$ and $\boldsymbol{x}_2 = (-\sin\theta, \cos\theta)$? Which of these properties can be predicted in advance?

$$A = A^{\mathrm{T}} \qquad A^2 = I \qquad \det A = -1 \qquad \text{pivot are + and } - \qquad A^{-1} = A$$

Solution All those properties can be predicted! With real eigenvalues $1, -1$ and orthonormal $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, the matrix $A = Q\Lambda Q^{\mathrm{T}}$ must be symmetric. The eigenvalues 1 and -1 tell us that $A^2 = I$ (since $\lambda^2 = 1$) and $A^{-1} = A$ (same thing) and $\det A = -1$. The two pivots must be positive and negative like the eigenvalues, since A is symmetric.

The matrix will be a reflection. Vectors in the direction of $\boldsymbol{x}_1$ are unchanged by A (since $\lambda = 1$). Vectors in the perpendicular direction are reversed (since $\lambda = -1$). The reflection $A = Q\Lambda Q^{\mathrm{T}}$ is across the "θ-line". Write c for $\cos\theta$ and s for $\sin\theta$:

$$A = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} c^2 - s^2 & 2cs \\ 2cs & s^2 - c^2 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

Notice that $\boldsymbol{x} = (1, 0)$ goes to $A\boldsymbol{x} = (\cos 2\theta, \sin 2\theta)$ on the 2θ-line. And $(\cos 2\theta, \sin 2\theta)$ goes back across the θ-line to $\boldsymbol{x} = (1, 0)$.

6.4 B Find the eigenvalues and eigenvectors (discrete sines and cosines) of A_3 and B_4.

$$A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad B_4 = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix}$$

The $-1, 2, -1$ pattern in both matrices is a "second difference". This is like a second derivative. Then $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $B\boldsymbol{x} = \lambda\boldsymbol{x}$ are like $d^2x/dt^2 = \lambda x$. This has eigenvectors $x = \sin kt$ and $x = \cos kt$ that are the bases for Fourier series.

A_n and B_n lead to "discrete sines" and "discrete cosines" that are the bases for the *Discrete Fourier Transform*. This DFT is absolutely central to all areas of digital signal processing. The favorite choice for JPEG in image processing has been B_8 of size $n = 8$.

Solution The eigenvalues of A_3 are $\lambda = 2 - \sqrt{2}$ and 2 and $2 + \sqrt{2}$ (see **6.3 B**). Their sum is 6 (the trace of A_3) and their product is 4 (the determinant). The eigenvector matrix gives the "Discrete Sine Transform" and the eigenvectors fall onto sine curves.

$$\textbf{Sines} = \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix} \qquad \textbf{Cosines} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \sqrt{2}-1 & -1 & 1-\sqrt{2} \\ 1 & 1-\sqrt{2} & -1 & \sqrt{2}-1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Sine matrix = Eigenvectors of A_3 **Cosine matrix = Eigenvectors of B_4**

The eigenvalues of B_4 are $\lambda = 2 - \sqrt{2}$ and 2 and $2 + \sqrt{2}$ and 0 (the same as for A_3, plus the zero eigenvalue). The trace is still 6, but the determinant is now zero. The eigenvector matrix C gives the 4-point "Discrete Cosine Transform". The graph on the Web shows how the first two eigenvectors fall onto cosine curves. (So do all the eigenvectors of B.) These eigenvectors match cosines at the *halfway points* $\pi/8, 3\pi/8, 5\pi/8, 7\pi/8$.

Problem Set 6.4

1 Which of these matrices ASB will be symmetric with eigenvalues 1 and -1?

$$\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}1&\\&-1\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}\quad\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}1&\\&-1\end{bmatrix}\begin{bmatrix}1&0\\-1&1\end{bmatrix}\quad\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}1&\\&-1\end{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix}$$

$B = A^{\mathrm{T}}$ doesn't do it. $B = A^{-1}$ doesn't do it. $B =$ ___ $=$ ___ will succeed. So B must be an _____ matrix.

2 Suppose $S = S^{\mathrm{T}}$. When is ASB also symmetric with the same eigenvalues as S?

(a) Transpose ASB to see that it stays symmetric when $B =$ _____.

(b) ASB is similar to S (same eigenvalues) when $B =$ _____.

Put (a) and (b) together. The symmetric matrices similar to S look like (___)S(___).

3 Write A as $S + N$, symmetric matrix S plus skew-symmetric matrix N:

$$A = \begin{bmatrix}1&2&4\\4&3&0\\8&6&5\end{bmatrix} = S + N \qquad (S^{\mathrm{T}} = S \text{ and } N^{\mathrm{T}} = -N).$$

For any square matrix, $S = \frac{1}{2}(A + A^{\mathrm{T}})$ and $N =$ _____ add up to A.

4 If C is symmetric prove that $A^{\mathrm{T}}CA$ is also symmetric. (Transpose it.) When A is 6 by 3, what are the shapes of C and $A^{\mathrm{T}}CA$?

5 Find the eigenvalues and the unit eigenvectors of

$$S = \begin{bmatrix}2&2&2\\2&0&0\\2&0&0\end{bmatrix}.$$

6 Find an orthogonal matrix Q that diagonalizes $S = \begin{bmatrix}-2&6\\6&7\end{bmatrix}$. What is Λ?

7 Find an orthogonal matrix Q that diagonalizes this symmetric matrix:

$$S = \begin{bmatrix}1&0&2\\0&-1&-2\\2&-2&0\end{bmatrix}.$$

8 Find *all* orthogonal matrices that diagonalize $S = \begin{bmatrix}9&12\\12&16\end{bmatrix}$.

9 (a) Find a symmetric matrix $\begin{bmatrix}1&b\\b&1\end{bmatrix}$ that has a negative eigenvalue.

(b) How do you know it must have a negative pivot?

(c) How do you know it can't have two negative eigenvalues?

10 If $A^3 = 0$ then the eigenvalues of A must be ____. Give an example that has $A \neq 0$. But if A is symmetric, diagonalize it to prove that A must be a zero matrix.

11 If $\lambda = a + ib$ is an eigenvalue of a real matrix A, then its conjugate $\overline{\lambda} = a - ib$ is also an eigenvalue. (If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then also $A\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}$: a conjugate pair λ and $\overline{\lambda}$.) Explain why every real 3 by 3 matrix has at least one real eigenvalue.

12 Here is a quick "proof" that the eigenvalues of every real matrix A are real:

$$\textbf{False proof}\quad A\boldsymbol{x} = \lambda\boldsymbol{x} \quad \text{gives} \quad \boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = \lambda\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \quad \text{so} \quad \lambda = \frac{\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}} = \frac{\text{real}}{\text{real}}.$$

Find the flaw in this reasoning—a hidden assumption that is not justified. You could test those steps on the 90° rotation matrix $[\,0 \;\; -1;\;\; 1 \;\; 0\,]$ with $\lambda = i$ and $\boldsymbol{x} = (i, 1)$.

13 Write S and B in the form $\lambda_1\boldsymbol{x}_1\boldsymbol{x}_1^{\mathrm{T}} + \lambda_2\boldsymbol{x}_2\boldsymbol{x}_2^{\mathrm{T}}$ of the spectral theorem $Q\Lambda Q^T$:

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \qquad (\text{keep } \|\boldsymbol{x}_1\| = \|\boldsymbol{x}_2\| = 1).$$

14 Every 2 by 2 symmetric matrix is $\lambda_1\boldsymbol{x}_1\boldsymbol{x}_1^{\mathrm{T}} + \lambda_2\boldsymbol{x}_2\boldsymbol{x}_2^{\mathrm{T}} = \lambda_1 P_1 + \lambda_2 P_2$. Explain $P_1 + P_2 = \boldsymbol{x}_1\boldsymbol{x}_1^{\mathrm{T}} + \boldsymbol{x}_2\boldsymbol{x}_2^{\mathrm{T}} = I$ from columns times rows of Q. Why is $P_1P_2 = 0$?

15 What are the eigenvalues of $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$? Create a 4 by 4 antisymmetric matrix ($A^{\mathrm{T}} = -A$) and verify that all its eigenvalues are imaginary.

16 (Recommended) This matrix M is antisymmetric and also ____. Then all its eigenvalues are pure imaginary and they also have $|\lambda| = 1$. ($\|M\boldsymbol{x}\| = \|\boldsymbol{x}\|$ for every $\boldsymbol{x}$ so $\|\lambda\boldsymbol{x}\| = \|\boldsymbol{x}\|$ for eigenvectors.) Find all four eigenvalues from the trace of M:

$$M = \frac{1}{\sqrt{3}}\begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \quad \text{can only have eigenvalues } i \text{ or } -i.$$

17 Show that this A (**symmetric but complex**) has only one line of eigenvectors:

$$A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \text{ is not even diagonalizable: eigenvalues } \lambda = 0, 0.$$

$A^{\mathrm{T}} = A$ is not such a special property for complex matrices. The good property is $\overline{A}^{\mathrm{T}} = A$ (Section 9.2). Then all λ's are real and the eigenvectors are orthogonal.

18 Even if A is rectangular, the block matrix $S = \begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}$ is symmetric:

$$S\boldsymbol{x} = \lambda\boldsymbol{x} \quad \text{is} \quad \begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} = \lambda\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} \quad \text{which is} \quad \begin{matrix} A\boldsymbol{z} = \lambda\boldsymbol{y} \\ A^{\mathrm{T}}\boldsymbol{y} = \lambda\boldsymbol{z}. \end{matrix}$$

(a) Show that $-\lambda$ is also an eigenvalue, with the eigenvector $(\boldsymbol{y}, -\boldsymbol{z})$.

(b) Show that $A^{\mathrm{T}}A\boldsymbol{z} = \lambda^2\boldsymbol{z}$, so that λ^2 is an eigenvalue of $A^{\mathrm{T}}A$.

(c) If $A = I$ (2 by 2) find all four eigenvalues and eigenvectors of S.

19 If $A = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ in Problem 18, find all three eigenvalues and eigenvectors of S.

20 ***Another proof that eigenvectors are perpendicular when $S = S^{\mathrm{T}}$.*** Two steps:

1. Suppose $S\boldsymbol{x} = \lambda\boldsymbol{x}$ and $S\boldsymbol{y} = 0\boldsymbol{y}$ and $\lambda \neq 0$. Then $\boldsymbol{y}$ is in the nullspace and $\boldsymbol{x}$ is in the column space. They are perpendicular because _____. Go carefully—why are these subspaces orthogonal?
2. If $S\boldsymbol{y} = \beta\boldsymbol{y}$, apply that argument to $S - \beta I$. One eigenvalue of $S - \beta I$ moves to zero. The eigenvectors $\boldsymbol{x}, \boldsymbol{y}$ stay the same—so they are perpendicular.

21 Find the eigenvector matrices Q for S and X for B. Show that X doesn't collapse at $d = 1$, even though $\lambda = 1$ is repeated. Are those eigenvectors perpendicular?

$$S = \begin{bmatrix} 0 & d & 0 \\ d & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -d & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{bmatrix} \qquad \text{have} \quad \lambda = 1, d, -d.$$

22 Write a 2 by 2 *complex* matrix with $\overline{S}^{\mathrm{T}} = S$ (a "Hermitian matrix"). Find λ_1 and λ_2 for your complex matrix. Check that $\overline{\boldsymbol{x}}_1^{\mathrm{T}}\boldsymbol{x}_2 = 0$ (this is complex orthogonality).

23 ***True*** (with reason) ***or false*** (with example).

(a) A matrix with real eigenvalues and n real eigenvectors is symmetric.

(b) A matrix with real eigenvalues and n orthonormal eigenvectors is symmetric.

(c) The inverse of an invertible symmetric matrix is symmetric.

(d) The eigenvector matrix Q of a symmetric matrix is symmetric.

24 (A paradox for instructors) If $AA^{\mathrm{T}} = A^{\mathrm{T}}A$ then A and A^{T} share the same eigenvectors (true). A and A^{T} always share the same eigenvalues. Find the flaw in this conclusion: A and A^{T} must have the same X and same Λ. Therefore A equals A^{T}.

25 (Recommended) Which of these classes of matrices do A and B belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Which of these factorizations are possible for A and B: $\boldsymbol{LU}$, $\boldsymbol{QR}$, $\boldsymbol{X\Lambda X^{-1}}$, $\boldsymbol{Q\Lambda Q^{\mathrm{T}}}$?

26 What number b in $A = \left[\begin{smallmatrix}2 & b\\1 & 0\end{smallmatrix}\right]$ makes $A = Q\Lambda Q^{\mathrm{T}}$ possible? What number will make it impossible to diagonalize A? What number makes A singular?

27 Find all 2 by 2 matrices that are orthogonal and also symmetric. Which two numbers can be eigenvalues of those two matrices?

28 This A is nearly symmetric. But its eigenvectors are far from orthogonal:

$$A = \begin{bmatrix} 1 & 10^{-15} \\ 0 & 1+10^{-15} \end{bmatrix} \quad \text{has eigenvectors} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [?]$$

What is the angle between the eigenvectors?

29 (MATLAB) Take two symmetric matrices with different eigenvectors, say $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 8 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Graph the eigenvalues $\lambda_1(A+tB)$ and $\lambda_2(A+tB)$ for $-8 < t < 8$. Peter Lax says on page 113 of *Linear Algebra* that λ_1 and λ_2 appear to be on a collision course at certain values of t. "Yet at the last minute they turn aside." How close does λ_1 come to λ_2 ?

Challenge Problems

30 *For complex matrices*, the symmetry $S^{\mathrm{T}} = S$ that produces real eigenvalues must change in Section 9.2 to $\overline{\boldsymbol{S}}^{\mathbf{T}} = \boldsymbol{S}$. From $\det(S - \lambda I) = 0$, find the eigenvalues of the 2 by 2 **Hermitian matrix** $\boldsymbol{S} = [4 \quad 2+i; \quad 2-i \quad 0] = \overline{\boldsymbol{S}}^{\mathbf{T}}$.

31 **Normal matrices** have $\overline{N}^{\mathrm{T}} N = N\overline{N}^{\mathrm{T}}$. For real matrices, this is $N^{\mathrm{T}}N = NN^{\mathrm{T}}$. Normal includes symmetric, skew-symmetric, and orthogonal (with real λ, imaginary λ, and $|\lambda| = 1$). Other normal matrices can have any complex eigenvalues.

Key point: ***Normal matrices have*** n ***orthonormal eigenvectors***. Those vectors $\boldsymbol{x}_i$ probably will have complex components. In that complex case (Chapter 9) orthogonality means $\overline{\boldsymbol{x}}_i^{\mathrm{T}} \boldsymbol{x}_j = 0$. Inner products (dot products) $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$ become $\overline{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{y}$.

