GAITSGORY'S CENTRAL FUNCTOR AND THE ARKHIPOV–BEZRUKAVNIKOV EQUIVALENCE IN MIXED CHARACTERISTIC

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ABSTRACT. We show that the nearby cycles functor for the p-adic Hecke stack at Iwahori level is perverse t-exact, by developing a theory of Wakimoto filtrations, and that it lifts to the $\mathbb{E}_1$-center. We apply these tools to construct the Arkhipov–Bezrukavnikov functor for p-adic affine flag varieties, and prove that it is an equivalence for type A groups.

CONTENTS

1. INTRODUCTION

Let F be a p-adic field with ring of integers O and residue field k, and let G be a connected reductive F-group with Iwahori O-model $\mathcal{I}$. The goal of this paper is to define a p-adic analogue of the Gaitsgory central functor [Gai01] sending perverse sheaves on the Hecke stack $\mathrm{Hk}_{G,C}$ over the completed algebraic closure C of F to central perverse sheaves on the Hecke stack $\mathrm{Hk}_{\mathcal{I},\bar{k}}$, and also of the Arkhipov–Bezrukavnikov functor [AB09] relating coherent sheaves on the dual Springer variety $\hat{\mathcal{N}}_{\mathrm{Spr}}$ to constructible étale sheaves on $\mathrm{Hk}_{\mathcal{I},\bar{k}}$. During the introduction, we will assume for simplicity that G is split and that the coefficients of our sheaves equal $\bar{\mathbb{Q}}_\ell$, as these hypothesis hold for most of the paper. We begin by recalling some of the representation-theoretic aspects at the level of Grothendieck groups.

1.1. **Hecke algebras.** We assume G is a pinned split connected reductive F-group, i.e. equipped with a choice of Borel subgroup B, a maximal split torus $T \subset B \subset G$, and pinning isomorphisms for the root groups attached to positive simple roots with respect to B. The corresponding Iwahori–Weyl group $W = N(F)/T(O)$, where N is the normalizer of $T \subset G$, admits a length function ℓ which makes (W, ℓ) into a quasi-Coxeter group.

Let $\mathbb{H}$ denote the affine Hecke algebra. Recall that the Iwahori-Matsumoto presentation, see [IM65], defines $\mathbb{H}$ as the $\mathbb{Z}[q^{\pm 1/2}]$-algebra generated by a basis $\{T_w | w \in W\}$ modulo relations $T_w T_{w'} = T_{ww'}$ if $\ell(w) + \ell(w') = \ell(ww')$, and $(T_s + q^{-1/2})(T_s - q^{1/2}) = 0$ for all length 1 elements s. Let $I \subset G(F)$ be the Iwahori subgroup associated with B. Then, the Iwahori-Hecke algebra $\mathcal{H} := C_c(I \backslash G(F)/I)$ is the space of compactly supported smooth functions on $I \backslash G(F)/I$. Fixing a Haar measure on $G(F)$ so that I has measure 1, convolution of functions equips $\mathcal{H}$ with the structure of a unital associative algebra. The affine Hecke algebra specializes to the Iwahori-Hecke algebra via the isomorphism

$$\mathbb{H} \otimes_{\mathbb{Z}[q^{\pm 1}]} \bar{\mathbb{Q}}_\ell \simeq \mathcal{H},$$

where $\bar{\mathbb{Q}}_\ell$ is regarded as a $\mathbb{Z}[q^{\pm 1/2}]$-algebra by mapping q to the cardinality of k (and thus choosing a square root of this integer in $\bar{\mathbb{Q}}_\ell$). Let $\mathbb{H}_f \subset \mathbb{H}$ be the finite Hecke algebra associated with the finite Weyl group and the set of simple reflections. The *antispherical module* is defined as

$$\mathbb{M}^{\mathrm{as}} := \mathbb{Z}[q^{\pm 1/2}]^{\mathrm{sgn}} \otimes_{\mathbb{H}_f} \mathbb{H},$$

where $T_w \in \mathbb{H}_f$ acts by multiplication by $(-1)^{\ell(w)} q^{1/2}$ on $\mathbb{Z}[q^{\pm 1/2}]^{\mathrm{sgn}}$. We also define the antispherical module for $\mathcal{H}$ to be $\mathcal{M}^{\mathrm{as}} := \mathbb{M} \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \bar{\mathbb{Q}}_\ell$.

According to Grothendieck's sheaf-function dictionary, the space of functions on the set of $\mathbb{F}_q$-points of a scheme has the category of complexes of coherent or constructible sheaves as its geometric counterpart. Let $\hat{G}$ denote the Langlands dual group of G over $\bar{\mathbb{Q}}_\ell$, and $\hat{U} \subset \hat{B} \subset \hat{G}$ be a Borel subgroup and its unipotent radical. Recall the Springer resolution of the nilpotent cone $\hat{\mathcal{N}} \subset \mathrm{Lie}(\hat{G})$

$$p_{\mathrm{Spr}} : \hat{\mathcal{N}}_{\mathrm{Spr}} = \hat{G} \times^{\hat{B}} \mathrm{Lie}\,\hat{U} \to \hat{\mathcal{N}}.$$

The Steinberg variety is defined as $\hat{\mathrm{St}} := \hat{\mathcal{N}}_{\mathrm{Spr}} \times_{\hat{\mathcal{N}}} \hat{\mathcal{N}}_{\mathrm{Spr}}$. Kazhdan-Lusztig [KL87] showed that the affine Hecke algebra is isomorphic to the Grothendieck group $K_0([\hat{G} \times \mathbb{G}_m \backslash \hat{\mathrm{St}}])$, where the latter has an algebra structure induced by convolution. In particular, the antispherical module $\mathbb{M}^{\mathrm{as}}$ is identified with $K_0([\hat{G} \times \mathbb{G}_m \backslash \hat{\mathcal{N}}_{\mathrm{Spr}}])$. If we forget the $\mathbb{G}_m$-equivariance, then we recover both the Iwahori-Hecke algebra $\mathcal{H}$ and its anti-spherical module $\mathcal{M}^{\mathrm{as}}$.

On the other hand, it follows from the work of Iwahori–Matsumoto [IM65] that the Iwahori-Hecke algebra coincides with $K_0(\mathcal{P}(\mathrm{Hk}_\mathcal{I}))$ where $\mathcal{P}$ denotes the category of perverse sheaves on the Hecke stack $\mathrm{Hk}_\mathcal{I} = L^+\mathcal{I} \backslash LG / L^+\mathcal{I}$. The natural action of $\mathcal{H}$ on $\mathcal{M}^{\mathrm{as}}$ induces a surjective map $\mathcal{H} \to \mathcal{M}^{\mathrm{as}}$ with kernel generated by the Kazhdan-Lusztig basis elements indexed by the $w \in W$, which are not minimal in their left W_{fin}-coset. This leads us to consider the *antispherical category* of perverse sheaves $\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_\mathcal{I})$ given as the Serre quotient by the IC-sheaves index by those w. Another approach to realize $\mathcal{M}^{\mathrm{as}}$ is as the I-invariants of the compact induction to $G(F)$ of a generic character χ of the unipotent radical I_u^{op} of the opposite Iwahori subgroup. This is the so-called Iwahori–Whittaker model and its categorification plays an important role later in our arguments.

We have observed that there is an abundance of spaces and sheaves that seem related to affine Hecke algebras and their anti-spherical modules. In the next sections, we will explain how to upgrade these isomorphisms to equivalences of stable ∞-categories. A guiding principle for this is the fact that there are certain objects which serve as building blocks for the various categories and we must track down where they get sent to. This is motivated by Bernstein's construction of translation elements θ_ν in the affine Hecke algebra whose trace along the finite Weyl orbits are central, and also of an isomorphism between the spherical Hecke algebra and the center of the Iwahori-Hecke algebra.

1.2. **The central functor.** The first goal of this paper is to fully develop the central functor $\mathcal{Z}$ for p-adic groups, in analogy with Gaitsgory's central functor from [Gai01] in the function field

case. The correct geometric setup for this construction is naturally the world of diamonds or more generally v-sheaves on perfectoids of characteristic p. Indeed, we have a Hecke stack $\mathrm{Hk}_{\mathcal{I}}$ defined over $\mathrm{Spd}(O)$ and we can define a nearby cycles functor $R\Psi$ by pull-push along the geometric fibers over a complete algebraic closure C of F and the residue field k. Indeed, this was already partially exploited in [AGLR22] to define certain complexes of sheaves $\mathcal{Z}(V) = R\Psi(\mathrm{Sat}(V))$, where Sat denotes the geometric Satake equivalence of Fargues–Scholze [FS21]. It was proved in [AGLR22] that these complexes are algebraic and constructible, that they carry certain centrality isomorphisms, and finally that they are supported at the V-admissible locus $\mathcal{A}_{\mathcal{I},V}$.

Two very important properties of the functor $\mathcal{Z}$ remained however elusive in [AGLR22], namely verifying that the centrality isomorphisms of $\mathcal{Z}$ satisfy various expected compatibilities that make it into a central functor, and that it lands in the category of perverse sheaves. Our most important results in this direction can be resumed as follows:

Theorem 1.1 (Theorems 4.15 and 4.17, Corollaries 4.19 and 4.20). *The functor $\mathcal{Z}\colon \mathrm{Rep}(\hat{G}) \to \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{I},k})$ lifts to an $\mathbb{E}_2$-monoidal functor towards the $\mathbb{E}_1$-center of the right side. Moreover, each $\mathcal{Z}(V)$ is perverse, convolution-exact, has unipotent monodromy, and admits a Wakimoto filtration whose associated graded is canonically isomorphic to $\mathcal{I}(V_{|\hat{T}})$.*

Let us explain a bit of the notions and ingredients that go into the above theorem. Our treatment of the centrality of $\mathcal{Z}$ is to our best knowledge the only one that uses the machinery of stable ∞-categories, which entails additional higher coherent homotopies. An important technical tool is the notion of an abstract six-functor formalism in the sense of Mann [Man22a, Man22b], which allows us to work at the level of the category of correspondences. Once we are there, we perform the usual fusion trick of looking at the disjoint locus of $(\mathrm{Spd}(O_C))^2$ and conclude the desired monoidality via full faithfulness of pullback away from the diagonal for those sheaves which are perverse over $(\mathrm{Spd}C)^2$. To obtain this full faithfulness, we apply a certain calculation of nearby cycles of kimberlites from [GL22].

Trying to prove perversity of $\mathcal{Z}(V)$ was the genesis of this project. Contrary to the function field case, we cannot rely on Artin vanishing to provide us with this crucial fact. Instead, we look at the Wakimoto functor $\mathcal{I}\colon \mathrm{Rep}(\hat{T}) \to \mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ following [AB09], but defined instead at the level of complexes as in [AR]. The centrality of $\mathcal{Z}(V)$ implies that it lies in the full subcategory generated by the essential image of $\mathcal{I}$ under extensions. Each graded piece can then be recovered by invoking geometric Satake and a certain orthogonality with respect to the constant terms $\mathrm{CT}_{B^{\mathrm{op}}}$, which proves perversity and the existence of a Wakimoto filtration all at once. This differs considerably from the strategy in [AB09], which exploits both perversity and convolution-exactness (known in equicharacteristic by Artin vanishing). We have learned that work in preparation of Cass–Scholbach–van den Hoven on the motivic counterpart of the central functor building on [CvdHS22] has benefited from our proof. From the Wakimoto filtration, we can deduce the convolution-exactness and unipotency of the monodromy operator induced by the Galois group (note that $\mathrm{Sat}(V)$ descends to $\mathrm{Spd}(F)$ with trivial inertia action). In the meantime, the Wakimoto filtration has been decisively used in [GL22] to give a new proof of unibranchness (i.e., topological normality) of local models in complete generality.

1.3. **The AB functor and Iwahori–Whittaker sheaves.** The next part of our paper carries out the construction of the various functors from [AB09] and, except for treating ∞-categorical questions carefully, does not significantly diverge from it. Recall that we work here with the Springer stack $[\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]$ that resolves the corresponding nilpotent stack of $\hat{G}$. We have the following result:

Theorem 1.2 (Proposition 5.7). *There is a monoidal functor $\mathcal{F}\colon \mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$ extending $\mathcal{Z} \times \mathcal{I}$.*

Let us briefly describe the construction of $\mathcal{F}$ following [AB09]. The main idea consists in defining an analogous functor starting from the quotient stack $[\hat{G} \times \hat{T} \backslash \hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}]$ containing $[\hat{G} \backslash \hat{\mathcal{N}}_{\mathrm{Spr}}]$ as a dense open stack, where $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}$ is the affine hull of the canonical $\hat{T}$-torsor over $\hat{\mathcal{N}}_{\mathrm{Spr}}$. The projective objects of the category of coherent sheaves on the enlarged stack can be mapped to $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ by using the functors $\mathcal{Z}$, $\mathcal{I}$, and the nilpotent monodromy endomorphism of the former, after verifying the Plücker relations. After deriving this functor to perfect complexes on the affine stack, we are reduced to showing vanishing on complexes supported at the complement of $[\hat{G} \backslash \hat{\mathcal{N}}_{\mathrm{Spr}}]$.

Next, we study the category $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}\mathcal{W}})$ of Iwahori–Whittaker sheaves. This is the stable ∞-category of $(L^+ \mathcal{I}_{\mathrm{u}}^{\mathrm{op}}, \mathcal{L})$-twisted equivariant constructible sheaves on $\mathrm{Fl}_{\mathcal{I}}$, where $\mathcal{L}$ is a certain character sheaf obtained via the Artin–Schreier cover. It also carries a perverse t-structure and the category $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}\mathcal{W}})$ is a highest weight category in the sense of Beilinson–Ginzburg–Soergel [BGS96], with simple and tilting objects indexed by $\mathbb{X}_*(T)$. We get a perverse t-exact averaging functor $\mathrm{av}_{\mathcal{I}\mathcal{W}} \colon \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}\mathcal{W}})$ given by left convolution against the simple object attached to $0 \in \mathbb{X}_*(T)$ and we denote by $\mathcal{Z}_{\mathcal{I}\mathcal{W}}$, resp. $\mathcal{F}_{\mathcal{I}\mathcal{W}}$, the composition $\mathrm{av}_{\mathcal{I}\mathcal{W}} \circ \mathcal{Z}$, resp. $\mathrm{av}_{\mathcal{I}\mathcal{W}} \circ \mathcal{F}$.

Theorem 1.3 (Theorem 7.4). *If G is of type A, then $\mathcal{Z}_{\mathcal{I}\mathcal{W}}(V)$ is tilting.*

Notice that in [AB09] there is no assumption on the type of G. Unfortunately, we are missing a crucial ingredient replacing the $\mathbb{G}_{m,k}$-action given by rotating the uniformizer, which is impossible in the p-adic setting. Let us explain how the proof works, so as to better understand the impact of this gap. First of all, the tilting property propagates under convolution and can be verified on adjoint quotients, so we may assume V is either minuscule or quasi-minuscule. In the minuscule case, all the weights are comprised in a finite Weyl orbit, so one can easily verify the given property. In the quasi-minuscule case, we must still handle the (co)restriction of $\mathcal{Z}_{\mathcal{I}\mathcal{W}}(V)$ to the weight 0 Iwahori–Whittaker cell. Here, the vanishing can be achieved by calculating the alternating sum of the Ext groups via an argument on Grothendieck groups and finally bound a $\mathrm{Hom}(\mathcal{Z}_{\mathcal{I}\mathcal{W}}(1), \mathcal{Z}_{\mathcal{I}\mathcal{W}}(V))$ accordingly. Now, this bound is achieved in [AB09] via the theory of the regular quotient described below together with the fact that the monodromy operator is defined for every sheaf in $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ as it is induced by the $\mathbb{G}_{m,k}$-action given by rotation. For type A groups, we can realise every representation up to central isogeny as a direct summand of a minuscule representation, so we still get the tilting property.

The last step in proving that $\mathcal{F}_{\mathcal{I}\mathcal{W}}$ is an equivalence (now, necessarily only in type A) revolves around the regular orbit $\mathcal{O}_{\mathrm{r}} \subset \hat{\mathcal{N}}$ inside the nilpotent cone. The Springer resolution is an isomorphism above this $\hat{G}$-orbit, and hence we should find a category of étale sheaves that plays a similar role. For this, we look at the Serre quotient $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$ of perverse sheaves on the Hecke stack obtained by modding out IC sheaves with positive-dimensional support.

Theorem 1.4 (Proposition 8.3, Proposition 10.8). *If G is of type A, then there is a symmetric monoidal equivalence $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}}) \xrightarrow{\sim} \mathrm{Rep}(Z_{\hat{G}}(n_0))$, where n_0 is a regular nilpotent element.*

Together with the tilting property, this result is key in order to prove the Arkhipov–Bezrukavnikov equivalence, as it induces certain injections of Hom maps. Let us remark that the most delicate point in the above theorem consists in showing that n_0 is regular. For this, we use the theory of weights by descending $\mathcal{Z}(V)$ to a mixed sheaf and calculating its monodromy filtration. In [AB09], one applies Gabber's local weight-monodromy theorem, see [BB93], stating that the weight filtration equals the monodromy filtration, and then calculates the former via the affine Hecke algebra. This is not available for our nearby cycles, unless μ is minuscule, by work of Hansen–Zavyalov [HZ23] combined with the representability theorem in [AGLR22]. Again, we

can only easily reproduce this argument for type A groups. It would be possible to adapt a different argument due to Bezrukavnikov–Riche–Rider [BRR20], but this would lead us into some detours that seem unnecessary, as we do not have the tilting property for other groups.

Let us finish by stating the second main result of this paper, i.e. the AB equivalence for p-adic groups of type A:

Theorem 1.5 (Theorem 9.1). *If G is of type A, then the functor $\mathcal{F}_{\mathcal{IW}}\colon \mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{IW}})$ is an equivalence.*

We strongly believe that this result must also hold for general split connected reductive groups G. Yun–Zhu have announced in conference talks regarding work of preparation of Hemo–Zhu, see also [Zhu20], a proof of the full Bezrukavnikov equivalence [Bez16] for p-adic groups, that builds on a colimit presentation in terms of double quotients of parahoric jet groups due to Tao–Travkin [TT20]. Recently, Bando [Ban23] also gave a distinct proof of the Bezrukavnikov equivalence for p-adic groups by comparing constructible-étale sheaves in equi- and mixed characteristic via an ingenious geometric construction. However, these previous methods do not yield concrete knowledge about the central functor, whereas our paper places $\mathcal{Z}$ right at the center of it all. We also think that our functor $\mathcal{Z}$ will naturally appear in the picture if one studies étale sheaves on p-adic Hecke stacks, see, e.g., the unibranchness theorem of [GL22], and thus it must play a role in comparing the Zhu [Zhu20] and the Fargues–Scholze [FS21] variants of a categorical p-adic local Langlands correspondence. A natural task for the future will be to explain if and how all of the previous approaches fit together, namely by comparing a priori different central functors.

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1.5. **Notation.** Unfortunately, we will have to use a lot of notations. Thus, let's get over this and define the following objects, which will occur in the whole text.

First, let us discuss scheme-theoretic notations.

- p a prime,
- F a p-adic field with ring of integers O, and (perfect) residue field k.
- $\overline{F}$ an algebraic closure of F and $\Gamma := \mathrm{Gal}(\overline{F}/F)$ the absolute Galois group,
- $\check{\Gamma} \subseteq \Gamma$ the inertia subgroup and $\Gamma^{\mathrm{un}} := \Gamma/\check{\Gamma}$ the unramified quotient,
- $\check{F}$ the completion of the maximal unramified extension of F in $\overline{F}$, $\check{O} \subseteq \check{F}$ its ring of integers, and $\overline{k}$ the residue field of $\check{O}$,
- G a quasi-split reductive group over F,
- $S \subseteq T \subseteq B \subseteq G$ a maximal split torus $S \subseteq G$, T its centralizer (a maximal torus in G as G is quasi-split), and a Borel $B \subseteq G$ containing T,
- $B^- \subseteq G$ is the opposite Borel of B,

- $N := N_G(T)$ denotes the normalizer of T in G,
- $\mathcal{T}$ the connected Néron model of T over O,
- If H/F is a torus, then $\mathbb{X}_\bullet(H), \mathbb{X}^\bullet(H)$ denote the groups of (geometric) cocharacters/characters of H,
- $\mathbb{X}_\bullet := \mathbb{X}_\bullet(T)$, $\mathbb{X}^\bullet := \mathbb{X}_\bullet(T)$,
- $\bar{\mathbb{X}}_\bullet := \mathbb{X}_{\bullet,\check{\Gamma}}$, where the subscript $\check{\Gamma}$ denotes the coinvariants.
- $\mathbb{X}_\bullet^+$, $\mathbb{X}^{\bullet,+}$ denote the dominant cocharacters resp. dominant characters of T with respect to B,
- $\mathbb{X}_\bullet(S)^+$, $\mathbb{X}^{\bullet,+}(S)$ denote the dominant cocharacters resp. dominant characters of S with respect to B,

Next, let us introduce combinatorial notations.

- $W := N(\check{F})/\mathcal{T}(\check{O})$ the Iwahori-Weyl group of T, also called extended affine Weyl group,
- $\mathcal{A}(G,S) \cong \mathbb{X}_\bullet(S)_\mathbb{R}$ the appartment associated with S,
- $\mathbf{a} \subseteq \mathcal{A}(G,S)$ a fixed alcove,
- $\mathbb{S} \subseteq W$ the set of reflections at the walls of $\mathbf{a}$, also called the set of simple reflections,
- $W_{\mathrm{af}} \subseteq W$ the affine Weyl group, which is the Coxeter group generated by the simple reflections,
- $\Omega_\mathbf{a}$ the stabilizer of $\mathbf{a}$, which yields an isomorphism

$$W \cong W_{\mathrm{af}} \rtimes \Omega_\mathbf{a}. \tag{1.1}$$

- $\ell \colon W \to \mathbb{N}_{\geq 0}$ the length function on W, i.e., the unique function $\ell(-) \colon W \to \mathbb{N}_{\geq 0}$, which extends the length function $\ell(-) \colon W_{\mathrm{af}} \to \mathbb{N}_{\geq 0}$ on the Coxeter group W_{af}, such that $\ell(\tau) = 0$ for $\tau \in \Omega_\mathbf{a}$.
- $\leq$ is the Bruhat order on W, i.e., $w \leq w'$ for $w = (w_{\mathrm{af}}, \tau), w' = (w'_{\mathrm{af}}, \tau') \in W \cong W_{\mathrm{af}} \rtimes \pi_1(G)_{\check{\Gamma}}$ if and only if $\tau = \tau'$ and $w_{\mathrm{af}} \leq w'_{\mathrm{af}}$ for the Bruhat order $\leq$ on W_{af} coming from its Coxeter structure,
- $W_{\mathrm{fin}} = N(\check{F})/T(\check{F})$ the finite Weyl group, which sits in an exact sequence

$$1 \to \bar{\mathbb{X}}_\bullet \to W \to W_{\mathrm{fin}} \to 1. \tag{1.2}$$

- $t_{\bar\nu} \in W$ is the translation element associated with $\bar\nu \in \bar{\mathbb{X}}_\bullet$.
- $w_{\bar\nu}$ denotes the minimal length element in the coset $W_{\mathrm{fin}} t_{\bar\nu}$ for $\bar\nu \in \bar{\mathbb{X}}_\bullet$.

Now let us define notations related to affine flag varieties and perfect geometry.

- $\mathrm{Alg}_k^{\mathrm{perf}}$ the category of perfect k-algebras,
- for a scheme X over k we denote by X^{pf} its perfection,
- for $R \in \mathrm{Alg}_k^{\mathrm{perf}}$ we set $W_O(R) := O \otimes_{W(k)} W(-)$, the ring of O-Witt vectors,
- if X/O is an affine scheme of finite type, then $L^+X \colon (\mathrm{Alg}^{\mathrm{perf}}) \to (\mathrm{Sets})$, $R \mapsto X(W_O(R))$ is the positive loop functor for X,
- if Z/F is an affine scheme of finite type, then $LZ \colon (\mathrm{Alg}^{\mathrm{perf}}) \to (\mathrm{Sets})$, $R \mapsto Z(W_O(R)[1/p])$ is the loop functor for Z,
- if $\mathcal{G}/O$ is a parahoric model of G, then the quotient of étale sheaves $\mathrm{Fl}_\mathcal{G} := L\mathcal{G}/L^+\mathcal{G}$ is the (partial) affine flag variety for $\mathcal{G}$,
- the quotient stack $\mathrm{Hk}_\mathcal{G} = [L^+\mathcal{G}\backslash\mathrm{Fl}_\mathcal{G}]$ in the étale topology is the Hecke stack for $\mathcal{G}$,
- $\mathcal{I}/O$ the Iwahori group scheme for G associated with the alcove $\mathfrak{a}$,
- $\check{\mathcal{I}} := \mathcal{I} \otimes_O \check{O}$, which is an Iwahori group scheme for $\check{G} := G \otimes_O \check{O}$,
- $I := L^+\mathcal{I}$, $\check{I} := L^+\check{\mathcal{I}}$.

Next, let us introduce some cohomological notations.

- $\ell \neq p$ a prime,
- Λ an algebraic extension of $\mathbb{F}_\ell$ or $\mathbb{Q}_\ell$.

- $\mathcal{D}_{\text{ét}}(-) := \mathcal{D}_{\text{ét}}(-, \Lambda)$ denotes the ∞-derived category of "étale sheaves of Λ-modules" on a perfect scheme, or a small v-stack.
- $\mathcal{D}_{\text{ét}}(\text{Hk}_{\mathcal{I}}) := \mathcal{D}_{\text{ét}}(\text{Hk}_{\mathcal{I}}, \Lambda)^{\text{bd}}$ denotes the ∞-category of "étale sheaves of Λ-modules" on $\text{Hk}_{\mathcal{I}}$, whose support is a finite subset of the underlying topological space of $\text{Hk}_{\mathcal{I}}$.

Finally, we collect our notations for the "coherent" side. Note that we consider these objects usually under the assumption that Λ is a field extension of $\mathbb{Q}_\ell$.

- $\hat{G}$ denotes the dual group of G over Λ,
- $\hat{T} \subseteq \hat{G}$ denotes the dual torus to T, and we identify $\mathbb{X}^\bullet(\hat{T}) \cong \mathbb{X}_\bullet(T)$,
- $\hat{G}' := \hat{G} \times \hat{T}$,
- $\hat{T} \subseteq \hat{B} \subseteq \hat{G}$ denotes the Borel subgroup with dominant characters identifying with $\mathbb{X}_\bullet(T)^+$,
- $\hat{U} \subseteq \hat{B}$ is the unipotent radical of $\hat{B}$ with Lie algebra $\text{Lie}(\hat{U})$,
- $\hat{\mathfrak{g}} := \text{Lie}(\hat{G})$ denotes the Lie algebra of $\hat{G}$,
- $\hat{\mathcal{N}} \subseteq \hat{\mathfrak{g}}$ is the nilpotent cone, i.e., the closed subscheme of nilpotent elements,
- $p_{\text{Spr}} \colon \hat{\mathcal{N}}_{\text{Spr}} := \hat{G} \times^{\hat{B}} \text{Lie}(\hat{U}) \to \hat{\mathcal{N}}$ denotes the Springer resolution of the nilpotent cone,
- $\hat{\mathcal{N}}_{\text{Spr}}^{\text{qaf}} := \hat{G} \times^{\hat{U}} \text{Lie}(\hat{U}) \to \hat{\mathcal{N}}_{\text{Spr}}$ denotes the canonical $\hat{T}$-torsor over $\hat{\mathcal{N}}_{\text{Spr}}$,
- $\hat{\mathcal{X}} := \text{Spec}(\mathcal{O}(\hat{G}/\hat{U}))$ is the affine closure of the quasi-affine scheme $\hat{G}/\hat{U}$,

2. Geometry of the affine flag variety

In this section, we want to recall the geometry of the (Witt vector) affine flag variety

$$\text{Fl}_{\mathcal{I}}, \tag{2.1}$$

which was first considered as an algebraic space in [Zhu17, Section 1.4]. Its representability by an ind-(perfected projective k-scheme) was then proven in [BS17, Corollary 9.6] via reduction to $G = \text{GL}_n$ and the construction of the determinant line bundle there. Let us note that

$$\text{Fl}_{\mathcal{I}} \times_{\text{Spec}(k)} \text{Spec}(\overline{k}) \cong \text{Fl}_{\breve{\mathcal{I}}}. \tag{2.2}$$

Hence, geometric questions for $\text{Fl}_{\mathcal{I}}$ often reduce to the case $F = \breve{F}$.

2.1. **Schubert varieties and convolution.** The geometry of the Iwahori orbits on the affine flag variety is summarized in the next lemma.

Lemma 2.1.   (1) *The map* $N(\breve{F}) \to \text{Fl}_{\breve{\mathcal{I}}}$, $n \mapsto n \cdot \breve{I}$ *induces a bijection*

$$W = N(\breve{F})/\mathcal{T}(\breve{O}) \to [\breve{I} \backslash \text{Fl}_{\breve{\mathcal{I}}}] \tag{2.3}$$

on underlying topological spaces, i.e, the $\breve{I}$-orbits $\text{Fl}_{\breve{I},w} = \breve{I} \cdot w$ on $\text{Fl}_{\breve{I}}$ are indexed by W.

(2) *The $\breve{I}$-orbits on $\text{Fl}_{\breve{I}}$ form a stratification of $\text{Fl}_{\breve{\mathcal{I}}}$, i.e., the closure $\text{Fl}_{\breve{\mathcal{I}},w}$ of a Schubert cell $\text{Fl}_{\breve{\mathcal{I}},w}$ is a union of Schubert cells. More precisely,*

$$\text{Fl}_{\breve{\mathcal{I}}, \leq w} = \bigcup_{w' \leq w} \text{Fl}_{\breve{\mathcal{I}}, w'} \tag{2.4}$$

for the Bruhat order on W.

(3) *The map $N(\breve{F}) \to \text{Fl}_{\breve{\mathcal{I}}}$ descends to a bijection*

$$W_{\Gamma-\text{orbits}} \to \text{Hk}_{\mathcal{I}} \tag{2.5}$$

on topological spaces, i.e., the I-orbits $\text{Fl}_{\mathcal{I}, \Gamma \cdot w}$ on $\text{Fl}_{\mathcal{I}}$ are indexed by Γ-orbits $\Gamma \cdot w$ in W.

(4) *If $w \in W$, then*

$$\text{Fl}_{\mathcal{I}, \leq \Gamma \cdot w} := \overline{\text{Fl}_{\mathcal{I}, \Gamma \cdot w}} = \bigcup_{w' \in W, \; w' \leq \gamma \cdot w \text{ for some } \gamma \in \Gamma} \text{Fl}_{\mathcal{I}, \Gamma \cdot w'}. \tag{2.6}$$

Proof. Statement (1) is essentially the Bruhat decomposition, see [BT72, Théorème 6.5] or [KP23, Theorem 7.8.1],

$$W \cong \mathcal{I}(\breve{F}) \backslash G(\breve{F}) / \mathcal{I}(\breve{F}) \tag{2.7}$$

(applied over all formally unramified extensions of $\breve{O}$). Then (2) follows by considering convolution and the Demazure resolutions, cf. [Zhu17, Section 1.4] for details. Statement (3), (4) follow by descent along $\mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(k)$. $\square$

Definition 2.2. The (perfect) projective schemes $\mathrm{Fl}_{\breve{\mathcal{I}}, \leq w}, \mathrm{Fl}_{\mathcal{I}, \leq \Gamma \cdot w}$ are called Schubert varieties, while their open and dense subschemes $\mathrm{Fl}_{\breve{\mathcal{I}}, w}, \mathrm{Fl}_{\mathcal{I}, \Gamma \cdot w}$ are called Schubert cells.

For simplicity, we will often write

$$\mathrm{Fl}_{\mathcal{I}, w} := \mathrm{Fl}_{\mathcal{I}, \Gamma \cdot w}, \mathrm{Fl}_{\mathcal{I}, \leq w} := \mathrm{Fl}_{\mathcal{I}, \leq \Gamma \cdot w} \tag{2.8}$$

for $w \in W$.

The discrepancy of I- and $\breve{I}$-orbits is irrelevant under certain assumptions on G.