The test for n ***orthonormal columns in*** Q ***becomes*** $\overline{Q}^{\mathrm{T}}Q = I$ ***instead of*** $Q^{\mathrm{T}}Q = I$.

N has n **orthonormal eigenvectors** $(N = Q\Lambda\overline{Q}^{\mathrm{T}})$ if and only if N is **normal**.

(a) Start from $N = Q\Lambda\overline{Q}^{\mathrm{T}}$ with $\overline{Q}^{\mathrm{T}}Q = I$. Show that $\overline{N}^{\mathrm{T}}N = N\overline{N}^{\mathrm{T}}$: N is normal.

(b) Now start from $\overline{N}^{\mathrm{T}}N = N\overline{N}^{\mathrm{T}}$. Schur found $A = QT\overline{Q}^{\mathrm{T}}$ for every matrix A, with a *triangular* T. For normal matrices $A = N$ we must show (in 3 steps) that this triangular matrix T will actually be diagonal. Then $T = \Lambda$.

Step 1. Put $N = QT\overline{Q}^{\mathrm{T}}$ into $\overline{N}^{\mathrm{T}}N = N\overline{N}^{\mathrm{T}}$ to find $\overline{T}^{\mathrm{T}}T = T\overline{T}^{\mathrm{T}}$.

Step 2. Suppose $T = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ has $\overline{T}^{\mathrm{T}}T = T\overline{T}^{\mathrm{T}}$. Prove that $b = 0$.

Step 3. Extend Step 2 to size n. *Any normal triangular* T *must be diagonal.*

32 If $\lambda_{\max}$ is the largest eigenvalue of a symmetric matrix S, no diagonal entry can be larger than $\lambda_{\max}$. What is the first entry a_{11} of $S = Q\Lambda Q^{\mathrm{T}}$? Show why $a_{11} \le \lambda_{\max}$.

33 Suppose $A^{\mathrm{T}} = -A$ (real *antisymmetric* matrix). Explain these facts about A:

(a) $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 0$ for every real vector $\boldsymbol{x}$.

(b) The eigenvalues of A are pure imaginary.

(c) The determinant of A is positive or zero (not negative).

For (a), multiply out an example of $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$ and watch terms cancel. Or reverse $\boldsymbol{x}^{\mathrm{T}}(A\boldsymbol{x})$ to $-(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{x}$. For (b), $A\boldsymbol{z} = \lambda\boldsymbol{z}$ leads to $\overline{\boldsymbol{z}}^{\mathrm{T}}A\boldsymbol{z} = \lambda\overline{\boldsymbol{z}}^{\mathrm{T}}\boldsymbol{z} = \lambda\|\boldsymbol{z}\|^2$. Part (a) shows that $\overline{\boldsymbol{z}}^{\mathrm{T}}A\boldsymbol{z} = (\boldsymbol{x} - i\boldsymbol{y})^{\mathrm{T}}A(\boldsymbol{x} + i\boldsymbol{y})$ has zero real part. Then (b) helps with (c).

34 If S is symmetric and all its eigenvalues are $\lambda = 2$, how do you know that S must be $2I$? Key point: Symmetry guarantees that $S = Q\Lambda Q^{\mathrm{T}}$. What is that Λ?

35 *Which symmetric matrices S are also orthogonal*? Then $S^{\mathrm{T}} = S^{-1}$.

(a) Show how symmetry and orthogonality lead to $S^2 = I$.

(b) What are the possible eigenvalues of this S?

(c) What are the possible eigenvalue matrices Λ? Then S must be $Q\Lambda Q^{\mathrm{T}}$ for those Λ and any orthogonal Q.

36 If S is symmetric, show that $A^{\mathrm{T}}SA$ is also symmetric (take the transpose of $A^{\mathrm{T}}SA$). Here A is m by n and S is m by m. Are eigenvalues of S = eigenvalues of $A^{\mathrm{T}}SA$?

In case A is square and invertible, $A^{\mathrm{T}}SA$ is called *congruent* to S. They have the same number of positive, negative, and zero eigenvalues: ***Law of Inertia***.

37 Here is a way to show that a is *in between* the eigenvalues λ_1 and λ_2 of S:

$$S = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \qquad \begin{array}{l} \det(S - \lambda I) = \lambda^2 - a\lambda - c\lambda + ac - b^2 \\ \text{is a parabola opening upwards (because of } \lambda^2) \end{array}$$

Show that $\det(S - \lambda I)$ is negative at $\lambda = a$. So the parabola crosses the axis left and right of $\lambda = a$. It crosses at the two eigenvalues of S so they must enclose a.

The $n-1$ eigenvalues of A always fall between the n eigenvalues of $S = \begin{bmatrix} A & \boldsymbol{b} \\ \boldsymbol{b}^{\mathrm{T}} & c \end{bmatrix}$.

6.5 Positive Definite Matrices

1 **Symmetric** S : all eigenvalues $> \mathbf{0} \Leftrightarrow$ all pivots $> \mathbf{0} \Leftrightarrow$ all upper left determinants $> \mathbf{0}$.

2 The matrix S is then **positive definite**. The energy test is $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} > 0$ for all vectors $\boldsymbol{x} \neq \mathbf{0}$.

3 One more test for positive definiteness : $S = A^{\mathrm{T}} A$ with independent columns in A.

4 **Positive semidefinite** S allows $\lambda = 0$, pivot $= 0$, determinant $= 0$, energy $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} = 0$.

5 The equation $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} = 1$ gives an ellipse in $\mathbf{R}^n$ when S is symmetric positive definite.

This section concentrates on ***symmetric matrices that have positive eigenvalues***. If symmetry makes a matrix important, this extra property (*all* $\lambda > 0$) makes it truly special. When we say special, we don't mean rare. Symmetric matrices with positive eigenvalues are at the center of all kinds of applications. They are called ***positive definite***.

The first problem is to recognize positive definite matrices. You may say, just find all the eigenvalues and test $\lambda > 0$. That is exactly what we want to avoid. Calculating eigenvalues is work. When the λ's are needed, we can compute them. But if we just want to know that all the λ's are positive, there are faster ways. Here are two goals of this section:

- To find *quick tests* on a symmetric matrix that guarantee *positive eigenvalues*.
- To explain important applications of positive definiteness.

Every eigenvalue is real because the matrix is symmetric.

Start with $\mathbf{2}$ ***by*** $\mathbf{2}$. ***When does*** $S = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ ***have*** $\boldsymbol{\lambda_1} > \mathbf{0}$ ***and*** $\boldsymbol{\lambda_2} > \mathbf{0}$?

Test: The eigenvalues of S are positive if and only if $\boldsymbol{a} > \mathbf{0}$ ***and*** $\boldsymbol{ac} - \boldsymbol{b}^{\mathbf{2}} > \mathbf{0}$.

$$S_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ is } \textbf{not} \text{ positive definite because } ac - b^2 = 1 - 4 < 0$$

$$S_2 = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix} \text{ is positive definite because } a = 1 \text{ and } ac - b^2 = 6 - 4 > 0$$

$$S_3 = \begin{bmatrix} -1 & 2 \\ 2 & -6 \end{bmatrix} \text{ is } \textbf{not} \text{ positive definite (even with } \det A = +2) \text{ because } a = -1$$

The eigenvalues 3 and -1 of S_1 confirm that S_1 is *not* positive definite. Positive trace $3 - 1 = 2$, but negative determinant $(3)(-1) = -3$. And $S_3 = -S_2$ is *negative* definite. Two positive eigenvalues for S_2, two negative eigenvalues for S_3.

Proof that the 2 *by* 2 *test is passed when* $\lambda_1 > 0$ *and* $\lambda_2 > 0$. Their product $\lambda_1 \lambda_2$ is the determinant so $ac - b^2 > 0$. Their sum $\lambda_1 + \lambda_2$ is the trace so $a + c > 0$. Then a and c are both positive (if a or c is not positive, $ac - b^2 > 0$ will fail). Problem 1 reverses the reasoning to show that the tests $a > 0$ and $ac > b^2$ guarantee $\lambda_1 > 0$ and $\lambda_2 > 0$.

This test uses the 1 by 1 determinant a and the 2 by 2 determinant $ac - b^2$. When S is 3 by 3, det $S > 0$ is the third part of the test. The next test requires *positive pivots*.

Test: The eigenvalues of S are positive if and only if the pivots are positive:

$$a > 0 \qquad \text{and} \qquad \frac{ac - b^2}{a} > 0.$$

$a > 0$ is required in both tests. So $ac > b^2$ is also required, for the determinant test and now the pivot test. The point is to recognize that ratio as the *second pivot* of S:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \xrightarrow[\text{The multiplier is } b/a]{\text{The first pivot is } a} \begin{bmatrix} a & b \\ 0 & c - \frac{b}{a}b \end{bmatrix} \qquad \textbf{\textit{The second pivot is}} \quad \boldsymbol{c - \frac{b^2}{a} = \frac{ac - b^2}{a}}$$

This connects two big parts of linear algebra. ***Positive eigenvalues mean positive pivots and vice versa***. Each pivot is a ratio of upper left determinants. The pivots give a quick test for $\lambda > 0$, and they are a lot faster to compute than the eigenvalues. It is very satisfying to see pivots and determinants and eigenvalues come together in this course.

3 by 3 example $\quad S = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ is positive definite $\quad$ **eigenvalues** $1, 1, 4$ **determinants** 2 and 3 and 4 **pivots** 2 and $3/2$ and $4/3$

$S - I$ will be *semidefinite*: eigenvalues $0, 0, 3$. $\quad S - 2I$ is *indefinite* because $\lambda = -1, -1, 2$.

Now comes a different way to look at symmetric matrices with positive eigenvalues.

Energy-based Definition

From $S\boldsymbol{x} = \lambda\boldsymbol{x}$, multiply by $\boldsymbol{x}^{\mathrm{T}}$ to get $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} = \lambda\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$. The right side is a positive λ times a positive number $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = \|\boldsymbol{x}\|^2$. So the left side $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ is positive for any eigenvector.

Important point: The new idea is that ***$\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ is positive for all nonzero vectors $\boldsymbol{x}$***, not just the eigenvectors. In many applications this number $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ (or $\frac{1}{2}\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$) is the **energy** in the system. The requirement of positive energy gives *another definition* of a positive definite matrix. I think this energy-based definition is the fundamental one.

Eigenvalues and pivots are two equivalent ways to test the new requirement $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} > 0$.

Definition *S is positive definite if $x^{\mathrm{T}}Sx > 0$ for every nonzero vector x:*

$$\textbf{2 by 2} \qquad \boxed{x^{\mathrm{T}}Sx} = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \boxed{ax^2 + 2bxy + cy^2 > 0.} \qquad (1)$$

The four entries a, b, b, c give the four parts of $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$. From a and c come the pure squares ax^2 and cy^2. From b and b off the diagonal come the cross terms bxy and byx (the same). Adding those four parts gives $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$. This energy-based definition leads to a basic fact:

If S and T are symmetric positive definite, so is $S + T$.

Reason: $\boldsymbol{x}^{\mathrm{T}}(S+T)\boldsymbol{x}$ is simply $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}+\boldsymbol{x}^{\mathrm{T}}T\boldsymbol{x}$. Those two terms are positive (for $\boldsymbol{x} \neq \mathbf{0}$) so $S + T$ is also positive definite. The pivots and eigenvalues are not easy to follow when matrices are added, but the energies just add.

$\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ also connects with our final way to recognize a positive definite matrix. Start with any matrix A, possibly rectangular. We know that $S = A^{\mathrm{T}}A$ is square and symmetric. More than that, S will be positive definite when A has independent columns:

Test: If the columns of A are independent, then $S = A^{\mathrm{T}}A$ is positive definite.

Again eigenvalues and pivots are not easy. But the number $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ is the same as $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{x}$. $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{x}$ is exactly $(A\boldsymbol{x})^{\mathrm{T}}(A\boldsymbol{x}) = \|A\boldsymbol{x}\|^2$—another important proof by parenthesis! That vector $A\boldsymbol{x}$ is not zero when $\boldsymbol{x} \neq \mathbf{0}$ (this is the meaning of independent columns). Then $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ is the positive number $\|A\boldsymbol{x}\|^2$ and the matrix S is positive definite.

Let me collect this theory together, into *five equivalent statements* of positive definiteness. You will see how that key idea connects the whole subject of linear algebra: pivots, determinants, eigenvalues, and least squares (from $A^{\mathrm{T}}A$). Then come the applications.

When a symmetric matrix S has one of these five properties, it has them all:

1. All ***n pivots*** of S are positive.
2. All ***n upper left determinants*** are positive.
3. All ***n eigenvalues*** of S are positive.
4. $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ is positive except at $\boldsymbol{x} = \mathbf{0}$. This is the ***energy-based*** definition.
5. S equals $A^{\mathrm{T}}A$ for a matrix A with ***independent columns***.

The "upper left determinants" are 1 by 1, 2 by 2, ..., n by n. The last one is the determinant of the complete matrix S. This theorem ties together the whole linear algebra course.

Example 1 Test these symmetric matrices S and T for positive definiteness:

$$S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}.$$

Solution The pivots of S are 2 and $\frac{3}{2}$ and $\frac{4}{3}$, all positive. Its upper left determinants are 2 and 3 and 4, all positive. The eigenvalues of S are $2-\sqrt{2}$ and 2 and $2+\sqrt{2}$, all positive. That completes tests **1**, **2**, and **3**. Any one test is decisive!

I have three candidates A_1, A_2, A_3 to suggest for $S = A^{\mathrm{T}}A$. They all show that S is positive definite. A_1 is a first difference matrix, 4 by 3, to produce $-1, 2, -1$ in S:

$$\boldsymbol{S = A_1^{\mathrm{T}} A_1} \qquad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

The three columns of A_1 are independent. Therefore S is positive definite.

A_2 comes from $S = LDL^{\mathrm{T}}$ (the symmetric version of $S = LU$). Elimination gives the pivots $2, \frac{3}{2}, \frac{4}{3}$ in D and the multipliers $-\frac{1}{2}, 0, -\frac{2}{3}$ in L. **Just put $\boldsymbol{A_2 = L\sqrt{D}}$.**

$$LDL^{\mathrm{T}} = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} = (L\sqrt{D})(L\sqrt{D})^{\mathrm{T}} = A_2^{\mathrm{T}} A_2.$$

A_2 is the Cholesky factor of S

This triangular choice of A has square roots (not so beautiful). It is the "Cholesky factor" of S and the MATLAB command is $A = \mathrm{chol}(S)$. In applications, the rectangular A_1 is how we build S and this Cholesky A_2 is how we break it apart.

Eigenvalues give the symmetric choice $\boldsymbol{A_3 = Q\sqrt{\Lambda}Q^{\mathrm{T}}}$. This is also successful with $A_3^{\mathrm{T}} A_3 = Q\Lambda Q^{\mathrm{T}} = S$. All tests show that the $-1, 2, -1$ matrix S is positive definite.

To see that the energy $\boldsymbol{x}^{\mathrm{T}} S\boldsymbol{x}$ is positive, we can write it as a sum of squares. The three choices A_1, A_2, A_3 give three different ways to split up $\boldsymbol{x}^{\mathrm{T}} S\boldsymbol{x}$:

$$\boldsymbol{x}^{\mathrm{T}} S\boldsymbol{x} = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2 \qquad \textbf{Rewrite with squares}$$

$$||A_1\boldsymbol{x}||^2 = x_1^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + x_3^2 \qquad \textbf{Using differences in } \boldsymbol{A_1}$$

$$||A_2\boldsymbol{x}||^2 = 2\left(x_1 - \tfrac{1}{2}x_2\right)^2 + \tfrac{3}{2}\left(x_2 - \tfrac{2}{3}x_3\right)^2 + \tfrac{4}{3}\,x_3^2 \qquad \textbf{Using } \boldsymbol{S = LDL^{\mathrm{T}}}$$

$$||A_3\boldsymbol{x}||^2 = \lambda_1(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{x})^2 + \lambda_2(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{x})^2 + \lambda_3(\boldsymbol{q}_3^{\mathrm{T}}\boldsymbol{x})^2 \qquad \textbf{Using } \boldsymbol{S = Q\Lambda Q^{\mathrm{T}}}$$

Now turn to T (top of this page). The $(1,3)$ and $(3,1)$ entries move away from 0 to b. This b must not be too large! *The determinant test is easiest.* The 1 by 1 determinant is 2, the 2 by 2 determinant T is still 3. The 3 by 3 determinant involves b:

$$\textbf{Test on } \boldsymbol{T} \qquad \det T = 4 + 2b - 2b^2 = (1+b)(4-2b) \quad \text{must be positive.}$$

At $b = -1$ and $b = 2$ we get $\det T = 0$. *Between $b = -1$ and $b = 2$ this matrix T is positive definite.* The corner entry $b = 0$ in the matrix S was safely between -1 and 2.

Positive Semidefinite Matrices

Often we are at the edge of positive definiteness. The determinant is zero. The smallest eigenvalue is zero. The energy in its eigenvector is $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}0\boldsymbol{x} = 0$. These matrices on the edge are called *positive semidefinite*. Here are two examples (not invertible):

$$S = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \text{ and } T = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \text{ are positive semidefinite.}$$

S has eigenvalues 5 and 0. Its upper left determinants are 1 and 0. Its rank is only 1. This matrix S factors into $A^{\mathrm{T}}A$ with **dependent columns** in A:

$$\begin{array}{l} \textbf{Dependent columns in } \boldsymbol{A} \\ \textbf{Positive semidefinite } \boldsymbol{S} \end{array} \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = A^{\mathrm{T}}A.$$

If 4 is increased by any small number, the matrix S will become positive definite.

The cyclic T also has zero determinant (computed above when $b = -1$). T is singular. The eigenvector $\boldsymbol{x} = (1, 1, 1)$ has $T\boldsymbol{x} = \mathbf{0}$ and energy $\boldsymbol{x}^{\mathrm{T}}T\boldsymbol{x} = 0$. Vectors $\boldsymbol{x}$ in all other directions do give positive energy. This T can be written as $A^{\mathrm{T}}A$ in many ways, but A will always have *dependent* columns, with $(1, 1, 1)$ in its nullspace:

$$\begin{array}{l} \textbf{Second differences } \boldsymbol{T} \\ \textbf{from first differences } \boldsymbol{A} \\ \textbf{Cyclic } \boldsymbol{T} \textbf{ from cyclic } \boldsymbol{A} \end{array} \quad \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Positive semidefinite matrices have all $\lambda \geq 0$ and all $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} \geq 0$. Those weak inequalities ($\geq$ **instead of** $>$) include positive definite S and also the singular matrices at the edge.

The Ellipse $ax^2 + 2bxy + cy^2 = 1$

Think of a tilted ellipse $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} = 1$. Its center is $(0, 0)$, as in Figure 6.7a. Turn it to line up with the coordinate axes (X and Y axes). That is Figure 6.7b. These two pictures show the geometry behind the factorization $S = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$:

1. The tilted ellipse is associated with S. Its equation is $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} = 1$.

2. The lined-up ellipse is associated with Λ. Its equation is $\boldsymbol{X}^{\mathrm{T}}\Lambda\boldsymbol{X} = 1$.

3. The rotation matrix that lines up the ellipse is the eigenvector matrix Q.

Example 2 Find the axes of this tilted ellipse $5x^2 + 8xy + 5y^2 = 1$.

Solution Start with the positive definite matrix that matches this equation:

$$\text{The equation is} \quad \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1. \qquad \text{The matrix is} \quad S = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}.$$

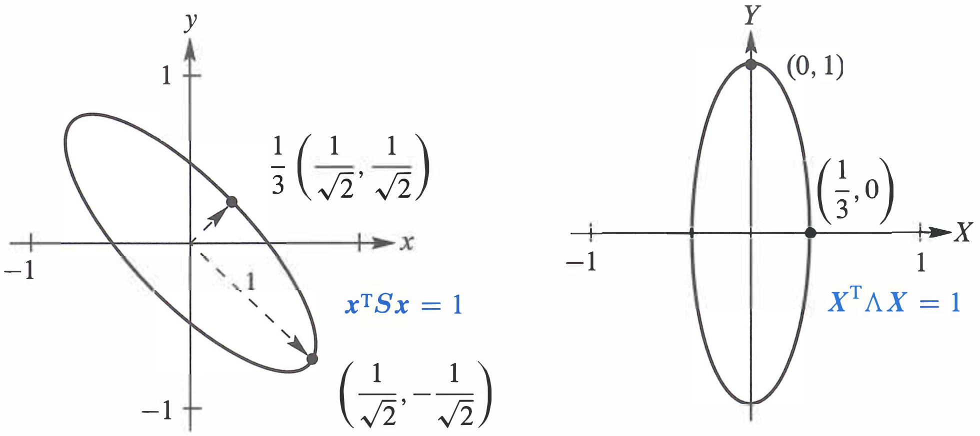


Figure 6.7: The tilted ellipse $5x^2 + 8xy + 5y^2 = 1$. Lined up it is $9X^2 + Y^2 = 1$.

The eigenvectors are $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]$. Divide by $\sqrt{2}$ for unit vectors. Then $S = Q\Lambda Q^T$:

Eigenvectors in Q
Eigenvalues 9 and 1

$$\begin{bmatrix}5 & 4\\4 & 5\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}\mathbf{9} & 0\\0 & \mathbf{1}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}.$$

Now multiply by $\begin{bmatrix}x & y\end{bmatrix}$ on the left and $\begin{bmatrix}x\\y\end{bmatrix}$ on the right to get $\boldsymbol{x}^T S\boldsymbol{x} = (\boldsymbol{x}^T Q)\Lambda(Q^T\boldsymbol{x})$:

$$\boldsymbol{x}^T S\boldsymbol{x} = \textbf{sum of squares} \quad 5x^2 + 8xy + 5y^2 = 9\left(\frac{x+y}{\sqrt{2}}\right)^2 + 1\left(\frac{x-y}{\sqrt{2}}\right)^2. \tag{2}$$

The coefficients are not the pivots 5 and 9/5 from D, they are the eigenvalues 9 and 1 from Λ. Inside the squares are the eigenvectors $\boldsymbol{q}_1 = (1,1)/\sqrt{2}$ and $\boldsymbol{q}_2 = (1,-1)/\sqrt{2}$.

The axes of the tilted ellipse point along those eigenvectors. This explains why $S = Q\Lambda Q^T$ is called the "principal axis theorem"—it displays the axes. Not only the axis directions (from the eigenvectors) but also the axis lengths (from the eigenvalues). To see it all, use capital letters for the new coordinates that line up the ellipse:

$$\textbf{Lined up} \qquad \frac{x+y}{\sqrt{2}} = X \quad \text{and} \quad \frac{x-y}{\sqrt{2}} = Y \quad \text{and} \quad \mathbf{9X^2 + Y^2 = 1}.$$

The largest value of X^2 is $1/9$. The endpoint of the shorter axis has $X = 1/3$ and $Y = 0$. Notice: The *bigger* eigenvalue λ_1 gives the *shorter* axis, of half-length $1/\sqrt{\lambda_1} = 1/3$. The smaller eigenvalue $\lambda_2 = 1$ gives the greater length $1/\sqrt{\lambda_2} = 1$.

In the xy system, the axes are along the eigenvectors of S. In the XY system, the **axes are along the eigenvectors of Λ**—the coordinate axes. All comes from $S = Q\Lambda Q^T$.

$\boldsymbol{S = Q\Lambda Q^{\mathrm{T}}}$ is positive definite when all $\lambda_i > 0$. The graph of $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} = 1$ is an ellipse:

$$\textbf{Ellipse} \quad \begin{bmatrix} x & y \end{bmatrix} Q\Lambda Q^{\mathrm{T}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X & Y \end{bmatrix} \Lambda \begin{bmatrix} X \\ Y \end{bmatrix} = \boldsymbol{\lambda_1 X^2 + \lambda_2 Y^2 = 1}. \tag{3}$$

The axes point along eigenvectors of S. The half-lengths are $1/\sqrt{\lambda_1}$ and $1/\sqrt{\lambda_2}$.

$S = I$ gives the circle $x^2 + y^2 = 1$. If one eigenvalue is negative (exchange 4's and 5's in S), the ellipse changes to a *hyperbola*. The sum of squares becomes a *difference of squares*: $9X^2 - Y^2 = 1$. For a negative definite matrix like $S = -I$, with both λ's negative, the graph of $-x^2 - y^2 = 1$ has no points at all.

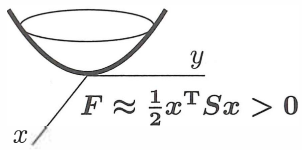
If S is n by n, $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} = 1$ is an "ellipsoid" in $\mathbf{R}^n$. Its axes are the eigenvectors of S.

Important Application: Test for a Minimum

Does $F(x, y)$ have a minimum if $\partial F/\partial x = 0$ and $\partial F/\partial y = 0$ at the point $(x, y) = (0, 0)$?

For $f(x)$, the test for a minimum comes from calculus: df/dx is zero and $d^2f/dx^2 > 0$. Two variables in $F(x, y)$ produce a symmetric matrix S. It contains *four second derivatives*. **Positive $\boldsymbol{d^2 f/dx^2}$ changes to positive definite $\boldsymbol{S}$:**

Second derivatives $\quad S = \begin{bmatrix} \partial^2 F/\partial x^2 & \partial^2 F/\partial x \partial y \\ \partial^2 F/\partial y \partial x & \partial^2 F/\partial y^2 \end{bmatrix}$



$\boldsymbol{F(x, y)}$ has a minimum if $\boldsymbol{\partial F/\partial x = \partial F/\partial y = 0}$ and $\boldsymbol{S}$ is positive definite.

Reason: S reveals the all-important terms $ax^2 + 2bxy + cy^2$ near $(x, y) = (0, 0)$. The second derivatives of F are $2a, 2b, 2b, 2c$. For $F(x, y, z)$ the matrix S will be 3 by 3.

■ REVIEW OF THE KEY IDEAS ■

1. Positive definite matrices have positive eigenvalues and positive pivots.
2. A quick test is given by the upper left determinants: $a > 0$ and $ac - b^2 > 0$.
3. The graph of the energy $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x}$ is then a "bowl" going up from $\boldsymbol{x} = \boldsymbol{0}$:

   $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} = ax^2 + 2bxy + cy^2$ is positive except at $(x, y) = (0, 0)$.
4. $S = A^{\mathrm{T}} A$ is automatically positive definite if A has independent columns.
5. The ellipsoid $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} = 1$ has its axes along the eigenvectors of S. Lengths $1/\sqrt{\lambda}$.
6. Minimum of $F(x, y)$ if $\dfrac{\partial F}{\partial x} = \dfrac{\partial F}{\partial y} = 0$ and 2nd derivative matrix is positive definite.

■ WORKED EXAMPLES ■

6.5 A The great factorizations of a symmetric matrix are $S = LDL^{\mathrm{T}}$ from pivots and multipliers, and $S = Q\Lambda Q^{\mathrm{T}}$ from eigenvalues and eigenvectors. Try these n by n tests on pascal(6) and ones(6) and hilb(6) and other matrices in MATLAB's gallery.

pascal(6) is positive *definite* because all its pivots are 1 (Worked Example **2.6 A**).

ones(6) is positive *semidefinite* because its eigenvalues are $0, 0, 0, 0, 0, 6$.

H=hilb(6) is positive *definite* even though eig(H) shows eigenvalues very near zero.

Hilbert matrix $\boldsymbol{x}^{\mathrm{T}}H\boldsymbol{x} = \int_0^1 (x_1 + x_2 s + \cdots + x_6 s^5)^2\, ds > 0$, $H_{ij} = 1/(i + j - 1)$.

rand(6) + rand(6)′ can be positive definite or not. *Experiments gave only* 2 *in* 20000.

$n = 20000$; $p = 0$; for $k = 1\!:\!n$, $A =$ rand(6); $p = p +$ all(eig$(A + A') > 0$); end, $p \,/\, n$

6.5 B **When is the symmetric block matrix** $M = \begin{bmatrix} A & B \\ B^{\mathrm{T}} & C \end{bmatrix}$ **positive definite?**

Solution Multiply the first row of M by $B^{\mathrm{T}}A^{-1}$ and subtract from the second row, to get a block of zeros. The *Schur complement* $S = C - B^{\mathrm{T}}A^{-1}B$ appears in the corner:

$$\begin{bmatrix} I & 0 \\ -B^{\mathrm{T}}A^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ B^{\mathrm{T}} & C \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & C - B^{\mathrm{T}}A^{-1}B \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & B \\ 0 & \boldsymbol{S} \end{bmatrix} \tag{4}$$

Those two blocks A and S must be positive definite. Their pivots are the pivots of M.

6.5 C Find the eigenvalues of the $-1, 2, -1$ tridiagonal n by n matrix S (my favorite).

Solution The best way is to guess λ and $\boldsymbol{x}$. Then check $S\boldsymbol{x} = \lambda\boldsymbol{x}$. Guessing could not work for most matrices, but special cases are a big part of mathematics (pure and applied).