Lemma 2.3. *The following conditions are equivalent:*

(1) *The Γ-action on W is trivial.*
(2) *Γ acts trivially on $\bar{\mathbb{X}}_\bullet$.*
(3) *G is residually split, i.e., for every facet in the building of G the reductive quotient of the special fiber of the associated parahoric group scheme is split over k.*

Proof. By [KP23, Proposition 7.10.10] the group G is residually split if and only if Γ acts trivially on $\bar{\mathbb{X}}_\bullet$. If Γ acts trivially on W, then as well on $\bar{\mathbb{X}}_\bullet \subseteq W$. Assume now that Γ acts trivially on $\bar{\mathbb{X}}_\bullet$. As W is generated by W_{af} and $\bar{\mathbb{X}}_\bullet$ it suffices to show that Γ acts trivially on W_{af}. But W_{af} embeds Γ-equivariantly into the group of affine transformations on $\bar{\mathbb{X}}_{\bullet, \mathbb{R}}$, and the Γ-action on the latter is trivial. $\square$

If $n \geq 1$, the contracted product

$$\underset{n-\text{factors}}{\mathrm{Fl}_{\mathcal{I}} \tilde{\times} \cdots \tilde{\times} \mathrm{Fl}_{\mathcal{I}}} := LG \times^I \cdots \times^I LG/I \tag{2.9}$$

is called the n-fold convolution product of $\mathrm{Fl}_{\mathcal{I}}$. The multiplication morphism

$$m := m_{\mathrm{Fl}_{\mathcal{I}}} \colon \mathrm{Fl}_{\mathcal{I}} \tilde{\times} \cdots \tilde{\times} \mathrm{Fl}_{\mathcal{I}} \to \mathrm{Fl}_{\mathcal{I}}, \ \overline{(g_1, \ldots, g_n)} \mapsto g_1 \cdots g_n I \tag{2.10}$$

has interesting geometric properties. If $X_1, \ldots, X_n \subseteq \mathrm{Fl}_{\mathcal{I}}$ are (locally) closed I-stable subschemes and $Y_1, \ldots, Y_n \subseteq LG$ their preimages, then we set

$$X_1 \tilde{\times} \ldots \tilde{\times} X_n := Y_1 \times^I \ldots \times^I Y_n / I, \tag{2.11}$$

which is a (locally) closed subscheme of $\mathrm{Fl}_{\mathcal{I}} \tilde{\times} \ldots \tilde{\times} \mathrm{Fl}_{\mathcal{I}}$.

Lemma 2.4. *Let $w_1, \ldots, w_n \in W$ and assume that $\ell(w_1 \cdots w_n) = \ell(w_1) + \cdots + \ell(w_n)$.*

(1) *The map*

$$\mathrm{Fl}_{\mathcal{I}, w_1} \tilde{\times} \cdots \tilde{\times} \mathrm{Fl}_{\mathcal{I}, w_n} \to \mathrm{Fl}_{\mathcal{I}} \tag{2.12}$$

has image in $\mathrm{Fl}_{\mathcal{I}, w_1 \cdots w_n}$ and induces an isomorphism

$$\mathrm{Fl}_{\mathcal{I}, w_1} \tilde{\times} \cdots \tilde{\times} \mathrm{Fl}_{\mathcal{I}, w_n} \cong \mathrm{Fl}_{\mathcal{I}, w_1 \cdots w_n}. \tag{2.13}$$

(2) *We have*

$$m(\mathrm{Fl}_{\mathcal{I}, \leq w_1} \tilde{\times} \cdots \tilde{\times} \mathrm{Fl}_{\mathcal{I}, \leq w_n}) \subseteq \mathrm{Fl}_{\mathcal{I}, \leq w_1 \cdots w_n} \tag{2.14}$$

and the map

$$\mathrm{Fl}_{\mathcal{I}, w_1} \tilde{\times} \cdots \tilde{\times} \mathrm{Fl}_{\mathcal{I}, w_n} \to \mathrm{Fl}_{\mathcal{I}, \leq w_1 \cdots w_n} \tag{2.15}$$

is (perfectly) proper and birational.

(3) *If $w \in W$ and $\tau \in \Omega_{\mathbf{a}}$, then*

$$\mathrm{Fl}_{\breve{\mathcal{I}},w} \to \mathrm{Fl}_{\breve{\mathcal{I}},w\tau}, \ gI \mapsto g\tau I \tag{2.16}$$

is an isomorphism.

Proof. It suffices to check the statements in the case that $F = \breve{F}$. Using induction on n, one reduces the first statement to the case that $n = 2$. Now,

$$\mathcal{I}(\breve{O})w_1\mathcal{I}(\breve{O}) \cdot \mathcal{I}(\breve{O})w_2\mathcal{I}(\breve{O}) \cong \mathcal{I}(\breve{O})w_1w_2\mathcal{I}(\breve{O}) \tag{2.17}$$

by the assumption $\ell(w_1)+\ell(w_2) = \ell(w_1w_2)$: indeed, this follows from the theory of Tits systems, which we can apply by [BT72, Théorème 6.5] and [BT84, Corollaire 6.4.7]. This implies the claim when also applied over all formally unramified extensions of $\breve{O}$. The second statement follows from the first, and the third follows from the first and second as $I(\breve{O})\tau I(\breve{O}) = \tau I(\breve{O})$ for $\tau \in \Omega_{\mathbf{a}}$. $\qquad\square$

For example, if $w = s_1 \ldots s_n\tau$ is a reduced word, with s_i being simple reflections and τ stabilizing $\mathbf{a}$, then $\mathrm{Fl}_{\breve{\mathcal{I}},\leq s_1} \tilde{\times} \ldots \tilde{\times} \mathrm{Fl}_{\breve{\mathcal{I}},\leq s_n} \tilde{\times} \mathrm{Fl}_{\breve{\mathcal{I}},\leq\tau} \to \mathrm{Fl}_{\breve{\mathcal{I}},\leq w}$ defines the Demazure resolution. Studying Demazure resolutions yields the following important geometric consequences.

Lemma 2.5. *Let $w \in W$.*
(1) $\mathrm{Fl}_{\breve{\mathcal{I}},w} \cong \mathbb{A}_{\breve{k}}^{\ell(w),\mathrm{pf}}$, *in particular* $\dim \mathrm{Fl}_{\mathcal{I},w} = \ell(w)$.
(2) *If $\ell(w) = 0$, i.e., $w = \tau \in \Omega_{\mathbf{a}}$, then $\mathrm{Fl}_{\mathcal{I},\tau} = \mathrm{Fl}_{\mathcal{I},\leq\tau}$ and $\mathrm{Fl}_{\mathcal{I},\tau} \cong \mathrm{Spec}(l)$ for some finite extension l/k.*
(3) *If $\ell(w) = 1$, i.e., $w = s\tau$ with s a simple reflection and $\tau \in \Omega_{\mathbf{a}}$, then*

$$\mathrm{Fl}_{\breve{\mathcal{I}},\leq s} \cong \mathbb{P}_{\breve{k}}^{1,\mathrm{pf}} \tag{2.18}$$

and

$$\mathrm{Fl}_{\mathcal{I},\leq s} = \mathrm{Fl}_{\mathcal{I},\Gamma\cdot s} \tag{2.19}$$

is a finite configuration of Brauer-Severi curves with intersections occuring at points $\mathrm{Fl}_{\mathcal{I},\tau'}$ for $\tau' \in \Omega_{\mathbf{a}}$.

Proof. Using 2.4 one reduces to the case that $w = s$ is a simple reflection. In this case, one checks $\mathrm{Fl}_{\breve{\mathcal{I}},\leq s} \cong \mathbb{P}_{\breve{k}}^{1,\mathrm{pf}}$ by hand, cf. [Zhu17, Section 1.4]. The remaining assertions follow. $\qquad\square$

Let us note that the morphism

$$(\mathrm{pr},m)\colon \mathrm{Fl}_{\mathcal{I}} \tilde{\times} \mathrm{Fl}_{\mathcal{I}} \to \mathrm{Fl}_{\mathcal{I}} \times \mathrm{Fl}_{\mathcal{I}}, \ \overline{(g_1,g_2)} \mapsto (\overline{g_1}, \overline{g_1g_2}) \tag{2.20}$$

is an isomorphism, i.e., convolution products (for the full affine flag variety) are secretly just products. Given now I-stable locally closed subschemes $X_1, X_2, Y \subseteq \mathrm{Fl}_{\mathcal{I}}$ such that $m(X_1 \tilde{\times} X_2) \subseteq Y$, we can factor

$$m\colon X_1 \tilde{\times} X_2 \to Y \tag{2.21}$$

as

$$X_1 \tilde{\times} X_2 \xrightarrow{(\mathrm{pr},m)} X_1 \times Y \xrightarrow{\text{projection}} Y. \tag{2.22}$$

Regarding convolution we get the following important affineness statement.

Lemma 2.6. *Let $w \in W$. Let $X \subseteq \mathrm{Fl}_{\mathcal{I}}$ be a closed I-stable subscheme, which is a union of finitely many I-orbits. Then the maps*

$$m\colon X \tilde{\times} \mathrm{Fl}_{\mathcal{I},w} \to \mathrm{Fl}_{\mathcal{I}}, \ m\colon \mathrm{Fl}_{\mathcal{I},w} \tilde{\times} X \to \mathrm{Fl}_{\mathcal{I}} \tag{2.23}$$

are affine. In particular, the inclusion $j_w\colon \mathrm{Fl}_{\mathcal{I},w} \to \mathrm{Fl}_{\mathcal{I}}$ is affine for any $w \in W$.

The affineness of j_w follows as well from 2.5 because $\mathrm{Fl}_{\mathcal{I},\leq w} \to \mathrm{Fl}_{\mathcal{I}}$ is a closed immersion and $j_w\colon \mathrm{Fl}_{\mathcal{I},w} \to \mathrm{Fl}_{\mathcal{I},\leq w}$ is affine because $\mathrm{Fl}_{\mathcal{I},\leq w}$ is separated.

Proof. We may assume that $F = \check{F}$. The proof of [AR, Lemma 4.1.6] applies, and we recall its argument. Using the above remarks, we can write $m\colon \mathrm{Fl}_{\mathcal{I},w}\tilde{\times}X \to \mathrm{Fl}_{\mathcal{I}}$ as the composition

$$\mathrm{Fl}_{\mathcal{I},w}\tilde{\times}X \to \mathrm{Fl}_{\mathcal{I},w}\tilde{\times}\mathrm{Fl}_{\mathcal{I}} \overset{(\mathrm{pr},m)}{\cong} \mathrm{Fl}_{\mathcal{I},w} \times \mathrm{Fl}_{\mathcal{I}} \xrightarrow{\text{projection}} \mathrm{Fl}_{\mathcal{I}} \tag{2.24}$$

of morphisms of ind-perfect schemes. The first morphism is affine as $X \subseteq \mathrm{Fl}_{\mathcal{I}}$ is a closed immersion, and the second is affine by 2.5. $\qquad\square$

We need the following result later on.

Lemma 2.7. *Let $X \subseteq \mathrm{Fl}_{\mathcal{I}}$ be a finite union of I-orbits. Then there exists a finite subset $S_X \subset W$ such that for any $w \in W$ we have*

$$m(X\tilde{\times}\mathrm{Fl}_{\mathcal{I},w}) \subset \cup_{x\in S_X}\mathrm{Fl}_{\mathcal{I},xw}, \quad m(\mathrm{Fl}_{\mathcal{I},w}\tilde{\times}X) \subset \cup_{x\in S_X}\mathrm{Fl}_{\mathcal{I},wx}.$$

Proof. The proof is via an induction argument on $\ell(w)$ which is similar to the equal characteristic setting, [AR, Lemma 4.4.2]. We sketch the proof here. Of course, it suffices to treat the case $X = \mathrm{Fl}_{\mathcal{I},w}$ for some $w \in W$. Let first $X = \mathrm{Fl}_{\mathcal{I},w}$ for some $w \in W$ with $\ell(w) = 0$. Take $S_X = \{w\}$, then the statement holds by noting that $\ell(xw) = \ell(wx) = \ell(x)$ for any $x \in W$.

Assume now $X = \mathrm{Fl}_{\mathcal{I},w}$ for some $w \in W$ with $\ell(w) > 0$. Write $w = w_1 s_1 = s_2 w_2$, where $\ell(w_1) = \ell(w_2) = \ell(w) - 1$. The induction hypothesis implies that there exist finite subsets $S_{X_1}, S_{X_2} \subset W$ such that

$$m(\mathrm{Fl}_{\mathcal{I},w_1}\tilde{\times}\mathrm{Fl}_{\mathcal{I},w'}) \subset \cup_{x\in S_{X_1}}\mathrm{Fl}_{\mathcal{I},xw'}, \quad m(\mathrm{Fl}_{\mathcal{I},w'}\tilde{\times}\mathrm{Fl}_{\mathcal{I},w_2}) \subset \cup_{x\in S_{X_2}}\mathrm{Fl}_{\mathcal{I},w'x},$$

for any $w' \in W$. Note that for any $w'' \in W$,

$$\mathrm{Fl}_{\mathcal{I},s_1}\tilde{\times}\mathrm{Fl}_{\mathcal{I},w''} \subset \mathrm{Fl}_{\mathcal{I},w''} \sqcup \mathrm{Fl}_{\mathcal{I},s_1w''}, \quad \mathrm{Fl}_{\mathcal{I},w''}\tilde{\times}\mathrm{Fl}_{\mathcal{I},s_2} \subset \mathrm{Fl}_{\mathcal{I},w''} \sqcup \mathrm{Fl}_{\mathcal{I},w''s_2},$$

and

$$\mathrm{Fl}_{\mathcal{I},w_1}\tilde{\times}\mathrm{Fl}_{\mathcal{I},s_1} \simeq \mathrm{Fl}_{\mathcal{I},s_2}\tilde{\times}\mathrm{Fl}_{\mathcal{I},w_2} \simeq \mathrm{Fl}_{\mathcal{I},w}$$

by 2.4. We conclude that $S_X := S_{X_1} \cup S_{X_1}s_1 \cup S_{X_2} \cup s_2S_{X_2}$ is the desired finite subset, thereby concluding the proof. $\qquad\square$

2.2. Constant terms and semi-infinite orbits.
Let $U \subseteq B$ be the unipotent radical. Then we get the Iwasawa decompositions ([KP23, Theorem 3.3.3])

$$N(F)/\mathcal{T}(O) \simeq U(F)\backslash G(F)/\mathcal{I}(O), \; w \mapsto U(F)w\mathcal{I}(O). \tag{2.25}$$

and

$$W = N(\check{F})/\mathcal{T}(\check{O}) \simeq U(\check{F})\backslash G(\check{F})/\mathcal{I}(\check{O}), \; w \mapsto U(\check{F})w\mathcal{I}(\check{O}). \tag{2.26}$$

Geometrically, this yields the semi-infinite orbits.

Definition 2.8. For $w \in W$ we set $\mathcal{S}_w := LU \cdot w \subseteq \mathrm{Fl}_{\check{\mathcal{I}}}$.

By [AGLR22, Section 5] the $\mathcal{S}_w, w \in W$, are represented by locally closed ind-(perfect schemes) and coincide with the connected components of the attractor $\mathrm{Fl}_{\check{\mathcal{I}}}^+$ for a regular action by $\mathbb{G}_m$. More precisely, take a regular coweight $\chi\colon \mathbb{G}_{m,F} \to S$, i.e., such that the centralizer of χ in G is T (for example, the sum of all positive coroots). Then B is the attractor locus

$$G^+ = \{g \in G \mid t \mapsto \chi(t)g\chi(t)^{-1} \text{ extends to } \mathbb{A}^1_F\} \tag{2.27}$$

for the conjugation action of $\mathbb{G}_{m,F}$ on G. More generally, if $B \subseteq P \subseteq G$ is any parabolic subgroup, then there exists a character $\psi\colon \mathbb{G}_{m,F} \to S$ such that $P = G^+$ is the attractor locus for the $\mathbb{G}_{m,F}$-action on G by conjugation. The centralizer M of ψ is then a Levi subgroup of P.

The cocharacter χ extends to a group homomorphism $\mathbb{G}_{m,O} \to \mathcal{S}$ by the universal property of connected Néron models. By conjugation, we deduce a $L^+\mathbb{G}_{m,O}$-action on $\mathrm{Fl}_{\mathcal{I}}$ and we restrict it to $\mathbb{G}_{m,k}$ along the Teichmüller lift map. Over $\mathrm{Spec}(\bar{k})$, we get the decomposition

$$\mathrm{Fl}_{\breve{\mathcal{I}}}^+ = \coprod_{w \in W} \mathcal{S}_w, \text{ where } \mathcal{S}_w := LU \cdot w \subset \mathrm{Fl}_{\breve{\mathcal{I}}}, \tag{2.28}$$

of the attractor locus, cf. [AGLR22, Section 5]. Similarly, the repeller locus $\mathrm{Fl}_{\mathcal{I}}^-$, i.e., the attractor locus for the inverted $\mathbb{G}_m$-action, decomposes (over $\mathrm{Spec}(\bar{k})$) as

$$\mathrm{Fl}_{\breve{\mathcal{I}}}^- = \coprod_{w \in W} \mathcal{S}_w^-, \text{ where } \mathcal{S}_w^- := LU^- \cdot w \subset \mathrm{Fl}_{\breve{\mathcal{I}}}, \tag{2.29}$$

where U^- denotes the unipotent radical of the opposite Borel B^- of B. The semi-infinite orbits are relevant for computing constant term functors. Let

$$i^+ : \mathrm{Fl}_{\mathcal{I}}^+ \to \mathrm{Fl}_{\mathcal{I}}, \ i^- : \mathrm{Fl}_{\mathcal{I}}^- \to \mathrm{Fl}_{\mathcal{I}} \tag{2.30}$$

be the inclusions (=disjoint union of disjoint locally closed immersions). Let $\mathrm{Fl}_{\mathcal{I}}^0 \subseteq \mathrm{Fl}_{\mathcal{I}}$ be the fixed point locus of $\mathbb{G}_m$ and let

$$q^+ : \mathrm{Fl}_{\mathcal{I}}^+ \to \mathrm{Fl}_{\mathcal{I}}^0, \ q^- : \mathrm{Fl}_{\mathcal{I}}^- \to \mathrm{Fl}_{\mathcal{I}}^0 \tag{2.31}$$

be the natural morphism given by evaluating at $0 \in \mathbb{A}_k^1$ resp. $\infty \in \mathbb{A}_k^1$.

Remark 2.9. The natural inclusion $\mathrm{Fl}_{\mathcal{T}} \to \mathrm{Fl}_{\mathcal{I}}^0$ induced by the $\mathbb{G}_m$-equivariant morphism $\mathcal{T} \to \mathcal{I}$, with $\mathbb{G}_m$ acting trivially on $\mathcal{T}$, is not an isomorphism. Namely, $\mathrm{Fl}_{\mathcal{T}}$ is the ind-étale perfect k-scheme associated with the $\Gamma/\breve{\Gamma}$-set $\bar{\mathbb{X}}_\bullet$, while $\mathrm{Fl}_{\mathcal{I}}^0$ is associated with the $\Gamma/\breve{\Gamma}$-set W.

Following [AGLR22, Section 6.3] we can now define the constant term functor (associated with B and $\mathcal{I}$).

Definition 2.10. We set

$$\mathrm{CT}_B := Rq_!^+ \circ i^{+,*} : \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{D}_{\text{ét}}(L^+\mathcal{T} \backslash \mathrm{Fl}_{\mathcal{I}}^0) \tag{2.32}$$

By Braden's theorem the natural map $Rq_!^+ \circ i^{+,*} \to Rq_*^- \circ Ri^{-,!}$ is an isomorphism, cf. [FS21, Theorem IV.6.5], [AGLR22, Section 6]. This implies excellent formal properties of the constant term functor.

Remark 2.11. Let $A \in \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$ and $w \in W = \mathrm{Fl}_{\mathcal{I}}(\bar{k})$. By proper base change the fiber of $\mathrm{CT}_B(A)$ over w is calculated by $R\Gamma_c(\mathcal{S}_w, A_{|\mathcal{S}_w})$.

To use the formula in 2.11 we establish the following lemma.

Lemma 2.12. *Let $w \in W$ be such that $w(b) - b \in \bar{\mathbb{X}}_\bullet^+$, where b denotes the barycenter of the standard alcove $\mathbf{a}$.*

(1) $\mathrm{Fl}_{\breve{\mathcal{I}},w} = L^+\mathcal{U} \cdot w$, *where $\mathcal{U} \subseteq \mathcal{I}$ denotes the scheme-theoretic closure of U in $\mathcal{I}$.*
(2) $\mathrm{Fl}_{\breve{\mathcal{I}},w} = \mathcal{S}_w \cap \mathrm{Fl}_{\breve{\mathcal{I}},\leq w}$.

Proof. The first claim is equation (5.11) in the proof of [AGLR22, Lemma 5.3]. The first claim implies $\mathrm{Fl}_{\breve{\mathcal{I}},w} \subseteq \mathcal{S}_w \cap \mathrm{Fl}_{\breve{\mathcal{I}},\leq w}$ as $\mathcal{S}_w = LU \cdot w$. Let $x \in \mathrm{Fl}_{\breve{\mathcal{I}},\leq w} \setminus \mathrm{Fl}_{\breve{\mathcal{I}},w}$. By (perfect) properness of $\mathrm{Fl}_{\breve{\mathcal{I}},\leq w}$ the orbit map $\gamma : \mathbb{G}_m \to \mathrm{Fl}_{\breve{\mathcal{I}},\leq w}$, $t \mapsto \chi(t)x$ extends to a $\mathbb{G}_m$-equivariant map $\tilde{\gamma} : \mathbb{A}_{\bar{k}}^1 \to \mathrm{Fl}_{\breve{\mathcal{I}},\leq w}$. As $\mathrm{Fl}_{\breve{\mathcal{I}},w}$ is open in $\mathrm{Fl}_{\breve{\mathcal{I}},\leq w}$ and $W \subseteq \mathrm{Fl}_{\breve{\mathcal{I}}}$ is exactly the set of $\mathbb{G}_m$-fixed points, 2.1 implies that $x \in \mathcal{S}_{w'}$ for $w' := \tilde{\gamma}(0) < w$ because $\mathcal{S}_{w'}$ is exactly the subscheme of points contracting to w' under the $\mathbb{G}_m$-action. $\square$

Next we describe the closure relations for the stratification of $\mathrm{Fl}_{\breve{I}}$ by the $\mathcal{S}_w, w \in W$. As this is a geometric questions, we may assume $F = \breve{F}$ for this. Then we have to define the dominant cocharacters $\bar{\mathbb{X}}_\bullet^+$ in $\bar{\mathbb{X}}_\bullet$. Recall that $\mathbb{X}_\bullet(S)^+$ denotes the (B-)dominant cocharacters for S. As we assumed $F = \breve{F}$, we get that

$$\mathbb{X}_\bullet(S)_\mathbb{Q} \cong \bar{\mathbb{X}}_{\bullet,\mathbb{Q}}, \tag{2.33}$$

and thus we can define $\bar{\mathbb{X}}_\bullet^+$ as the preimage of $\mathbb{X}_\bullet(S)_\mathbb{Q}^+$ under the map $\bar{\mathbb{X}}_\bullet \to \bar{\mathbb{X}}_{\bullet,\mathbb{Q}}$. Given $\bar{\mathbb{X}}_\bullet$ we can now define the semi-infinite Bruhat order $\preceq$ on W, which depends on B. Namely, set $w \preceq w'$ if and only if for the Bruhat order $t_{n\bar{\nu}} w \leq t_{n\bar{\nu}} w'$ for all $\bar{\nu} \in \bar{\mathbb{X}}_\bullet^+$ and $n \gg 0$.

Lemma 2.13. *For $w, w' \in W$ we have $\mathcal{S}_w \subseteq \overline{\mathcal{S}_{w'}}$ if and only if $w \preceq w'$.*

Proof. This is [AGLR22, Proposition 5.4], where $\preceq$ is denoted by $\leq^{\frac{\infty}{2}}$. $\square$

If $w = t_{\bar{\mu}}, w' = t_{\bar{\nu}}$, then $w \preceq w'$ if and only if $\bar{\nu} - \bar{\mu}$ lies in $\bar{\mathbb{X}}_\bullet^+$. In particular, on $\bar{\mathbb{X}}_\bullet^+$ the two orders $\leq, \preceq$ agree. We will constantly use the equality $\ell(t_{\bar{\nu}}) = \langle 2\bar{\rho}, \bar{\nu} \rangle$ for $\bar{\nu}$ dominant, see [Zhu14, Lemma 9.1].

3. Cohomology of the affine flag variety

In this section, we want to study cohomology of constructible sheaves on the Hecke stack $\mathrm{Hk}_\mathcal{I}$. In particular, we will

(1) introduce Wakimoto-filtered complexes in mixed characteristic, following [AB09], [AR] and [Zhu14],
(2) calculate the constant terms of Wakimoto sheaves,
(3) show that central objects for convolution are Wakimoto-filtered.

In this section we assume that $F = \breve{F}$, and thus in particular that k is algebraically closed.

3.1. (Co)standard functors.
The considerations that are going to come have an easy shadow on Grothendieck groups. Recall that we always assume our ring of coefficients Λ to be a field.

For $w \in W$ let $j_w \colon \mathrm{Fl}_{\mathcal{I},w} \to \mathrm{Fl}_\mathcal{I}$ be the locally closed immersion. Note that j_w is I-equivariant, and hence descents to a morphism

$$j_w \colon [\mathrm{Fl}_{\mathcal{I},w}/I] \to \mathrm{Hk}_\mathcal{I} \tag{3.1}$$

of stacks, that we will usually denote in the same way.

Define the standard object

$$\Delta_w := j_{w,!}(\Lambda)[\ell(w)] \in \mathcal{D}_{\text{ét}}(\mathrm{Hk}_\mathcal{I}) \tag{3.2}$$

and the costandard object

$$\nabla_w := Rj_{w,*}(\Lambda)[\ell(w)] \in \mathcal{D}_{\text{ét}}(\mathrm{Hk}_\mathcal{I}) \tag{3.3}$$

associated with $w \in W$.

Let

$$\mathcal{D}_{\text{cons}}(\mathrm{Hk}_\mathcal{I}) \subseteq \mathcal{D}_{\text{ét}}(\mathrm{Hk}_\mathcal{I}) \tag{3.4}$$

be the full subcategory of objects $\mathcal{F} \in \mathcal{D}_{\text{ét}}(\mathrm{Hk}_\mathcal{I})$ such that the stalk $\mathcal{F}_w$ of $\mathcal{F}$ is perfect for all $w \in W$. Let

$$K_0(\mathrm{Hk}_\mathcal{I}) \tag{3.5}$$

be the Grothendieck group of $\mathcal{D}_{\text{cons}}(\mathrm{Hk}_\mathcal{I})$. Because for each $w \in W$ the stabilizer I_w in I is connected, $K_0(\mathrm{Hk}_\mathcal{I})$ is equivalently the Grothendieck group of the category of I-equivariant sheaves on $\mathrm{Fl}_\mathcal{I}$. Consequently, $K_0(\mathrm{Hk}_\mathcal{I})$ is a free abelian group on the classes of the intersection complexes $\mathrm{IC}_w = j_{!*}(\Lambda)$ of $\mathrm{Fl}_{\mathcal{I},\leq w}$.

Via convolution $K_0(\mathrm{Hk}_\mathcal{I})$ is naturally a ring, cf. [AR, Section 5.2]. In fact, it identifies with the integral group ring of W as we recall now.

Lemma 3.1. *The maps*

$$\theta \colon K_0(\mathrm{Hk}_{\mathcal{I}}) \to \mathbb{Z}[W], \ [\mathcal{F}] \mapsto \sum_{w \in W} (-1)^{\ell(w)} \chi(\mathrm{Fl}_{\mathcal{I},w}, j_w^* \mathcal{F}) w \qquad (3.6)$$

and

$$\vartheta \colon \mathbb{Z}[W] \to K_0(\mathrm{Hk}_{\mathcal{I}}), \ w \mapsto (-1)^{\ell(w)}[\nabla_w] \qquad (3.7)$$

are inverse ring isomorphisms.

Here, $\chi(\mathrm{Fl}_{\mathcal{I},w}, j_w^* \mathcal{F})$ denotes the Euler characteristic. In the equal characteristic case, a proof is given in [AR, Lemma 5.2.1].

Proof. If $s \in W$ and $\ell(s) = 1$, then by 2.5 $\mathrm{Fl}_{\mathcal{I}, \leq s} \cong \mathbb{P}_k^{1, \mathrm{pf}}$ and we get distinguished triangles

$$\mathrm{IC}_e \to \Delta_s \to \mathrm{IC}_s \xrightarrow{+1}, \quad \mathrm{IC}_s \to \nabla_s \to \mathrm{IC}_e \xrightarrow{+1}$$

because IC_s identifies with the underived pushforward $j_{s,*}(\Lambda)$. Thus, $\theta([\nabla_s]) = \theta([\Delta_s]) = s$ and $\theta(\vartheta(s)) = s$. It then follows from 3.2 below that θ, ϑ are ring homomorphisms and in fact $\theta([\Delta_w]) = \theta([\nabla_w]) = w$ for any $w \in W$. Now, ϑ is surjective because the $[\Delta_w] = [\nabla_w]$ generate $K_0(\mathrm{Hk}_{\mathcal{I}})$. This finishes the proof. □

We will now study convolutions Δ_w, ∇_w for $w \in W$. We do this in a slightly more general situation.

Recall that $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$ has a natural monoidal structure in the sense of [Lur17, Definition 4.1.1.10] as we will see later on, see §4.2. For now, it suffices to construct the underlying bifunctor as follows. First, consider the diagram

$$\mathrm{Hk}_{\mathcal{I}} \times \mathrm{Hk}_{\mathcal{I}} \xleftarrow{\ p\ } [I \backslash LG \times^I LG / I] \xrightarrow{\ m\ } \mathrm{Hk}_{\mathcal{I}}$$

<div align="center">

</div>

$$(3.8)$$

of ind-v-stacks on perfect schemes, with $\mathrm{pr}_1, \mathrm{pr}_2$ the two projections onto the first and second factors, respectively, p the natural morphism, and m the (quotient by the left I-action of the) convolution morphism discussed in 2.1. Now, for any $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$, define

$$\mathcal{F}_1 * \mathcal{F}_2 := Rm_!(p^*(\mathcal{F}_1 \boxtimes^{\mathbb{L}} \mathcal{F}_2)). \qquad (3.9)$$

Note that m is (perfectly) proper and thus $Rm_! = Rm_*$. The full subcategory $\mathcal{D}_{\text{cons}}(\mathrm{Hk}_{\mathcal{I}})$ is stable under convolution.

Note that the full subcategory of $\mathcal{D}_{\text{cons}}(\mathrm{Hk}_{\mathcal{I}})$ supported at the origin identifies with $\mathcal{D}_{\text{cons}}([*/L^+\mathcal{I}])$. Convolution restricts to the usual tensor product on these sheaves which is symmetric monoidal. Furthermore, since $L^+\mathcal{I}$ is an extension of its reductive quotient, which is naturally isomorphic to the special fiber S_k of the connected Néron model $\mathcal{S}$ of S, by a connected pro-unipotent group. By [FS21, Proposition VI.4.1], we can therefore identify $\mathcal{D}_{\text{cons}}([*/L^+\mathcal{I}])$ with $\mathcal{D}_{\text{cons}}([*/S_k])$ via pullback along $[*/L^+\mathcal{I}] \to [*/S_k]$. This is convenient, because for every $w \in W$ the category of constructible sheaves supported at the Schubert cell $\mathrm{Fl}_{\mathcal{I},w}$ also identifies with $\mathcal{D}_{\text{cons}}([*/S_k])$, since S_k maps isomorphically to the reductive quotient of the stabilizer group $L^+\mathcal{I} \cap wL^+\mathcal{I}w^{-1}$.