The key is hidden in a differential equation. The second difference matrix S is like a *second derivative*, and those eigenvalues are much easier to see:

Eigenvalues $\boldsymbol{\lambda_1, \lambda_2, \ldots}$ **Eigenfunctions** $\boldsymbol{y_1, y_2, \ldots}$	$\dfrac{d^2y}{dx^2} = \lambda y(x)$ **with** $\begin{matrix} y(0) = 0 \\ y(1) = 0 \end{matrix}$	(5)

Try $y = \sin cx$. Its second derivative is $y'' = -c^2 \sin cx$. So the eigenvalue in (5) will be $\lambda = -c^2$, provided $y = \sin cx$ satisfies the end point conditions $y(0) = 0 = y(1)$.

Certainly $\sin 0 = 0$ (this is where cosines are eliminated). At the other end $x = 1$, we need $y(1) = \sin c = 0$. The number c must be $k\pi$, a multiple of π. Then λ is $-k^2\pi^2$:

Eigenvalues $\boldsymbol{\lambda = -k^2\pi^2}$

Eigenfunctions $\boldsymbol{y = \sin k\pi x}$

$$\frac{d^2}{dx^2} \sin k\pi x = -k^2\pi^2 \sin k\pi x. \tag{6}$$

Now we go back to the matrix S and guess its eigenvectors. They come from $\sin k\pi x$ at n points $x = h, 2h, \ldots, nh$, equally spaced between 0 and 1. The spacing Δx is $h = 1/(n+1)$, so the $(n+1)$st point has $(n+1)h = 1$. Multiply that sine vector $\boldsymbol{x}$ by S:

Eigenvalue of S is positive $\quad S\boldsymbol{x} = \lambda_k \boldsymbol{x} = (2 - 2\cos k\pi h)\, \boldsymbol{x}$

Eigenvector of S is sine vector $\quad \boldsymbol{x} = (\sin k\pi h, \ldots, \sin nk\pi h)$. (7)

Problem Set 6.5

Problems 1–13 are about tests for positive definiteness.

1 Suppose the 2 by 2 tests $a > 0$ and $ac - b^2 > 0$ are passed. Then $c > b^2/a > 0$.

(i) λ_1 and λ_2 have the *same sign* because their product $\lambda_1\lambda_2$ equals ____ .

(i) That sign is positive because $\lambda_1 + \lambda_2$ equals ____ .

Conclusion: The tests $a > 0, ac - b^2 > 0$ guarantee positive eigenvalues λ_1, λ_2.

2 Which of S_1, S_2, S_3, S_4 has two positive eigenvalues? Use a test, don't compute the λ's. Also find an $\boldsymbol{x}$ so that $\boldsymbol{x}^{\mathrm{T}}S_1\boldsymbol{x} < 0$, so S_1 is not positive definite.

$$S_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \quad S_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix} \quad S_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix} \quad S_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}.$$

3 For which numbers b and c are these matrices positive definite?

$$S = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \qquad S = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} \qquad S = \begin{bmatrix} c & b \\ b & c \end{bmatrix}.$$

With the pivots in D and multiplier in L, factor each A into LDL^{T}.

4 What is the function $f = ax^2 + 2bxy + cy^2$ for each of these matrices? Complete the square to write each f as a sum of one or two squares $f = d_1(\quad)^2 + d_2(\quad)^2$.

$$S_1 = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \qquad f = [x \;\; y] \begin{bmatrix} & S & \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

5 Write $f(x, y) = x^2 + 4xy + 3y^2$ as a *difference* of squares and find a point (x, y) where f is negative. No minimum at $(0, 0)$ even though f has positive coefficients.

6 The function $f(x, y) = 2xy$ certainly has a saddle point and not a minimum at $(0, 0)$. What symmetric matrix S produces this f? What are its eigenvalues?

7 Test to see if $A^{\mathrm{T}}A$ is positive definite in each case: A needs independent columns.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

8 The function $f(x, y) = 3(x + 2y)^2 + 4y^2$ is positive except at $(0, 0)$. What is the matrix in $f = [\,x \;\; y\,]S[\,x \;\; y\,]^{\mathrm{T}}$? Check that the pivots of A are 3 and 4.

9 Find the 3 by 3 matrix S and its pivots, rank, eigenvalues, and determinant:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2.$$

10 Which 3 by 3 symmetric matrices S and T produce these quadratics?

$$\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} = 2\left(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3\right). \quad \text{Why is } S \text{ positive definite?}$$
$$\boldsymbol{x}^{\mathrm{T}} T \boldsymbol{x} = 2\left(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_1 x_3 - x_2 x_3\right). \quad \text{Why is } T \text{ semidefinite?}$$

11 Compute the three upper left determinants of S to establish positive definiteness. Verify that their ratios give the second and third pivots.

$$\textbf{Pivots = ratios of determinants} \qquad S = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}.$$

12 For what numbers c and d are S and T positive definite? Test their 3 determinants:

$$S = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

13 Find a matrix with $a > 0$ and $c > 0$ and $a + c > 2b$ that has a negative eigenvalue.

Problems 14–20 are about applications of the tests.

14 *If S is positive definite then S^{-1} is positive definite.* Best proof: The eigenvalues of S^{-1} are positive because _____. *Second proof* (only for 2 by 2):

$$\text{The entries of } S^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \quad \text{pass the determinant tests} \quad \underline{\qquad}.$$

15 *If S and T are positive definite, their sum $S + T$ is positive definite.* Pivots and eigenvalues are not convenient for $S + T$. Better to use $\boldsymbol{x}^{\mathrm{T}}(S + T)\boldsymbol{x} > 0$. Also $S = A^{\mathrm{T}} A$ and $T = B^{\mathrm{T}} B$ give $S + T = [\,\mathbf{A}\ \mathbf{B}\,]^{\mathrm{T}} \left[\begin{smallmatrix} \mathbf{A} \\ \mathbf{B} \end{smallmatrix}\right]$ with independent columns.

16 A positive definite matrix cannot have a zero (or even worse, a negative number) on its main diagonal. Show that this matrix fails to have $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} > 0$:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is not positive when } (x_1, x_2, x_3) = (\quad, \quad, \quad).$$

17 A diagonal entry s_{jj} of a symmetric matrix cannot be smaller than all the λ's. If it were, then $S - s_{jj} I$ would have _____ eigenvalues and would be positive definite. But $S - s_{jj} I$ has a _____ on the main diagonal.

18 If $S\boldsymbol{x} = \lambda\boldsymbol{x}$ then $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} =$ ____. Why is this number positive when $\lambda > 0$?

19 Reverse Problem 18 to show that *if all* $\lambda > 0$ *then* $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} > 0$. We must do this for **every nonzero** $\boldsymbol{x}$, not just the eigenvectors. So write $\boldsymbol{x}$ as a combination of the eigenvectors and *explain why all "cross terms" are* $\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$. Then $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ is

$$(c_1\boldsymbol{x}_1+\cdots+c_n\boldsymbol{x}_n)^{\mathrm{T}}(c_1\lambda_1\boldsymbol{x}_1+\cdots+c_n\lambda_n\boldsymbol{x}_n) = c_1^2\lambda_1\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{x}_1+\cdots+c_n^2\lambda_n\boldsymbol{x}_n^{\mathrm{T}}\boldsymbol{x}_n > 0.$$

20 Give a quick reason why each of these statements is true:

(a) Every positive definite matrix is invertible.

(b) The only positive definite projection matrix is $P = I$.

(c) A diagonal matrix with positive diagonal entries is positive definite.

(d) A symmetric matrix with a positive determinant might not be positive definite!

Problems 21–24 use the eigenvalues; Problems 25–27 are based on pivots.

21 For which s and t do S and T have all $\lambda > 0$ (therefore positive definite)?

$$S = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{bmatrix}.$$

22 From $S = Q\Lambda Q^{\mathrm{T}}$ compute the positive definite symmetric square root $Q\sqrt{\Lambda}Q^{\mathrm{T}}$ of each matrix. Check that this square root gives $A^{\mathrm{T}}A = S$:

$$S = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}.$$

23 You may have seen the equation for an ellipse as $x^2/a^2 + y^2/b^2 = 1$. What are a and b when the equation is written $\lambda_1 x^2 + \lambda_2 y^2 = 1$? The ellipse $9x^2 + 4y^2 = 1$ has axes with half-lengths $a =$ ____ and $b =$ ____.

24 Draw the tilted ellipse $x^2 + xy + y^2 = 1$ and find the half-lengths of its axes from the eigenvalues of the corresponding matrix S.

25 With positive pivots in D, the factorization $S = LDL^{\mathrm{T}}$ becomes $L\sqrt{D}\sqrt{D}L^{\mathrm{T}}$. (Square roots of the pivots give $D = \sqrt{D}\sqrt{D}$.) Then $C = \sqrt{D}L^{\mathrm{T}}$ yields the ***Cholesky factorization*** $A = C^{\mathrm{T}}C$ which is "symmetrized LU":

$$\text{From} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{find } S. \qquad \text{From} \quad S = \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} \quad \text{find } C = \textbf{chol}(S).$$

26 In the Cholesky factorization $S = C^{\mathrm{T}}C$, with $C = \sqrt{D}L^{\mathrm{T}}$, the square roots of the pivots are on the diagonal of C. Find C (upper triangular) for

$$S = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix}.$$

27 The symmetric factorization $S = LDL^{\mathrm{T}}$ means that $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}LDL^{\mathrm{T}}\boldsymbol{x}$:

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 0 \\ b/a & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & (ac-b^2)/a \end{bmatrix}\begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

The left side is $ax^2 + 2bxy + cy^2$. The right side is $a\left(x + \frac{b}{a}y\right)^2 +$ ____ y^2. The second pivot completes the square! Test with $a = 2$, $b = 4$, $c = 10$.

28 Without multiplying $S = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, find

(a) the determinant of S (b) the eigenvalues of S
(c) the eigenvectors of S (d) a reason why S is symmetric positive definite.

29 For $F_1(x,y) = \frac{1}{4}x^4 + x^2y + y^2$ and $F_2(x,y) = x^3 + xy - x$ find the second derivative matrices S_1 and S_2:

$$\textbf{Test for minimum} \quad S = \begin{bmatrix} \partial^2F/\partial x^2 & \partial^2F/\partial x\partial y \\ \partial^2F/\partial y\partial x & \partial^2F/\partial y^2 \end{bmatrix} \text{ is positive definite}$$

S_1 is positive definite so F_1 is concave up (= convex). Find the minimum point of F_1. *Find the saddle point of* F_2 (look only where first derivatives are zero).

30 The graph of $z = x^2 + y^2$ is a bowl opening upward. *The graph* of $z = x^2 - y^2$ *is a saddle*. The graph of $z = -x^2 - y^2$ is a bowl opening downward. What is a test on a, b, c for $z = ax^2 + 2bxy + cy^2$ to have a saddle point at $(x, y) = (0, 0)$?

31 Which values of c give a bowl and which c give a saddle point for the graph of $z = 4x^2 + 12xy + cy^2$? Describe this graph at the borderline value of c.

The Minimum of a Function $F(x, y, z)$

What tests would you expect for a minimum point? First come zero slopes:

$$\textbf{First derivatives are zero} \quad \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = \mathbf{0} \text{ at the minimum point.}$$

Next comes the linear algebra version of the usual calculus test $d^2f/dx^2 > 0$:

$$\textbf{Second derivative matrix } \boldsymbol{S} \textbf{ is positive definite} \quad S = \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{bmatrix}$$

Here $\boldsymbol{F_{xy}} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial F}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial F}{\partial x}\right) = \boldsymbol{F_{yx}}$ is a 'mixed" second derivative.

Challenge Problems

32 A *group* of nonsingular matrices includes AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these are groups (as in 2.7.37)?

Invent a "subgroup" of two of these groups (not I by itself = the smallest group).

(a) Positive definite symmetric matrices S.

(b) Orthogonal matrices Q.

(c) All exponentials e^{tA} of a fixed matrix A.

(d) Matrices P with positive eigenvalues.

(e) Matrices D with determinant 1.

33 When S and T are symmetric positive definite, ST might not even be symmetric. But its eigenvalues are still positive. Start from $ST\boldsymbol{x} = \lambda\boldsymbol{x}$ and take dot products with $T\boldsymbol{x}$. Then prove $\lambda > 0$.

34 Write down the 5 by 5 sine matrix Q from Worked Example **6.5 C**, containing the eigenvectors of S when $n = 5$ and $h = 1/6$. Multiply SQ to see the five λ's.

The sum of λ's should equal the trace 10. Their product should be $\det S = 6$.

35 Suppose C is positive definite (so $\boldsymbol{y}^{\mathrm{T}}C\boldsymbol{y} > 0$ whenever $\boldsymbol{y} \neq \mathbf{0}$) and A has independent columns (so $A\boldsymbol{x} \neq \mathbf{0}$ whenever $\boldsymbol{x} \neq \mathbf{0}$). Apply the energy test to $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}CA\boldsymbol{x}$ to show that $\boldsymbol{S = A^{\mathrm{T}}CA}$ ***is positive definite: the crucial matrix in engineering***.

36 **Important**! Suppose S is positive definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

(a) What are the eigenvalues of the matrix $\lambda_1 I - S$? Is it positive semidefinite?

(b) How does it follow that $\lambda_1\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \geq \boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ for every $\boldsymbol{x}$?

(c) Draw this conclusion: **The maximum value of $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ is** ____.

37 For which a and c is this matrix positive definite ? For which a and c is it positive semidefinite (this includes definite) ?

$$S = \begin{bmatrix} a & a & a \\ a & a+c & a-c \\ a & a-c & a+c \end{bmatrix}$$

All 5 tests are possible.
The energy $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$ equals
$a\,(x_1 + x_2 + x_3)^2 + c\,(x_2 - x_3)^2$.

Table of Eigenvalues and Eigenvectors

How are the properties of a matrix reflected in its eigenvalues and eigenvectors? This question is fundamental throughout Chapter 6. A table that organizes the key facts may be helpful. Here are the special properties of the eigenvalues λ_i and the eigenvectors $\boldsymbol{x}_i$.