Moreover, the above abstract nonsense allows us to regard standard and costandard objects as functors by tensoring. Indeed, we define the *standard* and *costandard* functors:

$$\Delta_w \colon \mathcal{D}_{\text{cons}}([*/S_k]) \to \mathcal{D}_{\text{cons}}(\mathrm{Hk}_{\mathcal{I}}), \ M \mapsto \Delta_w * M, \qquad (3.10)$$

$$\nabla_w \colon \mathcal{D}_{\text{cons}}([*/S_k]) \to \mathcal{D}_{\text{cons}}(\mathrm{Hk}_{\mathcal{I}}), \ M \mapsto \nabla_w * M, \qquad (3.11)$$

and one checks easily that there are isomorphisms $\Delta_w(M) \simeq j_{w!} M[\ell(w)]$ and $\nabla_w(M) \simeq Rj_{w*} M[\ell(w)]$ of functors. Since j_w is an affine morphism (Lemma 2.6), both functors are t-exact (by [BBDG18,

Corollaire 4.1.3.]) for the natural t-structure on $\mathcal{D}_{\mathrm{cons}}([*/S_k])$ and the perverse t-structure on $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$, cf. [AGLR22, Definition 6.8] for the latter.

We start with the following lemma on the convolution of standard and costandard objects.

Lemma 3.2. *For any $w_1, w_2 \in W$ such that $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$, there exist canonical isomorphisms*

$$\Delta_{w_1} * \Delta_{w_2} \simeq \Delta_{w_1 w_2}$$
$$\nabla_{w_1} * \nabla_{w_2} \simeq \nabla_{w_1 w_2},$$

satisfying the obvious associativity constraint.

Proof. In equicharacteristic, this statement can be found in [AB09, Lemma 8(a)] with a proof given in [AR, Lemma 4.1.4 (1),(2)]. The same proof applies here. For any $w_1, w_2 \in W$ such that $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$, the convolution morphism $\mathrm{Fl}_{\mathcal{I},w_1} \tilde{\times} \mathrm{Fl}_{\mathcal{I},w_2} \to \mathrm{Fl}_{\mathcal{I},w_1 w_2}$ is an isomorphism by 2.4, so the constant complex $m_! \Lambda$ identifies with $\underline{\Lambda}$. This yields the desired isomorphisms after !- or *-extension and shifts. Indeed, $Rm_!(j_{w_1,!}\Lambda \tilde{\boxtimes} j_{w_2,!}\Lambda) \cong j_{w_1 w_2,!}\Lambda$ and $Rm_*(Rj_{w_1,*}\Lambda \tilde{\boxtimes} Rj_{w_2,*}\Lambda) \cong Rj_{w_1 w_2,*}\Lambda$. $\qquad\square$

Recall that in a monoidal category, an object is called left-invertible (resp. right-invertible) if multiplication on the left (resp. right) is an equivalence.

Lemma 3.3. *For any $w \in W$, the objects ∇_w and Δ_w are both left- and right-invertible in the monoidal category $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$. More concretely, there exist (non-canonical) isomorphisms*

$$\nabla_w * \Delta_{w^{-1}} \simeq \delta_e \simeq \Delta_w * \nabla_{w^{-1}}.$$

where $\delta_e := \Delta_e = \nabla_e$ with $e \in W$ is the identity element and δ_e is the unit object for convolution.

Proof. It is clear that δ_e is the unit for convolution. The existence of the searched for isomorphisms is stated over a Laurent series field in [AB09, Lemma 8(a)] and proved in [AR, Lemma 4.1.4(3)]. Again the same proof applies. The non-canonical isomorphism can be obtained by induction on $\ell(w)$ provided we construct them for all simple reflections s and $\tau \in \Omega_{\mathbf{a}}$. The case for $\tau \in \Omega_{\mathbf{a}}$ is clear as $\mathrm{Fl}_{\mathcal{I},\tau} = \mathrm{Fl}_{\mathcal{I},\leq \tau}$ by 2.5. Let $s \in \mathbb{S}$. Notice now that the twisted product $\mathrm{Fl}_{\mathcal{I},\leq s} \tilde{\times} \mathrm{Fl}_{\mathcal{I},\leq s}$ identifies with the direct product $\mathrm{Fl}_{\mathcal{I},\leq s} \times \mathrm{Fl}_{\mathcal{I},\leq s}$ via the morphism (pr_1, m), cf. 2.5. In particular, it is isomorphic to $\mathbb{P}_k^{1,\mathrm{pf}} \times_{\mathrm{Spec}(k)} \mathbb{P}_k^{1,\mathrm{pf}}$. Under this identification $\mathrm{Fl}_{\mathcal{I},e} \tilde{\times} \mathrm{Fl}_{\mathcal{I},\leq s}$ identifies with $\{\infty\} \times_{\mathrm{Spec}(k)} \mathbb{P}_k^{1,\mathrm{pf}}$ and $\mathrm{Fl}_{\mathcal{I},\leq s} \tilde{\times} \{\infty\}$ with the diagonal $\Delta_{\mathbb{P}_k^1}^{\mathrm{pf}}$. Here, $\infty \in \mathbb{P}_k^{1,\mathrm{pf}} \cong \mathrm{Fl}_{\mathcal{I},\leq s}$ denotes the point eI. From here the argument from [AR, Lemma 4.1.4.(3)] applies (using that perfection does not alter the étale site). $\qquad\square$

Remark 3.4. As noted in the discussion after [AR, Remark 4.1.5], the isomorphisms in 3.3 depend on various choices, e.g., a decomposition of w into elements of length ≤ 1. For functorial purposes, one can often work with any inverse of ∇_w, while using $\Delta_{w^{-1}}$ for practical computations.

As usual one is interested in understanding what happens on the abelian subcategory $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ arising as the heart of the perverse t-structure. While it is not stable under the monoidal structure of $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$, we can still benefit from the following fact.

Lemma 3.5. *For any $w_1, w_2 \in W$, we have $\Delta_{w_1} * \nabla_{w_2}, \nabla_{w_1} * \Delta_{w_2} \in \mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$.*

Proof. By 2.6 the restrictions $m' \colon \mathrm{Fl}_{\mathcal{I},w_1} \tilde{\times} \mathrm{Fl}_{\mathcal{I},\leq w_2} \to \mathrm{Fl}_{\mathcal{I}}$ and $m'' \colon \mathrm{Fl}_{\mathcal{I},\leq w_1} \tilde{\times} \mathrm{Fl}_{\mathcal{I},w_2} \to \mathrm{Fl}_{\mathcal{I}}$ of the convolution map are affine. Now,

$$\Delta_{w_1} * \nabla_{w_2} = Rm'_!(\Lambda \tilde{\boxtimes} \nabla_{w_2}) \tag{3.12}$$

with $\Lambda \tilde{\boxtimes} \nabla_{w_2}$ perverse by our assumption and Lemma 2.6. Thus $\Delta_{w_1}(M_1) * \nabla_{w_2}(M_2)$ is concentrated in non-negative perverse degrees because !-pushforward of affine morphisms is left exact for the perverse t-structure, cf. [BBDG18, Corollaire 4.1.2]. On the other hand,

$$\Delta_{w_1} * \nabla_{w_2} = Rm''_*(\Delta_{w_1} \tilde{\boxtimes} \Lambda) \tag{3.13}$$

is concentrated in non-positive perverse degrees by [BBDG18, Théorème 4.1.1]. This finishes the proof. $\qquad\square$

We also note the following.

Lemma 3.6. *For any $w_1, w_2 \in W$, the perverse sheaves $\Delta_{w_1} * \nabla_{w_2}, \nabla_{w_1} * \Delta_{w_2}$ are both supported on $\mathrm{Fl}_{\mathcal{I}, \leq w_1 w_2}$, and restrict to $\Lambda[\ell(w_1 w_2)]$ on $\mathrm{Fl}_{\mathcal{I}, w_1 w_2}$.*

Proof. The proof is similar to the equal characteristic case, cf. [AR, 4.1.10], and we sketch it here. The Euler characteristic

$$\theta \colon K_0(\mathrm{Hk}_{\mathcal{I}}) \longrightarrow \mathbb{Z}[W], \quad [\mathcal{F}] \longmapsto \sum_{w \in W} (-1)^{\ell(w)} \chi(\mathrm{Fl}_{\mathcal{I}, w}, j_w^* \mathcal{F}) w \tag{3.14}$$

defines a ring homomorphism. By the proof of 3.1 we know that $\theta([\Delta_w]) = \theta([\nabla_w]) = w$ for any $w \in W$. Now let $w \in W$ be any element such that $\mathrm{Fl}_{\mathcal{I}, w}$ is open in the support of $\Delta_{w_1} * \nabla_{w_2}$. Then the coefficient of w in $\theta([\Delta_{w_1} * \nabla_{w_2}]) \in \mathbb{Z}[W]$ does not vanish. By perversity, see Lemma 3.5, and $\mathcal{I}$-equivariance it is a non-zero multiple of the Euler characteristic of the cohomology of $\Lambda[\ell(w)]$ on $\mathrm{Fl}_{\mathcal{I}, w} \cong \mathbb{A}_k^{\ell(w), \mathrm{pf}}$. Now,

$$\theta([\Delta_{w_1} * \nabla_{w_2}]) = \theta([\Delta_{w_1}]) \theta([\nabla_{w_2}]) = w_1 w_2, \tag{3.15}$$

and thus $w = w_1 w_2$ and $\Delta_{w_1} * \nabla_{w_2}$ is supported on $\mathrm{Fl}_{\mathcal{I}, \leq w_1 w_2}$. By perversity and I-equivariance, we must have that

$$j_{w_1 w_2}^*(\Delta_{w_1} * \nabla_{w_2}) \cong \Lambda^{\oplus m}[\ell(w_1 w_2)] \tag{3.16}$$

for some $m \geq 1$. But the coefficient of $w_1 w_2$ is 1, so $m = 1$ as desired. The statement for $\nabla_{w_1} * \Delta_{w_2}$ follows similarly. $\qquad\square$

If $\mathcal{F} \in \mathcal{D}_{\mathrm{\acute{e}t}}(\mathrm{Hk}_{\mathcal{I}})$, then

$$\mathrm{supp}(\mathcal{F}) = \{ w \in W \mid j_w^* \mathcal{F} \neq 0 \} \tag{3.17}$$

is the support of $\mathcal{F}$, and

$$\mathrm{cosupp}(\mathcal{F}) = \{ w \in W \mid j_w^! \mathcal{F} \neq 0 \} \tag{3.18}$$

its cosupport.

We need the following geometric consequence of Lemma 2.7. In equal characteristic this is [AR, Proposition 4.4.4.].

Proposition 3.7. *For any $\mathcal{F} \in \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$, there exists a finite subset $A_{\mathcal{F}} \subset W$ such that for any $w \in W$,*

(1) $\mathrm{supp}(\Delta_w * \mathcal{F}) \subseteq w \cdot A_{\mathcal{F}}, \quad \mathrm{cosupp}(\nabla_w * \mathcal{F}) \subseteq w \cdot A_{\mathcal{F}},$
(2) $\mathrm{supp}(\mathcal{F} * \Delta_w) \subseteq A_{\mathcal{F}} \cdot w, \quad \mathrm{cosupp}(\mathcal{F} * \nabla_w) \subseteq A_{\mathcal{F}} \cdot w.$

Proof. Let $X \subseteq \mathrm{Fl}_{\mathcal{I}}$ be a closed finite union of I-orbits such that $\mathrm{supp}(\mathcal{F}) \subseteq X$. Set $A_{\mathcal{F}} := S_X$ with S_X as in 2.7, i.e.,

$$m(X \tilde{\times} \mathrm{Fl}_{\mathcal{I}, w}) \subseteq A_{\mathcal{F}} \cdot w \tag{3.19}$$

and

$$m(\mathrm{Fl}_{\mathcal{I}, w} \tilde{\times} X) \subseteq w \cdot A_{\mathcal{F}} \tag{3.20}$$

for all $w \in W$. Now the proper base change theorem implies that $\mathrm{supp}(\mathcal{F} * \Delta_w) \subseteq A_{\mathcal{F}} \cdot w$ and $\mathrm{supp}(\Delta_w * \mathcal{F}) \subseteq w \cdot A_{\mathcal{F}}$ for any $w \in W$. Because we assumed that X is closed, we can also use that Rm_* commutes with !-restrictions (by the adjoint version of the proper base change

theorem) to see that $\mathrm{cosupp}(\mathcal{F} * \nabla_w) \subseteq A_{\mathcal{F}} \cdot w$ and $\mathrm{cosupp}(\nabla_w * \mathcal{F}) \subseteq w \cdot A_{\mathcal{F}}$ for any $w \in W$. This finishes the proof. $\qquad\square$

Regarding the products $\Delta_{w_1} * \Delta_{w_2}$, $\nabla_{w_1} * \nabla_{w_2}$ for $w_1, w_2 \in W$ we note the following.

Lemma 3.8. *Let $w_1, w_2 \in W$, then*

    (1) $\Delta_{w_1} * \Delta_{w_2}$ *lies in the smallest full subcategory of $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$, which is closed under extensions, and contains $\Delta_w[n]$ for $w \in W$ and $n \in \mathbb{Z}_{\leq 0}$.*

    (2) $\nabla_{w_1} * \nabla_{w_2}$ *lies in the smallest full subcategory of $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$, which is closed under extensions, and contains $\nabla_w[n]$ for $w \in W$ and $n \in \mathbb{Z}_{\geq 0}$.*

Proof. Given the results of this section the argument of [AR, Lemma 6.5.8] applies. $\qquad\square$

Let $\Omega_{\mathbf{a}} \subset W$ be the stabilizer of the fundamental alcove $\mathbf{a}$, i.e., the subset of length 0 elements.

Lemma 3.9. *Given $w \in W$, let $\tau \in \Omega_{\mathbf{a}}$ be the unique element contained in $\mathrm{Fl}_{\mathcal{I}, \leq w}$. Then, the sheaf IC_τ appears with multiplicity 1 in the Jordan–Hölder series of ∇_w and equals its top. Dually, IC_τ appears with multiplicity 1 inside Δ_w as its socle.*

Proof. This follows from the same proof of [BBM04, Lemma 2.1]. The idea is to argue by induction on the length of w. Besides the combinatorics of Coxeter groups, one only has to know that $\mathrm{Fl}_{\mathcal{I}} \to \mathrm{Fl}_{\mathcal{J}_s}$ is a $\mathbb{P}^{1,\mathrm{pf}}_k$-bundle locally for the étale topology that actually splits over Schubert cells (use root groups to see this latter property). Here, $\mathcal{I} \to \mathcal{J}_s$ is the minimal parahoric fixing the wall of the fundamental alcove $\mathbf{a}$ fixed by s. $\qquad\square$

3.2. **Wakimoto sheaves.** Let $w_1, w_2 \in W$. In general,
$$\Delta_{w_1} * \Delta_{w_2} \not\cong \Delta_{w_1 w_2}, \ \nabla_{w_1} * \nabla_{w_2} \not\cong \nabla_{w_1 w_2} \tag{3.21}$$
unless $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$, cf. 3.2. In this subsection, we want to remedy this fact by introducing objects
$$\mathcal{J}_{\bar{\nu}} \in \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}}) \tag{3.22}$$
for $\bar{\nu} \in \bar{\mathbb{X}}_{\bullet}$ (recall the embedding $\bar{\mathbb{X}}_{\bullet} \to W$, $\bar{\nu} \mapsto t_{\bar{\nu}}$) such that
$$\mathcal{J}_{\bar{\nu}_1} * \mathcal{J}_{\bar{\nu}_2} \cong \mathcal{J}_{\bar{\nu}_1 + \bar{\nu}_2} \tag{3.23}$$
for all $\bar{\nu}_1, \bar{\nu}_2 \in \bar{\mathbb{X}}_{\bullet}$, and
$$\mathcal{J}_{\bar{\nu}} \cong \Delta_{t_{\bar{\nu}}} \tag{3.24}$$
if $\bar{\nu} \in -\bar{\mathbb{X}}_{\bullet}^+$. Note that by Lemma 3.3 this already forces
$$\mathcal{J}_{\bar{\nu}} \cong \nabla_{t_{\bar{\nu}}} \tag{3.25}$$
if $\bar{\nu} \in \bar{\mathbb{X}}_{\bullet}^+$. In fact, we must have
$$\mathcal{J}_{\bar{\nu}} \cong \Delta_{t_{\bar{\nu}_2}} * \nabla_{\bar{\nu}_1} \tag{3.26}$$
if we write $\bar{\nu} = \bar{\nu}_1 - \bar{\nu}_2$ with $\bar{\nu}_1, \bar{\nu}_2 \in \bar{\mathbb{X}}_{\bullet}^+$ (which is always possible). Note that $\ell(t_{\bar{\nu}_1}) + \ell(t_{\bar{\nu}_2}) = \ell(t_{\bar{\nu}_1} t_{\bar{\nu}})$ if $\bar{\nu}_1, \bar{\nu}_2 \in \bar{\mathbb{X}}_{\bullet}^+$. Hence, Lemma 3.2 implies that the above formula for $\mathcal{J}_{\bar{\nu}}$ is independent (up to isomorphism) of $\bar{\nu}_1, \bar{\nu}_2$. To get a more canonical construction of $\mathcal{J}_{\bar{\nu}}$, we will adopt the definition from [AR, Section 4.2.1].

Definition 3.10. Let $\bar{\nu} \in \bar{\mathbb{X}}_{\bullet}$. The *Wakimoto sheaf* $\mathcal{J}_{\bar{\nu}}$ is the object in $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$ corepresenting the functor
$$\mathcal{F} \mapsto \mathrm{colim}\,\mathrm{Hom}(\nabla_{t_{\bar{\nu}_1}}, \mathcal{F} * \nabla_{t_{\bar{\nu}_2}}), \tag{3.27}$$
with the colimit running over all $\bar{\nu}_1, \bar{\nu}_2 \in \bar{\mathbb{X}}_{\bullet}^+$ such that $\bar{\nu} = \bar{\nu}_1 - \bar{\nu}_2$. The transition morphisms in the colimit are given by convolution (and using the canonical isomorphisms in Lemma 3.2).

Note that all the transition morphisms in the colimit are isomorphisms. In particular, we can conclude (by invertibility of $\nabla_{t_{\bar\nu_2}}$), cf. Lemma 3.3) that

$$\mathcal{I}_{\bar\nu} \cong \Delta_{t_{\bar\nu_2}} * \nabla_{t_{\bar\nu_1}} \tag{3.28}$$

as desired. More generally, we can use the fact that $\mathcal{D}_{\mathrm{cons}}([*/S_k])$ acts on $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$ to deduce a functor

$$\mathcal{I}_{\bar\nu} \colon \mathcal{D}_{\mathrm{cons}}([*/S_k]) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}}), \ M \mapsto \mathcal{I}_{\bar\nu} * M \tag{3.29}$$

between the two categories via evaluation at the Wakimoto sheaf. This will be called the Wakimoto functor and still be denoted by $\mathcal{I}_{\bar\nu}$ by abuse of notation.

Remark 3.11. The Wakimoto sheaves $\mathcal{I}_{\bar\nu}$ were introduced by Mirković for geometrizing Bernstein elements in the affine Hecke algebra, see [AR, Section 5.1].

Given a subset $\Omega \subset \bar{\mathbb{X}}_\bullet$, it will also be convenient to define the Ω-Wakimoto functor

$$\mathcal{I}_\Omega = \oplus_{\bar\nu \in \Omega} \mathcal{I}_{\bar\nu} \colon \mathcal{D}_{\mathrm{cons}}([\underline{\Omega}/S_k]) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}}) \tag{3.30}$$

as the direct sum of the $\mathcal{I}_{\bar\nu}$ for $\bar\nu \in \Omega$, where $\underline{\Omega} = \bigsqcup_\Omega \operatorname{Spec} k$ regarded as an ind-scheme, so that complexes of étale sheaves have compact support. If $\Omega = \bar{\mathbb{X}}_\bullet$ is the total set, then we simply write $\mathcal{I}$ for $\mathcal{I}_{\bar{\mathbb{X}}_\bullet}$, which is monoidal by Lemma 3.2. Indeed, we can identify $\mathcal{D}_{\mathrm{cons}}([\underline{\bar{\mathbb{X}}}_\bullet/S_k])$ with the full subcategory of compact objects of the product in Cat_∞ of the monoidal 1-category $\bar{\mathbb{X}}_\bullet$ with the stable ∞-category $\mathcal{D}_{\mathrm{cons}}([*/S_k])$. We see that the first category maps monoidally to the abelian category $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ via the Wakimoto sheaves $\mathcal{I}_{\bar\nu}$, see [AR, Section 4.2.3], whereas the second maps monoidally to the $\mathbb{E}_1$-center $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$. This implies the claim by the universal property of centers, see [Lur17, Section 5.3.1].

Lemma 3.12. *The Wakimoto functors satisfy the following properties:*
  (1) *For any $\bar\nu \in \bar{\mathbb{X}}_\bullet$, $\mathcal{I}_{\bar\nu}$ is t-exact for the perverse t-structure.*
  (2) *For any $\bar\nu \in \bar{\mathbb{X}}_\bullet$, $\mathcal{I}_{\bar\nu}$ is supported on $\mathrm{Fl}_{\leq\bar\nu}$ and $j_{\bar\nu}^* \mathcal{I}_{\bar\nu} \simeq \Lambda[\langle 2\bar\rho, \bar\nu\rangle]$.*
  (3) *For any $\bar\mu, \bar\nu \in \bar{\mathbb{X}}_\bullet$, there exists a canonical isomorphism $\mathcal{I}_{\bar\mu} * \mathcal{I}_{\bar\nu} \simeq \mathcal{I}_{\bar\mu+\bar\nu}$.*
  (4) *For any $\bar\mu, \bar\nu \in \bar{\mathbb{X}}_\bullet$ with $t_{\bar\nu} \not\preceq t_{\bar\mu}$, we have $R\mathrm{Hom}_{\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})}(\mathcal{I}_{\bar\mu}, \mathcal{I}_{\bar\nu}) = 0$.*

Proof. The first statement follows from 3.5, the second from 3.6 and the third is implicit in the discussion of monoidality of $\mathcal{I}$. Let us discuss the forth statement. Using (3), the invertibity of $\mathcal{I}_{\bar\nu}(\Lambda)$ and the definition of the semi-infinite Bruhat order $\preceq$ reduces by suitable convolution to the case that $\bar\nu, \bar\mu \in \bar{\mathbb{X}}_\bullet$ are dominant. Then $\mathcal{I}_{\bar\mu}(M) = \nabla_{t_{\bar\mu}}(M), \mathcal{I}_{\bar\nu}(N) = \nabla_{t_{\bar\nu}}(N)$ and thus by 2.1

$$R\mathrm{Hom}_{\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})}(\nabla_{t_{\bar\mu}}(M), \nabla_{t_{\bar\nu}}(N)) \cong R\mathrm{Hom}_{\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})}(j_{t_{\bar\nu}}^* \nabla_{t_{\bar\nu}}, N[\ell(t_{\bar\nu})]) = 0 \tag{3.31}$$

if $t_{\bar\nu} \not\leq t_{\bar\mu}$ (for the Bruhat order $\leq$ or equivalently the semi-infinite order $\preceq$ as $\bar\nu, \bar\mu \in \bar{\mathbb{X}}_\bullet^+$). $\qquad\square$

For a stable ∞-category $\mathcal{D}$ and a set of objects $S \subset \mathrm{Ob}(\mathcal{D})$, let $\langle S \rangle$ be the smallest full subcategory of $\mathcal{D}$ whose objects include S and which is stable under cones and shifts.

Definition 3.13. Define the Wakimoto category as the full subcategory $\mathrm{Wak} := \langle \mathcal{I} \rangle$ of $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$ generated by the essential image of $\mathcal{I}$ under cones. An object $\mathcal{F} \in \mathrm{Ob}(\mathrm{Wak})$ is called Wakimoto filtered.

By 3.12 the category $\mathrm{Wak} \subseteq \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$ is stable under convolution. As it contains $\delta_e \cong \mathcal{I}_{\bar\nu}$ it is thus itself monoidal.

Remark 3.14. In the works [AB09, AR], their respective authors do not define the full subcategory $\mathrm{Wak} \subset \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$, but rather a full subcategory $\mathcal{P}(\mathrm{Wak}) \subset \mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ consisting of those perverse sheaves that admit a filtration by perverse sheaves with grading in the essential image of $\mathcal{I}$. Morally, one can try to think of $\mathcal{P}(\mathrm{Wak})$ as the heart of Wak, but it is not an abelian

category, only exact, and it is not true that every perverse sheaf that is Wakimoto filtered as a complex actually lies in $\mathcal{P}(\mathrm{Wak})$. Indeed, pick ν dominant with respect to B and let $\Delta_0 \subset \Delta_{-\nu}$ be the socle by [BBM04, Lemma 2.1]. Then, the cokernel lies in Wak, but its 0-th graded piece equals $\Delta_0[1]$, which is not perverse.

We give the following simple criterion for determing whether an object of $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$ lies in Wak.

Proposition 3.15. *Let $\mathcal{F} \in \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$. Then, the following are equivalent:*

   (1) *$\mathcal{F}$ is Wakimoto filtered;*
   (2) *$\mathrm{supp}(\mathcal{I}_{-\bar{\nu}} * \mathcal{F}) \subset \{t_{-\bar{\mu}} \colon \bar{\mu} \in \mathbb{X}_{\bullet}^+\}$ for all $\bar{\nu} \ll 0$;*
   (3) *$\mathrm{cosupp}(\mathcal{I}_{\bar{\nu}} * \mathcal{F}) \subset \{t_{\bar{\mu}} \colon \bar{\mu} \in \mathbb{X}_{\bullet}^+\}$ for all $\bar{\nu} \gg 0$.*

In particular, if $\mathcal{F}$ satisfies $\mathcal{I}_{\bar{\nu}} * \mathcal{F} \cong \mathcal{F} * \mathcal{I}_{\bar{\nu}}$ for all $\nu \in \bar{\mathbb{X}}_{\bullet}$, then it is Wakimoto filtered.

Here, the notation $\bar{\nu} \gg 0$ means that $\langle \bar{\nu}, \alpha \rangle \gg 0$ for all B-positive relative roots α of G, while $\bar{\nu} \ll 0$ means $-\bar{\nu} \gg 0$.

Proof. Assume that $\mathcal{F}$ is Wakimoto filtered, and let us check that it satisfies (2) and (3). We may then assume that $\mathcal{F} = \mathcal{I}_{\bar{\nu}'}$ for some $\nu' \in \bar{\mathbb{X}}_{\bullet}$. If now $\nu \gg 0$, then

$$\mathcal{I}_{-\bar{\nu}} * \mathcal{F} \cong \Delta_{t_{-\bar{\nu}+\nu'}} \tag{3.32}$$

and the support claim follows. Similarly, we can argue for (3). Let us now assume that $\mathcal{F}$ satisfies (2). We want to show that $\mathcal{F}$ is Wakimoto filtered. Replacing $\mathcal{F}$ by $\mathcal{I}_{-\bar{\nu}} * \mathcal{F}$ for some suitable $\nu \gg 0$, we may assume that

$$\mathrm{supp}(\mathcal{F}) \subseteq \{t_{-\bar{\mu}} : \mu \in \bar{\mathbb{X}}_{\bullet}^+\}. \tag{3.33}$$

It is then formal that $\mathcal{F}$ lies in $\langle \Delta_{t_{-\bar{\mu}}} : \mu \in \bar{\mathbb{X}}_{\bullet}^+ \rangle$, cf. [AR, Lemma 4.4.3]. But

$$\langle \Delta_{t_{-\bar{\mu}}} : \mu \in \bar{\mathbb{X}}_{\bullet}^+ \rangle \subseteq \mathrm{Wak} \tag{3.34}$$

by the construction of Wakimoto sheaves. The argument that (3) implies (1) is similar.

For the last claim, let $A_{\mathcal{F}} \subseteq W$ be as in Proposition 3.7, i.e.,

$$\mathrm{supp}(\Delta_w * \mathcal{F}) \subseteq w \cdot A_{\mathcal{F}}, \quad \mathrm{supp}(\mathcal{F} * \Delta_w) \subseteq A_{\mathcal{F}} \cdot w \tag{3.35}$$

for all $w \in W$. As $\mathcal{I}_{-\bar{\nu}} * \mathcal{F} \cong \mathcal{F} * \mathcal{I}_{-\bar{\nu}}$ for $\bar{\nu} \in \bar{\mathbb{X}}_{\bullet}$, we can conclude that for $\nu \gg 0$

$$\mathrm{supp}(\mathcal{I}_{-\bar{\nu}} * \mathcal{F}) \subset t_{-\bar{\nu}} A_{\mathcal{F}} \cap A_{\mathcal{F}} t_{-\bar{\nu}}. \tag{3.36}$$

Now, we claim that for $\nu \gg 0$

$$t_{-\bar{\nu}} A_{\mathcal{F}} \cap A_{\mathcal{F}} t_{-\bar{\nu}} \subseteq \{t_{-\bar{\mu}} : \bar{\mu} \in \mathbb{X}_{\bullet}^+\}, \tag{3.37}$$

which would finish the proof. To check the claim let us recall that

$$\bar{\mathbb{X}}_{\bullet}^+ \cong W_{\mathrm{fin}} \backslash W / W_{\mathrm{fin}}. \tag{3.38}$$

If now $w \in t_{-\bar{\nu}} A_{\mathcal{F}} \cap A_{\mathcal{F}} t_{-\bar{\nu}}$, then we can write $w = t_{-\bar{\nu}} w_1 = w_2 t_{-\bar{\nu}}$ for $w_1, w_2 \in A_{\mathcal{F}} \subseteq W$, i.e.,

$$t_{\bar{\nu}} = w_1^{-1} t_{\bar{\nu}} w_2. \tag{3.39}$$

Given (3.38) and writing w_1, w_2 as a product of a translation and an element in the finite Weyl group, we can conclude that $w_1, w_2 \in \bar{\mathbb{X}}_{\bullet}$, i.e.,

$$t_{-\bar{\nu}} A_{\mathcal{F}} \cap A_{\mathcal{F}} t_{-\bar{\nu}} \subseteq \bar{\mathbb{X}}_{\bullet}. \tag{3.40}$$

As $A_{\mathcal{F}}$ is finite, we can conclude that for $\nu \gg 0$ we even get

$$t_{-\bar{\nu}} A_{\mathcal{F}} \cap A_{\mathcal{F}} t_{-\bar{\nu}} \subseteq \{t_{-\bar{\mu}} : \bar{\mu} \in \bar{\mathbb{X}}_{\bullet}^+\} \tag{3.41}$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.16. In [AB09, Proposition 5] and [AR, Proposition 4.4.1], it is shown that a central perverse sheaf whose convolution functor is perverse t-exact lies in the category $\mathcal{P}(\text{Wak})$. The proof given in those references is considerably more complicated, because of the need to ensure that the graded sheaves are actually perverse. Our proof is much simpler due to taking place in the derived setting, and later we will see how to recover the extra degree information required for perversity for the essential image of the Gaitsgory's central functor $\mathcal{Z}$.

For an arbitrary subset $\Omega \subset \bar{\mathbb{X}}_\bullet$, we can also define the full subcategory $\text{Wak}_\Omega = \langle \mathcal{I}_\Omega \rangle$.

Proposition 3.17. *If $\Omega \subset \bar{\mathbb{X}}_\bullet$ is a lower poset (for $\preceq$), the inclusion $\text{Wak}_\Omega \to \text{Wak}$ has a right adjoint $\text{Wak} \to \text{Wak}_\Omega$, $\mathcal{F} \mapsto \mathcal{F}_\Omega$ such that the cone $\mathcal{G}$ of the adjunction unit $\mathcal{F}_\Omega \to \mathcal{F}$ lies in Wak and satisfies $\mathcal{G}_\Omega = 0$.*

Proof. Given $\mathcal{F} \in \text{Ob}(\text{Wak})$, we show the existence of a final morphism $\mathcal{F}_\Omega \to \mathcal{F}$, such that $\mathcal{F}_\Omega \in \text{Ob}(\text{Wak}_\Omega)$. By induction on the length of the filtration, we can write $\mathcal{F}$ as an extension of $\mathcal{G} = \mathcal{G}_\Omega$ by $\mathcal{I}_{\bar{\nu}}(M)$ for some $\bar{\nu} \in \bar{\mathbb{X}}_\bullet$ and $M \in \mathcal{D}^b_{\text{ét}}(\text{Spec } k)$. If $\bar{\nu} \in \Omega$, we are done. Otherwise, we have $\bar{\mu} \not\preceq \bar{\mu}$ for all $\mu \in \Omega$ by the lower set hypothesis on Ω, so we get $R\text{Hom}(\mathcal{I}_{\bar{\nu}}(M), \mathcal{G}) = 0$ by Lemma 3.12 and hence there exists a (unique) splitting $\mathcal{F} \simeq \mathcal{I}_{\bar{\nu}}(M) \oplus \mathcal{G}$. One concludes that $\mathcal{F}_\Omega$ exists and identifies with $\mathcal{G}$ via the given map to $\mathcal{F}$ (again by Lemma 3.12). $\qquad\square$

If Ω equals $\{\bar{\nu} \preceq \bar{\mu}\}$ resp. $\{\bar{\nu} \prec \bar{\mu}\}$ for some $\bar{\mu} \in \bar{\mathbb{X}}_\bullet$, we simply write $\text{Wak}_{\preceq \bar{\mu}}$ resp. $\text{Wak}_{\prec \bar{\mu}}$, instead of Wak_Ω. We can now define the Wakimoto gradeds for $\mathcal{F} \in \text{Wak}$.