Symmetric: $S^{\mathrm{T}} = S = Q\Lambda Q^{\mathrm{T}}$	real eigenvalues	orthogonal $\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Orthogonal: $Q^{\mathrm{T}} = Q^{-1}$	all $\lvert\lambda\rvert = 1$	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Skew-symmetric: $A^{\mathrm{T}} = -A$	imaginary λ's	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Complex Hermitian: $\overline{S}^{\mathrm{T}} = S$	real λ's	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Positive Definite: $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} > 0$	all $\lambda > 0$	orthogonal since $S^{\mathrm{T}} = S$
Markov: $m_{ij} > 0, \sum_{i=1}^{n} m_{ij} = 1$	$\lambda_{\max} = 1$	steady state $\boldsymbol{x} > 0$
Similar: $A = BCB^{-1}$	$\lambda(A) = \lambda(C)$	B times eigenvector of C
Projection: $P = P^2 = P^{\mathrm{T}}$	$\lambda = 1;\ 0$	column space; nullspace
Plane Rotation : cosine-sine	$e^{i\theta}$ and $e^{-i\theta}$	$\boldsymbol{x} = (1, i)$ and $(1, -i)$
Reflection: $I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$	$\lambda = -1;\ 1, .., 1$	$\boldsymbol{u}$; whole plane $\boldsymbol{u}^{\perp}$
Rank One: $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$	$\lambda = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{u};\ 0, .., 0$	$\boldsymbol{u}$; whole plane $\boldsymbol{v}^{\perp}$
Inverse: A^{-1}	$1/\lambda(A)$	keep eigenvectors of A
Shift: $A + cI$	$\lambda(A) + c$	keep eigenvectors of A
Stable Powers: $A^n \to 0$	all $\lvert\lambda\rvert < 1$	any eigenvectors
Stable Exponential: $e^{At} \to 0$	all Re $\lambda < 0$	any eigenvectors
Cyclic Permutation: $P_{i,i+1} = 1, P_{n1} = 1$	$\lambda_k = e^{2\pi ik/n}$ = roots of 1	$\boldsymbol{x}_k = (1, \lambda_k, \ldots, \lambda_k^{n-1})$
Circulant: $c_0I + c_1P + \cdots$	$\lambda_k = c_0 + c_1\, e^{2\pi ik/n} + \cdots$	$\boldsymbol{x}_k = (1, \lambda_k, \ldots, \lambda_k^{n-1})$
Tridiagonal: $-1, 2, -1$ on diagonals	$\lambda_k = 2 - 2\cos\frac{k\pi}{n+1}$	$\boldsymbol{x}_k = \left(\sin\frac{k\pi}{n+1}, \sin\frac{2k\pi}{n+1}, \ldots\right)$
Diagonalizable: $A = X\Lambda X^{-1}$	diagonal of Λ	columns of X are independent
Schur: $A = QTQ^{-1}$	diagonal of triangular T	columns of Q *if* $A^{\mathrm{T}}A = AA^{\mathrm{T}}$
Jordan: $A = BJB^{-1}$	diagonal of J	each block gives 1 eigenvector
SVD: $A = U\Sigma V^{\mathrm{T}}$	r singular values in Σ	eigenvectors of $A^{\mathrm{T}}A$, AA^{T} in V, U

Chapter 7

The Singular Value Decomposition (SVD)

7.1 Image Processing by Linear Algebra

1 An image is a large matrix of grayscale values, one for each pixel and color.

2 When nearby pixels are correlated (not random) the image can be compressed.

3 The SVD separates any matrix A into rank one pieces $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ = (**column**)(**row**).

4 The columns and rows are eigenvectors of symmetric matrices AA^{T} and $A^{\mathrm{T}}A$.

The singular value theorem for A is the eigenvalue theorem for $A^{\mathrm{T}}A$ and AA^{T}.

That is a quick preview of what you will see in this chapter. A has *two* sets of singular vectors (the eigenvectors of $A^{\mathrm{T}}A$ and AA^{T}). There is *one* set of positive singular values (because $A^{\mathrm{T}}A$ has the same positive eigenvalues as AA^{T}). A is often rectangular, but $A^{\mathrm{T}}A$ and AA^{T} are square, symmetric, and positive semidefinite.

The Singular Value Decomposition (SVD) separates any matrix into simple pieces.

Each piece is a column vector times a row vector. An m by n matrix has m times n entries (a big number when the matrix represents an image). But a column and a row only have **$m + n$ components, far less than m times n.** Those (column)(row) pieces are full size matrices that can be processed with extreme speed—they need only m *plus* n numbers.

Unusually, this image processing application of the SVD is coming before the matrix algebra it depends on. I will start with simple images that only involve one or two pieces. Right now I am thinking of an image as a large rectangular matrix. The entries a_{ij} tell the grayscales of all the pixels in the image. Think of a pixel as a small square, i steps across and j steps up from the lower left corner. Its grayscale is a number (often a whole number in the range $0 \leq a_{ij} < 256 = 2^8$). An all-white pixel has $a_{ij} = 255 = 11111111$. That number has eight 1's when the computer writes 255 in binary notation.

You see how an image that has m times n pixels, with each pixel using 8 bits (0 or 1) for its grayscale, becomes an m by n matrix with 256 possible values for each entry a_{ij}.

In short, an image is a large matrix. To copy it perfectly, we need $8\,(m)(n)$ bits of information. High definition television typically has $m = 1080$ and $n = 1920$. Often there are 24 frames each second and you probably like to watch in color (3 color scales). This requires transmitting $(3)(8)(48,470,400)$ bits per second. That is too expensive and it is not done. The transmitter can't keep up with the show.

When compression is well done, you can't see the difference from the original. *Edges in the image* (sudden changes in the grayscale) are the hard parts to compress.

Major success in compression will be impossible if every a_{ij} is an independent random number. We totally depend on the fact that *nearby pixels generally have similar grayscales*. An edge produces a sudden jump when you cross over it. Cartoons are more compressible than real-world images, with edges everywhere.

For a video, the numbers a_{ij} don't change much between frames. **We only transmit the small changes**. This is *difference coding* in the H.264 video compression standard (on this book's website). We compress each change matrix by linear algebra (and by nonlinear "quantization" for an efficient step to integers in the computer).

The natural images that we see every day are absolutely ready and open for compression—but that doesn't make it easy to do.

Low Rank Images (Examples)

The easiest images to compress are all black or all white or all a constant grayscale g. The matrix A has the same number g in every entry : $a_{ij} = g$. When $g = 1$ and $m = n = 6$, here is an extreme example of the central SVD dogma of image processing :

Example 1 **Don't send** $A = \begin{bmatrix} 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \end{bmatrix}$ **Send this** $A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1&1&1&1&1&1 \end{bmatrix}$

36 numbers become 12 numbers. With 300 by 300 pixels, $90,000$ numbers become 600. And if we define the all-ones vector $\boldsymbol{x}$ in advance, we only have to send **one number**. That number would be the constant grayscale g that multiplies $\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}$ to produce the matrix.

Of course this first example is extreme. But it makes an important point. If there are special vectors like $\boldsymbol{x} =$ **ones** that can usefully be defined in advance, then image processing can be extremely fast. The battle is between **preselected bases** (the Fourier basis allows speed-up from the FFT) and **adaptive bases** determined by the image. The SVD produces bases from the image itself—this is adaptive and it can be expensive.

I am not saying that the SVD always or usually gives the most effective algorithm in practice. The purpose of these next examples is instruction and not production.

Example 2
"ace flag"
French flag $\boldsymbol{A}$
Italian flag $\boldsymbol{A}$
German flag $\boldsymbol{A}^{\mathrm{T}}$

Don't send $A = \begin{bmatrix} a & a & c & c & e & e \\ a & a & c & c & e & e \\ a & a & c & c & e & e \\ a & a & c & c & e & e \\ a & a & c & c & e & e \\ a & a & c & c & e & e \end{bmatrix}$ **Send** $A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} a & a & c & c & e & e \end{bmatrix}$

This flag has 3 colors but it still has rank 1. We still have one column times one row. The 36 entries could even be all different, provided they keep that rank 1 pattern $A = \boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}}$. But when the rank moves up to $r = 2$, we need $\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$. Here is one choice:

Example 3
Embedded square

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is equal to $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$

The 1's and the 0 in A could be blocks of 1's and a block of 0's. *We would still have rank* 2. We would still only need two terms $\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}}$ and $\boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$. A 6 by 6 image would be compressed into 24 numbers. An N by N image (N^2 numbers) would be compressed into $4N$ numbers from the four vectors $\boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{u}_2, \boldsymbol{v}_2$.

Have I made the best choice for the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s ? This is *not* the choice from the SVD! I notice that $\boldsymbol{u}_1 = (1,1)$ is not orthogonal to $\boldsymbol{u}_2 = (1,0)$. And $\boldsymbol{v}_1 = (1,1)$ is not orthogonal to $\boldsymbol{v}_2 = (0,1)$. The theory says that orthogonality will produce a smaller second piece $c_2\boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$. (**The SVD chooses rank one pieces in order of importance.**)

If the rank of A is much higher than 2, as we expect for real images, then A will add up many rank one pieces. We want the small ones to be really small—they can be discarded with no loss to visual quality. Image compression becomes lossy, but good image compression is virtually undetectable by the human visual system.

The question becomes: **What are the orthogonal choices from the SVD**?

Eigenvectors for the SVD

I want to introduce the use of eigenvectors. But the eigenvectors of most images are not orthogonal. Furthermore the eigenvectors $\boldsymbol{x}_1, \boldsymbol{x}_2$ give only one set of vectors, and we want two sets ($\boldsymbol{u}$'s and $\boldsymbol{v}$'s). The answer to both of those difficulties is the SVD idea:

Use the eigenvectors $\boldsymbol{u}$ of $\boldsymbol{AA}^{\mathrm{T}}$ and the eigenvectors $\boldsymbol{v}$ of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$.

Since AA^{T} and $A^{\mathrm{T}}A$ are automatically symmetric (but not usually equal!) the $\boldsymbol{u}$'s will be one orthogonal set and the eigenvectors $\boldsymbol{v}$ will be another orthogonal set. We can and will make them all unit vectors: $||\boldsymbol{u}_i|| = 1$ and $||\boldsymbol{v}_i|| = 1$. Then our rank 2 matrix will be $A = \sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \sigma_2\boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$. The size of those numbers σ_1 and σ_2 will decide whether they can be ignored in compression. *We keep larger σ's, we discard small σ's.*

The $\boldsymbol{u}$'s from the SVD are called **left singular vectors** (unit eigenvectors of AA^{T}). The $\boldsymbol{v}$'s are **right singular vectors** (unit eigenvectors of $A^{\mathrm{T}}A$). The σ's are **singular values**, square roots of the equal eigenvalues of AA^{T} and $A^{\mathrm{T}}A$:

Choices from the SVD $\quad AA^{\mathrm{T}}\boldsymbol{u}_i = \sigma_i^2\boldsymbol{u}_i \quad A^{\mathrm{T}}A\boldsymbol{v}_i = \sigma_i^2\boldsymbol{v}_i \quad A\boldsymbol{v}_i = \sigma_i\boldsymbol{u}_i \qquad (1)$

In Example 3 (the embedded square), here are the symmetric matrices AA^{T} and $A^{\mathrm{T}}A$:

$$AA^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} \end{bmatrix} \qquad A^{\mathrm{T}}A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}.$$

Their determinants are 1, so $\lambda_1\lambda_2 = 1$. Their traces (diagonal sums) are 3:

$$\det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = \lambda^2 - 3\lambda + 1 = 0 \quad \text{gives} \quad \lambda_1 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{3-\sqrt{5}}{2}.$$

The square roots of λ_1 and λ_2 are $\sigma_1 = \dfrac{\sqrt{5}+1}{2}$ and $\sigma_2 = \dfrac{\sqrt{5}-1}{2}$ with $\sigma_1\,\sigma_2 = 1$.

The nearest rank 1 matrix to A will be $\sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}}$. The error is only $\sigma_2 \approx 0.6 =$ best possible.

The orthonormal eigenvectors of AA^{T} and $A^{\mathrm{T}}A$ are

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ \sigma_1 \end{bmatrix} \quad \boldsymbol{u}_2 = \begin{bmatrix} \sigma_1 \\ -1 \end{bmatrix} \quad \boldsymbol{v}_1 = \begin{bmatrix} \sigma_1 \\ 1 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -\sigma_1 \end{bmatrix} \quad \text{all divided by } \sqrt{1+\sigma_1^2}. \qquad (2)$$

Every reader understands that in real life those calculations are done by computers! (Certainly not by unreliable professors. I corrected myself using svd(A) in MATLAB.) And we can check that the matrix A is correctly recovered from $\sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \sigma_2\boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$:

$$A = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_1^{\mathrm{T}} \\ \boldsymbol{v}_2^{\mathrm{T}} \end{bmatrix} \quad \text{or more simply} \quad A\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} \sigma_1\boldsymbol{u}_1 & \sigma_2\boldsymbol{u}_2 \end{bmatrix} \qquad (3)$$

Important The key point is not that images tend to have low rank. **No**: Images mostly have full rank. But they do have **low effective rank**. This means: Many singular values are small and can be set to zero. *We transmit a low rank approximation.*

Example 4 Suppose the flag has two triangles of different colors. The lower left triangle has 1's and the upper right triangle has 0's. The main diagonal is included with the 1's. Here is the image matrix when $n = 4$. It has full rank $r = 4$ so it is invertible:

Triangular flag matrix

$$A = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

With full rank, A has a full set of n singular values σ (all positive). The SVD will produce n pieces $\sigma_i\, \boldsymbol{u}_i\, \boldsymbol{v}_i^{\mathrm{T}}$ of rank one. Perfect reproduction needs all n pieces.

In compression *small* σ's can be discarded with no serious loss in image quality. We want to understand and plot the σ's for $n = 4$ and also for large n. Notice that Example 3 was the special case $n = 2$ of this triangular Example 4.

Working by hand, we begin with AA^{T} (a computer would proceed differently):

$$AA^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \quad \text{and} \quad (AA^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}} A^{-1} = \begin{bmatrix} \mathbf{2} & -1 & 0 & 0 \\ -1 & \mathbf{2} & -1 & 0 \\ 0 & -1 & \mathbf{2} & -1 \\ 0 & 0 & -1 & \mathbf{1} \end{bmatrix}. \quad (4)$$

That $-1, 2, -1$ inverse matrix is included because its eigenvalues all have the form $2 - 2\cos\theta$. So we know the λ's for AA^{T} and the σ's for A:

$$\lambda = \frac{1}{2 - 2\cos\theta} = \frac{1}{4\sin^2(\theta/2)} \quad \text{gives} \quad \sigma = \sqrt{\lambda} = \frac{1}{2\sin(\theta/2)}. \quad (5)$$

The n different angles θ are equally spaced, which makes this example so exceptional:

$$\theta = \frac{\pi}{2n+1}, \frac{3\pi}{2n+1}, \ldots, \frac{(2n-1)\pi}{2n+1} \quad \left(n = 4 \text{ includes } \theta = \frac{3\pi}{9} \text{ with } 2\sin\frac{\theta}{2} = 1\right).$$

That special case gives $\lambda = 1$ as an eigenvalue of AA^{T} when $n = 4$. So $\sigma = \sqrt{\lambda} = 1$ is a singular value of A. You can check that the vector $\boldsymbol{u} = (1, 1, 0, -1)$ has $AA^{\mathrm{T}}\boldsymbol{u} = \boldsymbol{u}$ (a truly special case).