Definition 3.18. For any $\bar{\mu} \in \bar{\mathbb{X}}_\bullet$, we define the endofunctor

$$\text{gr}_{\bar{\mu}} \colon \text{Wak} \to \text{Wak}, \mathcal{F} \mapsto \text{cone}(\mathcal{F}_{\prec \bar{\mu}} \to \mathcal{F}_{\preceq \bar{\mu}}). \tag{3.42}$$

We also define $\text{gr} := \oplus_{\bar{\mu} \in \bar{\mathbb{X}}_\bullet} \text{gr}_{\bar{\mu}} \colon \text{Wak} \to \text{Wak}$.

Note that $\text{gr}_{\bar{\nu}}(\mathcal{F})$ lies in the essential image of the functor $\mathcal{I}_{\bar{\nu}} \colon \mathcal{D}_{\text{cons}}([*/S_k]) \to \text{Wak}$. In the next subsection, we will show that this functor is fully faithful by explicitly constructing an inverse via constant terms of the opposite Borel, see 3.21. In particular, we can essentially uniquely lift $\text{gr}_{\bar{\nu}}(\mathcal{F})$ to an element of $\mathcal{D}_{\text{cons}}([*/S_k])$ and can make the following definition.

Definition 3.19. Let $\mathcal{F} \in \text{Wak}$, we define

$$\text{Grad}_{\bar{\nu}}(\mathcal{F}) \in \mathcal{D}_{\text{cons}}([*/S_k]) \tag{3.43}$$

to be the canonical object such that $\mathcal{I}_{\bar{\nu}}(\text{Grad}_{\bar{\nu}}(\mathcal{F}))$ identifies with $\text{gr}_{\bar{\nu}}(\mathcal{F})$.

3.3. **Cohomology of Wakimoto filtered objects.** We now analyze the cohomology of objects in Wak. First, we show that convolution with Wakimoto sheaves induces a shift.

Proposition 3.20. *For any $\bar{\nu} \in \bar{\mathbb{X}}_\bullet$ and $\mathcal{F} \in \mathcal{D}_{\text{cons}}(\text{Hk}_{\mathcal{I}})$, there is a canonical isomorphism*

$$R\Gamma(\text{Fl}_{\mathcal{I}}, \mathcal{F} * \mathcal{I}_{\bar{\nu}}) \simeq R\Gamma(\text{Fl}_{\mathcal{I}}, \mathcal{F})[\langle 2\rho, \nu \rangle]. \tag{3.44}$$

Proof. Let us first assume that $\bar{\nu} \in \bar{\mathbb{X}}^+_\bullet$, which implies

$$\mathcal{I}_{\bar{\nu}}(\Lambda) = \nabla_{t_{\bar{\nu}}} = Rj_{w,*}(\Lambda)[\ell(t_{\bar{\nu}})]. \tag{3.45}$$

The map $(\text{pr}_1, m) \colon \text{Fl}_{\mathcal{I}} \tilde{\times} \text{Fl}_{\mathcal{I}} \to \text{Fl}_{\mathcal{I}} \times \text{Fl}_{\mathcal{I}}$ is an isomorphism. The second projection $\text{Fl}_{\mathcal{I}} \tilde{\times} \text{Fl}_{\mathcal{I}} \to \text{Hk}_{\mathcal{I}}$ is transformed to the map

$$\pi \colon \text{Fl}_{\mathcal{I}} \times \text{Fl}_{\mathcal{I}} \to \text{Hk}_{\mathcal{I}}, \ (\bar{g}, \bar{h}) \mapsto \overline{g^{-1}h}. \tag{3.46}$$

By definition $\mathcal{F} * \mathcal{I}_{\bar{\nu}}(\Lambda) \cong R\text{pr}_{2,*}(\text{pr}_1^*(\mathcal{F}) \otimes_\Lambda^L \pi^* \nabla_{\bar{\nu}}(\Lambda))$ and thus we get

$$R\Gamma(\text{Fl}_{\mathcal{I}}, \mathcal{F} * \mathcal{I}_{\bar{\nu}}(\Lambda))$$
$$\simeq R\Gamma(\text{Fl}_{\mathcal{I}} \times \text{Fl}_{\mathcal{I}}, \text{pr}_1^*(\mathcal{F}) \otimes_\Lambda^L \pi^* \nabla_{t_{\bar{\nu}}})$$
$$\simeq R\Gamma(\text{Fl}_{\mathcal{I}}, \mathcal{F} \otimes_\Lambda^L R\text{pr}_{1,*} \pi^* \nabla_{t_{\bar{\nu}}})$$

Using that $R\Gamma(\mathrm{Fl}_{\mathcal{G},\leq t_{\bar{\nu}}}, \nabla_{t_{\bar{\nu}}}) \cong \Lambda[\ell(t_{\bar{\nu}})]$ one checks that the pullback of $R\mathrm{pr}_{1,*}\pi^*\nabla_{t_{\bar{\nu}}}$ along $LG \to$ $\mathrm{Fl}_{\mathcal{I}}$ is isomorphic to $\Lambda[\ell(t_{\bar{\nu}})]$. As the object $R\mathrm{pr}_{1,*}\pi^*\nabla_{t_{\bar{\nu}}} \in \mathcal{D}_{\mathrm{\acute{e}t}}(\mathrm{Fl}_{\mathcal{G}})$ is LG-equivariant (because π and pr_1 are) we can conclude that $R\mathrm{pr}_{1,*}(\pi^*\nabla_{t_{\bar{\nu}}}) \cong \Lambda[\ell(t_{\bar{\nu}})]$. Moreover, we normalize this isomorphism such that over $1{\cdot}I \in \mathrm{Fl}_{\mathcal{I}}$ it reduces to the canonical isomorphism $R\Gamma(\mathrm{Fl}_{\mathcal{G},\leq t_{\bar{\nu}}}, \nabla_{t_{\bar{\nu}}}) \cong$ $R\Gamma(\mathrm{Fl}_{\mathcal{I},t_{\bar{\nu}}}, \Lambda[\ell(t_{\bar{\nu}})]) \cong \Lambda[\ell(t_{\bar{\nu}})]$ (induced by adjunction). With this convention, the resulting isomorphism

$$R\Gamma(\mathrm{Fl}_{\mathcal{I}}, \mathcal{F} * \mathcal{I}_{\bar{\nu}}) \cong R\Gamma(\mathrm{Fl}_{\mathcal{I}}, \mathcal{F})[\ell(t_{\bar{\nu}})] \tag{3.47}$$

for $\bar{\nu} \in \mathbb{X}_{\bullet}^+$ is additive in $\bar{\nu}$. Thus, it can be extended to the desired natural isomorphism

$$R\Gamma(\mathrm{Fl}_{\mathcal{I}}, \mathcal{F} * \mathcal{I}_{\bar{\nu}}) \cong R\Gamma(\mathrm{Fl}_{\mathcal{I}}, \mathcal{F})[\langle 2\bar{\rho}, \bar{\nu}\rangle], \tag{3.48}$$

using [Zhu14, Lemma 9.1.] to see that $\ell(t_{\bar{\nu}}) = \langle 2\rho, \nu\rangle$ if $\bar{\nu}$ is dominant. $\qquad\square$

We immediately deduce the following two corollaries.

Corollary 3.21. *There is a canonical isomorphism*

$$R\Gamma(\mathrm{Fl}_{\mathcal{I}}, \mathcal{I}_{\bar{\nu}}(M)) \simeq M[\langle 2\bar{\rho}, \bar{\nu}\rangle] \tag{3.49}$$

for $M \in \mathcal{D}([*/S_k])$ and $\bar{\nu} \in \bar{\mathbb{X}}_{\bullet}$.

Proof. This follows from Corollary 3.20 by setting $\mathcal{F} = \mathcal{I}_0(\Lambda) = \delta_e$. $\qquad\square$

Corollary 3.22. *For any $\mathcal{F} \in \mathcal{P}(\mathrm{Wak})$, there is a canonical isomorphism*

$$H^n(\mathrm{Fl}_{\mathcal{I}}, \mathcal{F}) \simeq \bigoplus_{\langle 2\bar{\rho}, \bar{\nu}\rangle = -n} \mathrm{Grad}_{\bar{\nu}}(\mathcal{F}). \tag{3.50}$$

Proof. This result is analogous to [AR, Proposition 4.5.4]. The existence of a canonical isomorphism follows from Corollary 3.21 by using a filtration of $\mathcal{F}$ by Wakimoto sheaves. Note that the associated graded of $\mathcal{F}$ is perverse, so we conclude that, whenever $\bar{\nu}_1 \preceq \bar{\nu}_2$, then the cohomology complexes $R\Gamma(\mathrm{Fl}_{\mathcal{I}}, \mathrm{gr}_{\bar{\nu}_i}(\mathcal{F}))$ sit in different degrees with the same parity. This implies that the connecting homomorphisms of the associated long exact sequences vanish. $\qquad\square$

We wish to determine the Λ-module $\mathrm{Grad}_{\bar{\mu}}(\mathcal{F})$ in a functorial manner. For this we calculate constant terms.

Proposition 3.23. *For any $\mathcal{F} \in \mathcal{D}_{\mathrm{\acute{e}t}}(\mathrm{Hk}_{\mathcal{I}})$, $w \in W$ and $\bar{\nu} \in \mathbb{X}_{\bullet}$, there is a canonical identification*

$$\mathrm{CT}_{B^-}(\mathcal{I}_{\bar{\nu}} * \mathcal{F})_{t_{\bar{\nu}}w} \simeq \mathrm{CT}_{B^-}(\mathcal{F})_w[\langle 2\rho, \nu\rangle] \tag{3.51}$$

between stalks of constant term complexes.

Proof. First, we assume $\bar{\nu} \in \bar{\mathbb{X}}_{\bullet}^+$, so that $\mathcal{I}_{\bar{\nu}}(\Lambda) = \nabla_{t_{\bar{\nu}}}(\Lambda)$. Notice that

$$\nabla_{t_{\bar{\nu}}}(\Lambda) * \mathcal{F} \simeq Rm_*(\Lambda\tilde{\boxtimes}\bar{\nu}(\pi)_*\mathcal{F})[\langle 2\rho, \nu\rangle].$$

By Braden's theorem, the left side of (3.51) naturally identifies with cohomology supported at the corresponding LU-orbit $\mathcal{S}_{t_{\bar{\nu}}w}$. Since $\mathcal{S}_{t_{\bar{\nu}}}$ contains $\mathrm{Fl}_{\mathcal{I},t(\lambda)}$ by Lemma 2.12, the pullback of $\mathcal{S}_{t_{\bar{\nu}}w}$ to $\mathrm{Fl}_{\mathcal{I},\leq t(\lambda)}\tilde{\times}\mathrm{Fl}_{\mathcal{I}}$ identifies with $L^+\mathcal{U}\bar{\nu}(\pi)L^+\mathcal{U} \times^{L^+\mathcal{U}} \mathcal{S}_w$. By abuse of notation, we denote the latter scheme by $\mathrm{Fl}_{t(\lambda)}\tilde{\times}\mathcal{S}_w$, even though the twisted product is not for the Iwahori group.

Amassing all this information, we get that

$$R\Gamma_c(\mathcal{S}_{t_{\bar{\nu}}w}^-, \mathcal{I}_{\bar{\nu}}(\Lambda) * \mathcal{F}) \simeq R\Gamma(\mathrm{Fl}_{\mathcal{I},t_{\bar{\nu}}}\tilde{\times}\mathcal{S}_w, \Lambda\tilde{\boxtimes}\bar{\nu}(\pi)_*Ri_w^!\mathcal{F})[\langle 2\rho, \nu\rangle], \tag{3.52}$$

where we also applied base change to commute $Ri_w^!$ and Rm_*. Because $\mathrm{Fl}_{\mathcal{I},t_{\bar{\nu}}}$ is an orbit under the pro-unipotent group $L^+\mathcal{U}$, see again Lemma 2.12, the twisted product does not alter the cohomology complex, thereby yielding the desired claim. $\qquad\square$

Corollary 3.24. *We have a canonical isomorphism $\mathrm{Grad}_{\bar{\nu}}(\mathcal{F})[\langle 2\rho, \nu\rangle] \simeq \mathrm{CT}_{B^-}(\mathcal{F})_{t_{\bar{\nu}}}$.*

Proof. This follows by induction on $\bar{\nu}$, by considering the filtration $\mathcal{F}_{\preceq\bullet}$ and applying Proposition 3.23. $\qquad\square$

Remark 3.25. The corollary above tells us when $\mathrm{Grad}_{\bar{\nu}}(\mathcal{F})$ is perverse with some ease for $\mathcal{F} \in \mathrm{Wak}$. This corollary together with geometric Satake and constant terms is what will allow us to show that the central functor $\mathcal{Z}$ actually factors through $\mathcal{P}(\mathrm{Wak})$, thus bypassing the strategy of [AB09, Theorem 4, Proposition 5] and [AR, Proposition 4.4.1]. Indeed, if we know that $\mathrm{CT}_{B^-}(\mathcal{F})$ is perverse, then its associated graded is perverse, and we can write the Wakimoto complex $\mathcal{F}$ as an extension of perverse Wakimoto sheaves, so it lies in $\mathcal{P}(\mathrm{Wak})$.

We end this section by discussing the monoidal structure of the functor $\mathrm{Grad} := \bigoplus_{\bar{\nu}\in\bar{\mathbb{X}}_\bullet} \mathrm{Grad}_{\bar{\nu}}$ restricted to $\mathcal{P}(\mathrm{Wak})$.[1]

Lemma 3.26. *The full subcategories $\mathcal{P}(\mathrm{Wak}) \subset \mathrm{Wak}$ are stable under convolution.*

Proof. By induction on the number of non-zero graded pieces, we reduce to the case of the convolution of two Wakimoto complexes, but this is Lemma 3.12. $\qquad\square$

Proposition 3.27. *For any $\mathcal{F}, \mathcal{G} \in \mathcal{P}(\mathrm{Wak})$, and $\bar{\nu}_1, \bar{\nu}_2 \in \mathbb{X}_\bullet$ with $\bar{\nu} := \bar{\nu}_1 + \bar{\nu}_2$, there is a canonical morphism*

$$\beta_{\bar{\nu}_1,\bar{\nu}_2} : \mathrm{gr}_{\bar{\nu}_1}(\mathcal{F}) * \mathrm{gr}_{\bar{\nu}_2}(\mathcal{G}) \to \mathrm{gr}_{\bar{\nu}}(\mathcal{F} * \mathcal{G}). \tag{3.53}$$

such that $\bigoplus_{\bar{\nu}_1+\bar{\nu}_2=\bar{\nu}} \beta_{\bar{\nu}_1,\bar{\nu}_2}$ is an isomorphism.

Proof. The statement is proved by induction on the number of non-vanishing $\mathrm{Grad}_{\bar{\nu}}$, similarly to the equicharacteristic case [AB09, Proposition 6a)] and [AR, Lemma 4.7.4, Proposition 4.7.5]. $\qquad\square$

Corollary 3.28. *For any $\mathcal{F}, \mathcal{G} \in \mathcal{P}(\mathrm{Wak})$, and $\bar{\nu}_1, \bar{\nu}_2 \in \bar{\mathbb{X}}_\bullet$ with $\bar{\nu} := \bar{\nu}_1 + \bar{\nu}_2$, there is a canonical morphism*

$$\alpha_{\bar{\nu}_1,\bar{\nu}_2} : \mathrm{Grad}_{\bar{\nu}_1}(\mathcal{F}) \otimes_\Lambda^L \mathrm{Grad}_{\bar{\nu}_2}(\mathcal{G}) \to \mathrm{Grad}_{\bar{\nu}}(\mathcal{F} * \mathcal{G}). \tag{3.54}$$

such that $\bigoplus_{\bar{\nu}_1+\bar{\nu}_2=\bar{\nu}} \alpha_{\bar{\nu}_1,\bar{\nu}_2}$ is an isomorphism. Consequently, for any such $\mathcal{F}$ and $\mathcal{G}$, there is a canonical isomorphism

$$\mathrm{Grad}(\mathcal{F}) \otimes_\Lambda^L \mathrm{Grad}(\mathcal{G}) \cong \mathrm{Grad}(\mathcal{F} * \mathcal{G}) \tag{3.55}$$

inside the category $\mathcal{D}_{\mathrm{\acute{e}t}}(\bar{\underline{\mathbb{X}}}_\bullet)$.

Proof. This follows directly from proposition 3.27 and lemma 3.12 (3). $\qquad\square$

4. CENTRAL FUNCTOR

4.1. **Background.** We introduce the spaces that underlie the construction of the Gaitsgory's central functor $\mathcal{Z}$ in mixed characteristic. The spaces are not of classical nature, and live in the world of v-stacks created by Scholze [Sch17, SW20]. We recall their basic properties, following [FS21, AGLR22].

We use the notation introduced in 1.5, but additionally assume that G is residually split, which implies that each I-orbit in $\mathrm{Fl}_\mathcal{I}$ is already defined over $\mathrm{Spec}(k)$, cf. 2.3.

Definition 4.1. The Hecke stack $\mathrm{Hk}_{\mathcal{I},O}$ is the v-stack sending a characteristic p affinoid perfectoid space $\mathrm{Spa}(R, R^+)$ to the groupoid of
- untilts $\mathrm{Spa}(R^\sharp, R^{\sharp,+})$ of $\mathrm{Spa}(R, R^+)$ over $\mathcal{O}$,
- $\mathcal{I}$-torsors $\mathcal{P}_1$ and $\mathcal{P}_2$ on $\mathrm{Spec}(B_{\mathrm{dR}}^+(R^\sharp))$ together with an isomorphism

$$\gamma : \mathcal{P}_1|_{\mathrm{Spec}(B_{\mathrm{dR}}(R^\sharp))} \cong \mathcal{P}_2|_{\mathrm{Spec}(B_{\mathrm{dR}}(R^\sharp))}. \tag{4.1}$$

We refer to [SW20, Section 20.3] for the definition of the rings $B_{\mathrm{dR}}^+(R^\sharp)$ and $B_{\mathrm{dR}}(R^\sharp)$.

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[1]We however don't discuss an $\mathbb{E}_1$-monoidal structure of the functor Grad on Wak.

An alternative way to define $\mathrm{Hk}_{\mathcal{I},O}$ is as the v-stack quotient

$$\mathrm{Hk}_{\mathcal{I},O} = [L_O^+\mathcal{I} \setminus \mathrm{Gr}_{\mathcal{I},\mathcal{O}}] \tag{4.2}$$

where $L_O^+\mathcal{I}$ is the jet group over O, i.e., the v-group sheaf over $\mathrm{Spd}(O)$ with value $\mathcal{I}(B_{\mathrm{dR}}^+(R^\sharp))$ on untilts $\mathrm{Spa}(R^\sharp, R^{\sharp,+})$ over O, and

$$\mathrm{Gr}_{\mathcal{I},O} := L_O\mathcal{I}/L_O^+\mathcal{I} \tag{4.3}$$

with $L_O\mathcal{I}$ the loop group over O, i.e., the v-group sheaf over $\mathrm{Spd}(O)$ with value $\mathcal{I}(B_{\mathrm{dR}}(R^\sharp))$ on $\mathrm{Spa}(R^\sharp, R^{\sharp,+})$, cf. [AGLR22, Lemma 4.10].

Note that over the generic fiber $\eta = \mathrm{Spd}(F)$, $\mathrm{Gr}_{\mathcal{I},O}$ identifies with the B_{dR}-affine Grassmannian $\mathrm{Gr}_{G,F}$. On the other hand, over the special fiber $s = \mathrm{Spd}(k)$, $\mathrm{Gr}_{\mathcal{I},O}$ becomes isomorphic to the analytification $\mathrm{Fl}_{\mathcal{I}}^\Diamond$ of the Witt vector affine flag variety $\mathrm{Fl}_{\mathcal{I}}$.

Now we pick a complete algebraically closed extension C of F with residue field $\bar{k}$, and let $\bar{\eta} = \mathrm{Spd}(C)$, $\bar{s} = \mathrm{Spd}(\bar{k})$. Consider the natural diagram

$$\mathrm{Hk}_{\mathcal{I},C} \overset{j}{\hookrightarrow} \mathrm{Hk}_{\mathcal{I},O_C} \overset{i}{\hookleftarrow} \mathrm{Hk}_{\mathcal{I},\bar{k}}, \tag{4.4}$$

where j is the open immersion of the generic fiber and i the closed immersion of the special fiber. This induces a nearby cycles functor, see [AGLR22, Section 6.5],

$$R\Psi := i^*Rj_* : \mathcal{D}_{\mathrm{\acute{e}t}}(\mathrm{Hk}_{\mathcal{I},C}) \longrightarrow \mathcal{D}_{\mathrm{\acute{e}t}}(\mathrm{Hk}_{\mathcal{I},\bar{k}}) \tag{4.5}$$

between the stable ∞-categories of derived étale sheaves in the sense of Scholze [Sch17, Definition 14.13, Lemma 17.1] with bounded support as in Fargues–Scholze [FS21, Chapter VI]. For ℓ-adic coefficients, we follow the same conventions of passing to the limit as in [Sch17, Section 27] and then inverting ℓ, compare with [AGLR22, Section 6.5].

An important property of $R\Psi$ is that it preserves universal local acyclicity in the sense of [FS21, Section IV.2.1], see also [AGLR22, Section 6] for our convention for non-torsion coefficients Λ. Below, we denote by $\mathcal{D}_{\mathrm{ula}}(X/S) \subset \mathcal{D}_{\mathrm{\acute{e}t}}(X)$ the full subcategory of universally locally acyclic sheaves (or, if the base is understood, simply $\mathcal{D}_{\mathrm{ula}}(X)$).

Proposition 4.2. *Nearby cycles $R\Psi$ restrict to a functor*

$$\mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{I},C}) \to \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{I},\bar{k}}). \tag{4.6}$$

Proof. This is [AGLR22, Corollary 6.14]. □

Recall that in the previous sections of the paper, we introduced a Hecke stack $\mathrm{Hk}_{\mathcal{I}}^{\mathrm{sch}}$ as a perfect k-stack. Its associated v-sheaf under the functor $\diamond$ is the fiber over $\mathrm{Spd}k$ of the analytic Hecke stack $\mathrm{Hk}_{\mathcal{I}}^{\mathrm{an}}$ that we defined over $\mathrm{Spd}O$. There is a natural comparison map of sheaves due to [Sch17, Section 27]

Proposition 4.3. *The natural comparison functor*

$$c : \mathcal{D}_{\mathrm{\acute{e}t}}(\mathrm{Hk}_{\mathcal{I},\bar{k}}^{\mathrm{sch}}) \to \mathcal{D}_{\mathrm{\acute{e}t}}(\mathrm{Hk}_{\mathcal{I},\bar{k}}^{\mathrm{an}}) \tag{4.7}$$

is an equivalence carrying $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I},\bar{k}}^{\mathrm{sch}})$ to $\mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{I},\bar{k}}^{\mathrm{an}})$.

Proof. For the definition of the comparison functor, we refer to [Sch17, Section 27] and [AGLR22, Appendix A]. The above result is [AGLR22, Propositions 6.7 and A.5]. □

This result also highlights the importance of ula sheaves as singling out constructible sheaves over a base field. From now on, we will not make a stark distinction between $\mathrm{Hk}_{\mathcal{I}}^{\mathrm{sch}}$ and $\mathrm{Hk}_{\mathcal{I},k}^{\mathrm{an}}$ and will simply omit the superscript when writing down its derived category of sheaves.

Definition 4.4. We define the central functor (for the Witt vector affine flag variety) as the composition

$$\mathcal{Z}(-) : \operatorname{Rep}(\hat{G}) \xrightarrow{\sim} \mathcal{P}_{\mathrm{ula}}(\mathrm{Hk}_{G,C}) \xrightarrow{R\Psi} \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{I},\bar{k}}) \tag{4.8}$$

Here, the first arrow comes from the geometric Satake equivalence of [FS21, Chapter VI], with the Satake category consisting of ula perverse sheaves on $\mathrm{Hk}_{G,C}$ (which are additionally Λ-flat if Λ is assumed to be a ℓ-power torsion ring). The second arrow are just nearby cycles which respect the ula property by Proposition 4.2. Often below, we will find it convenient to still abusively denote by $\mathcal{Z}$ the nearby cycles functor $R\Psi \colon \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{G,C}) \to \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{I},\bar{k}})$.

Remark 4.5. As explained in [AGLR22, Section 8] the nearby cycles functor is Galois equivariant. More precisely, given $A \in \mathcal{P}_{\mathrm{ula}}(\mathrm{Hk}_{G,E})$ for some finite extension E/F, then $R\Psi(A_C)$ has a natural Γ_E-action that is equivariant with respect to the residual action of Γ_{k_E}. Here, $\Gamma_E \subseteq \Gamma$ denotes the Galois group of E, and Γ_{k_E} the one for the residue field k_E of E.

4.2. **Convolution and fusion.** In this section, we are going to discuss in detail the convolution and fusion products.

Definition 4.6. Given a finite linearly ordered set $J = \{j_1 < \cdots < j_n\}$, we define the convolution Hecke stack $\mathrm{Hk}_{\mathcal{I}}^J$ to be the v-sheaf over $\operatorname{Spd} O$ which classifies successive modifications of $\mathcal{I}$-torsors over B_{dR}^+, indexed by the elements $j_i \in J$. More precisely, for a given $f \colon S \to \operatorname{Spd} O$ the groupoid $\mathrm{Hk}_{\mathcal{I}}^J(S)$ is given by $\mathcal{I}$-torsors $\mathcal{P}_{j_1}, \ldots, \mathcal{P}_{j_n}$ on $B_{\mathrm{dR}}^+(S)$ with modifications $\mathcal{P}_{j_1} \dashrightarrow \mathcal{P}_{j_2}, \ldots, \mathcal{P}_{j_{n-1}} \dashrightarrow \mathcal{P}_{j_n}$ defined on $B_{\mathrm{dR}}(S)$.

One often finds the expression $\mathrm{Hk}_{\mathcal{I}}^J = \mathrm{Hk}_{\mathcal{I}}^{j_1} \tilde{\times} \ldots \tilde{\times} \mathrm{Hk}_{\mathcal{I}}^{j_n}$ to denote the convolution Hecke stack. We have already seen that there is a natural correspondence with $n = 2$ inducing the convolution product $* \colon \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I},S}) \times \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I},S}) \to \mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I},S})$ for any $S \to \operatorname{Spd} O$, see 3.1. We wish to enhance this operation to a monoidal structure of ∞-categories in the sense of [Lur17, Definition 4.1.1.10]. This will be quite technical, and we recommend the unaccostumed reader to try to ignore the heavy language at first, and focus on the geometry at hand. After each categorical proof, we also provide an explanation of our constructions at the level of 1-morphisms of correspondences, which should prove helpful.

Let us recall some of the notions from [Lur17, Section 4.1]. First, we have the (nerve of) the 1-category $\operatorname{Comm}^\otimes$ (also denoted by Fin_* or $\mathbb{E}_\infty^\otimes$ in [Lur17]) whose objects are finite pointed sets $\langle n \rangle = \{0, 1, \ldots, n\}$ with base point 0 and whose morphisms $\langle n \rangle \to \langle m \rangle$ preserve 0. A symmetric monoidal ∞-category $\mathcal{C}$ is given by a cocartesian fibration $\mathcal{C}^\otimes \to \operatorname{Comm}^\otimes$ of ∞-operads, see [Lur17, Example 2.1.2.18], [Lur17, Definition 2.1.1.10], in particular, $\mathcal{C}_{[n]}^\otimes \simeq \mathcal{C}^n$ in a natural manner. Similarly, we have the ∞-operad $\operatorname{Assoc}^\otimes$ (which is equivalent to some other common ∞-operads denoted by $\mathbb{A}_\infty^\otimes$ or $\mathbb{E}_1^\otimes$ in [Lur17]) given as the 1-category whose objects are pointed finite set $\langle n \rangle$ and morphisms $\langle n \rangle \to \langle m \rangle$ are pointed maps equipped with a total order on the non-pointed fibers and composition is given by the lexicographical order, see [Lur17, Remark 4.1.1.4]. A monoidal ∞-category is a cocartesian fibration $\mathcal{C}^\otimes \to \operatorname{Assoc}^\otimes$ of ∞-operads and its underlying ∞-category $\mathcal{C}$ is the fiber of the fibration over $\langle 1 \rangle$. When the cocartesian fibration is clear from the context, we will often abuse language and refer to $\mathcal{C}$ as a monoidal ∞-category. Such a datum induces by [Lur17, Propositions 2.4.1.7, 2.4.2.5] a map $\operatorname{Assoc}^\otimes \to \operatorname{Cat}_\infty^\times$ that preserves inert morphisms in the sense of [Lur17, Definitions 2.1.1.8, 2.1.2.3] and also cocartesian morphisms, in particular $\mathcal{C}^\otimes \to \operatorname{Assoc}^\otimes$ induces an associative algebra in $\operatorname{Cat}_\infty$. Here, $\operatorname{Cat}_\infty^\times$ denotes $\operatorname{Cat}_\infty$ with its cartesian symmetric monoidal structure, cf. [Lur17, Section 2.4.1]. Note that in general maps of ∞-operads are not necessarily monoidal, but rather only lax-monoidal, see [Lur17, Definition 2.1.3.7].

In order to produce the desired map that will induce a monoidal structure on $\mathcal{D}_{\text{ét}}(\text{Hk}_{\mathcal{I},S})$, we recall that following [Man22a, Definition A.5.2] we dispose of a symmetric monoidal ∞-category $\text{Corr}(\text{vSt})$ of correspondences on v-stacks. The 6-functor formalism defined in [Sch17] for torsion coefficients can be reinterpreted as in [Man22a, Definition A.5.6] thanks to [Man22b, Theorem 5.11] via an operadic map

$$\mathcal{D}_{\text{ét}}^{\otimes} \colon \text{Corr}^{\otimes}(\text{vSt})_{\ell\text{-fine}} \to \text{Cat}_{\infty}^{\times}, \tag{4.9}$$

where the ℓ-fine subscript indicates that we restrict to the full subcategory of $\text{Corr}(\text{vSt})$ whose correspondences have ℓ-fine maps to the right. We extend it to ℓ-adic coefficients via the naive construction of taking limits and tensoring with $\mathbb{Q}$, instead of using nuclear ℓ-adic sheaves, compare with [Sch17, Section 26] and [Man22b, page 6]. Note also that $\mathcal{D}_{\text{ét}}$ is a map of ∞-operads, and not symmetric monoidal (only lax symmetric monoidal). Despite this, all the maps obtained below between ∞-operads of either correspondences or sheaves will turn out to be monoidal.

Now, we are going to enhance $*$ to a monoidal structure on the ∞-category, by constructing a map $\mathcal{H}_S^{\otimes} \colon \text{Assoc}^{\otimes} \to \text{Corr}^{\otimes}(\text{vSt})$ of ∞-operads that commutes with the maps towards $\text{Comm}^{\otimes}$ and recovers the convolution $*$ on $\text{Hk}_{\mathcal{I},S}$ via evaluation on the active morphism $\langle 2 \rangle \to \langle 1 \rangle$ (with order $1 < 2$). We were crucially assisted in this task by discussions with Heyer, Mann, and Zhao. Note that there is an obvious isomorphism

$$\text{Hk}_{\mathcal{I},S} := L_S^+\mathcal{I}\backslash L_S\mathcal{I}/L_S^+\mathcal{I} \simeq [*/L_S^+\mathcal{I}] \times_{[*/L_S\mathcal{I}]} [*/L_S^+\mathcal{I}] \tag{4.10}$$

One can therefore realize $\text{Hk}_{\mathcal{I},S}$ as the internal endomorphism object of $[*/L_S^+\mathcal{I}]$ in the ∞-category $\text{Corr}(\text{vSt}_{/[*/L_S\mathcal{I}]})$, and hence it inherits a natural ∞-monoidal structure in the category of correspondences by forgetting the slice over $[*/L_S\mathcal{I}]$. In fact, the functor $\text{Corr}(\text{vSt}_{/[*/L_S\mathcal{I}]}) \to \text{Corr}(\text{vSt})$ is naturally symmetric monoidal. Note that the associated planar ∞-operad in the sense of [Lur17, Definition 4.1.3.2] is nothing other than the Čech nerve of the natural map $[*/L_S^+\mathcal{I}] \to [*/L_S\mathcal{I}]$.