The important point is to graph the n singular values of A. Those numbers drop off (unlike the eigenvalues of A, which are all 1). But the dropoff is not steep. So the SVD gives only moderate compression of this triangular flag. *Great compression for Hilbert.*

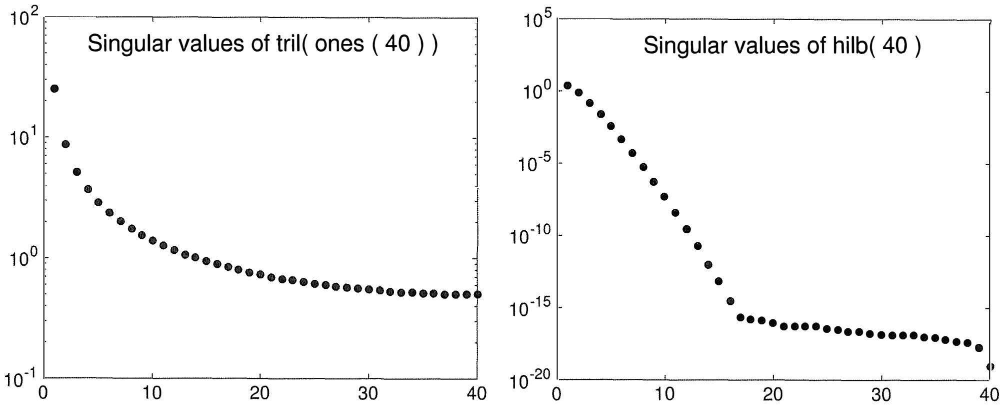


Figure 7.1: Singular values of the triangle of 1's in Examples 3-4 (not compressible) and the evil Hilbert matrix $H(i,j) = (i+j-1)^{-1}$ in Section 8.3: compress it to work with it.

Your faithful author has continued research on the ranks of flags. Quite a few are based on horizontal or vertical stripes. Those have *rank one*—all rows or all columns are multiples of the *ones* vector $(1, 1, \ldots, 1)$. Armenia, Austria, Belgium, Bulgaria, Chad, Colombia, Ireland, Madagascar, Mali, Netherlands, Nigeria, Romania, Russia (and more) have three stripes. Indonesia and Poland have two ! Libya was the extreme case in the Gadaffi years 1977 to 2011 (*the whole flag was green*).

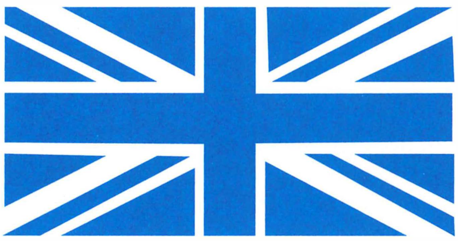
At the other extreme, many flags include diagonal lines. Those could be long diagonals as in the British flag. Or they could be short diagonals coming from the edges of a star—as in the US flag. The text example of a triangle of ones shows how those flag matrices will have large rank. The rank increases to infinity as the pixel sizes get small.

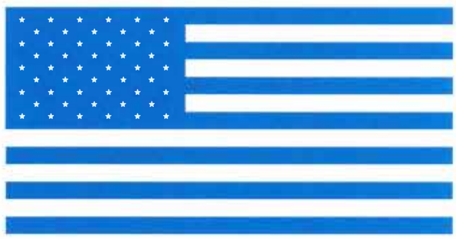
Other flags have circles or crescents or various curved shapes. Their ranks are large and also increasing to infinity. These are still compressible! The compressed image won't be perfect but our eyes won't see the difference (with enough terms $\sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^{\mathrm{T}}$ from the SVD). Those examples actually bring out the main purpose of image compression:

Visual quality can be preserved even with a big reduction in the rank.

For fun I looked back at the flags with finite rank. They can have stripes and they can also have crosses—provided the edges of the cross are horizontal or vertical. Some flags have a thin outline around the cross. This artistic touch will increase the rank. Right now my champion is the flag of Greece shown below, with a cross and also stripes. Its rank is **three** by my counting (three different columns). I see no US State Flags of finite rank !

The reader could google "national flags" to see the variety of designs and colors. I would be glad to know any finite rank examples with rank > 3. Good examples of all kinds will go on the book's website **math.mit.edu/linearalgebra** (and flags in full color).





Problem Set 7.1

1 What are the ranks r for these matrices with entries i times j and i plus j? Write A and B as the sum of r pieces $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ of rank one. Not requiring $\boldsymbol{u}_1^{\mathrm{T}}\boldsymbol{u}_2 = \boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{v}_2 = \mathbf{0}$.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

2 We usually think that the identity matrix I is as simple as possible. But why is I completely incompressible? *Draw a rank* 5 *flag with a cross.*

3 These flags have rank 2. Write A and B in any way as $\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$.

$$\boldsymbol{A}_{\textbf{Sweden}} = \boldsymbol{A}_{\textbf{Finland}} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix} \qquad \boldsymbol{B}_{\textbf{Benin}} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

4 Now find the trace and determinant of BB^{T} and $B^{\mathrm{T}}B$ in Problem 3. The singular values of B are close to $\sigma_1^2 = 28 - \frac{1}{14}$ and $\sigma_2^2 = \frac{1}{14}$. Is B compressible or not?

5 Use $[U, S, V] = \mathsf{svd}\,(A)$ to find two orthogonal pieces $\sigma\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ of $\boldsymbol{A}_{\textbf{Sweden}}$.

6 Find the eigenvalues and the singular values of this 2 by 2 matrix A.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \quad \text{with} \quad A^{\mathrm{T}}A = \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} \quad \text{and} \quad AA^{\mathrm{T}} = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}.$$

The eigenvectors $(1, 2)$ and $(1, -2)$ of A are not orthogonal. How do you know the eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ of $A^{\mathrm{T}}A$ are orthogonal? Notice that $A^{\mathrm{T}}A$ and AA^{T} have the same eigenvalues (25 and 0).

7 How does the second form $\boldsymbol{AV} = \boldsymbol{U\Sigma}$ in equation (3) follow from the first form $\boldsymbol{A} = \boldsymbol{U\Sigma V}^{\mathbf{T}}$? That is the most famous form of the SVD.

8 The two columns of $AV = U\Sigma$ are $A\boldsymbol{v}_1 = \sigma_1\boldsymbol{u}_1$ and $A\boldsymbol{v}_2 = \sigma_2\boldsymbol{u}_2$. So we hope that

$$A\boldsymbol{v}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \sigma_1 \\ 1 \end{bmatrix} = \sigma_1\begin{bmatrix} 1 \\ \sigma_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -\sigma_1 \end{bmatrix} = \sigma_2\begin{bmatrix} \sigma_1 \\ -1 \end{bmatrix}$$

The first needs $\sigma_1 + 1 = \sigma_1^2$ and the second needs $1 - \sigma_1 = -\sigma_2$. Are those true?

9 The MATLAB commands $A = \mathsf{rand}\,(20, 40)$ and $B = \mathsf{randn}\,(20, 40)$ produce 20 by 40 random matrices. The entries of A are between 0 and 1 with uniform probability. The entries of B have a normal "bell-shaped" probability distribution. Using an svd command, find and graph their singular values σ_1 to σ_{20}. Why do they have 20 σ's?

7.2 Bases and Matrices in the SVD

1 The SVD produces **orthonormal basis** of $\boldsymbol{v}$'s and $\boldsymbol{u}$'s for the four fundamental subspaces.

2 Using those bases, A becomes a diagonal matrix Σ and $\boldsymbol{Av_i = \sigma_i u_i}$ **:** $\boldsymbol{\sigma_i =}$ **singular value**.

3 The two-bases diagonalization $\boldsymbol{A = U\Sigma V^{\mathrm{T}}}$ often has more information than $A = X\Lambda X^{-1}$.

4 $U\Sigma V^{\mathrm{T}}$ separates A into rank-1 matrices $\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}} + \cdots + \sigma_r \boldsymbol{u}_r \boldsymbol{v}_r^{\mathrm{T}}$. $\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}}$ is the largest!

The Singular Value Decomposition is a highlight of linear algebra. A is any m by n matrix, square or rectangular. Its rank is r. We will diagonalize this A, but not by $X^{-1}AX$. The eigenvectors in X have three big problems: They are usually not orthogonal, there are not always enough eigenvectors, and $A\boldsymbol{x} = \lambda \boldsymbol{x}$ requires A to be a square matrix. The ***singular vectors*** of A solve all those problems in a perfect way.

Let me describe what we want from the SVD : **the right bases for the four subspaces**. Then I will write about the steps to find those basis vectors **in order of importance**.

The price we pay is to have **two sets of singular vectors**, $\boldsymbol{u}$'s and $\boldsymbol{v}$'s. The $\boldsymbol{u}$'s are in $\mathbf{R}^m$ and the $\boldsymbol{v}$'s are in $\mathbf{R}^n$. They will be the columns of an m by m matrix $\boldsymbol{U}$ and an n by n matrix $\boldsymbol{V}$. I will first describe the SVD in terms of those basis vectors. Then I can also describe the SVD in terms of the orthogonal matrices U and V.

(using vectors) The $\boldsymbol{u}$'s and $\boldsymbol{v}$'s give bases for the four fundamental subspaces :

$\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$	is an orthonormal basis for the **column space**
$\boldsymbol{u}_{r+1}, \ldots, \boldsymbol{u}_m$	is an orthonormal basis for the **left nullspace** $\boldsymbol{N}(A^{\mathrm{T}})$
$\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$	is an orthonormal basis for the **row space**
$\boldsymbol{v}_{r+1}, \ldots, \boldsymbol{v}_n$	is an orthonormal basis for the **nullspace** $\boldsymbol{N}(A)$.

More than just orthogonality, these basis vectors diagonalize the matrix A :

"A is diagonalized" $\quad A\boldsymbol{v}_1 = \sigma_1 \boldsymbol{u}_1 \quad A\boldsymbol{v}_2 = \sigma_2 \boldsymbol{u}_2 \quad \ldots \quad A\boldsymbol{v}_r = \sigma_r \boldsymbol{u}_r \qquad (1)$

Those **singular values** $\boldsymbol{\sigma_1}$ **to** $\boldsymbol{\sigma_r}$ will be positive numbers : *σ_i is the length of $A\boldsymbol{v}_i$*. The σ's go into a diagonal matrix that is otherwise zero. That matrix is Σ.

(using matrices) Since the $\boldsymbol{u}$'s are orthonormal, the matrix U_r with those r columns has $U_r^{\mathrm{T}} U_r = I$. Since the $\boldsymbol{v}$'s are orthonormal, the matrix V_r has $V_r^{\mathrm{T}} V_r = I$. Then the equations $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$ tell us column by column that $\boldsymbol{AV_r = U_r \Sigma_r}$:

$$\begin{array}{c} (m \text{ by } n)(n \text{ by } r) \\ \boldsymbol{AV_r = U_r\Sigma_r} \\ (m \text{ by } r)(r \text{ by } r) \end{array} \qquad A \begin{bmatrix} \boldsymbol{v}_1 & \cdot & \cdot & \boldsymbol{v}_r \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 & \cdot & \cdot & \boldsymbol{u}_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}. \qquad (2)$$

This is the heart of the SVD, but there is more. Those $\boldsymbol{v}$'s and $\boldsymbol{u}$'s account for the row space and column space of A. We have $n - r$ more $\boldsymbol{v}$'s and $m - r$ more $\boldsymbol{u}$'s, from the nullspace $\boldsymbol{N}(A)$ and the left nullspace $\boldsymbol{N}(A^{\mathrm{T}})$. They are automatically orthogonal to the first $\boldsymbol{v}$'s and $\boldsymbol{u}$'s (because the whole nullspaces are orthogonal). We now include all the $\boldsymbol{v}$'s and $\boldsymbol{u}$'s in V and U, so these matrices become *square*. ***We still have*** $\boldsymbol{AV = U\Sigma}$.

$$\begin{array}{c}(m \text{ by } n)(n \text{ by } n)\\ \boldsymbol{AV} \text{ equals } \boldsymbol{U\Sigma}\\ (m \text{ by } m)(m \text{ by } n)\end{array} \quad \boldsymbol{A}\begin{bmatrix} & & & & \\ \boldsymbol{v}_1 & \cdot & \cdot\, \boldsymbol{v}_r & \cdot & \cdot\, \boldsymbol{v}_n \\ & & & & \end{bmatrix} = \begin{bmatrix} & & & & \\ \boldsymbol{u}_1 & \cdot & \cdot\, \boldsymbol{u}_r & \cdot & \cdot\, \boldsymbol{u}_m \\ & & & & \end{bmatrix}\begin{bmatrix} \boldsymbol{\sigma_1} & & \\ & \ddots & \\ & & \boldsymbol{\sigma_r} \\ & & \end{bmatrix} \tag{3}$$

The new Σ is m by n. It is just the r by r matrix in equation (2) with $m - r$ extra zero rows and $n - r$ new zero columns. The real change is in the shapes of U and V. Those are square matrices and $V^{-1} = V^{\mathrm{T}}$. So $AV = U\Sigma$ becomes $\boldsymbol{A = U\Sigma V^{\mathrm{T}}}$. This is the ***Singular Value Decomposition***. I can multiply columns $\boldsymbol{u}_i\sigma_i$ from $U\Sigma$ by rows of V^{T}:

$$\textbf{SVD} \qquad \boxed{\boldsymbol{A = U\Sigma V^{\mathrm{T}} = u_1\sigma_1 v_1^{\mathrm{T}} + \cdots + u_r\sigma_r v_r^{\mathrm{T}}}.} \tag{4}$$

Equation (2) was a "reduced SVD" with bases for the row space and column space. Equation (3) is the full SVD with nullspaces included. They both split up A into the same r matrices $\boldsymbol{u}_i\sigma_i\boldsymbol{v}_i^{\mathrm{T}}$ of rank one. Column times row is the fourth way to multiply matrices.

We will see that each σ_i^2 is an eigenvalue of $A^{\mathrm{T}}A$ and also AA^{T}. When we put the singular values in descending order, $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_r > 0$, the splitting in equation (4) gives the r rank-one pieces of A ***in order of importance***. This is crucial.

Example 1 When is $A = U\Sigma V^{\mathrm{T}}$ (singular values) the *same* as $X\Lambda X^{-1}$ (eigenvalues)?

Solution A needs orthonormal eigenvectors to allow $X = U = V$. A also needs eigenvalues $\lambda \geq 0$ if $\Lambda = \Sigma$. So A must be a ***positive semidefinite (or definite) symmetric matrix***. Only then will $A = X\Lambda X^{-1}$ which is also $Q\Lambda Q^{\mathrm{T}}$ coincide with $A = U\Sigma V^{\mathrm{T}}$.