Let us try to understand more closely what the map $\mathcal{H}_S^{\otimes} \colon \text{Assoc}^{\otimes} \to \text{Corr}^{\otimes}(\text{vSt})$ induced by the above monoidal structure on $\text{Hk}_{\mathcal{I},S}$ looks like. We send an object $\langle n \rangle$ to the fiber product $\text{Hk}_{\mathcal{I},S}^n$ over S and the morphism $\alpha \colon \langle n \rangle \to \langle m \rangle$ to the correspondence

$$\text{Hk}_{\mathcal{I},S}^n \leftarrow \text{Hk}_{\mathcal{I},S}^{\alpha} \to \text{Hk}_{\mathcal{I},S}^m, \tag{4.11}$$

where the middle term is the m-indexed product of the convolution Hecke stacks in the sense of 4.6 with superscripts ranging over the ordered fibers of α, the left map is the natural projection and the right map is the product of the natural multiplication. Note that the left map is a torsor for a power of $L_S^+\mathcal{I}$, and thus pro-smooth, whereas the right map is fibered in powers of $\text{Gr}_{\mathcal{I},S}$ and hence it is ind-proper. One can also write down the image under $\mathcal{H}_S^{\otimes}$ of arbitrary n-morphisms of the 1-category $\text{Assoc}^{\otimes}$, which are in bijection with sequences of composable morphisms.

In order to be able to apply the functor $\mathcal{D}_{\text{ét}}^{\otimes}$, we have to replace the convolution Hecke stacks by finite-dimensional truncations so that the maps to the right become ℓ-fine, but here we will ignore this subtlety and refer to [XZ17, Definition 5.1.2, Subsection 5.1.7] for a detailed treatment. We denote by $\mathcal{D}_{\text{ét}}^{\otimes}(\text{Hk}_{\mathcal{I},S})$ the monoidal ∞-category obtained from composing $\mathcal{H}_S^{\otimes}$ and $\mathcal{D}_{\text{ét}}^{\otimes}$ (after taking appropriate truncations, so that this becomes legitimate). This clearly refines the convolution product $*$, as seen by taking one of the two active maps $\langle 2 \rangle \to \langle 1 \rangle$.

Lemma 4.7. *The full subcategory $\mathcal{D}_{\text{ula}}(\text{Hk}_{\mathcal{I},S})$ is stable under convolution.*

Proof. The reason is ula sheaves are preserved under smooth pullback, exterior products, and proper pushforward, which are the operations involved in the convolution product. □

By [Lur17, Proposition 2.2.1.1] on full subcategories of ∞-operads, we have a monoidal ∞-category $\mathcal{D}_{\text{ula}}(\text{Hk}_{\mathcal{I},S})$ giving rise to convolution.

Lemma 4.8. *Given a map $f\colon T \to S$, the pullback functor f^* is monoidal, i.e., it enhances essentially uniquely to a $\mathbb{E}_1$-monoidal map $f^{*,\otimes}\colon \mathcal{D}^{\otimes}_{\text{ét}}(\text{Hk}_{\mathcal{I},S}) \to \mathcal{D}^{\otimes}_{\text{ét}}(\text{Hk}_{\mathcal{I},T})$ (and similarly for ula sheaves).*

Proof. This is immediate because the pullback functor $f^*\colon \text{Corr}(\text{vSt}_S) \to \text{Corr}(\text{vSt}_T)$ is symmetric monoidal. $\qquad\square$

Proposition 4.9. *The functor $\mathcal{Z}\colon \mathcal{D}_{\text{ula}}(\text{Hk}_{\mathcal{I},C}) \to \mathcal{D}_{\text{ula}}(\text{Hk}_{\mathcal{I},k})$ is monoidal, i.e., it enhances essentially uniquely to a $\mathbb{E}_1$-monoidal map $\mathcal{Z}^{\otimes}\colon \mathcal{D}^{\otimes}_{\text{ula}}(\text{Hk}_{\mathcal{I},C}) \to \mathcal{D}^{\otimes}_{\text{ula}}(\text{Hk}_{\mathcal{I},k})$.*

Proof. Recall that $\mathcal{Z} = i^* R j_*$, where j and i denote the inclusion of the generic and special fibers of $\text{Hk}_{\mathcal{I},O_C}$. We have seen that both pullback functors j^* and i^* are monoidal, thanks to Lemma 4.8. We claim that on ULA objects, j^* induces an isomorphism of ∞-operads. This can be checked at the level of underlying ∞-categories by [Lur17, Remark 2.1.3.8], and that statement is [AGLR22, Proposition 6.12]. $\qquad\square$

There is a more general version of the Hecke stack that can be obtained by not taking $\text{Spd}\,O$ as the base, but allowing products with itself over $\text{Spd}\,k$.

Definition 4.10. Let $S_i \to \text{Spd}\,O$, $i = 1, \ldots, d$ be finitely many v-sheaves over O. We define the Hecke stack $\text{Hk}_{\mathcal{I},S}$ with $S = S_1 \times \cdots \times S_d$ as the classifying stack of modifications of $\mathcal{I}$-bundles over the completion of the relative curve $\mathcal{Y}_S$ at the union of the d Cartier divisors specified by the d projections $S \to S_i$, see [FS21, Definition VI.1.6].

A similar variant exists for the convolution Hecke stacks, where one allows compositions of several modifications instead of modifying simultaneously at several divisors. We are now able to recall the fusion interpretation from [FS21, Section VI.9] that refines the convolution product and induces symmetry constraints on perverse sheaves. Recall that a perverse t-structure on $\mathcal{D}_{\text{ét}}(\text{Hk}_{G,C})$ was defined by Fargues–Scholze in [FS21, Definition/Proposition VI.7.1].

During the rest of this subsection and the next one, we are going to abbreviate the categories $\mathcal{D}_{\text{ula}}(\text{Hk}_{\mathcal{I},S})$ by $\mathcal{C}_S$, where S is some v-sheaf over $(\text{Spd}\,O)^n$. If S is the product of the v-sheaf associated with Huber rings (R_i, R_i°) over O, then we will write $\mathcal{C}_{R_1 \times \cdots \times R_n}$ for $\mathcal{C}_S$ so as to highlight each of the factors. The full subcategory of perverse sheaves will be abbreviated by $\mathcal{P}_S$ and $\mathcal{P}_{R_1 \times \cdots \times R_n}$, respectively.

Proposition 4.11. *The full subcategory $\mathcal{P}_C \subset \mathcal{C}_C$ of perverse sheaves is stable under convolution and it extends to a symmetric monoidal ∞-category.*

Proof. Stability under convolution can be found in [FS21, Proposition VI.8.1] and the symmetric monoidal structure follows from [FS21, Definition/Proposition VI.9.4]. We explain the second part, which will prove useful later on. We have to prove that the bifunctor

$$\mathcal{P}_C \times \mathcal{P}_C \to \mathcal{P}_C \tag{4.12}$$

is monoidal, where the left side carries the monoidal structure. This will endow $\mathcal{P}_C$ with the structure of a braided monoidal category (but we will not check explicitly that it is symmetric). To get the braiding, we extend the map into two commutative triangles in Cat_∞:

$$\tag{4.13}$$

where the vertical arrows are the pullbacks to the obvious strata of $(\mathrm{Spd}\, C)^2$ given by the diagonal and its complement, and the middle map is induced by the fusion correspondence

$$\mathrm{Hk}^2_{G,C} \leftarrow \mathrm{Hk}_{G,C} \tilde{\times} \mathrm{Hk}_{G,C} \rightarrow \mathrm{Hk}_{G,C^2}. \tag{4.14}$$

The vertical maps are clearly monoidal and the upper one is fully faithful by [FS21, Proposition VI.9.3], so it suffices to see that the upper diagonal map is monoidal. Indeed, the loop groups $L^+_{C^2}\mathcal{I}$ and $L_{C^2}\mathcal{I}$ naturally factor into a product away from the diagonal, so we get an induced map $\mathcal{H}^\otimes_C \times \mathcal{H}^\otimes_C \leftarrow \mathcal{H}^{\otimes,\neq}_{C^2}$ of functors $\mathrm{Assoc}^\otimes \rightarrow \mathrm{Corr}^\otimes(\mathrm{vSt})$ regarded as a correspondence to the left by functoriality of endomorphisms objects. This yields our desired monoidal map upon applying $\mathcal{D}^\otimes$ and restricting to the monoidal subcategories of perverse sheaves. $\qquad\square$

For the reader's convenience, let us explain more informally what is happening in the above proof. Let $\alpha\colon \langle n \rangle \rightarrow \langle m \rangle$ be a 1-morphism in $\mathrm{Assoc}^\otimes$. Notice that we have a composition of two correspondences, namely the fusion and the diagonal ones:

$$\mathrm{Hk}^n_{\mathcal{I},C} \leftarrow \mathrm{Hk}^\alpha_{\mathcal{I},C^n} \rightarrow \mathrm{Hk}^m_{\mathcal{I},C^n} \leftarrow \mathrm{Hk}^m_{\mathcal{I},C} \tag{4.15}$$

where the first two maps are the natural pro-smooth projection and ind-proper multiplication, and the last is a diagonal closed immersion. It is clear that the fiber product is the usual correspondence defining the monoidal structure on $\mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{I},S})$. Now the advantage of the first correspondence lies in the fact that, after excluding the partial diagonals, the stack $\mathrm{Hk}^\alpha_{\mathcal{I},S}$ decomposes as a product of regular Hecke stacks, so that the order of the modifications (in other words the ordering on the fibers of α) no longer matters. If we restrict to the full subcategory $\mathcal{P}_C \subset \mathcal{C}_C$ of perverse sheaves, then pullback $\mathcal{P}_{C^n} \rightarrow \mathcal{P}^{\neq}_{C^n}$ away from the union of the partial diagonals of $(\mathrm{Spd}\, C)^n$ is fully faithful, see [FS21, Proposition VI.9.3]. This yields the various symmetry constraints, as desired.

4.3. **Associative center.** Let $\mathcal{C}$ be a monoidal ∞-category. One may attach to $\mathcal{C}$ another monoidal ∞-category called its associative center and denoted $\mathcal{Z}(\mathcal{C})$. Observe that the ∞-category $\mathrm{End}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ is left-tensored over $\mathcal{C}$ via the latter's monoidal structure. We define $\mathcal{Z}(\mathcal{C}) := \mathrm{End}_{\mathcal{C}\times\mathcal{C}}(\mathcal{C})$ of $\mathcal{C}$-bilinear endomorphisms in the sense of [Lur17, Definition 4.6.2.7]. Since these are monoidal ∞-categories with tensor structure given by composition, and $\mathcal{C}$-bilinearity is stable under composition, we see that the full subcategory $\mathcal{Z}(\mathcal{C})$ inherits a monoidal structure. It comes equipped with a natural monoidal map $\mathrm{ev}_1\colon \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ given by evaluation at the monoidal unit. Also note that this definition coincides by [Lur17, Theorem 4.4.1.28, Theorem 5.3.1.30] with the center of an associative algebra of Cat_∞ in the sense of [Lur17, Definition 5.3.1.12].

Theorem 4.12. *The monoidal functor $\mathcal{Z}\colon \mathcal{C}_C \rightarrow \mathcal{C}_k$ lifts monoidally to the center $\mathcal{Z}(\mathcal{C}_k)$.*

Proof. According to [Lur17, Definition 5.3.1.12], this amounts to showing that the left action morphism $\mathcal{Z}_{\mathrm{lc}} := * \circ (\mathcal{Z}, \mathrm{id})\colon \mathcal{C}_C \times \mathcal{C}_k \rightarrow \mathcal{C}_k$ is monoidal, where the left side is an associative algebra in Cat_∞ by multiplying coordinates separately. We consider the following union of two commutative squares in Cat_∞:

$$
\begin{array}{ccc}
\mathcal{C}_C \times \mathcal{C}_k & \longrightarrow & \mathcal{C}_{C\times k} \\
\uparrow & & \uparrow \\
\mathcal{C}_{O_C} \times \mathcal{C}_k & \longrightarrow & \mathcal{C}_{O_C \times k} \\
\downarrow & & \downarrow \\
\mathcal{C}_k \times \mathcal{C}_k & \longrightarrow & \mathcal{C}_{k^2}
\end{array}
\tag{4.16}
$$

where the vertical maps are pullback functors and hence clearly monoidal, and the horizontal maps are given by the fusion product (and therefore are not a priori monoidal). Note also

that $\mathcal{C}_{k^2} = \mathcal{C}_k$ as we take products over k itself and so the lower horizontal map is simply convolution. Since the left upper morphism is an equivalence, we recover $\mathcal{Z}_{\mathrm{lc}}$ by taking an inverse and composing across the left lower edge of the diagram. Since the right upper map is fully faithful by Lemma 4.13 below, it suffices to monoidally enhance the upper horizontal map. But this follows as in the case of the fusion map $\mathcal{P}_C^2 \to \mathcal{P}_{C^2}^{\neq}$ of perverse sheaves away from the diagonal: indeed, the loop groups $L_{C \times k}^+ \mathcal{I}$ and $L_{C \times k} \mathcal{I}$ split as a direct product of the loop groups over C and k, so we get an equivalence by functoriality of endomorphism objects. $\qquad\square$

Again for the reader's convenience, we repeat our explanation of our reasoning in terms of 1-morphisms of $\mathrm{Assoc}^{\otimes}$. We have to see that morphisms $\alpha\colon \langle n \rangle \to \langle m \rangle$ in $\mathrm{Assoc}^{\otimes}$ are naturally intertwined with $\mathcal{Z}_{\mathrm{lc}}$, i.e., that the diagram below

$$\begin{array}{ccc}
\mathcal{C}_C^n \times \mathcal{C}_k^n & \longrightarrow & \mathcal{C}_C^m \times \mathcal{C}_k^m \\
\downarrow & & \downarrow \\
\mathcal{C}_k^n & \longrightarrow & \mathcal{C}_k^m
\end{array} \tag{4.17}$$

commutes, where the vertical maps are powers of $\mathcal{Z}_{\mathrm{lc}}$ and the horizontal ones are induced by α.

Notice that the composition across the right arises from the composition of correspondences

$$\mathrm{Hk}_{\mathcal{I},O_C}^n \times \mathrm{Hk}_{\mathcal{I},k}^n \leftarrow \mathrm{Hk}_{\mathcal{I},O_C^n \times k^n}^{\gamma[m] \circ \alpha[2]} \to \mathrm{Hk}_{\mathcal{I},O_C^n \times k^n}^m \tag{4.18}$$

where $\gamma : \langle 2 \rangle \to \langle 1 \rangle$ is active carrying the usual order, $\gamma[m]\colon \langle 2m \rangle \to \langle m \rangle$ denotes its concatenation, and similarly for $\alpha[2]\colon \langle 2n \rangle \to \langle 2m \rangle$. The composition across the left arises instead from the correspondence

$$\mathrm{Hk}_{\mathcal{I},O_C}^n \times \mathrm{Hk}_{\mathcal{I},k}^n \leftarrow \mathrm{Hk}_{\mathcal{I},O_C^n \times k^n}^{\alpha \circ \gamma[n]} \to \mathrm{Hk}_{\mathcal{I},O_C^n \times k^n}^m. \tag{4.19}$$

Indeed, we can invoke the monoidal equivalence $\mathcal{C}_C \simeq \mathcal{C}_{O_C}$ proved in Proposition 4.9 and [AGLR22, Proposition 6.12], apply the monoidal functor $\mathcal{D}$ to the previous correspondences, and then compose with the pullback i^*.

In order to verify that these maps are naturally isomorphic, we must be able to swap the contribution of each O_C-factor adjacent to a k-factor. Thanks again to the equivalence $\mathcal{C}_{O_C} \simeq \mathcal{C}_C$ of [AGLR22, Proposition 6.12] and the fully faithful embeddings $\mathcal{C}_{O_C^n \times k^n} \subset \mathcal{C}_{C^n \times k^n}$ of Lemma 4.13 proved below, we are reduced to comparing the maps after taking the pullback functor j^*. But since $\mathrm{Spd}\, C$ and $\mathrm{Spd}\, k$ map disjointly to $\mathrm{Spd}\, O$, both convolution Hecke stacks become isomorphic to $\mathrm{Hk}_{G,C^n}^{\alpha} \times \mathrm{Hk}_{\mathcal{I},k^n}^{\alpha}$, so the result is clear.

The following lemmas were used in the proof of Theorem 4.12:

Lemma 4.13. *The natural map $j^*\colon \mathcal{C}_{O_C^n \times k^m} \to \mathcal{C}_{C^n \times k^m}$ is fully faithful for any $n, m \geq 0$.*

Proof. We must show that the unit $A = Rj_*j^*A$ for every ULA sheaf. This follows from the ula property and the next lemma. $\qquad\square$

Recall our shorthand notation for the various functors and categories defined over products of O_C, C, and k. In order to avoid cumbersome notation below involving Spd and lots of brackets, we apply this convention now to the point functor, so that $*_{O_C^n} := (\mathrm{Spd}\, O_C)^n$

Lemma 4.14. *If $j\colon *_{C^n} := (\mathrm{Spd}\, C)^n \to (\mathrm{Spd}\, O_C)^n =: *_{O_C^n}$, then $Rj_*\Lambda = \Lambda$.*

Proof. For reasons that will become clear during our induction argument, we replace the exponent n by a finite set J during our proof. If $|J| = 1$, this follows already from [GL22, Theorem 4.7] applied to the kimberlite $*_{O_C}$, since its reduction equals $*_k$ and hence nearby cycles are per definition algebraic, so they can be calculated via the étale site for kimberlites, which is trivial.

If $|J| = 2$, then we first compute the stalk of $Rj_*\Lambda$ at $*_{k \times C}$. We know that partially compactly supported cohomology vanishes by [FS21, Theorem IV.5.3], so $R\Gamma(*_{O_C \times C}, j_!\Lambda) = 0$, compare

with [FS21, Proposition V.4.2, Remark V.4.3]. This means that our sought stalk is given by $R\Gamma(*_{C^2}, \Lambda)$ which coincides with Λ thanks to [Sch17, Theorem 19.5]. It remains to compute the stalk at the reduction $*_k$ of the kimberlite $*_{O_C^2}$, so we apply [GL22, Theorem 4.7] once again.

Finally, in the general case, we can stratify $*_{O_C^J}$ by locally closed subsets of the form $*_{C^K}$ where $K \subset J$. We prove the equality $Rj_*\Lambda = \Lambda$ on the analytic strata (i.e., with K being non-empty) by descending induction on the cardinality of K. If $K = J$, there is nothing to show. Otherwise, consider the open set $*_{O_C^{J\backslash K} \times C^K}$ and observe again by [FS21, Theorem IV.5.3] that $R\pi_*i_!\Lambda = 0$ where

$$*_{O_C^{J\backslash K'} \times C^{K'}} \xrightarrow{i} *_{O_C^{J\backslash K} \times C^K} \xrightarrow{\pi} *_{O_C^{J\backslash K'} \times C^K}, \tag{4.20}$$

and $K \subset K'$ has singleton complement. This implies the claim regarding the stratum $*_{C^K}$ again thanks to [Sch17, Theorem 19.5]. As for the non-analytic point $*_k$ of $*_{O_C^J}$, we invoke [GL22, Theorem 4.7] again for the last time. $\square$

Next, we prove that the symmetry constraints that appear in the full subcategory $\mathcal{P}_C \subset \mathcal{C}_C$ of perverse sheaves are compatible with the braidings in the associative center $\mathcal{Z}(\mathcal{C}_k)$. While $\mathcal{P}_C$ is symmetric monoidal, $\mathcal{Z}(\mathcal{C}_k)$ is not. Instead, the associative center carries a structure over the ∞-operad $\mathbb{E}_2^{\otimes}$ of little squares, see [Lur17, Definition 5.1.0.2]. This arises more formally as the tensor product in Op_∞ of $\mathbb{E}_1^{\otimes}$ with itself, see [Lur17, Theorem 5.1.2.2]. Here, we identify $\mathbb{E}_1^{\otimes}$ with $\mathrm{Assoc}^{\otimes}$ via [Lur17, Example 5.1.0.7]. Our assertion that associative centers carry an $\mathbb{E}_2^{\otimes}$-structure is [Lur17, Remark 5.3.1.13], which explains that they can be regarded as associative algebras in the category of associative algebras of Cat_∞, the extra associative structure arising by bilinearity.

Theorem 4.15. *The composite $\mathcal{P}_C \subset \mathcal{C}_C \xrightarrow{\mathcal{Z}} \mathcal{Z}(\mathcal{C}_k)$ is an $\mathbb{E}_2$-monoidal map.*

Proof. Our goal is verifying that the monoidal map $\mathcal{P}_C \to \mathcal{Z}(\mathcal{C}_k)$ actually respects the extra monoidal structures on both sides in the ∞-category of associative algebras. This amounts to checking by the universal property of the center that the following commutative square

$$\begin{array}{ccc} \mathcal{P}_C \times \mathcal{P}_C & \longrightarrow & \mathcal{P}_C \times \mathcal{C}_k \\ \downarrow & & \downarrow \\ \mathcal{P}_C & \longrightarrow & \mathcal{C}_k \end{array} \tag{4.21}$$

in Cat_∞ is actually a commutative square in $\mathrm{Alg}_{\mathbb{E}_1^{\otimes}}(\mathrm{Cat}_\infty^{\times})$, where the maps are the obvious ones induced by convolution or $\mathcal{Z}$ and their monoidal enhancements were defined in Proposition 4.11 and Theorem 4.12.

Let us recapitulate how the braiding isomorphisms were constructed. For $\mathcal{P}_C$, we saw during Proposition 4.11 how to define a monoidal structure on the left vertical map via a pair of commuting triangles. In the special fiber, we saw during Theorem 4.12 how to define a monoidal structure on the right vertical map (actually, we took the larger category $\mathcal{C}_C$ instead of just $\mathcal{P}_C$) via a pair of commuting squares.

We must now perform these constructions at once in such a way that they are intertwined. Indeed, we have the following pair of commuting triangles

$$
\begin{array}{ccc}
& & \mathcal{C}^{\neq}_{O^2_C} \\
& \nearrow & \uparrow \\
\mathcal{C}_{O_C} \times \mathcal{C}_{O_C} & \longrightarrow & \mathcal{C}_{O^2_C} \\
& \searrow & \downarrow \\
& & \mathcal{C}_{O_C}
\end{array}
\tag{4.22}
$$

which relate to the previously constructed diagrams via natural pullback functors and passing to certain full subcategories. More precisely, restricting to C^2 and to perverse sheaves recovers the diagram (4.13), while restricting to $O_C \times k$ recovers the diagram (4.16) up to composing across the upper left and the lower left corners. Now, the upper vertical map is not fully faithful, and so we need to restrict to a full subcategory of sheaves where that happens. It suffices to take the category $\mathcal{E}_{O^2_C}$ of sheaves which are perverse over C^2 by Lemma 4.16 below. $\qquad\square$

Lemma 4.16. *Denote by $\mathcal{E}_{O^n_C}$ the ∞-category given as the fiber product $\mathcal{C}_{O^n_C} \times_{\mathcal{C}_{C^n}} \mathcal{P}_{C^n}$. Then, the pullback functor $\mathcal{E}_{O^n_C} \to \mathcal{E}^{\neq}_{O^n_C}$ is fully faithful.*

Proof. We have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{E}_{O^n_C} & \longrightarrow & \mathcal{E}^{\neq}_{O^n_C} \\
\downarrow & & \downarrow \\
\mathcal{P}_{C^n} & \longrightarrow & \mathcal{P}^{\neq}_{C^n}
\end{array}
\tag{4.23}
$$

The left arrow is fully faithful, because it is base changed from $\mathcal{C}_{O^n_C} \subset \mathcal{C}_{C^n}$ along $\mathcal{P}_{C^n} \to \mathcal{C}_{C^n}$ as proved in Lemma 4.13. The bottom arrow is fully faithful by [FS21, Lemma VI.9.3]. To show full faithfulness of the right arrow, it suffices to handle the map $\mathcal{C}^{\neq}_{O^n_C} \to \mathcal{C}^{\neq}_{C^n}$. By the ula property, we are reduced to showing that the derived pushforward of the constant sheaf along $*^{\neq}_{C^n} \to *^{\neq}_{O^n_C}$ is constant, which is also a consequence of Lemma 4.14. In particular, the top arrow is fully faithful. $\qquad\square$

4.4. **Perversity.** Recall that for every algebraically closed field C, we have a perverse t-structure on $\mathcal{D}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I},C})$ given by strata dimension, see [FS21, Section VI.7] and [AGLR22]. This restricts to a t-structure on the full subcategory $\mathcal{D}_{\text{ula}}(\mathrm{Hk}_{\mathcal{I},C})$ of ula sheaves, since Λ is a field[2]. It would be possible to define a relative perverse t-structure as in [FS21, Definition/Proposition VI.7.1], at least after restricting to ula sheaves, but we will not pursue this avenue here.

Our main result is the perverse t-exactness of $\mathcal{Z}$.

Theorem 4.17. *Let $B \subset G$ be an arbitrary Borel subgroup. The complex $\mathcal{Z}(V)$ is a Wakimoto-filtered perverse sheaf with graded isomorphic to $\mathcal{I}(V|_{\hat{T}^\Gamma})$.*

Proof. By Theorem 4.12, we see that $\mathcal{Z}(V)$ lies in the essential image of the obvious evaluation functor coming from the associative center $\mathcal{Z}(\mathcal{D}_{\text{ula}}(\mathrm{Hk}_{\mathcal{I},k}))$. By Proposition 3.15, this implies

---

[2]Otherwise, the truncation functors do not generally preserve perfect complexes, an issue that already arises for the natural t-structure.

that $\mathcal{Z}(V)$ lies in the full subcategory Wak for our choice of Borel subgroup $B \subset G$. It remains to see that there is a canonical isomorphism

$$\mathrm{CT}_{B^-}(\mathcal{Z}(V))_{\bar{\nu}} \simeq V(w_0 \bar{\nu})[\langle 2\rho, \nu \rangle], \tag{4.24}$$

where w_0 denotes the longest element of the finite absolute Weyl group of G. Indeed, we would then know by Corollary 3.24 that $\mathcal{Z}(V)$ is a perverse sheaf, because the same would hold for its Wakimoto grading. But notice that constant terms of $\mathcal{Z}(V)$ can be calculated applying geometric Satake in the generic fiber, see [AGLR22, Corollary 6.14, Equation (6.32)], which yields the desired answer. $\qquad\square$

Remark 4.18. There appears to be a discrepancy between the isomorphism $\mathrm{grad} \circ \mathcal{Z}(V) \simeq V|_{\hat{T}^{\Gamma}}$ and (4.24) due to the appearance of the longest element w_0 in the latter formula. However, this is due to the fact that we were implicitly using an identification of T with the universal Cartan of G, compare with [AR, Remark 1.1.10]. Conjugating the identification by w_0 will not change the $\hat{T}^I$-grading coming from geometric Satake, but will change the one coming from the Wakimoto filtration, thereby fixing the issue.

Next, we deduce a few important consequences from this theorem. We say that a central perverse sheaf A is convolution exact if its left (equivalently right) convolution functor $\ell_A : \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{I},k}) \to \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_{\mathcal{I},k})$ is t-exact for the perverse t-structure.

Corollary 4.19. *The central perverse sheaf $\mathcal{Z}(V)$ is convolution exact.*

Proof. Given an element w of the affine Weyl group, we can find a Borel $B \subset G$ such that $\ell(t_{\bar{\nu}} w) = \ell(t_{\bar{\nu}}) + \ell(w)$ for all $\bar{\nu} \gg 0$ with respect to B. Indeed, we can consider a minimal gallery from the $\mathcal{I}(O)$-stable alcove $\mathbf{a}$ to its Weyl translate $w\mathbf{a}$, and simply take B as the Borel corresponding to a Weyl chamber containing the vector given as the difference of the barycenters of $\mathbf{a}$ and $w\mathbf{a}$.

Now consider the complexes $\mathcal{J}_{\bar{\nu}} * \nabla_w$ for arbitrary $\bar{\nu}$, and notice that it equals the perverse sheaf $\Delta_{t_{-\bar{\nu}'}} * \nabla_{t_{\bar{\nu}''} w}$ if we choose $\bar{\nu} = \bar{\nu}'' - \bar{\nu}'$ and $\bar{\nu}'' \gg 0$. Here, we applied Lemma 3.12 and Lemma 3.8. Now, Theorem 4.17 states that the perverse sheaf $\mathcal{Z}(V)$ admits a filtration with subquotients isomorphic to a direct sum of $\mathcal{J}_{\bar{\nu}}$, hence implying that $\mathcal{Z}(V) * \nabla_w$ is perverse for any V. By a dual argument, the same result holds for Δ_w. Finally, we apply the fact that the iterated extensions of the non-positive shifts of ∇_w (resp. non-negative shifts of Δ_w) span the non-negative part ${}^p\mathcal{D}_{\mathrm{ula}}^{\geq 0}(\mathrm{Hk}_{\mathcal{I},k})$ (resp. the non-positive part ${}^p\mathcal{D}_{\mathrm{ula}}^{\leq 0}(\mathrm{Hk}_{\mathcal{I},k})$) of the perverse t-structure to deduce that $\ell_{\mathcal{Z}(V)}$ is indeed perverse t-exact. $\qquad\square$

In the following, we say that an endomorphism $\varphi : A \to A$ of an object A in an abelian category $\mathcal{C}$ is unipotent if $(\varphi - 1)^n = 0$ for some positive integer n. We say φ is quasi-unipotent if a power of φ is unipotent. Recall that $\mathcal{Z}(V)$ carries a natural I_E-action, where E is the reflex field of the representation V and $I_E \subseteq \Gamma_E$ the inertia subgroup, see Remark 4.5.

Corollary 4.20. *The I_E-action on the perverse sheaf $\mathcal{Z}(V)$ is given by quasi-unipotent automorphisms. Moreover, there exists a finite index subgroup $I' \subset I_E$ such that the action factors through its maximal pro-ℓ quotient. If G is split, then $I' = I_E = I$ acts unipotently on $\mathcal{Z}(V)$.*

Proof. Since I_E fixes a Borel subgroup $B \subset G$ defined over F, we conclude the I_E-action on $\mathcal{Z}(V)$ preserves the Wakimoto filtration and it acts on $\mathrm{Grad}_{\bar{\nu}}(\mathcal{Z}(V)) \simeq V(w_0 \bar{\nu})$, compare with Theorem 4.17, via its natural action on the given weight space. Since $V(w_0 \bar{\nu})$ equals the sum of the $V(w_0 \nu)$ for all lifts ν of $\bar{\nu}$, we see that I_E acts on the Wakimoto sheaves by permuting those weight spaces. Let F' be a splitting field of G and note that its absolute Galois group I' acts trivially on V. In particular, the I'-action on $\mathcal{Z}(V)$ is unipotent. Note, moreover, that

both the pro-p wild inertia, and the remaining prime-to-ℓ tame quotient must map trivially to an unipotent ℓ-adic group, so the I'-action factors through its maximal pro-ℓ quotient. $\qquad\square$

In particular, if Λ is an algebraic extension of $\mathbb{Q}_\ell$ and given an isomorphism between $\mathbb{Z}_\ell$ and the maximal pro-ℓ quotient of I', we deduce the existence of a canonical nilpotent morphism

$$\mathbf{n}_V : \mathcal{Z}(V) \longrightarrow \mathcal{Z}(V) \tag{4.25}$$

such that the action of $\gamma' \in I'$ on $\mathcal{Z}(V)$ is given by $\exp(t_\ell(\gamma')\mathbf{n}_V)$, where $t_\ell : I' \to \mathbb{Z}_\ell$ is the natural quotient map.

Corollary 4.21. *The isomorphism of functors* $\mathrm{Grad} \circ \mathcal{Z}(V) \simeq V|_{\hat{T}^I}$ *is monoidal.*

Proof. We first explain how to construct the monoidal structure of the restriction functor $V \mapsto V_{|\hat{T}^I}$ geometrically using constant terms following [Yu22, §6]. Namely, for any $\mathcal{A}, \mathcal{B} \in \mathcal{P}(\mathrm{Hk}_{G,C})$, we have isomorphisms

$$\mathrm{CT}_{B^-}(\mathcal{A} * \mathcal{B})_\nu \cong R\Gamma_c(S_\nu^-, \mathcal{A} * \mathcal{B}) \tag{4.26}$$

$$\cong \bigoplus_{\nu_1 + \nu_2 = \nu} R\Gamma_c(S_{\nu_1}^- \tilde{\times} S_{\nu_2}^-, \mathcal{A}\tilde{\boxtimes}\mathcal{B}) \tag{4.27}$$

$$\cong \bigoplus_{\nu_1 + \nu_2 = \nu} R\Gamma_c(S_{\nu_1}^-, \mathcal{A}) \otimes R\Gamma_c(S_{\nu_2}^-, \mathcal{B})) \tag{4.28}$$

$$\cong \bigoplus_{\nu_1 + \nu_2 = \nu} \mathrm{CT}_{B^-}(\mathcal{A})_{\nu_1} \otimes \mathrm{CT}_{B^-}(\mathcal{B})_{\nu_2}. \tag{4.29}$$

by combining [BR18, Theorem 5.9] and [Yu22, Lemma 6.1], and this monoidal structure coincides with the natural one on $V \mapsto V_{\hat{T}^I}$ under geometric Satake. However, strictly speaking, this construction is not quite complete, because on the one hand [Zhu17, Yu22] works with the Witt vector affine Grassmannian instead of the B_{dR}^+-affine Grassmannian, and moreover it would not be immediate that this monoidal structure is compatible with that of [FS21]. In order to fill this gap, we must use the equivalence between the Satake categories over C and $\overline{k}$ for the split group G_C in [FS21, VI.6.7] and a theorem of Bando [Ban22, Ban23] showing that it is monoidal, i.e., that pulling back the Satake category of [Zhu17, Yu22] to the context of [FS21] yields the same monoidal structure on perverse sheaves.

As nearby cycles commute with constant terms [FS21, Proposition IV.6.12], we get an equivalence $\mathcal{Z}_T \circ \mathrm{CT}_{B^-} \simeq \mathrm{CT}_{B^-} \circ \mathcal{Z}_G$, where the indices T and G denote the underlying group of the Hecke stack for which we take nearby cycles. In particular, we obtain a monoidal structure on the functor $\mathrm{CT}_{B^-} \circ \mathcal{Z}_G$ by composing the above monoidal structure on CT_{B^-} with the one on $\mathcal{Z}_T$ (which clearly coincides with the restriction along $\hat{T}^I \subseteq \hat{T}$ under geometric Satake for T resp. $\mathcal{T}$). Looking back to the construction of the monoidal structure on Wakimoto gradeds in Corollary 3.28, it made inductive use of the isomorphisms from Corollary 3.22 and Corollary 3.24. These resulted as well from decomposing twisted products of semi-infinite orbits, and so this monoidal structure on $\mathrm{CT}_{B^-} \circ \mathcal{Z}_G$ must coincide with the previous one above, which was constructed using geometric Satake. $\qquad\square$

4.5. **Highest weight arrows.** Let $\mu \in \bar{\mathbb{X}}_\bullet$ be a dominant coweight with respect to B. For a $\hat{G}_\Lambda$-representation V with a single highest weight μ, we see that $\mathcal{Z}(V)$ is supported on the μ-admissible locus $\mathcal{A}_{\mathcal{I},\mu}$, cf. [AGLR22, Theorem 6.16], which equals the union of the $\mathcal{I}(O)$-orbits of the translations $t(\bar{\nu})$ associated with weights ν of V. We are going to define a canonical map

$$\mathfrak{f}_V : \mathcal{Z}(V) \to \mathrm{gr}_{\bar{\mu}} \mathcal{Z}(V) \tag{4.30}$$

called the *highest weight arrow*, which geometrizes the projection onto the $\bar{\mu}$-weight space.

First, observe that we have the adjunction unit

$$\mathcal{Z}(V) \to Rj_{\bar{\mu},*}j_{\bar{\mu}}^{*}\mathcal{Z}(V). \tag{4.31}$$

But the restriction of $\mathcal{Z}(V)$ to the I-orbit $\mathrm{Fl}_{\mathcal{I},\bar{\mu}}$ is isomorphic to the local system with value $R\Gamma(\mathrm{Fl}_{\mathcal{I},\bar{\mu}}, \mathcal{Z}(V))$. On the other hand, we know by Lemma 2.12 that $\mathrm{Fl}_{\mathcal{I},\bar{\mu}}$ coincides with the intersection $\mathcal{A}_{\mathcal{I},\mu} \cap \mathcal{S}_{t(\bar{\mu})} = \mathrm{Fl}_{\mathcal{I},\leq\bar{\mu}} \cap \mathcal{S}_{t(\bar{\mu})}$. Therefore, Corollary 3.24 tells us that $j_{\bar{\mu}}^{*}\mathcal{Z}(V) \simeq \mathrm{Grad}_{\bar{\mu}}\mathcal{Z}(V)[\langle 2\rho, \mu\rangle]$ in natural fashion. In particular, we get $Rj_{\bar{\mu},*}j_{\bar{\mu}}^{*}\mathcal{Z}(V) \simeq \mathrm{gr}_{\bar{\mu}}\mathcal{Z}(V)$ and we obtain the desired highest weight arrow.

Proposition 4.22. *The highest weights arrows are symmetric monoidal, i.e., for V (resp. W) a representation of $\hat{G}_{\Lambda}$ with a single highest weight μ (resp. ν), there are natural identifications $\mathfrak{f}_{V} * \mathfrak{f}_{W} \simeq \mathfrak{f}_{V\otimes W} \simeq \mathfrak{f}_{W} * \mathfrak{f}_{V}$ in the sense that the diagram*

$$
\begin{array}{ccccc}
\mathcal{Z}(V) * \mathcal{Z}(W) & \xrightarrow{\;\sim\;} & \mathcal{Z}(V \otimes W) & \xleftarrow{\;\sim\;} & \mathcal{Z}(W) * \mathcal{Z}(V) \\
\Big\downarrow{\mathfrak{f}_{V}*\mathfrak{f}_{W}} & & \Big\downarrow{\mathfrak{f}_{V\otimes W}} & & \Big\downarrow{\mathfrak{f}_{W}*\mathfrak{f}_{V}} \\
\mathrm{gr}_{\bar{\mu}}\mathcal{Z}(V) * \mathrm{gr}_{\bar{\nu}}\mathcal{Z}(W) & \xrightarrow{\;\sim\;} & \mathrm{gr}_{\overline{\mu+\nu}}\mathcal{Z}(V \otimes W) & \xleftarrow{\;\sim\;} & \mathrm{gr}_{\bar{\nu}}\mathcal{Z}(W) * \mathrm{gr}_{\bar{\mu}}\mathcal{Z}(V)
\end{array}
\tag{4.32}
$$

is commutative, where the horizontal isomorphisms in the first row stem from Proposition 4.9, and the isomorphisms in the second row are given by

$$\mathrm{gr}_{\bar{\mu}}\mathcal{Z}(V) * \mathrm{gr}_{\bar{\nu}}\mathcal{Z}(W) \simeq \mathrm{gr}_{\overline{\mu+\nu}}(\mathcal{Z}(V) * \mathcal{Z}(W)) \simeq \mathrm{gr}_{\overline{\mu+\nu}}(\mathcal{Z}(V \otimes W)) \tag{4.33}$$

with the first isomorphism given by Proposition 3.27.

Proof. By 4.21 we know that the composition $\mathrm{Grad} \circ \mathcal{Z}$ identifies with the restriction functor from $\mathrm{Rep}_{\Lambda}(\hat{G})$ to $\mathrm{Rep}_{\Lambda}(\hat{T}^{I})$ as a tensor functor, so it is symmetric monoidal. Indeed, the monoidal structure of Sat comes from the monoidality of constant terms in the generic fiber Gr_{G} which is compatible with the one in the special fiber $\mathrm{Fl}_{\mathcal{I}}$, which was used in Corollary 3.28. Finally, we just have to remark that the adjunction unit is naturally symmetric monoidal as are the isomorphisms $Rj_{\bar{\mu},*}j_{\bar{\mu}}^{*}\mathcal{Z}(V) \simeq \mathrm{gr}_{\bar{\mu}}\mathcal{Z}(V)$. $\qquad\square$

We also have the relation of $\mathfrak{f}_{V}$ with the monodromy operator.

Lemma 4.23. *Let V be a representation of $\hat{G}_{\Lambda}$, then we have*

$$\mathfrak{f}_{V} \circ \mathbf{n}_{V} = 0. \tag{4.34}$$

Proof. By definition, $\mathfrak{f}_{V}$ is the quotient map of $\mathcal{Z}(V)$ towards the final subquotient of the Wakimoto filtration, upon which I' acts trivially by geometric Satake, see 4.20. $\qquad\square$

Moreover, we also have that n_{V} is monoidal with respect to V. Note that here the tensor product of two nilpotent operators n_{A} and n_{B} of objects A and B of a monoidal category is given by $n_{A} \otimes 1 + 1 \otimes n_{B}$.

Lemma 4.24. *The nilpotent endomorphisms $\mathbf{n}_{V}$ for $V \in \mathrm{Rep}_{\Lambda}\hat{G}$ form a nilpotent monoidal endomorphism $\mathbf{n}$ of $\mathcal{Z} \colon \mathrm{Rep}_{\Lambda}\hat{G}) \to \mathcal{P}(\mathrm{Hk}_{\mathcal{I},k})$.*

Proof. It is enough to observe that the monoidal structure in Proposition 4.9 is I'-equivariant, but this follows directly from the construction. $\qquad\square$

4.6. **Mixed variant.** In this subsection, we are going to upgrade our previous work to the setting of mixed sheaves. We consider a p-adic field F with ring of integers O, a finite residue field k of cardinality q, and an absolute Galois group Γ. We continue to fix a quasi-split and residually split F-group G with a Iwahori O-model $\mathcal{I}$. In this subsection, we assume furthermore that Λ is an algebraic extension of $\mathbb{Q}_\ell$ and contains a preferred choice of square-root $\sqrt{q}$.

We need to introduce the Γ-equivariant derived category of étale sheaves on our preferred spaces. Note that the Deligne topos $X \times_s \eta$ for a finite type k-scheme X with compatible Γ-action defined in [SGA73], see also [HZ23], is the same as the étale topos of the stack $[\Gamma \backslash X_{\bar{k}}]$. We usually consider its stable derived category $\mathcal{D}_{\text{ét}}([\Gamma \backslash X_{\bar{k}}])$ which is equivalent to the stable derived category $\mathcal{D}_{\text{ét}}(X \times_s \eta)$ of Deligne defined in [SGA73], compare with the definitions in [HZ23, Appendix A]. Recall that we have a decisive notion of a mixed complex $A \in \mathcal{D}_{\text{ét}}([\Gamma \backslash X_{\bar{k}}])$ of weight $\leq w$ (resp. $\geq w$) in the sense of [HZ23, Definition 2.4.4]. The condition $\leq w$ is defined by requiring that $\mathcal{H}^i(\sigma^* A)$ have weights bounded by w in the sense of Deligne, where $\sigma\colon X \to [\Gamma \backslash X_{\bar{k}}]$ is induced by a section of the morphism of sets $\Gamma \to \text{Gal}_k$. The weight bound is ultimately independent from σ, see [HZ23, Section 2.4] for a discussion. The condition $\geq w$ is defined in terms of $\leq w$ and Verdier duality for $X_{\bar{k}}$.

Again, we can define the mixed standard functor from $\mathcal{D}_{\text{ét}}([\Gamma \backslash *])$ towards $\mathcal{D}_{\text{ét}}([\Gamma \backslash \text{Hk}_{\mathcal{I}, \bar{k}}])$

$$\Delta_w^{\text{mix}}\colon M \mapsto j_{w!} M \langle \ell(w) \rangle, \tag{4.35}$$

where $\langle d \rangle$ denotes the shift-twist operator $[d](\frac{d}{2})$, and the mixed costandard functor

$$\nabla_w^{\text{mix}}\colon M \mapsto R j_{w*} M \langle \ell(w) \rangle. \tag{4.36}$$

both of which preserve mixed perverse sheaves by Weil II. Lemma 3.2 and Lemma 3.3 generalize to the current setting, so that we can define the mixed Wakimoto functor $\mathcal{I}_{\bar{\nu}}^{\text{mix}}\colon \mathcal{D}_{\text{ét}}([\Gamma \backslash *]) \to \mathcal{D}_{\text{ét}}([\Gamma \backslash \text{Hk}_{\mathcal{I}, \bar{k}}])$ mapping a weighted complex M of Λ-modules to the object representing

$$\mathcal{F} \mapsto \text{colim} \operatorname{Hom}(\nabla_{t_{\bar{\nu}_1}}^{\text{mix}}(M), \mathcal{F} * \nabla_{t_{\bar{\nu}_2}}^{\text{mix}}(\Lambda)), \tag{4.37}$$

where $\bar{\nu}_1, \bar{\nu}_2 \in \bar{\mathbb{X}}_\bullet^+$ run over all those elements such that $\bar{\nu} = \bar{\nu}_1 - \bar{\nu}_2$. Again this sends a mixed weighted Λ-module to a mixed perverse sheaf, since the mixedness property is preserved under derived pushforward and pullback, whereas perversity was already verified in Lemma 3.6. We can define a notion of a mixed Wakimoto complex, as lying in the full subcategory Wak^{mix} spanned by the essential image of $\mathcal{I}_{\bar{\nu}}^{\text{mix}}$ on mixed Λ-modules. Similarly, we can generalize Proposition 3.23 to the mixed setting, in such a way that it allows us to determine the Wakimoto grading of such an object. The full subcategory $\mathcal{P}(\text{Wak}^{\text{mix}})$ consists of the objects in Wak^{mix} whose gradeds are all perverse.

Note that by [AGLR22, Section 8], the functor of nearby cycles upgrades to the mixed setting

$$R\Psi^{\text{mix}} := (i^{\text{mix}})^* R(j^{\text{mix}})_* \colon \mathcal{D}_{\text{ét}}([\Gamma \backslash \text{Hk}_{G,C}]) \to \mathcal{D}_{\text{ét}}([\Gamma \backslash \text{Hk}_{\mathcal{I}, \bar{k}}]) \tag{4.38}$$

and composition with the functor $\text{Rep}_\Lambda(^L G) \to \mathcal{P}(\text{Hk}_{G,F})$ defines the mixed central functor $\mathcal{Z}^{\text{mix}}(-)$.

Theorem 4.25. *The mixed central functor $\mathcal{Z}^{\text{mix}}$ lands in $\mathcal{P}(\text{Wak}^{\text{mix}})$. Concretely, the Wakimoto gradeds of $\mathcal{Z}^{\text{mix}}(V)$ are canonically isomorphic to $\mathcal{I}_{\bar{\nu}}^{\text{mix}}(V(w_0 \bar{\nu}))$.*

Proof. The arguments of Theorem 4.17 apply in this case as well. $\qquad \square$

Recall that there exists a unique exhaustive and separated filtration $\text{Fil}_i^M \mathcal{Z}(V)$ (called the monodromy filtration) on the perverse sheaf $\mathcal{Z}(V)$ such that $\mathbf{n}_V$ is a filtered operator of degree -2 inducing isomorphisms $\mathbf{n}_V^i\colon \text{Gr}_i^M \mathcal{Z}(V) \simeq \text{Gr}_{-i}^M \mathcal{Z}(V)$. This filtration descends by functoriality to the corresponding mixed object $\mathcal{Z}^{\text{mix}}(V)$. On the other hand, the mixed perverse sheaf $\mathcal{Z}^{\text{mix}}(V)$ admits a filtration $\text{Fil}_i^W \mathcal{Z}^{\text{mix}}(V)$ in mixed perverse sheaves whose weights are at most

i and whose gradeds $\mathrm{Gr}_i^{\mathrm{W}} \mathcal{Z}^{\mathrm{mix}}(V)$ are purely of weight i, see [BBDG18, Théorème 5.3.5] and [HZ23, Theorem 2.6.8]. We say following [HZ23] that $\mathcal{Z}^{\mathrm{mix}}(V)$ is monodromy-pure of weight 0 if these two filtrations coincide. We have the following local weight-monodromy conjecture:

Conjecture 4.26. *The mixed perverse sheaf $\mathcal{Z}^{\mathrm{mix}}(V)$ is monodromy-pure of weight 0.*

For finite-type schemes over a field, it is known that nearby cycles send pure sheaves of weight 0 to monodromy-pure sheaves of weight 0, by a theorem of Gabber [BB93, Theorem 5.1.2]. In mixed characteristic, this was partially generalized by Hansen–Zavyalov [HZ23] assuming the existence of an étale cover by rigid-analytic tubes that admit an étale map to a disk.

Proposition 4.27. *If G is split and every non-zero weight of V is minuscule, then Conjecture 4.26 holds true for $\mathcal{Z}(V)$.*

Proof. By semi-simplicity of the Satake category in characteristic 0, we may assume $V = V_\mu$ is the simple representation with highest weight μ. In particular, we know by the proof of [AGLR22, Theorem 7.21, 7.23], that the local model $M_{\mathcal{I},\mu}$ -defined as the v-sheaf closure of the Schubert cell for μ- is representable by a flat projective scheme $M_{\mathcal{I},\mu}^{\mathrm{sch}}$ over O. By functoriality, it also maps to the local model $\mathcal{G}/\mathcal{P}_\mu^-$ at hyperspecial level $\mathcal{G}$, which is smooth over O. Since the transition map is an isomorphism in the generic fiber, we deduce by pull-back an étale cover of G/P_μ^- by rigid-analytic tubes admitting étale maps to a disk. Therefore, we can apply [HZ23, Theorem 4.4.4]. $\square$

5. COHERENT FUNCTOR

In this section, we assume that G is split, that Λ is an algebraic extension of $\mathbb{Q}_\ell$, and that $\mathcal{I}$ is the Iwahori O-model obtained as the dilatation of a split model G_O along the closed subgroup $B_k \to G_k$. Consider the Springer resolution

$$p_{\mathrm{Spr}} : \hat{\mathcal{N}}_{\mathrm{Spr}} = \hat{G} \times^{\hat{B}} \mathrm{Lie}\,\hat{U} \to \hat{\mathcal{N}} \subset \mathrm{Lie}\,\hat{G} \qquad (5.1)$$

of the nilpotent cone $\hat{\mathcal{N}}$ defined over Λ. Observe that there are natural functors $\mathrm{Rep}_\Lambda \hat{G} \to \mathrm{Coh}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ given by $V \mapsto \mathcal{O} \boxtimes V$ and $\mathrm{Rep}_\Lambda \hat{T} \to \mathrm{Coh}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ given by $\nu \mapsto \mathcal{O}(\nu)$, where $[\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]$ denotes the quotient stack. We aim to construct a monoidal functor

$$\mathcal{F} : \mathrm{Perf}(\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}) \to \mathcal{D}_{\mathrm{ét}}(\mathrm{Hk}_{\mathcal{I}}) \qquad (5.2)$$

of monoidal, stable ∞-categories. Here, the domain of $\mathcal{F}$ is the category of perfect complex on a smooth Artin Λ-stack, thus equivalently, the ∞-derived category of coherent sheaves, and the source of $\mathcal{F}$ is the ∞-derived category of étale Λ-sheaves on a perfect Artin k-stack. The functor $\mathcal{F}$ is supposed to extend both the Wakimoto functor $\mathcal{J}$ and the central functor $\mathcal{Z}$ in the sense that the composition of $\mathcal{F}$ with the functor $V \mapsto V \boxtimes \mathcal{O}$ on $\mathrm{Rep}_\Lambda(\hat{G})$ resp. the functor $\nu \mapsto \mathcal{O}(\nu)$ on $\mathrm{Rep}_\Lambda(\hat{T})$ is equivalent to $\mathcal{Z}$ resp. $\mathcal{J}$.

5.1. **Generalities on coherent sheaves.** Throughout this section, we continue to assume Λ is an algebraic extension of $\mathbb{Q}_\ell$ and we let $X = Y/H$ be the quotient stack of a finitely presented quasi-affine Λ-scheme acted upon by a reductive group H over Λ. Let us recall how to define the derived category $\mathcal{D}_{\mathrm{qc}}(X)$ of quasi-coherent sheaves on X. Recall that the category Mod_Y of $\mathcal{O}_Y$-module sheaves is Grothendieck abelian in the sense of [Lur17, Definition 1.3.5.1]. By [Lur17, Definition 1.3.5.8], this abelian category induces a stable ∞-category $\mathcal{D}(\mathrm{Mod}_Y)$ of $\mathcal{O}_Y$-modules on Y. It is naturally endowed with a t-structure in the sense of [Lur17, Definition 1.2.1.4] defined by non-vanishing degrees of its cohomology functors, see [Lur17, Definition 1.3.5.16]. Hence, we can define $\mathcal{D}_{\mathrm{qc}}(Y)$ (resp. $\mathcal{D}_{\mathrm{coh}}(Y)$) as the full subcategory spanned by complexes whose cohomologies are quasi-coherent (resp. coherent) $\mathcal{O}_Y$-modules. We now define $\mathcal{D}_{\mathrm{qc}}(X)$ (resp. $\mathcal{D}_{\mathrm{coh}}(X)$) as the

limit of the simplicial object in ∞-categories $[n] \mapsto \mathcal{D}_{\mathrm{qc}}(Y_X^n)$ (resp. $[n] \mapsto \mathcal{D}_{\mathrm{coh}}(Y_X^n)$). The resulting ∞-categories are stable and carry natural t-structures, whose hearts will be denoted $\mathrm{QCoh}(X)$, resp. $\mathrm{Coh}(X)$.

We will decorate the derived categories by the superscripts $^b, ^+, ^-$ to denote the full subcategories of bounded, left-bounded, and right-bounded complexes. The full subcategory $\mathrm{Perf}(Y)$ of perfect complexes is spanned by bounded complexes with finite Tor-amplitude (i.e., those which are represented by finite complexes of vector bundles as Y is quasi-affine), and we define $\mathrm{Perf}(X)$ again by descent.

Notice that the Grothendieck abelian category $\mathrm{QCoh}(X)$ induces a stable ∞-category $\mathcal{D}(\mathrm{QCoh}(X))$ again by an application of [Lur17, Definition 1.3.5.8] with a natural t-structure. It will be often useful to relate this to $\mathcal{D}_{\mathrm{qc}}(X)$. There is an induced t-exact functor $\mathcal{D}(\mathrm{QCoh}(X)) \to \mathcal{D}_{\mathrm{qc}}(X)$ and under our assumptions, we get:

Proposition 5.1. *The functor* $\mathcal{D}(\mathrm{QCoh}(X)) \to \mathcal{D}_{\mathrm{qc}}(X)$ *is an equivalence.*

Proof. In virtue of the equivalence of [Lur17, Remark 1.2.1.18], it suffices to verify that the functor induces an equivalence of bounded categories. Essential surjectivity can be tested at the triangulated level, i.e., by taking homotopy categories. Similarly, full faithfulness amounts to checking isomorphism of homotopy groups of mapping spaces, which can be expressed in terms of Ext groups by [Lur17, Notation 1.1.2.17], so we can also verify it at the triangulated level. Since $\mathcal{D}_{\mathrm{qc}}(X)$ is compactly generated by [HR17, Theorem B], the claim now follows from [HNR19, Theorem 1.2]. $\square$

In order to understand right-bounded complexes in the affine case, the following lemma is decisive.

Lemma 5.2. *If* Y *is affine, then the abelian category* $\mathrm{Coh}(X)$ *has enough projectives. In particular, the t-exact functor* $\mathcal{D}^-(\mathrm{Coh}(X)) \to \mathcal{D}_{\mathrm{coh}}^-(X)$ *is an equivalence.*

Proof. Let R, resp. A be the ring of global section on Y resp. H. The category of finitely generated R-modules (which is equivalent to $\mathrm{Coh}(Y)$) has enough projectives by considering the collection of free R-modules. Notice that the functor of taking H-invariants on R-modules is exact by assumption on H. We can deduce that the H-equivariant free R-module $V \otimes M$ with V being a finite dimensional representation of H is projective in $\mathrm{Coh}(X)$. This collection of projectives is enough, as each coherent sheaf on X is surjected upon by the n-fold sum of the regular representation $A \otimes R$ for $n \gg 0$, and we can find a finite representation $V \subset A$ which completes the job, by finiteness of the underlying R-module of the initial coherent sheaf on X. $\square$

If the stack X is smooth, then we actually get an equality $\mathcal{D}_{\mathrm{coh}}^b(X) = \mathrm{Perf}(X)$ of full subcategories. This motivates our construction of the AB functor via the following equivariant analogue of the localization theorem originally due to Thomason–Trobaugh [TT90] and Neeman [Nee92].

Proposition 5.3. *Let* $U \subset X$ *be an open immersion with closed complement* Z*. Then* $\mathrm{Perf}(U)$ *is the idempotent-completion of the quotient* $\mathrm{Perf}(X)/\mathrm{Perf}(X)_Z$*, where the denominator indicates the full subcategory spanned by complexes supported in* Z*.*

Proof. This is [KR18, Theorem 3.4, Equation (3.6)] for the underlying triangulated categories, which implies the statement in general. Let us explain how one obtains the result. First, it is clear that $\mathcal{D}_{\mathrm{qc}}(U)$ is a localization of $\mathcal{D}_{\mathrm{qc}}(X)$ with kernel $\mathcal{D}_{\mathrm{qc}}(X)_Z$, because restriction admits a right adjoint given by pushforward with unit being an equivalence. Finally, since each of the categories involved are compactly generated by [HR17, Theorem B] with compact objects given exactly by perfect complexes by [HR17, Lemma 4.4], we can apply the localization theorem, see [HR17, Theorem 3.12], to obtain the claim. $\square$

5.2. **Coherent sheaves on the Springer variety.** Recall that the variety $\hat{G}/\hat{U}$, which is a $\hat{T}$-torsor over $\hat{G}/\hat{B}$, is quasi-affine[3], so it embeds openly in the spectrum $\hat{\mathcal{X}}$ of its global sections $\mathcal{O}(\hat{G}/\hat{U})$. In turn, these admit the following explicit description as a graded Λ-algebra

$$\mathcal{O}(\hat{G}/\hat{U}) = \bigoplus_{\mu \in \mathbb{X}_\bullet^+} V_\mu \tag{5.3}$$

where V_μ denotes the highest weight representation of highest weight μ and multiplication is given by the obvious maps $V_{\mu_1} \otimes V_{\mu_2} \to V_{\mu_1 + \mu_2}$, see [AR, Lemma 6.2.1]. In particular, the above Λ-algebra is finitely generated.

Similarly, we can define the following $\hat{T}$-torsor

$$\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{qaf}} = \hat{G} \times^{\hat{U}} \mathrm{Lie}(\hat{U}) \tag{5.4}$$

over the Springer resolution, which is a quasi-affine scheme with an action of $\hat{G}' := \hat{G} \times \hat{T}$. The Lie algebra $\hat{\mathfrak{g}}$ of $\hat{G}$ acts naturally via derivations on the structure sheaf of $\hat{G}/\hat{U}$, see [AR, Equation (6.2.8)] and we can associate to it the so-called infinitesimal universal stabilizer $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}$ as the closed subscheme of $\hat{\mathfrak{g}} \times \hat{\mathcal{X}}$ given by the image of the derivation map. Note that, even though the intersection of $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}$ with $\hat{\mathfrak{g}} \times \hat{G}/\hat{U}$ is exactly $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{qaf}}$, it is not generally true that $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}$ coincides with the scheme-theoretic closure of the locally closed immersion $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{qaf}} \to \hat{\mathfrak{g}} \times \hat{\mathcal{X}}$. The latter is an integral variety admitting $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{qaf}}$ as a dense open subset, with ideal of definition given by the kernel of $\mathcal{O}(\hat{\mathfrak{g}}) \times \mathcal{O}(\hat{\mathcal{X}}) \to \mathcal{O}(\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{qaf}})$.

We have two distinguished collections of generators for the derived category of $[\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]$.

Lemma 5.4. *The derived category $\mathcal{D}_{\mathrm{coh}}^b([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ is spanned by the set of the line bundles $\mathcal{O}(\nu)$ for $\nu \in \mathbb{X}_\bullet$, or by the set of the vector bundles $V \otimes \mathcal{O}(\nu)$ for $V \in \mathrm{Rep}\,\hat{G}$ and $\nu \in \mathbb{X}_\bullet^+$.*

Proof. This is [Bez09, Lemma 21]. See also [AR, Lemma 6.2.7]. $\qquad\square$

Originally, it was claimed in [AB09, Lemma 20] that the triangulated category $\mathcal{D}_{\mathrm{coh}}^b([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ is a Verdier quotient of $\mathrm{Perf}([\hat{G}'\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}])$. Upon expanding the argument in [AR, Proposition 6.2.8], we noticed that it seemed to rely on density of $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{qaf}} \subset \hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}$, which unfortunately fails in general. Instead, we will argue below via [AR, Remark 6.3.10].

5.3. **Construction of the AB functor.** As in this whole section, we assume that $\mathcal{I}$ is the standard Iwahori attached to the fixed Borel B of the pinned split group G. We recall also the notation $G' = G \times T$ and $\hat{G}' = \hat{G} \times \hat{T}$. First, we start with the functor

$$\mathcal{Z}' := \mathcal{Z} \times \mathcal{I} \colon \mathrm{Rep}_\Lambda(\hat{G}') \to \mathcal{D}_{\mathrm{ula}}(\mathrm{Hk}_\mathcal{I}) \tag{5.5}$$

which has a natural monoidal structure[4]and factors through the full subcategory of $\mathcal{P}(\mathrm{Wak})$ consisting of Wakimoto-filtered perverse sheaves. However, this is still not good enough, because the convolution of Wakimoto-filtered perverse sheaves is not *symmetric* in general.

In order to fix this, we consider the (non-full!) subcategory $\mathcal{C}$ of $\mathcal{P}(\mathrm{Hk}_\mathcal{I})$ whose objects are those in the image of $\mathcal{Z}'$ and whose morphisms commute with the images along $\mathcal{Z}'$ of the symmetry isomorphisms of $\mathrm{Rep}_\Lambda(\hat{G}')$. This is a symmetric monoidal category by definition, see [AR, Lemma 6.3.3]. Consider the following Λ-algebra

$$A = \mathrm{Hom}_{\mathrm{Ind}\,\mathcal{C}}(1_\mathcal{C}, \mathcal{Z}'(\mathcal{O}(\hat{G}'))) \tag{5.6}$$

---

[3]By the construction of quotients via fixed vectors in representations, any quotient of an affine scheme of finite type over a field by a unipotent group scheme is quasi-affine. For details on $\hat{G}/\hat{U}$ see [AR, Subsection 6.2.1].

[4]Even in the $\mathbb{E}_1$-monoidal sense: The functors $\mathcal{Z}$ and $\mathcal{I}$ are $\mathbb{E}_1$-monoidal, and $\mathcal{Z}$ is central, see 4.12. This implies the existence of $\mathcal{Z}'$ by the definition of $\mathbb{E}_1$-centers, see [Lur17, Definition 5.3.1.2].

where the multiplication is induced by that of the group $\hat{G}'$, and $\mathcal{O}(\hat{G}')$ is a $\hat{G}'$-representation via conjugation. By [AR, Proposition 6.3.5], this defines an identification between $\mathcal{C}$ and the category of free A-modules with $\hat{G}'$-equivariant structure of the form $V \otimes_\Lambda A$ where V is a finite dimensional Λ-representation of $\hat{G}'$.

Next, we construct a Λ-algebra homomorphism

$$\mathcal{O}(\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}) \to A \tag{5.7}$$

that is equivariant with respect to the $\hat{G}'$-module structures. Via the $\hat{G}'$-equivariant embedding $\mathcal{O}(\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}) \to \mathfrak{g} \times \hat{\mathcal{X}}$, we start by handling each of these two factors separately (following closely the respective part in [AR, Section 6.3]).