Example 2 If $A = \boldsymbol{xy}^{\mathrm{T}}$ (rank 1) with unit vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, what is the SVD of A?

Solution The reduced SVD in (2) is exactly $\boldsymbol{xy}^{\mathrm{T}}$, with rank $r = 1$. It has $\boldsymbol{u}_1 = \boldsymbol{x}$ and $\boldsymbol{v}_1 = \boldsymbol{y}$ and $\sigma_1 = 1$. For the full SVD, complete $\boldsymbol{u}_1 = \boldsymbol{x}$ to an orthonormal basis of $\boldsymbol{u}$'s, and complete $\boldsymbol{v}_1 = \boldsymbol{y}$ to an orthonormal basis of $\boldsymbol{v}$'s. No new σ's, only $\sigma_1 = 1$.

Proof of the SVD

We need to show how those amazing $\boldsymbol{u}$'s and $\boldsymbol{v}$'s can be constructed. The $\boldsymbol{v}$'s will be **orthonormal eigenvectors of** $\boldsymbol{A^{\mathrm{T}}A}$. This must be true because we are aiming for

$$\boldsymbol{A^{\mathrm{T}}A} = (U\Sigma V^{\mathrm{T}})^{\mathrm{T}}(U\Sigma V^{\mathrm{T}}) = V\Sigma^{\mathrm{T}}U^{\mathrm{T}}U\Sigma V^{\mathrm{T}} = \boldsymbol{V\Sigma^{\mathrm{T}}\Sigma V^{\mathrm{T}}}. \tag{5}$$

On the right you see the eigenvector matrix V for the symmetric positive (semi) definite matrix $A^{\mathrm{T}}A$. And $(\Sigma^{\mathrm{T}}\Sigma)$ must be the eigenvalue matrix of $(A^{\mathrm{T}}A)$: *Each* σ^2 *is* $\lambda(A^{\mathrm{T}}A)$!

Now $Av_i = \sigma_i u_i$ tells us the unit vectors u_1 to u_r. This is the key equation (1). The essential point—the whole reason that the SVD succeeds—is that those unit vectors u_1 to u_r are automatically orthogonal to each other (*because the v's are orthogonal*):

Key step $\quad$ $i \neq j$

$$u_i^{\mathrm{T}} u_j = \left(\frac{Av_i}{\sigma_i}\right)^{\mathrm{T}} \left(\frac{Av_j}{\sigma_j}\right) = \frac{v_i^{\mathrm{T}} A^{\mathrm{T}} A v_j}{\sigma_i \sigma_j} = \frac{\sigma_j^2}{\sigma_i \sigma_j} v_i^{\mathrm{T}} v_j = \textbf{zero}. \qquad (6)$$

The v's are eigenvectors of $A^{\mathrm{T}}A$ (symmetric). They are orthogonal and now the u's are also orthogonal. *Actually those u's will be eigenvectors of AA^{T}.*

Finally we complete the v's and u's to n v's and m u's with any orthonormal bases for the nullspaces $N(A)$ and $N(A^{\mathrm{T}})$. We have found V and Σ and U in $A = U\Sigma V^{\mathrm{T}}$.

An Example of the SVD

Here is an example to show the computation of all three matrices in $A = U\Sigma V^{\mathrm{T}}$.

Example 3 Find the matrices U, Σ, V for $A = \begin{bmatrix} \mathbf{3} & \mathbf{0} \\ \mathbf{4} & \mathbf{5} \end{bmatrix}$. The rank is $r = \mathbf{2}$.

With rank 2, this A has positive singular values σ_1 and σ_2. We will see that σ_1 is larger than $\lambda_{\max} = 5$, and σ_2 is smaller than $\lambda_{\min} = 3$. Begin with $A^{\mathrm{T}}A$ and AA^{T}:

$$A^{\mathrm{T}}A = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} \qquad AA^{\mathrm{T}} = \begin{bmatrix} 9 & 12 \\ 12 & 41 \end{bmatrix}$$

Those have the same trace (50) and the same eigenvalues $\sigma_1^2 = 45$ and $\sigma_2^2 = 5$. The square roots are $\sigma_1 = \sqrt{\mathbf{45}}$ and $\sigma_2 = \sqrt{\mathbf{5}}$. Then $\sigma_1\sigma_2 = 15$ and this is the determinant of A.

A key step is to find the eigenvectors of $A^{\mathrm{T}}A$ (with eigenvalues 45 and 5):

$$\begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{45} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \mathbf{5} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Then v_1 and v_2 are those orthogonal eigenvectors rescaled to length 1. Divide by $\sqrt{2}$.

Right singular vectors $v_1 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad v_2 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad$ **Left singular vectors** $u_i = \dfrac{Av_i}{\sigma_i}$

Now compute Av_1 and Av_2 which will be $\sigma_1 u_1 = \sqrt{45}\, u_1$ and $\sigma_2 u_2 = \sqrt{5}\, u_2$:

$$\begin{aligned} Av_1 &= \frac{3}{\sqrt{2}}\begin{bmatrix} 1 \\ 3 \end{bmatrix} &= \sqrt{45}\frac{\mathbf{1}}{\sqrt{\mathbf{10}}}\begin{bmatrix} \mathbf{1} \\ \mathbf{3} \end{bmatrix} &= \sigma_1\, u_1 \\ Av_2 &= \frac{1}{\sqrt{2}}\begin{bmatrix} -3 \\ 1 \end{bmatrix} &= \sqrt{5}\,\frac{\mathbf{1}}{\sqrt{\mathbf{10}}}\begin{bmatrix} \mathbf{-3} \\ \mathbf{1} \end{bmatrix} &= \sigma_2\, u_2 \end{aligned}$$

The division by $\sqrt{10}$ makes u_1 and u_2 orthonormal. Then $\sigma_1 = \sqrt{45}$ and $\sigma_2 = \sqrt{5}$ as expected. The Singular Value Decomposition of A is U times Σ times V^{T}.

$$\boldsymbol{U} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sqrt{45} & \\ & \sqrt{5} \end{bmatrix} \qquad \boldsymbol{V} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \tag{7}$$

U and V contain orthonormal bases for the column space and the row space (both spaces are just $\mathbf{R}^2$). The real achievement is that those two bases diagonalize A: AV equals $U\Sigma$. The matrix A splits into a combination of two rank-one matrices, columns times rows:

$$\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}} + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^{\mathrm{T}} = \frac{\sqrt{45}}{\sqrt{20}}\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{3} \end{bmatrix} + \frac{\sqrt{5}}{\sqrt{20}}\begin{bmatrix} \mathbf{3} & \mathbf{-3} \\ \mathbf{-1} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = A.$$

An Extreme Matrix

Here is a larger example, when the $\boldsymbol{u}$'s and the $\boldsymbol{v}$'s are just columns of the identity matrix. So the computations are easy, but keep your eye on the *order of the columns*. The matrix A is badly lopsided (strictly triangular). All its eigenvalues are zero. AA^{T} is not close to $A^{\mathrm{T}}A$. The matrices U and V will be permutations that fix these problems properly.

$$A = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{2} & 0 \\ 0 & 0 & 0 & \mathbf{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \text{eigenvalues } \lambda = 0,0,0,0 \text{ all zero!} \\ \text{only one eigenvector } (1,0,0,0) \\ \text{singular values } \sigma = \mathbf{3},\mathbf{2},\mathbf{1} \\ \text{singular vectors are columns of } I \end{array}$$

$A^{\mathrm{T}}A$ and AA^{T} are diagonal (with easy eigenvectors, but in different orders):

$$A^{\mathrm{T}}A = \begin{bmatrix} \mathbf{0} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{4} & 0 \\ 0 & 0 & 0 & \mathbf{9} \end{bmatrix} \qquad AA^{\mathrm{T}} = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{4} & 0 & 0 \\ 0 & 0 & \mathbf{9} & 0 \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$$

Their eigenvectors ($\boldsymbol{u}$'s for AA^{T} and $\boldsymbol{v}$'s for $A^{\mathrm{T}}A$) go in decreasing order $\sigma_1^2 > \sigma_2^2 > \sigma_3^2$ of the eigenvalues. Those eigenvalues are $\sigma^2 = 9, 4, 1$.

$$\boldsymbol{U} = \begin{bmatrix} 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \mathbf{3} & & & \\ & \mathbf{2} & & \\ & & \mathbf{1} & \\ & & & 0 \end{bmatrix} \qquad \boldsymbol{V} = \begin{bmatrix} 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \end{bmatrix}$$

Those first columns $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$ have 1's in positions 3 and 4. Then $\boldsymbol{u}_1\sigma_1\boldsymbol{v}_1^{\mathrm{T}}$ picks out the biggest number $A_{34} = 3$ in the original matrix A. The three rank-one matrices in the SVD come (for this extreme example) exactly from the numbers 3, 2, 1 in A.

$$\boldsymbol{A} = \boldsymbol{U\Sigma V}^{\mathbf{T}} = \mathbf{3}\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \mathbf{2}\boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}} + \mathbf{1}\boldsymbol{u}_3\boldsymbol{v}_3^{\mathrm{T}}.$$

Note Suppose I remove the last row of A (all zeros). Then A is a 3 by 4 matrix and AA^{T} is 3 by 3—its fourth row and column will disappear. We still have eigenvalues $\lambda = 1, 4, 9$ in $A^{\mathrm{T}}A$ and AA^{T}, producing the same singular values $\sigma = 3, 2, 1$ in Σ.

Removing the zero row of A (now 3×4) just removes the last row of Σ and also the last row and column of U. Then $(3 \times 4) = U\Sigma V^{\mathrm{T}} = (3 \times 3)(3 \times 4)(4 \times 4)$. The SVD is totally adapted to rectangular matrices.

A good thing, because the rows and columns of a data matrix A often have completely different meanings (like a spreadsheet). If we have the grades for all courses, there would be a column for each student and a row for each course: The entry a_{ij} would be the grade. Then $\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}}$ could have $\boldsymbol{u}_1$ = **combination course** and $\boldsymbol{v}_1$ = **combination student**. And σ_1 would be the grade for those combinations: the highest grade.

The matrix A could count the frequency of key words in a journal: A different article for each column of A and a different word for each row. The whole journal is indexed by the matrix A and the most important information is in $\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}}$. Then σ_1 is the largest frequency for a hyperword (the word combination $\boldsymbol{u}_1$) in the hyperarticle $\boldsymbol{v}_1$.

Section 7.3 will apply the SVD to finance and genetics and search engines.

Singular Value Stability versus Eigenvalue Instability

The 4 by 4 example A provides an example (an extreme case) of the instability of eigenvalues. **Suppose the 4,1 entry barely changes** from zero to $1/60,000$. The rank is now 4.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ \dfrac{1}{60,000} & 0 & 0 & 0 \end{bmatrix}$$

That change by only $1/60,000$ produces a much bigger jump in the eigenvalues of A

$\lambda = 0, 0, 0, 0$ to $\boldsymbol{\lambda} = \dfrac{\mathbf{1}}{\mathbf{10}}, \dfrac{\boldsymbol{i}}{\mathbf{10}}, \dfrac{\mathbf{-1}}{\mathbf{10}}, \dfrac{-\boldsymbol{i}}{\mathbf{10}}$

The four eigenvalues moved from zero onto a circle around zero. The circle has radius $\frac{1}{10}$ when the new entry is only $1/60,000$. This shows serious instability of eigenvalues when AA^{T} is far from $A^{\mathrm{T}}A$. At the other extreme, if $A^{\mathrm{T}}A = AA^{\mathrm{T}}$ (a "normal matrix") the eigenvectors of A are orthogonal and the eigenvalues of A are totally stable.

By contrast, **the singular values of any matrix are stable**. They don't change more than the change in A. In this example, the new singular values are $\mathbf{3}, \mathbf{2}, \mathbf{1}$, and $\mathbf{1/60,000}$. The matrices U and V stay the same. The new fourth piece of A is $\sigma_4 \boldsymbol{u}_4 \boldsymbol{v}_4^{\mathrm{T}}$, with fifteen zeros and that small entry $\sigma_4 = 1/60,000$.

Singular Vectors of A and Eigenvectors of $S = A^{\mathrm{T}}A$

Equations (5–6) "proved" the SVD *all at once*. The singular vectors $\boldsymbol{v}_i$ are the eigenvectors $\boldsymbol{q}_i$ of $S = A^{\mathrm{T}}A$. The eigenvalues λ_i of S are the same as σ_i^2 for A. The rank r of S equals the rank of A. The expansions in eigenvectors and singular vectors are perfectly parallel.

Symmetric S	$S = Q\Lambda Q^{\mathrm{T}} = \lambda_1 \boldsymbol{q}_1 \boldsymbol{q}_1^{\mathrm{T}} + \lambda_2 \boldsymbol{q}_2 \boldsymbol{q}_2^{\mathrm{T}} + \cdots + \lambda_r \boldsymbol{q}_r \boldsymbol{q}_r^{\mathrm{T}}$
Any matrix A	$A = U\Sigma V^{\mathrm{T}} = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}} + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^{\mathrm{T}} + \cdots + \sigma_r \boldsymbol{u}_r \boldsymbol{v}_r^{\mathrm{T}}$

The $\boldsymbol{q}$'s are orthonormal, the $\boldsymbol{u}$'s are orthonormal, the $\boldsymbol{v}$'s are orthonormal. Beautiful.

But I want to look again, for two good reasons. One is to fix a weak point in the eigenvalue part, where Chapter 6 was not complete. If λ is a *double* eigenvalue of S, we can and must find *two* orthonormal eigenvectors. The other reason is to see how the SVD picks off the largest term $\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}}$ before $\sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^{\mathrm{T}}$. We want to understand the eigenvalues λ (of S) and the singular values σ (of A) **one at a time instead of all at once**.

Start with the largest eigenvalue λ_1 of S. It solves this problem:

$$\boldsymbol{\lambda_1} = \textbf{maximum ratio } \frac{\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}} \boldsymbol{x}}. \quad \text{The winning vector is } \boldsymbol{x} = \boldsymbol{q}_1 \text{ with } S\boldsymbol{q}_1 = \lambda_1 \boldsymbol{q}_1. \tag{8}$$

Compare with the largest singular value σ_1 of A. It solves this problem:

$$\boldsymbol{\sigma_1} = \textbf{maximum ratio } \frac{||A\boldsymbol{x}||}{||\boldsymbol{x}||}. \quad \text{The winning vector is } \boldsymbol{x} = \boldsymbol{v}_1 \text{ with } A\boldsymbol{v}_1 = \sigma_1 \boldsymbol{u}_1. \tag{9}$$

This "one at a time approach" applies also to λ_2 and σ_2. But not all $\boldsymbol{x}$'s are allowed:

$$\boldsymbol{\lambda_2} = \textbf{maximum ratio } \frac{\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}} \boldsymbol{x}} \text{ among all } \boldsymbol{x}\text{'s with } \boldsymbol{q}_1^{\mathrm{T}} \boldsymbol{x} = 0. \quad \boldsymbol{x} = \boldsymbol{q_2} \text{ will win.} \tag{10}$$

$$\boldsymbol{\sigma_2} = \textbf{maximum ratio } \frac{||A\boldsymbol{x}||}{||\boldsymbol{x}||} \text{ among all } \boldsymbol{x}\text{'s with } \boldsymbol{v}_1^{\mathrm{T}} \boldsymbol{x} = 0. \quad \boldsymbol{x} = \boldsymbol{v_2} \text{ will win.} \tag{11}$$

When $S = A^{\mathrm{T}}A$ we find $\lambda_1 = \sigma_1^2$ and $\lambda_2 = \sigma_2^2$. Why does this approach succeed?