For any $\hat{G}$-representation V, we extend it to a $\hat{G}'$-representation $V' = V \boxtimes 1$ by letting $\hat{T}$ act trivially and consider the logarithm of the monodromy n_V acting on $\mathcal{Z}(V) = \mathcal{Z}'(V')$. The collection of these endomorphisms defines a map of Λ-algebras $\mathcal{O}(\hat{\mathfrak{g}}) \to A$ which is $\hat{G}'$-equivariant. For details we refer to [AR, Example 6.3.1] and Lemma 4.24.

Next, we need to define a map of $\hat{G}'$-modules $V_\nu \boxtimes -\nu \to A$ and the natural source for this is the highest weight arrow f_ν provided by the Wakimoto filtration, see 4.5. It defines a morphism in $\mathcal{C}$ by the already checked compatibilities, so applying the description of $\mathcal{C}$ in terms of A yields a map $V'_\nu \otimes A \to 1 \boxtimes \nu \otimes A$ which corresponds to our goal after twisting by ν and restricting the domain on the left.

In total, we have thus constructed a $\hat{G}'$-module homomorphism

$$\mathcal{O}(\hat{\mathfrak{g}} \times \hat{\mathcal{X}}) \to A. \tag{5.8}$$

However, we are still left with the task of showing that this factors over the coordinate ring of the affine enlargement $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}$, which is a closed subscheme of $\hat{\mathfrak{g}} \times \hat{\mathcal{X}}$ of the Springer bundle.

Lemma 5.5. *The $\hat{G}'$-equivariant map* (5.8) *factors uniquely through a $\hat{G}'$-equivariant map*

$$\mathcal{O}(\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}) \to A. \tag{5.9}$$

Proof. Here, we follow [AR, Lemma 6.3.7]. We know that the highest weight arrow is equivariant with respect to the monodromy operator. Passing to the logarithm, we see as in 4.23 that $f_\nu \circ \mathbf{n}_\nu = 0$. This equality holds true in the auxiliary category $\mathcal{C}$ (in fact, the monodromy action on $\mathcal{Z}$ factors through $\mathcal{C}$). Comparing with [AR, Example 6.3.1], we conclude from this identity that the definition ideal of $\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}$ inside $\mathfrak{g} \times \hat{\mathcal{X}}$ vanishes under the map to A. $\square$

So far, we have arrived at a functor

$$\tilde{F} \colon \mathrm{Coh}_{\mathrm{fr}}([\hat{G}'\backslash \hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}]) \to \mathcal{C}, \tag{5.10}$$

where $\mathrm{Coh}_{\mathrm{fr}}$ denotes the full subcategory of $\mathrm{Coh}([\hat{G}'\backslash \hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}])$ spanned by the objects $V \otimes \mathcal{O}$ for $V \in \mathrm{Rep}_\Lambda(\hat{G})$. Now we are going to show that this functor passes to the actual Springer resolution $\hat{\mathcal{N}}_{\mathrm{Spr}}$.

Lemma 5.6. *The functor $\tilde{F}$ composed with the Wakimoto grading functor* gr *from 3.18 identifies with the pullback functor of coherent sheaves along the morphism*

$$[\hat{T}\backslash e] \to [\hat{G}\backslash \hat{\mathcal{N}}_{\mathrm{Spr}}] \cong [\hat{G}'\backslash \hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{qaf}}] \subseteq [\hat{G}'\backslash \hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{qaf}}].$$

Here, $e := \mathrm{Spec}(\Lambda) \to \hat{\mathcal{N}}_{\mathrm{Spr}} = \hat{G} \times^{\hat{B}} \mathrm{Lie}(\hat{U})$ denotes the point $[(1, 0)]$.

Proof. We follow the proof in [AR, Lemma 6.3.8]. It suffices to understand the corresponding $\hat{T}$-equivariant map of Λ-algebras $\mathcal{O}(\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}) \to \Lambda$. But the monodromy acts trivially on the Wakimoto grading as we saw in 4.20, and the highest weight arrow is projects to V_λ to the highest weight space $V_\lambda(\lambda)$. Hence the sought homomorphism is just evaluation at the origin e. $\square$

Proposition 5.7. *There is a unique monoidal functor of stable ∞-categories up to equivalence*

$$\mathcal{F} \colon \mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}}) \tag{5.11}$$

extending $\tilde{F}$.

Proof. This is [AR, Proposition 6.3.9, Remark 6.3.10] in the triangulated setting and we follow their argument.

Since $\mathrm{Coh}_{\mathrm{fr}}([\tilde{G}'\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}])$ consists of compact, projective generators of $\mathrm{Coh}([\tilde{G}'\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}])$ by Lemma 5.2, left Kan extension of the composition $\mathrm{Coh}_{\mathrm{fr}}([\tilde{G}'\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}]) \xrightarrow{\tilde{F}} \mathcal{C} \to \mathcal{D}_{\mathrm{ét}}(\mathrm{Hk}_{\mathcal{I}})$ yields the exact "left-derived" functor $L\tilde{F} \colon \mathcal{D}^{\leq 0}(\mathrm{Coh}([\tilde{G}'\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}])) \to \mathcal{D}_{\mathrm{ét}}(\mathrm{Hk}_{\mathcal{I}})$ as in [Lur17, Theorem 1.3.3.2]. This functor formally extends to an exact functor on $\mathcal{D}^-(\mathrm{Coh}([\tilde{G}'\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}]))$. Thanks to the equivalence from Proposition 5.1 and after restricting to perfect complexes, we get a functor

$$\tilde{\mathcal{F}} \colon \mathrm{Perf}([\hat{G}'\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}]) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}}). \tag{5.12}$$

This functor is monoidal because it can be written as the composition of the monoidal functor $\mathcal{C}^b(\tilde{F})$, where $\mathcal{C}^b$ denotes the associated ∞-category of complexes, followed by the restricted realization functor $\mathcal{C}^b(\mathcal{P}(\mathrm{Wak})) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$, which is monoidal because $\mathcal{P}(\mathrm{Wak})$ is a full subcategory of $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$ closed under convolution. Proposition 5.3 implies that the category $\mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ is the idempotent-completion of the quotient of $\mathrm{Perf}([\hat{G}'\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}^{\mathrm{af}}])$ by the full subcategory of those perfect complexes supported on the complement. Since $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{I}})$ is idempotent-complete, we are reduced to showing that such a perfect complex lies in the kernel of $\tilde{\mathcal{F}}$. Since the image of $\tilde{\mathcal{F}}$ lies in Wak (this reduces to the same statement for $\tilde{F}$ as Wak is idempotent-complete), we can check acyclicity after passing to graded by first taking Wakimoto filtrations termwise and then inducting. But the grading functor corresponds at the coherent level to restriction to the origin of $\hat{\mathcal{N}}_{\mathrm{Spr}}$ by Lemma 5.6, hence the desired vanishing holds. $\qquad\square$

6. Iwahori–Whittaker averaging

We continue to work with a pinned split F-group G with a fixed maximal torus T and a Borel B containing T (in particular, we do not regard them as being defined over O, unless indexed by O). We let $\mathcal{I}$ be the standard Iwahori O-model of G, i.e., such that $\mathcal{I}(O)$ fixes the origin of the apartment $\mathcal{A}(G,T)$ induced by the pinning and is contained in the B-dominant Weyl chamber.

We let $\mathcal{I}^{\mathrm{op}}$ denote the parahoric O-model opposite to $\mathcal{I}$ with respect to the origin of the apartment and the Borel, and simply call it the *opposite Iwahori*. In other words, $\mathcal{I}^{\mathrm{op}}(O)$ fixes the alcove opposite to the one fixed by $\mathcal{I}(O)$. We define likewise the *pro-p Iwahori* $\mathcal{I}_{\mathrm{u}}$ as the unique smooth affine O-model of G with connected geometric fibers whose O-valued points are the kernel of $\mathcal{I}(O) \to \mathcal{I}_k^{\mathrm{red}}(k)$, where the $\mathcal{I}_k^{\mathrm{red}}$ is the reductive quotient of the special fiber of $\mathcal{I}$.

Our next task is to choose a Whittaker datum. Assume Λ is an algebraic extension of $\mathbb{Q}_\ell(\zeta)$ where $\zeta \in \bar{\mathbb{Q}}_\ell$ is a choice of a primitive p-th root of unity. We get the Artin–Schreier étale sheaf $\mathcal{L}_{\mathrm{AS}}$ on $\mathbb{G}_{a,k}$: this is the rank 1 direct summand of the pushforward $\pi_*\Lambda$ of the constant sheaf along the Artin–Schreier cover $\pi \colon \mathbb{G}_{a,k} \to \mathbb{G}_{a,k}$ arising as the ζ-eigenspace for the Galois action of $\mathbb{Z}/p\mathbb{Z}$. It is a character sheaf in the sense of Lusztig–Yun [LY20], i.e., we have isomorphisms $m^*\mathcal{L}_{\mathrm{AS}} \simeq \mathcal{L}_{\mathrm{AS}} \boxtimes \mathcal{L}_{\mathrm{AS}}$ and $e^*\mathcal{L}_{\mathrm{AS}} \cong \Lambda$ with respect to the multiplication m and unit e of $\mathbb{G}_{a,k}$, that satisfy associativity constraints (this is equivalent to the corresponding ∞-enhancement, because $\mathcal{P}(\mathbb{G}_{a,k})$ is an abelian category). Besides, it satisfies the following cohomological vanishing

$$R\Gamma(\mathbb{G}_{a,k}, \mathcal{L}_{\mathrm{AS}}) = R\Gamma_c(\mathbb{G}_{a,k}, \mathcal{L}_{\mathrm{AS}}) = 0, \tag{6.1}$$

which will turn out to be important later on.

Let U^{op} be the unipotent radical of the opposite Borel B^{op}. Consider the homomorphism $U_k^{\mathrm{op}} \to \mathbb{G}_{a,k}$ induced by the sum of the negative simple roots and let $\chi \colon L^+\mathcal{I}_u^{\mathrm{op}} \to \mathbb{G}_{a,k}$ be the

homomorphism resulting from pre-composing the first one with the natural projection $L^+\mathcal{I}_u^{\mathrm{op}} \to U_k^{\mathrm{op}}$. Taking the pullback of $\mathcal{L}_{\mathrm{AS}}$ along χ, we get a character sheaf on $\mathcal{L}_{\mathcal{IW}} \in \mathcal{D}_{\mathrm{cons}}(L^+\mathcal{I}_u^{\mathrm{op}})$. Indeed, this is the character sheaf attached to the cover $\pi_\chi \colon (L^+\mathcal{I}_u^{\mathrm{op}})_{\mathrm{AS}} \to L^+\mathcal{I}_u^{\mathrm{op}}$ deduced from $\pi \colon \mathbb{G}_{a,k} \to \mathbb{G}_{a,k}$ by pullback along χ.

Definition 6.1. The derived category $\mathcal{D}_{\mathrm{ét}}(\mathrm{Hk}_{\mathcal{IW}})$ of *Iwahori-Whittaker sheaves* is the ζ-isotypical component of the stable ∞-category $\mathcal{D}_{\mathrm{ét}}([(L^+\mathcal{I}_u^{\mathrm{op}})_{\mathrm{AS}}\backslash \mathrm{Fl}_{\mathcal{I}}])$.

In the above definition, we are using the fact that the Λ-linear stable ∞-category $\mathcal{D}_{\mathrm{ét}}([(L^+\mathcal{I}_u^{\mathrm{op}})_{\mathrm{AS}}\backslash \mathrm{Fl}_{\mathcal{I}}])$ has a $\mathbb{Z}/p\mathbb{Z}$-action coming from $\mathbb{Z}/p\mathbb{Z} \simeq \ker(\pi_\chi)$ and that it decomposes as a direct sum of full subcategories where $\mathbb{Z}/p\mathbb{Z}$ acts via a $\Lambda^\times$-valued character, since Λ has characteristic 0 and $\mu_p \subseteq \Lambda$. Note that no underlying stack $\mathrm{Hk}_{\mathcal{IW}}$ seems to exist, but we find this shorthand notation useful, and hope it does not cause any confusion to the reader. We could also define $\mathcal{D}_{\mathrm{ét}}(\mathrm{Hk}_{\mathcal{IW}})$ as the ∞-category of $(L^+\mathcal{I}_u^{\mathrm{op}}, \mathcal{L}_{\mathcal{IW}})$-equivariant étale sheaves on $\mathrm{Fl}_{\mathcal{I}}$, obtained by twisting by the character sheaf $\mathcal{L}_{\mathcal{IW}}$ the cosimplicial diagram obtained by applying $\mathcal{D}_{\mathrm{ét}}$ to the Čech complex of $\mathrm{Fl}_{\mathcal{I}} \to [L^+\mathcal{I}_u^{\mathrm{op}}\backslash \mathrm{Fl}_{\mathcal{I}}]$. In the end, it turns out that all of this is unnecessarily complicated, because:

Proposition 6.2. *The forgetful functor* $\mathcal{D}_{\mathrm{ét}}(\mathrm{Hk}_{\mathcal{IW}}) \to \mathcal{D}_{\mathrm{ét}}(\mathrm{Fl}_{\mathcal{I}})$ *is fully faithful.*

Proof. This is essentially [ARW16, Proposition A.5] and follows from the fact that $L^+\mathcal{I}_u^{\mathrm{op}}$ is pro-unipotent and hence so is its Artin–Schreier cover. Thus, we can apply [FS21, Proposition VI.4.1]. $\qquad\square$

The category of Iwahori–Whittaker sheaves inherits a perverse t-structure from its fully faithful embedding into $\mathcal{D}_{\mathrm{ét}}(\mathrm{Fl}_{\mathcal{I}})$, so that one can consider its heart $\mathcal{P}(\mathrm{Hk}_{\mathcal{IW}})$, called the category of Iwahori–Whittaker perverse sheaves. The ∞-category $\mathcal{D}_{\mathrm{ét}}(\mathrm{Hk}_{\mathcal{IW}})$ does not appear to be monoidal, but it is a right module of $\mathcal{D}_{\mathrm{ét}}(\mathrm{Hk}_{\mathcal{I}})$ in the sense of [Lur17, Definition 4.2.1.13]. Indeed, we invoke the natural isomorphism

$$[(L^+\mathcal{I}_u^{\mathrm{op}})_{\mathrm{AS}}\backslash \mathrm{Fl}_{\mathcal{I}}] \simeq [*/(L^+\mathcal{I}_u^{\mathrm{op}})_{\mathrm{AS}}] \times_{[*/LG]} [*/L^+\mathcal{I}] \tag{6.2}$$

to identify our stack as a homomorphism object in $\mathrm{Corr}(\mathrm{vSt}_{/[*/LG]})$ with a natural right module structure under the endomorphism object $\mathrm{Hk}_{\mathcal{I}} \cong [*/L^+\mathcal{I}] \times_{[*/LG]} [*/L^+\mathcal{I}]$. Taking the symmetric monoidal forgetful functor $\mathrm{Corr}(\mathrm{vSt}_{[*/LG]}) \to \mathrm{Corr}(\mathrm{vSt})$ and applying $\mathcal{D}_{\mathrm{ét}}^{\otimes}$, we deduce a right module structure on $\mathcal{D}_{\mathrm{ét}}([L^+\mathcal{I}_u^{\mathrm{op}}\backslash \mathrm{Fl}_{\mathcal{I}}])$ under $\mathcal{D}_{\mathrm{ét}}(\mathrm{Hk}_{\mathcal{I}})$ and this module structure is preserved under passing to direct summands. We begin our study of this ∞-category by classifying Iwahori–Whittaker local systems on $L^+\mathcal{I}_u^{\mathrm{op}}$-orbits in $\mathrm{Fl}_{\mathcal{I}}$. Note that the latter are exactly the $L^+\mathcal{I}^{\mathrm{op}}$-orbits and hence are in bijection with the Iwahori–Weyl group W as in Section 2. We will denote the corresponding $L^+\mathcal{I}^{\mathrm{op}}$-orbit of w by $\mathrm{Fl}_{\mathcal{I},w}^{\mathrm{op}} := L^+\mathcal{I}^{\mathrm{op}}wL^+\mathcal{I}/L^+\mathcal{I}$.

Lemma 6.3. *The orbit* $\mathrm{Fl}_{\mathcal{I},w}^{\mathrm{op}}$ *carries a Iwahori–Whittaker local system if and only if w has minimal length in its W_{fin}-left coset, i.e., $\ell(w_{\mathrm{fin}}w) \geq \ell(w)$ for all $w_{\mathrm{fin}} \in W_{\mathrm{fin}}$. If the latter condition holds, then the rank 1 Iwahori–Whittaker local system on* $\mathrm{Fl}_{\mathcal{I},w}^{\mathrm{op}}$ *is unique up to isomorphism.*

Proof. The Iwahori–Whittaker equivariant condition forces the stabilizer of the point $\dot{w}$ to be contained in the kernel of χ, and conversely such a containment would allow us to pullback the Artin–Schreier sheaf along χ to the desired orbit. This inclusion happens if and only if $w(\alpha_s)$ is a positive affine root where α_s is the positive simple affine root attached to any positive simple reflection $s \in W_{\mathrm{fin}}$. But this is equivalent to $sw > w$, i.e., that w is the minimal length representative of its W_{fin}-left coset. $\qquad\square$

Since the set of left W_{fin}-cosets of the Iwahori–Weyl group is enumerated by $\mathbb{X}_\bullet$, we will call $\mathcal{L}_\nu$ the unique Iwahori–Whittaker local system supported on the $L^+\mathcal{I}_u^{\mathrm{op}}$-orbit of w_ν, the minimal

length element in $W_{\mathrm{fin}}t_\nu$, according to the preceding statement. We also obtain the standard Iwahori–Whittaker equivariant sheaf

$$\Delta_\nu^{\mathcal{IW}} := (j_{w_\nu}^{\mathrm{op}})_! \mathcal{L}_\nu[\ell(w_\nu)], \tag{6.3}$$

where $j_{w_\nu}^{\mathrm{op}}$ is the inclusion of the $L^+\mathcal{I}^{\mathrm{op}}$-orbit and likewise the costandard Iwahori–Whittaker equivariant sheaf

$$\nabla_\nu^{\mathcal{IW}} := R(j_{w_\nu}^{\mathrm{op}})_* \mathcal{L}_\nu[\ell(w_\nu)], \tag{6.4}$$

both of which are supported on $\mathrm{Fl}_{\mathcal{I},\leq w_\nu}^{\mathrm{op}}$ and are perverse because orbits of solvable groups are affine, so we can invoke Artin vanishing, compare with [BBDG18, Corollaire 4.1.10]. We also have access to IC sheaves $\mathrm{IC}_\nu^{\mathcal{IW}}$ by taking the image of the natural map $\Delta_\nu^{\mathcal{IW}} \to \nabla_\nu^{\mathcal{IW}}$. Recall that in [BGS96, Subsection 3.2] a sufficient criterion for the existence of projective covers, injective hulls and tilting modules was given. We call an abelian category satisfying these axioms a highest weight category, see also [BR18, Subsection 1.12.3].

Proposition 6.4. *The category* $\mathcal{P}(\mathrm{Hk}_{\mathcal{IW}})$ *of Iwahori–Whittaker equivariant perverse sheaves is a highest weight category, whose underlying poset equals* $\mathbb{X}_\bullet \simeq W_{\mathrm{fin}}\backslash W$ *ordered by the quotient Bruhat order.*

Proof. The first part is a standard consequence of arguments by Beilinson–Ginzburg–Soergel, see [BGS96, Theorems 3.2.1 and 3.3.1]. As for the second claim, it suffices to identify the closure relations. It can be easily checked that the opposite Schubert variety $\mathrm{Fl}_{\mathcal{I},\leq w_\nu}^{\mathrm{op}}$ equals the $(G,\mathcal{I})$-Schubert variety $\mathrm{Fl}_{(G,\mathcal{I}),\leq\nu}$ in the notation of [AGLR22, Section 3] (up to the order of action), which in turn coincides with $\mathrm{Fl}_{\mathcal{I},\leq w_0 w_\nu}$, where $w_0 \in W_{\mathrm{fin}}$ is the longest element (there is a notational clash here, because w_ν evaluated at $\nu = 0$ is simply the identity). Indeed, they have the same dimension and there is an obvious inclusion $\mathrm{Fl}_{\mathcal{I},\leq w_\nu}^{\mathrm{op}} \subset \mathrm{Fl}_{(G,\mathcal{I}),\leq\nu}$, because $L^+G \supset L^+\mathcal{I}^{\mathrm{op}}W_{\mathrm{fin}}$. The closure relations follow then from the usual combinatorics of flag varieties as in [AGLR22, Section 3]. Indeed, the variety $\mathrm{Fl}_{(G,\mathcal{I}),\leq\nu_1}$ is contained in $\mathrm{Fl}_{(G,\mathcal{I}),\leq\nu_2}$ if and only if $\nu_1 \leq \nu_2$ for the quotient Bruhat order. $\square$

Notice that $\mathrm{Fl}_{\mathcal{I},\leq 1}^{\mathrm{op}} = \mathrm{Fl}_{\mathcal{I},\leq w_0} = G/B \subset \mathrm{Fl}_{\mathcal{I}}$, with opposite cells indexed by W_{fin}. This implies the equalities $\Delta_0^{\mathcal{IW}} = \nabla_0^{\mathcal{IW}} = \mathrm{IC}_0^{\mathcal{IW}}$ and we will denote this simple Iwahori-Whittaker perverse sheaf by Ξ. It allows us to define the Iwahori–Whittaker averaging functor

$$\mathrm{av}_{\mathcal{IW}} \colon \mathcal{D}_{\mathrm{\acute{e}t}}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{D}_{\mathrm{\acute{e}t}}(\mathrm{Hk}_{\mathcal{IW}}) \tag{6.5}$$

given by $A \mapsto \Xi * A$. To get a better understanding of this functor, we start by the following calculation:

Lemma 6.5. *The sheaf* $\mathrm{av}_{\mathcal{IW}}(\mathrm{IC}_w)$ *equals* $\mathrm{IC}_\nu^{\mathcal{IW}}$ *if* $w = w_\nu$ *for some* ν *and vanishes otherwise.*

Proof. If $sw < w$ for some simple reflection $s \in W_{\mathrm{fin}}$, we conclude that IC_w is equivariant for the left action of $L^+\mathcal{P}_s$, where $\mathcal{P}_s$ is the minimal standard parahoric with respect to $\mathcal{I}$ and the simple reflection s. Notice that by [BGM+19, Lemma 2.5] we have a natural isomorphism

$$\Xi *^{\mathcal{I}} \mathrm{IC}_w \simeq (R\pi_{s,*}\Xi) *^{\mathcal{J}_s} \mathrm{IC}_w \tag{6.6}$$

where the exponent stands for the fact that the first convolution is induced by contracting the $L^+\mathcal{I}$-action, and the second one by contracting the $L^+\mathcal{J}_s$-action. Here, $\pi_s : \mathrm{Fl}_{\mathcal{I}} \to \mathrm{Fl}_{\mathcal{P}_s}$ is the projection. In particular, it suffices to check the vanishing of $R\pi_{s,*}\Xi$. Note that Ξ is supported on $U^{\mathrm{op}}w_0 \subset G/B$ as the complement cannot support a non-zero Iwahori-Whittaker sheaf by 6.3. We can now see that the fiber of π_s over the image of $\mathrm{supp}\,\Xi$ is isomorphic to U_a^{op}, where a is the positive root associated with s. Since χ does not vanish on U_a^{op}, it follows by proper base change that $R\pi_{s,*}\Xi$ identifies with the cohomology $R\Gamma(\mathbb{G}_{a,k}, \mathcal{L}_{\mathrm{AS}})$ of the Artin–Schreier sheaf, i.e., it vanishes.

If $w = w_\nu$ is the minimal length representative of $W_{\mathrm{fin}} t_\nu$, we can check that the map $\mathrm{Fl}^{\mathrm{op}}_{\mathcal{I},1} \tilde{\times} \mathrm{Fl}_{\mathcal{I},w} \to \mathrm{Fl}^{\mathrm{op}}_{\mathcal{I},w}$ is an isomorphism. Indeed, both are affine spaces with the same dimension, and thus the given map is universally bijective by basic properties of Tits systems. In particular, we conclude that $\mathrm{av}_{\mathcal{IW}}(\mathrm{IC}_w)$ identifies with $\mathrm{IC}^{\mathcal{IW}}_\nu$. $\square$

Proposition 6.6. *The functor* $\mathrm{av}_{\mathcal{IW}}$ *is perverse t-exact.*

Proof. Since each half of the t-structure on $\mathcal{D}_{\mathrm{ét}}(\mathrm{Hk}_{\mathcal{I}})$ is spanned under extensions by appropriate shifts of the standard or costandard sheaves, it will suffice by symmetry to show that

$$\mathrm{av}_{\mathcal{IW}}(\Delta_w) = \Delta^{\mathcal{IW}}_\nu \tag{6.7}$$

for all $w \in W$, where $\nu \in \mathbb{X}_\bullet$ is in the same left W_{fin}-coset. When $w = w_\nu$ is the minimal length representative, this follows from the same argument of the previous lemma for IC sheaves. In general, consider an injection of perverse sheaves $\Delta_{w_\nu} \to \Delta_w$ as in [AB09, Lemma 3b], whose cone is spanned under extensions by $\mathrm{IC}_y * \Delta_{w_\nu}$ for non-trivial $y \in W_{\mathrm{fin}}$. The latter sheaves vanish under $\mathrm{av}_{\mathcal{IW}}$ by the previous lemma and we get the desired conclusion. $\square$

We are now ready to prove the main result regarding Iwahori–Whittaker averaging.

Theorem 6.7. *The averaging functor restricted to perverse sheaves factors through a fully faithful functor*

$$\mathrm{av}^{\mathrm{as}}_{\mathcal{IW}} \colon \mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}(\mathrm{Hk}_{\mathcal{IW}}) \tag{6.8}$$

where the left side is the Serre quotient of $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ *obtained by modding out the sheaves* IC_w *for all* $w \in W$ *which are not minimal in their left* W_{fin}*-cosets.*

Proof. Let $\mathcal{G}_{\geq 1} \to G$ be the dilatation along the identity subgroup in G_k so that $L^+\mathcal{G}_{\geq 1} = L^{\geq 1}G_O \subset L^+G_O$ is the first congruence subgroup of G. We have a natural map $\alpha \colon [L^+\mathcal{G}_{\geq 1} \backslash \mathrm{Fl}_{\mathcal{I}}] \to \mathrm{Hk}_{\mathcal{I}}$ and similarly a forgetful functor between stable ∞-categories $\beta^* \colon \mathcal{D}_{\mathrm{ét}}(\mathrm{Hk}_{\mathcal{IW}}) \to \mathcal{D}_{\mathrm{ét}}([L^+\mathcal{G}_{\geq 1} \backslash \mathrm{Fl}_{\mathcal{I}}])$ because the kernel of χ contains $L^{\geq 1}G_O$. We consider the induced functor

$$\mathrm{ind}_{\mathcal{IW}} := {}^p H^{-\mathrm{rk}G} \circ R\alpha_* \circ \beta^* \colon \mathcal{P}(\mathrm{Hk}_{\mathcal{IW}}) \to \mathcal{P}(\mathrm{Hk}_{\mathcal{I}}) \tag{6.9}$$

and claim that its composition with the quotient map $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}})$ yields a right inverse to $\mathrm{av}^{\mathrm{as}}_{\mathcal{IW}}$. First notice that $\mathrm{ind}_{\mathcal{IW}}(\Xi)$ is an extension of negative shifts of IC_w for $w \in W_{\mathrm{fin}}$, with the local system IC_1 appearing with multiplicity 1 (see [AR, Lemma 6.4.8]). If w is non-trivial and $\mathcal{F} \in \mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$, we can show that $\mathrm{IC}_w[n] * \mathcal{F}$ is $L^+\mathcal{P}_s$-equivariant for some simple reflection s, hence its perverse cohomology groups die under the quotient map $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}})$. If $w = 1$ and $n \neq 0$, then $\mathrm{IC}_1[n] * \mathcal{F}$ sits in non-zero perverse degree. In total, this yields an equivalence of functors $\mathrm{ind}^{\mathrm{as}}_{\mathcal{IW}} \circ \mathrm{av}^{\mathrm{as}}_{\mathcal{IW}} \simeq \mathrm{id}$. This implies that $\mathrm{av}^{\mathrm{as}}_{\mathcal{IW}}$ is injective on Ext groups. To see that $\mathrm{av}^{\mathrm{as}}_{\mathcal{IW}}$ is fully faithful, we argue by induction on the length of the objects being considered: the simple case is a consequence of Lemma 6.5, implying bijectivity of simple objects along $\mathrm{av}^{\mathrm{as}}_{\mathcal{IW}}$; the induction step follows from the 5-lemma and the injectivity on Ext-groups. $\square$

7. Tilting modules

We proved in Proposition 6.4 that the category of Iwahori-Whittaker perverse sheaves admits a highest weight category structure. It then makes sense to discuss *tilting objects* in this category. The aim of the current section is to show that the functor

$$\mathcal{Z}_{\mathcal{IW}} := \mathrm{av}_{\mathcal{IW}} \circ \mathcal{Z} \colon \mathrm{Rep}_\Lambda(\hat{G}) \to \mathcal{P}(\mathrm{Hk}_{\mathcal{IW}}) \tag{7.1}$$

lands on the full subcategory of tilting objects. This is related to the 'parabolic-singular' Koszul duality phenomenon studied by Beilinson–Ginzburg–Soergel [BGS96] for finite flag varieties and Bezrukavnikov–Yun [BY13] for Kac–Moody flag varieties.

We recall the following useful property.

Proposition 7.1. *For any* $\mathcal{F} \in \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{IW}})$, $\mathcal{F}$ *is tilting if and only if* $(j_{w_\nu}^{\mathrm{op}})^* \mathcal{F}$ *and* $R(j_{w_\nu}^{\mathrm{op}})^! \mathcal{F}$ *are both concentrated in degree* $-\ell(w_\nu)$ *for all* $\nu \in \mathbb{X}_\bullet$.

Proof. This follows from [BBM04, Proposition 1.3]. $\qquad\square$

7.1. **Multiplicities of tilting objects.** Let $\mathcal{F} \in \mathcal{P}(\mathrm{Hk}_{\mathcal{IW}})$ be a tilting object. Recall that the multiplicity of the standard (resp. costandard) objects $\Delta_\nu^{\mathcal{IW}}$ (resp. $\nabla_\nu^{\mathcal{IW}}$) in $\mathcal{F}$ is well-defined and we denote it by $(\mathcal{F} : \Delta_\nu^{\mathcal{IW}})$(resp. $(\mathcal{F} : \nabla_\nu^{\mathcal{IW}})$). It follows by orthogonality of $\Delta_\nu^{\mathcal{IW}}$ and $\nabla_\nu^{\mathcal{IW}}$ that

$$(\mathcal{F} : \Delta_\nu^{\mathcal{IW}}) \simeq \dim \mathrm{Hom}(\mathcal{F}, \nabla_\nu^{\mathcal{IW}}), \tag{7.2}$$

$$(\mathcal{F} : \nabla_\nu^{\mathcal{IW}}) \simeq \dim \mathrm{Hom}(\Delta_\nu^{\mathcal{IW}}, \mathcal{F}). \tag{7.3}$$

where the Hom spaces are taken inside $\mathcal{D}_{\mathrm{ét}}(\mathrm{Hk}_{\mathcal{IW}})$, compare with [BGS96, Theorem 3.2.1]. We use the same notation for $\mathcal{F} \in \mathcal{D}_{\mathrm{ét}}(\mathrm{Hk}_{\mathcal{IW}}$ as well.

Proposition 7.2. *For any* $V \in \mathrm{Rep}_\Lambda(\hat{G})$ *and any* $\mu \in \mathbb{X}_\bullet$, *we have*

$$\sum_{i \geq 0} (-1)^i (\mathcal{Z}_{\mathcal{IW}}(V)[-i] : \nabla_\nu^{\mathcal{IW}}) = \dim(V(\nu)) \tag{7.4}$$

$$\sum_{i \leq 0} (-1)^i (\mathcal{Z}_{\mathcal{IW}}(V)[i] : \Delta_\nu^{\mathcal{IW}}) = \dim(V(\nu)) \tag{7.5}$$

where $V(\nu)$ *denotes the* ν-*weight space of* V.