Start with the ratio $r(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} / \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x}$. This is called the *Rayleigh quotient*. To maximize $r(\boldsymbol{x})$, set its partial derivatives to zero: $\partial r / \partial x_i = 0$ for $i = 1, \ldots, n$. Those derivatives are messy and here is the result: one vector equation for the winning $\boldsymbol{x}$:

$$\textbf{The derivatives of } \boldsymbol{r(x)} = \frac{\boldsymbol{x}^{\mathbf{T}} \boldsymbol{S} \boldsymbol{x}}{\boldsymbol{x}^{\mathbf{T}} \boldsymbol{x}} \textbf{ are zero when} \qquad S\boldsymbol{x} = r(\boldsymbol{x})\boldsymbol{x}. \tag{12}$$

So the winning $\boldsymbol{x}$ is an eigenvector of S. The maximum ratio $r(\boldsymbol{x})$ is the largest eigenvalue λ_1 of S. All good. Now turn to A—and notice the connection to $S = A^{\mathrm{T}}A$!

$$\text{Maximizing } \frac{||A\boldsymbol{x}||}{||\boldsymbol{x}||} \text{ also maximizes } \left(\frac{||A\boldsymbol{x}||}{||\boldsymbol{x}||}\right)^2 = \frac{\boldsymbol{x}^{\mathrm{T}} A^{\mathrm{T}} A \boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}} \boldsymbol{x}} = \frac{\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}} \boldsymbol{x}}.$$

So the winning $\boldsymbol{x} = \boldsymbol{v}_1$ in (9) is the same as the top eigenvector $\boldsymbol{q}_1$ of $S = A^{\mathrm{T}}A$ in (8).

Now I have to explain why $\boldsymbol{q}_2$ and $\boldsymbol{v}_2$ are the winning vectors in (10) and (11). We know they are orthogonal to $\boldsymbol{q}_1$ and $\boldsymbol{v}_1$, so they are allowed in those competitions. These paragraphs can be optional for readers who aim to see the SVD in action (Section 7.3).

Start with any orthogonal matrix Q_1 that has $\boldsymbol{q}_1$ in its first column. The other $n-1$ orthonormal columns just have to be orthogonal to $\boldsymbol{q}_1$. Then use $S\boldsymbol{q}_1 = \lambda_1 \boldsymbol{q}_1$:

$$SQ_1 = S\begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \cdots \boldsymbol{q}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \cdots \boldsymbol{q}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \boldsymbol{w}^{\mathrm{T}} \\ \mathbf{0} & S_{n-1} \end{bmatrix} = Q_1 \begin{bmatrix} \lambda_1 & \boldsymbol{w}^{\mathrm{T}} \\ \mathbf{0} & S_{n-1} \end{bmatrix}. \quad (13)$$

Multiply by Q_1^{T}, remember $Q_1^{\mathrm{T}} Q_1 = I$, and recognize that $Q_1^{\mathrm{T}} S Q_1$ is symmetric like S:

$$\text{The symmetry of } Q_1^{\mathrm{T}} S Q_1 = \begin{bmatrix} \lambda_1 & \boldsymbol{w}^{\mathrm{T}} \\ \mathbf{0} & S_{n-1} \end{bmatrix} \text{ forces } \boldsymbol{w} = \mathbf{0} \text{ and } S_{n-1}^{\mathrm{T}} = S_{n-1}.$$

The requirement $\boldsymbol{q}_1^{\mathrm{T}} \boldsymbol{x} = 0$ has reduced the maximum problem (10) to size $n-1$. The largest eigenvalue of S_{n-1} will be the *second largest* for S. **It is $\boldsymbol{\lambda_2}$.** The winning vector in (10) will be the eigenvector $\boldsymbol{q}_2$ with $S\boldsymbol{q}_2 = \lambda_2 \boldsymbol{q}_2$.

We just keep going—or use the magic word *induction*—to produce all the eigenvectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ and their eigenvalues $\lambda_1, \ldots, \lambda_n$. The Spectral Theorem $S = Q\Lambda Q^{\mathrm{T}}$ is proved even with repeated eigenvalues. All symmetric matrices can be diagonalized.

Similarly the SVD is found one step at a time from (9) and (11) and onwards. Section 7.4 will show the geometry—we are finding the axes of an ellipse. Here I ask a different question: **How are the $\boldsymbol{\lambda}$'s and $\boldsymbol{\sigma}$'s actually computed**?

Computing the Eigenvalues of S and Singular Values of A

The singular values σ_i of A are the square roots of the eigenvalues λ_i of $S = A^{\mathrm{T}} A$. This connects the SVD to a *symmetric eigenvalue problem* (good). But in the end we don't want to multiply A^{T} times A (squaring is time-consuming: not good).

The first idea is ***to produce zeros in*** A ***and*** S ***without changing any*** σ's *and* λ's. Singular vectors and eigenvectors will change—no problem. The similar matrix $Q^{-1}SQ$ has the same λ's as S. If Q is orthogonal, this matrix is $Q^{\mathrm{T}}SQ$ and still symmetric.

Section 11.3 will show how to build Q from 2 by 2 rotations so that $Q^{\mathrm{T}}SQ$ is **symmetric and tridiagonal** (many zeros). But rotations can't get all the way to a diagonal matrix. To show all the eigenvalues of S needs a new idea and more work.

For the SVD, what is the parallel to $Q^{\mathrm{T}}SQ$? Now we don't want to change any singular values of A. Natural answer: You can multiply A by *two different orthogonal matrices* Q_1 *and* Q_2. Use them to produce zeros in $Q_1^{\mathrm{T}} A Q_2$. The σ's don't change :

$$(Q_1^{\mathrm{T}} A Q_2)^{\mathrm{T}} (Q_1^{\mathrm{T}} A Q_2) = Q_2^{\mathrm{T}} A^{\mathrm{T}} A Q_2 = Q_2^{\mathrm{T}} S Q_2 \text{ gives the same } \sigma(A) \text{ and } \lambda(S).$$

The freedom of two Q's allows us to reach $Q_1^{\mathrm{T}} A Q_2 =$ **bidiagonal matrix** (2 diagonals). This compares perfectly to $Q^{\mathrm{T}}SQ = 3$ diagonals. It is nice to notice the connection between them: (*bidiagonal*)$^{\mathrm{T}}$ (*bidiagonal*) $=$ *tridiagonal*.

The final steps to a *diagonal* Λ and a *diagonal* Σ need more ideas. This problem can't be easy, because underneath we are solving $\det(S - \lambda I) = 0$ for polynomials of degree $n = 100$ or 1000 or more. We certainly don't use those polynomials !

The favorite way to find λ's and σ's in **LAPACK** uses simple orthogonal matrices to approach $Q^{\mathrm{T}}SQ = \Lambda$ and $U^{\mathrm{T}}AV = \Sigma$. **We stop when very close to Λ and Σ.**

This 2-step approach (zeros first) is built into the commands **eig**(S) and **svd**(A).

■ REVIEW OF THE KEY IDEAS ■

1. The SVD factors A into $U\Sigma V^{\mathrm{T}}$, with r singular values $\sigma_1 \geq \ldots \geq \sigma_r > 0$.
2. The numbers $\sigma_1^2, \ldots, \sigma_r^2$ are the nonzero eigenvalues of AA^{T} and $A^{\mathrm{T}}A$.
3. The orthonormal columns of U and V are eigenvectors of AA^{T} and $A^{\mathrm{T}}A$.
4. Those columns hold orthonormal bases for the four fundamental subspaces of A.
5. Those bases diagonalize the matrix: $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$ for $i \leq r$. This is $\boldsymbol{AV} = \boldsymbol{U\Sigma}$.
6. $A = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}} + \cdots + \sigma_r \boldsymbol{u}_r \boldsymbol{v}_r^{\mathrm{T}}$ and σ_1 is the maximum of the ratio $||A\boldsymbol{x}|| \,/\, ||\boldsymbol{x}||$.

■ WORKED EXAMPLES ■

7.2 A Identify by name these decompositions of A into a sum of columns times rows:

1.	*Orthogonal* columns	$\boldsymbol{u}_1\sigma_1, \ldots, \boldsymbol{u}_r\sigma_r$	times	*orthonormal* rows	$\boldsymbol{v}_1^{\mathrm{T}}, \ldots, \boldsymbol{v}_r^{\mathrm{T}}$.
2.	*Orthonormal* columns	$\boldsymbol{q}_1, \ldots, \boldsymbol{q}_r$	times	*triangular* rows	$\boldsymbol{r}_1^{\mathrm{T}}, \ldots, \boldsymbol{r}_r^{\mathrm{T}}$.
3.	*Triangular* columns	$\boldsymbol{l}_1, \ldots, \boldsymbol{l}_r$	times	*triangular* rows	$\boldsymbol{u}_1^{\mathrm{T}}, \ldots, \boldsymbol{u}_r^{\mathrm{T}}$.

Where do the rank and the pivots and the singular values of A come into this picture?

Solution These three factorizations are basic to linear algebra, pure or applied:

1. **Singular Value Decomposition $\boldsymbol{A = U\Sigma V^{\mathrm{T}}}$**
2. **Gram-Schmidt Orthogonalization $\boldsymbol{A = QR}$**
3. **Gaussian Elimination $\boldsymbol{A = LU}$**

You might prefer to separate out singular values $\boldsymbol{\sigma_i}$ and heights $\boldsymbol{h_i}$ and pivots $\boldsymbol{d_i}$:

1. $A = U\Sigma V^{\mathrm{T}}$ with unit vectors in U and V. ***The r singular values $\boldsymbol{\sigma_i}$ are in $\boldsymbol{\Sigma}$.***
2. $A = QHR$ with unit vectors in Q and diagonal 1's in R. ***The r heights $\boldsymbol{h_i}$ are in $\boldsymbol{H}$.***
3. $A = LDU$ with diagonal 1's in L and U. ***The r pivots $\boldsymbol{d_i}$ are in $\boldsymbol{D}$.***

Each h_i tells the height of column i above the plane of columns 1 to $i-1$. The volume of the full n-dimensional box ($r = m = n$) comes from $A = U\Sigma V^{\mathrm{T}} = LDU = QHR$:

$$|\,\mathbf{det}\,\boldsymbol{A}\,| = |\,\textit{\textbf{product of}}\ \boldsymbol{\sigma}\text{'s}\,| = |\,\textit{\textbf{product of}}\ \mathrm{d}\text{'s}\,| = |\,\textit{\textbf{product of}}\ \boldsymbol{h}\text{'s}\,|.$$

7.2 B Show that $\sigma_1 \geq |\lambda|_{\max}$. The largest singular value dominates all eigenvalues.

Solution Start from $A = U\Sigma V^{\mathrm{T}}$. Remember that multiplying by an orthogonal matrix *does not change length*: $||Q\boldsymbol{x}|| = ||\boldsymbol{x}||$ because $||Q\boldsymbol{x}||^2 = \boldsymbol{x}^{\mathrm{T}}Q^{\mathrm{T}}Q\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = ||\boldsymbol{x}||^2$. This applies to $Q = U$ and $Q = V^{\mathrm{T}}$. In between is the diagonal matrix Σ.

$$||A\boldsymbol{x}|| = ||U\Sigma V^{\mathrm{T}}\boldsymbol{x}|| = ||\Sigma V^{\mathrm{T}}\boldsymbol{x}|| \leq \sigma_1 ||V^{\mathrm{T}}\boldsymbol{x}|| = \sigma_1 ||\boldsymbol{x}||. \tag{14}$$

An eigenvector has $||A\boldsymbol{x}|| = |\lambda|||\boldsymbol{x}||$. So (14) says that $|\lambda|||\boldsymbol{x}|| \leq \sigma_1 ||\boldsymbol{x}||$. Then $\boldsymbol{|\lambda| \leq \sigma_1}$.

Apply also to the unit vector $\boldsymbol{x} = (1, 0, \ldots, 0)$. Now $A\boldsymbol{x}$ is the first column of A. Then by inequality (14), this column has length $\leq \sigma_1$. Every entry must have $|a_{ij}| \leq \sigma_1$.

Equation (14) shows again that ***the maximum value of*** $||A\boldsymbol{x}||/||\boldsymbol{x}||$ ***equals*** $\boldsymbol{\sigma_1}$.

Section 11.2 will explain how the ratio $\sigma_{\max}/\sigma_{\min}$ governs the roundoff error in solving $A\boldsymbol{x} = \boldsymbol{b}$. MATLAB warns you if this "*condition number*" is large. Then $\boldsymbol{x}$ is unreliable.

Problem Set 7.2

1 Find the eigenvalues of these matrices. Then find singular values from $A^{\mathrm{T}}A$:

$$A = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \qquad\qquad A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$$

For each A, construct V from the eigenvectors of $A^{\mathrm{T}}A$ and U from the eigenvectors of AA^{T}. Check that $A = U\Sigma V^{\mathrm{T}}$.

2 Find $A^{\mathrm{T}}A$ and V and Σ and $\boldsymbol{u}_i = A\boldsymbol{v}_i/\sigma_i$ and the full SVD:

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = U\Sigma V^{\mathrm{T}}.$$

3 In Problem 2, show that AA^{T} is diagonal. Its eigenvectors $\boldsymbol{u}_1, \boldsymbol{u}_2$ are _____. Its eigenvalues σ_1^2, σ_2^2 are _____. The rows of A are orthogonal but they are not _____. So the columns of A are not orthogonal.

4 Compute $A^{\mathrm{T}}A$ and AA^{T} and their eigenvalues and unit eigenvectors for V and U.

$$\textbf{Rectangular matrix} \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Check $AV = U\Sigma$ (this decides $\pm$ signs in U). Σ has the same shape as A: 2×3.

5 (a) The row space of $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ is 1-dimensional. Find $\boldsymbol{v}_1$ in the row space and $\boldsymbol{u}_1$ in the column space. What is σ_1? Why is there no σ_2?