Proof. The proof follows the strategy of [AB09, Lemma 27] and [AR, Proposition 6.5.4] by Corollary 3.24, and Equation (6.7). $\qquad\square$

Corollary 7.3. *For any* $V \in \mathrm{Rep}_\Lambda(\hat{G})$,

(1) *if* $\mathcal{Z}_{\mathcal{IW}}(V)$ *is tilting, then*

$$(\mathcal{Z}_{\mathcal{IW}}(V) : \Delta_\mu^{\mathcal{IW}}) = (\mathcal{Z}_{\mathcal{IW}}(V) : \nabla_\mu^{\mathcal{IW}}) = \dim(V_\mu); \tag{7.6}$$

(2) *if* V *is the highest weight representation of highest weight* μ, *then* $\mathcal{Z}_{\mathcal{IW}}(V)$ *is supported on* $\mathrm{Fl}_{\mathcal{I}, w_\mu}^{\mathrm{op}}$.

Proof. Statements follow from [AR, Remark 6.5.5, Corollary 6.5.6]. $\qquad\square$

The rest of this section is devoted to the proof of the following theorem:

Theorem 7.4. *If* V *is a direct summand of a tensor product of minuscule representations, then* $\mathcal{Z}_{\mathcal{IW}}(V)$ *is tilting.*

In particular, we deduce that every $\mathcal{Z}_{\mathcal{IW}}(V)$ is tilting if every simple normal quotient of G is isomorphic to PGL_n for some (not fixed) positive integer n, because every weight is a sum of minuscule ones.

First, we handle minuscule representations, and for that we require the next lemma.

Lemma 7.5. *For any* $V \in \mathrm{Rep}(\hat{G})$, $\nu \in \mathbb{X}_\bullet$, $x \in W_{\mathrm{fin}}$, *and* $n \in \mathbb{Z}$, *we have isomorphisms*

$$\mathrm{Ext}^n(\Delta_\nu^{\mathcal{IW}}, \mathcal{Z}_{\mathcal{IW}}(V)) \simeq \mathrm{Ext}^n(\Delta_{x(\nu)}^{\mathcal{IW}}, \mathcal{Z}_{\mathcal{IW}}(V)),$$

$$\mathrm{Ext}^n(\nabla_\nu^{\mathcal{IW}}, \mathcal{Z}_{\mathcal{IW}}(V)) \simeq \mathrm{Ext}^n(\nabla_{x(\nu)}^{\mathcal{IW}}, \mathcal{Z}_{\mathcal{IW}}(V)).$$

Proof. The proof is similar to the arguments in [AR, Lemma 6.5.11] and we sketch it here. Without loss of generality, we assume ν to be dominant so that $w_\nu = t_\nu$, as the statement only

depends on its W_{fin}-orbit. We can find $y \in W_{\mathrm{fin}}$ with minimal length such that $t_\nu = w_{x(\nu)}y^{-1}$ is a minimal length decomposition, and $xy(\nu) = \nu$. Then

$$\Delta_\nu^{\mathcal{IW}} = \Delta_0^{\mathcal{IW}} * \Delta_{w_\nu} \simeq \Delta_0^{\mathcal{IW}} * \Delta_{w_{x(\nu)}} * \Delta_{y^{-1}} \simeq \Delta_{x(\nu)}^{\mathcal{IW}} * \Delta_{y^{-1}},$$

by Proposition 6.6 and Lemma 3.2. Then

$$\begin{aligned}
\mathrm{Ext}^n(\Delta_\nu^{\mathcal{IW}}, \mathcal{Z}_{\mathcal{IW}}(V)) &\simeq \mathrm{Ext}^n(\Delta_\nu^{\mathcal{IW}} * \Delta_y, \mathcal{Z}_{\mathcal{IW}}(V) * \Delta_y) \\
&\simeq \mathrm{Ext}^n(\Delta_{x(\nu)}^{\mathcal{IW}}, \mathcal{Z}_{\mathcal{IW}}(V) * \Delta_y) \\
&\simeq \mathrm{Ext}^n(\Delta_{x(\nu)}^{\mathcal{IW}}, \Delta_0^{\mathcal{IW}} * \mathcal{Z}(V) * \Delta_y) \\
&\simeq \mathrm{Ext}^n(\Delta_{x(\nu)}^{\mathcal{IW}}, \Delta_0^{\mathcal{IW}} * \Delta_y * \mathcal{Z}(V)) \\
&\simeq \mathrm{Ext}^n(\Delta_{x(\nu)}^{\mathcal{IW}}, \mathcal{Z}_{\mathcal{IW}}(V))
\end{aligned}$$

again by Proposition 6.6 and Lemma 3.2. The second isomorphism is proved analogously. $\qquad\square$

Proposition 7.6. *Let V be a simple representation of $\hat{G}$ with highest weight μ being a minuscule dominant coweight of G, then $\mathcal{Z}_{\mathcal{IW}}(V)$ is tilting.*

Proof. Because of Proposition 7.1, it suffices to show that $(j_{w_\nu}^{\mathrm{op}})^*(\mathcal{Z}_{\mathcal{IW}}(V))$ and $(j_{w_\nu}^{\mathrm{op}})^!(\mathcal{Z}_{\mathcal{IW}}(V))$ are both perverse sheaves. Since μ is minuscule, the only weights we have to check are in the W_{fin}-orbit of μ. By adjunction and Lemma 7.5, we are reduced to showing the statements above for $\nu = \mu$. Note that the support of $\mathcal{Z}(V)$ equals the μ-admissible locus $A_{I,\mu}$ by [AGLR22, Theorem 6.16], whose open $L^+\mathcal{I}$-orbits are indexed by $W_{\mathrm{fin}}\mu$. We deduce that the support of $\mathcal{Z}_{\mathcal{IW}}(V)$ equals $\mathrm{Fl}_{\mathcal{I}, \leq w_0 t_\mu}$ and hence the locally closed immersion $j_{w_\mu}^{\mathrm{op}}$ is actually open and dense. In particular, it is clear that $(j_{w_\mu}^{\mathrm{op}})^*(\mathcal{Z}_{\mathcal{IW}}(V))$ and $(j_{w_\mu}^{\mathrm{op}})^!(\mathcal{Z}_{\mathcal{IW}}(V))$ are both perverse. $\qquad\square$

Now, we deduce Theorem 7.4 by propagating the result via convolution.

Proposition 7.7. *Let $V, W \in \mathrm{Rep}(\hat{G})$ such that $\mathcal{Z}_{\mathcal{IW}}(V)$ and $\mathcal{Z}_{\mathcal{IW}}(W)$ are both tilting. Then so is $\mathcal{Z}_{\mathcal{IW}}(V \otimes W)$.*

Proof. It suffices to prove that $(j_{w_\nu}^{\mathrm{op}})^* \mathcal{Z}_{\mathcal{IW}}(V \otimes W)$ and $(j_{w_\nu}^{\mathrm{op}})^! \mathcal{Z}_{\mathcal{IW}}(V \otimes W)$ are both perverse for any $\nu \in \mathbb{X}_\bullet$. Since $\mathcal{Z}_{\mathcal{IW}}(V \otimes W)$ is perverse, $(j_{w_\nu}^{\mathrm{op}})^* \mathcal{Z}_{\mathcal{IW}}(V \otimes W)$ concentrates in perverse degress ≤ 0, and $(j_{w_\nu}^{\mathrm{op}})^!(\mathcal{Z}_{\mathcal{IW}}(V \otimes W))$ concentrates in perverse degrees ≥ 0. Note that if $\mathcal{Z}_{\mathcal{IW}}(V)$ is tilting, the object $\mathcal{Z}_{\mathcal{IW}}(V \otimes W) \cong \mathcal{Z}_{\mathcal{IW}}(V) * \mathcal{Z}(W)$ admits a filtration with subquotients $\Delta_\mu^{\mathcal{IW}} * \mathcal{Z}(W)$. By Proposition 6.6 and Theorem 4.12, $\Delta_\mu^{\mathcal{IW}} * \mathcal{Z}(W) \cong \mathcal{Z}_{\mathcal{IW}}(W) * \Delta_{w_\mu}$. Since $\mathcal{Z}_{\mathcal{IW}}(W)$ is tilting, then $\Delta_\mu^{\mathcal{IW}} * \mathcal{Z}(W)$ admits a filtration with subquotients $\mathrm{av}_{\mathcal{IW}}(\Delta_{w_\nu} * \Delta_{w_\mu})$. Lemma 3.8 and Proposition 6.6 imply that $(j_{w_\nu}^{\mathrm{op}})^*(\mathcal{Z}_{\mathcal{IW}}(V \otimes W))$ concentrates in non-negative perverse degrees. The statement for $(j_{w_\nu}^{\mathrm{op}})^!(\mathcal{Z}_{\mathcal{IW}}(V \otimes W))$ is proved similarly. $\qquad\square$

8. REGULAR QUOTIENT

During this section, we assume $\Lambda = \bar{\mathbb{Q}}_\ell$ is algebraically closed. Consider the Serre subcategory $\mathcal{P}_{>0}(\mathrm{Hk}_{\mathcal{I}}) \subset \mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ generated by IC sheaves with positive dimensional support and denote by Π^0 the natural quotient functor

$$\Pi^0 : \mathcal{P}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}}) := \mathcal{P}(\mathrm{Hk}_{\mathcal{I}})/\mathcal{P}_{>0}(\mathrm{Hk}_{\mathcal{I}}) \qquad (8.1)$$

to the Serre quotient.

Therefore the simple objects in $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$ are precisely given by the IC_τ where $\tau \in \Omega_{\mathbf{a}}$ stabilizes the base alcove. In particular, if G is semi-simple, $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$ has only finitely many simple objects.

Proposition 8.1. *The monoidal structure on $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ given by perverse truncated convolution $^p H^0((-) * (-))$ descends to an exact monoidal structure $\circledast$ on $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$.*

Proof. The proof follows the idea in [AR, Proposition 6.5.14] and we sketch it here. Let $A_1 = \text{IC}_w$ for some $w \in W$ with $\ell(w) > 0$. Then there exists a simple reflection s such that $\ell(sw) < \ell(w)$. Let $\mathcal{J}_s$ be the minimal parahoric containing $\mathcal{I}$ associated with s. Then A_1 is $\mathcal{J}_s$-equivariant. It follows that $A_1 * A_2$ is also $\mathcal{J}_s$-equivariant for any $A_2 \in \mathcal{P}(\text{Hk}_\mathcal{I})$, and so are its perverse cohomology sheaves. But a $\mathcal{J}_s$-equivariant perverse sheaf has equivariant composition factors, hence lies in $\mathcal{P}_{>0}(\text{Hk}_\mathcal{I})$. Varying w and by symmetry, we conclude that the monoidal structure given by $^pH^0(*)$ on $\mathcal{P}(\text{Hk}_\mathcal{I})$ descends to a monoidal structure $\circledast$ on $\mathcal{P}_0(\text{Hk}_\mathcal{I})$. In order to check exactness of $\circledast$, we must see that for arbitrary $x, y \in W$ the perverse cohomology sheaves in non-zero degree of a convolution product $\text{IC}_x * \text{IC}_y$ lie in $\mathcal{P}_{>0}(\text{Hk}_\mathcal{I})$. The only remaining case to analyze is when both elements have length 0, but in this case $\text{IC}_x * \text{IC}_y = \text{IC}_{xy}$. □

We have the following important result:

Lemma 8.2. *The functor $\mathcal{Z}_0 := \Pi^0 \circ \mathcal{Z} \colon \text{Rep}(\hat{G}) \to \mathcal{P}_0(\text{Hk}_\mathcal{I})$ is monoidal and central.*

Proof. By Proposition 8.1, we can construct the monoidality and centrality isomorphisms by applying those of Proposition 4.9 and Theorem 4.12 and then projecting towards $\mathcal{P}_0(\text{Hk}_\mathcal{I})$, compare with [AR, Lemma 6.5.15]. □

Note that for every $\hat{G}$-representation V, we have a nilpotent operator $\mathbf{n}_V^0 \colon \mathcal{Z}_0(V) \to \mathcal{Z}_0(V)$ arising from the logarithm of the monodromy of $\mathcal{Z}(V)$. Denote by $\mathcal{P}_0^c(\text{Hk}_\mathcal{I})$ the full subcategory of $\mathcal{P}_0(\text{Hk}_\mathcal{I})$ whose objects are the subquotients of $\mathcal{Z}^0(V), V \in \text{Rep}(\hat{G})$. The exactness of $\circledast$ and monoidality of $\mathcal{Z}_0$ imply that $\mathcal{P}_0^c(\text{Hk}_\mathcal{I})$ is closed under the monoidal structure. By definition, the functor $\mathcal{Z}_0$ naturally factors through a functor $\mathcal{Z}_0^c \colon \text{Rep}(\hat{G}) \to \mathcal{P}_0^c(\text{Hk}_\mathcal{I})$.

Proposition 8.3. *There exists a closed subgroup $H \subset \hat{G}$ such that we have*

(1) *an equivalence of monoidal categories*
$$\Phi^0 \colon (\mathcal{P}_0^c(\text{Hk}_\mathcal{I}), \circledast) \simeq (\text{Rep}(H), \otimes). \tag{8.2}$$

(2) *a nilpotent element $n_0 \in \hat{\mathfrak{g}}$ such that $H \subset Z_{\hat{G}}(n_0)$.*

(4) *an isomorphism of functors $\alpha \colon \Phi^0 \circ \mathcal{Z}_0^c \simeq \text{For}_H^{\hat{G}}$, carrying the monodromy operators $\mathbf{n}_V^0$ to the natural action of n_0 on V.*

Remark 8.4. If G is a type A group, then $\mathcal{P}_0^c(\text{Hk}_\mathcal{I}) = \mathcal{P}_I^0$, and $H = Z_{\hat{G}}(n_0)$. We do not need this in the proof of the main theorem and will postpone the discussion of this fact to §10 (cf. Proposition 10.8).

Proof. The above proposition is the mixed characteristic analogue of a particular case of [Bez04, Proposition 1, Theorem 3]. We sketch the proof here and refer further details to *loc.cit.* Note that we can regard the regular representation $\mathcal{O}(\hat{G})$ of the dual group as a ring object in $\text{Ind}(\text{Rep}(\hat{G}))$. Then $\mathcal{Z}_0(\mathcal{O}(\hat{G}))$ is a ring object in $\text{Ind}(\mathcal{P}_0^c(\text{Hk}_\mathcal{I}))$. Zorn's lemma implies that there exists a maximal left ideal subobject $\mathcal{J} \subset \mathcal{Z}_0(\mathcal{O}(\hat{G}))$, whose quotient will be denoted by $\mathcal{O}(H)$. The centrality of $\mathcal{Z}^0$ (cf. Lemma 8.2) implies that $\mathcal{O}(H)$ is also a ring object. Thus, we define $\mathcal{O}(H)$-Mod as the category of left $\mathcal{O}(H)$-modules in $\text{Ind}(\mathcal{P}_0^c(\text{Hk}_\mathcal{I}))$. Clearly $\mathcal{O}(H)$ is a simple object in the abelian category $\mathcal{O}(H)$-Mod. Hence, its endomorphism ring $K := \text{End}_{\mathcal{O}(H)}(\mathcal{O}(H))$ is a division algebra, and $V \mapsto V \otimes K$ defines an equivalence between the category of right finite K-modules and the full subcategory in $\mathcal{O}(H)$-Mod generated by $\mathcal{O}(H)$ under finite direct sums and subquotients. Now, we deduce that
$$K \simeq \text{Hom}_{\text{Ind}(\mathcal{P}_0^c(\text{Hk}_\mathcal{I}))}(\delta_0, \mathcal{O}(H)) \simeq \bar{\mathbb{Q}}_\ell. \tag{8.3}$$
because the left hand side is a countable $\bar{\mathbb{Q}}_\ell$-vector space and hence it must be algebraic. Now, we construct a monoidal fiber functor to invoke the Tannakian formalism.

Lemma 8.5. (1) *For any $A \in \mathcal{P}_0^c(\mathrm{Hk}_\mathcal{I})$, there exists a finite-dimensional vector space V such that $\mathcal{O}(H) \circledast A \simeq \mathcal{O}(H) \otimes V$ is an isomorphism of $\mathcal{O}(H)$-modules, where we endow V with the trivial $\mathcal{O}(H)$-action.*

(2) *The functor $\Phi_G : \mathcal{P}_0^c(\mathrm{Hk}_\mathcal{I}) \to \mathrm{Vect}_\Lambda$ defined by $A \mapsto \mathrm{Hom}(\mathcal{O}(H), \mathcal{O}(H) \circledast A)$ is an exact, faithful, and monoidal functor. In addition, $\Phi_G \circ \mathcal{Z}_0^c \simeq \mathrm{For}^{\hat{G}} : \mathrm{Rep}(\hat{G}) \to \mathrm{Vect}_{\bar{\mathbb{Q}}_\ell}$.*

Proof. To prove statement (1) in the above lemma, we first note that there is a canonical isomorphism $\mathcal{Z}_0(\mathcal{O}(\hat{G})) \circledast \mathcal{Z}_0(V) \simeq \mathcal{Z}_0(\mathcal{O}(\hat{G})) \otimes_\Lambda V$ of $\mathcal{Z}_0(\mathcal{O}(\hat{G}))$-modules for any $V \in \mathrm{Rep}(\hat{G})$. Quotienting out the maximal left idea $\mathcal{J}$, we conclude that $\mathcal{O}(H) \circledast \mathcal{Z}_0(V) \simeq \mathcal{O}(H) \otimes V$. The general situation follows from taking subquotients from both sides and we thus settle statement (1).

The exactness of Φ_G follows from that of $\mathcal{Z}_0$ and statement (1). Also, Equation (8.3) and statement (1) imply that $\mathcal{O}(H) \circledast A \cong \mathcal{O}(H) \otimes \Phi_G(A)$ for any $A \in \mathcal{P}_0^c(\mathrm{Hk}_\mathcal{I})$. Then for any $A_1, A_2 \in \mathcal{P}_0^c(\mathrm{Hk}_\mathcal{I})$, we have

$$\Phi_G(A_1 \circledast A_2) \simeq \mathrm{Hom}(\mathcal{O}(H), \mathcal{O}(H) \otimes \Phi_G(A_1 \circledast A_2)) \simeq \mathrm{Hom}(\mathcal{O}(H), \mathcal{O}(H) \circledast (A_1 \circledast A_2))$$

$$\simeq \mathrm{Hom}(\mathcal{O}(H), \Phi_G(A_1) \otimes (\mathcal{O}(H) \circledast A_2)) \simeq \Phi_G(A_1) \otimes \Phi_G(A_2).$$

Finally, it suffices to check that Φ_G sends non-zero objects to non-zero objects since it is exact. This can be checked on all simple objects $\Pi^0(\mathrm{IC}_\tau)$. The faithfulness then follows from the dualizability, in fact invertibility, of $\Pi^0(\mathrm{IC}_\tau)$ and the monoidal structure of Φ_G. $\qquad\square$

Lemma 8.5 allows us to apply the Tannakian formalism and obtain an equivalence of monoidal categories

$$\Psi : \mathcal{P}_0^c(\mathrm{Hk}_\mathcal{I}) \simeq \mathrm{Comod}_{\mathcal{A}(H)}, \tag{8.4}$$

where $\mathcal{A}(H)$ is a Λ-bialgebra and $\mathrm{Comod}_{\mathcal{A}(H)}$ is the category of Λ-finite $\mathcal{A}(H)$-comodules. In addition, the composition of this equivalence with the natural forgetful functor $\mathrm{Comod}_{\mathcal{A}(H)} \to \mathrm{Vect}_\Lambda$ equals Φ_G. By the Tannakian construction, the functor $\Psi \circ \mathcal{Z}_0^c : \mathrm{Rep}(\hat{G}) \to \mathrm{Comod}_{\mathcal{A}(H)}$ induces a surjective morphism of bialgebras $\mathcal{O}(G) \to \mathcal{A}(H)$. It follows from [Bez04, Lemma 3] that $\mathcal{A}(H)$ is commutative and $\mathrm{Spec}\,\mathcal{A}(H)$ is the desired group scheme H.

Now we construct n_0. Recall our construction of the nilpotent endomorphism $\mathbf{n}_V^0$ of $\mathcal{Z}_0(V)$ for any $V \in \mathrm{Rep}(\hat{G})$. By naturality and compatibility with the monoidal structure as in Lemma 4.24, we deduce a tensor endomorphism of the functor $\Phi_G \circ \mathcal{Z}_0^c \simeq \mathrm{For}^{\hat{G}}$. In particular, this gives rise to an element n_0 of $\hat{\mathfrak{g}}$ by the Tannakian formalism. On the other hand, $\Phi_G \circ \mathcal{Z}_0^c \simeq \mathrm{For}^H \circ \Psi \circ \mathcal{Z}_0^c \simeq \mathrm{For}^H \circ \mathrm{For}_H^{\hat{G}}$, and $(\mathbf{n}_V^0)_V$ induces an automorphism of $\mathrm{For}_H^{\hat{G}}$. Hence, $H \subset Z_{\hat{G}}(n_0)$. $\qquad\square$

Proposition 8.6. *If G is of type A, then the nilpotent element n_0 is regular.*

Proof. Our argument is similar to the one in [AR, Section 6.5.8] and uses weight theory. Recall that in Conjecture 4.26, we posited that the mixed sheaves $\mathcal{Z}_{\mathrm{mix}}(V)$ ought to be monodromy-pure of weight 0, as this is the case in equicharacteristic due to a theorem of Gabber whose proof was written up by Beilinson–Bernstein [BB93]. By Proposition 4.27, we know that this holds for minuscule representations. Note that the functor $\Pi^0 : \mathcal{P}(\mathrm{Hk}_\mathcal{I}) \to \mathcal{P}_0(\mathrm{Hk}_\mathcal{I})$ in (8.1) admits a mixed variant Π_{mix}^0, namely the quotient of mixed perverse sheaves by the ones with positive dimensional support. We claim that the images under Π_{mix}^0 are monodromy-pure of weight 0. In other words, we want to show that the weight filtration obtained on $\mathcal{Z}_0(V)$ via push-pull coincides with the monodromy filtration induced by n_0. It suffices to prove this when G is adjoint, and then we can check that both filtrations are monoidal on V, so we can propagate the claim starting from the minuscule case.

Now, we can check whether n_0 is regular by calculating the dimension of $\hat{\mathfrak{g}}^{n_0}$. Reading off the weight filtration $\mathcal{Z}_0(\hat{\mathfrak{g}})$ on the Iwahori–Hecke algebra, one sees that its i-th graded has dimension

equal to that of the sum of the weight spaces $\hat{\mathfrak{g}}(\nu)$ with $\langle 2\rho, \nu \rangle = i$. Since the weights of $\hat{\mathfrak{g}}$ are roots of $\hat{G}$, its non-zero gradeds are even integers, and hence $\dim(\hat{\mathfrak{g}}^{n_0}) = \dim(\hat{\mathfrak{g}}(0)) = \mathrm{rk}(G)$. $\quad\square$

9. Proof of the AB equivalence

At this point, we consider the composition of the two functors

$$\mathcal{F}_{\mathcal{IW}} := \mathrm{av}_{\mathcal{IW}} \circ \mathcal{F} \colon \mathrm{Perf}([\hat{G} \backslash \hat{\mathcal{N}}_{\mathrm{Spr}}]) \to \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{IW}}) \tag{9.1}$$

that we have extensively studied thus far. Our goal is to prove the Arkhipov–Bezrukavnikov equivalence below:

Theorem 9.1. *If G is of type A, then the functor $\mathcal{F}_{\mathcal{IW}}$ is an equivalence.*

One can immediately draw the following conclusion:

Corollary 9.2. *If G is of type A, then the functor $\mathrm{av}_{\mathcal{IW}}^{\mathrm{as}}$ from 6.7 is an equivalence of abelian categories.*

The strategy behind the proof of the theorem is as usual based on generators and relations. We start with the following lemma.

Lemma 9.3. *The ∞-category $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{IW}})$ is spanned by $\mathrm{av}_{\mathcal{IW}}(\mathcal{I}_\nu)$ for all $\nu \in \mathbb{X}_\bullet$ under cones and extensions.*

Proof. As in Lemmas 3.1 and 3.6, we can check that $\mathrm{av}_{\mathcal{IW}}(\mathcal{I}_\nu)$ has the same class in the Grothendieck group as $\Delta_\nu^{\mathcal{IW}}$. Taking its Euler characteristic, we deduce that it is supported on $\mathrm{Fl}_{\mathcal{I}, \leq w_0 w_\nu}$ and has generic rank 1. A standard induction argument now implies the spanning assertion. $\quad\square$

Lemma 9.4. *For any $V \in \mathrm{Rep}_\Lambda \hat{G}$, the map*

$$\mathrm{Hom}(\mathcal{O}_{\hat{\mathcal{N}}_{\mathrm{Spr}}}, V \otimes \mathcal{O}_{\hat{\mathcal{N}}_{\mathrm{Spr}}}) \to \mathrm{Hom}(\Xi, \mathcal{Z}_{\mathcal{IW}}(V)) \tag{9.2}$$

induced by $\mathcal{F}_{\mathcal{IW}}$ is injective.

Proof. Since $\mathrm{av}_{\mathcal{IW}}^{\mathrm{as}}$ is fully faithful, it suffices to check the injectivity on the anti-spherical category $\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}})$. We can also further reduce to verifying injectivity after passing to the quotient $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$ defined in the previous section. Now, we use the regular orbit $\hat{G}/Z_{\hat{G}}(n_0) \simeq \mathcal{O}_r \subset \hat{\mathcal{N}}$, together with the compatible isomorphism $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}}) \cong \mathrm{Rep}_\Lambda(H)$ for a certain subgroup $H \subset Z_{\hat{G}}(n_0)$. In terms of these data, the homomorphism of Hom-groups identifies with $V^{Z_{\hat{G}}(n_0)} \to V^H$, which is clearly injective. $\quad\square$

We deduce our last key calculation:

Corollary 9.5. *For any $V \in \mathrm{Rep}_\Lambda \hat{G}$, any $n \in \mathbb{Z}$ and $\nu \in \mathbb{X}_+$, the natural map*

$$\mathrm{Ext}^n(\mathcal{O}, V \otimes \mathcal{O}(\nu)) \to \mathrm{Ext}^n(\Xi, \mathcal{Z}_{\mathcal{IW}}(V) * \mathcal{I}_\nu) \tag{9.3}$$

is injective.

Proof. The left side identifies with $(V \otimes H^n(\hat{\mathcal{N}}_{\mathrm{Spr}}, \mathcal{O}(\lambda)))^{\hat{G}}$. The higher cohomology of $\mathcal{O}_{\hat{\mathcal{N}}_{\mathrm{Spr}}}(\lambda)$ vanishes, meaning we only need to consider the right side when $n = 0$. Since there exists an equivariant embedding $\mathcal{O}_{\hat{\mathcal{N}}_{\mathrm{Spr}}}(\lambda) \to W \otimes \mathcal{O}_{\hat{\mathcal{N}}_{\mathrm{Spr}}}$ for a certain $W \in \mathrm{Rep}_\Lambda \hat{G}$, the claim reduces to the preceding lemma. $\quad\square$

Finally, we can prove our main theorem, the AB equivalence.

Proof of Theorem 9.1. Applying [AR, Lemma 6.2.6] and the 5-lemma, fully faithfulness will follow from seeing that the injection (9.3) is bijective. Since both sides are finite dimensional Λ-vector spaces, it will be enough to check their dimensions match. Furthermore, once we know $\mathcal{F}_{\mathcal{IW}}$ is fully faithful, we conclude it is an equivalence as its image spans the Iwahori–Whittaker category.

Let us compute the dimension of the right side. After convolution on the right with $\mathcal{I}_{-\nu}(\Lambda)$, it vanishes if $n \neq 0$ by the tilting property of $\mathcal{Z}_{\mathcal{IW}}(V)$, see 7.4, and has otherwise dimension equal to that of the weight space $V(-\nu)$. As for the left side, we have already checked its vanishing if $n \neq 0$ and it has otherwise dimension equal to that of the weight space $V(\nu)$, as one checks via the cohomology of $\mathcal{O}(\nu)$ on the Springer resolution $\hat{\mathcal{N}}_{\mathrm{Spr}}$, compare with [AR, Subsection 6.6.3]. $\qquad\square$

10. Exotic t-structure on the Springer resolution

The equivalence in Theorem 9.1 allows us to transport the perverse t-structure on $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{IW}})$ to a t-structure which we call the *exotic t-structure* on $\mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$, at least when G is of type A. The exotic t-structure has been intrinsically studied in [Bez06, MR16]. In this section, we discuss the exotic t-structure obtained via our object $\mathrm{Fl}_{\mathcal{I}}$, and explain how it will be used to prove the assertions in Remark 8.4 for type A groups.

Recall the partial order $\leq$ on $\mathbb{X}_\bullet$ given by $\nu \leq \mu$ if and only if $\mu-\nu$ is a linear combination of positive roots. Note that the ∞-category $\mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ has finite cohomological dimension by either [BGS96, Corollary 3.2.2] or [DG13, Theorem 1.4.2], i.e., for any objects $\mathcal{A},\mathcal{B}$, the vector space $\oplus_i \mathrm{Ext}^i(\mathcal{A},\mathcal{B})$ is finite-dimensional. Then [Bez06, Lemma 5] implies that the line bundles $\mathcal{O}(\nu)$ form an *exceptional collection* indexed by $\nu \in \mathbb{X}_\bullet$ in the sense of *loc.cit* and generate $\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ under shifts and cones.

Choose a refinement of the Bruhat partial order $\leq$ on $\mathbb{X}_\bullet$ to a total order $\leq'$. Now, we can define the *exotic exceptional collection*

$$\{\nabla^{\mathrm{ex}}_\nu : \nu \in \mathbb{X}_\bullet\} \tag{10.1}$$

of $\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ as the collection of objects produced by *mutation* of $\{\mathcal{O}(\nu)|\nu \in \mathbb{X}_\bullet)\}$ in the sense of [AR, §7.1.2]. By [Bez06, Proposition 3], it in turn gives rise to the dual exotic exceptional collection

$$\{\Delta^{\mathrm{ex}}_\nu : \nu \in \mathbb{X}_\bullet\} \tag{10.2}$$

in the sense of *loc.cit.* Define ${}^{\mathrm{ex}}\mathcal{D}^{b,\geq 0}_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ (resp. ${}^{\mathrm{ex}}\mathcal{D}^{b,\leq 0}_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$) as the full subcategory generated under extensions by objects $\nabla^{\mathrm{ex}}_\nu[n]$ (resp. $\Delta^{\mathrm{ex}}_\nu[n]$) with $\nu \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\leq 0}$ (resp. $n \in \mathbb{Z}_{\geq 0}$). Then [Bez06, Proposition 4] shows that the above pair of full subcategories forms a bounded t-structure and we call it the *exotic t-structure*. We denote the heart of this t-structure by $\mathrm{ExCoh}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$.

Proposition 10.1. *If G is of type A, there are isomorphisms*

$$\mathcal{F}_{\mathcal{IW}}(\nabla^{\mathrm{ex}}_\nu) \cong \nabla^{\mathcal{IW}}_\nu, \tag{10.3}$$

$$\mathcal{F}_{\mathcal{IW}}(\Delta^{\mathrm{ex}}_\nu) \cong \Delta^{\mathcal{IW}}_\nu, \tag{10.4}$$

for any $\nu \in \mathbb{X}_\bullet$.

Proof. The proof follows the idea of [AR, Proposition 7.1.5] in the equicharacteristic situation and we sketch it here. In view of Theorem 9.1, it amounts to prove that the collection $\{\nabla^{\mathcal{IW}}_\nu|\nu \in \mathbb{X}_\bullet\}$ coincide with the collection of exceptional objects that come from the mutation of $\{\mathrm{av}_{\mathcal{IW}}(\mathcal{I}_\nu): \nu \in \mathbb{X}_\bullet\}$ with respect to the Bruhat order on $\mathbb{X}_\bullet$. This is shown by the closure relation of affine Schubert varieties proved in Lemma 2.1. The second isomorphism follows from the uniqueness of the dual exceptional collection. $\qquad\square$

We have the following immediate corollary.

Corollary 10.2. *Assume G is of type A. Then, the following hold:*

(1) *The functor $\mathcal{F}_{\mathcal{IW}}$ is t-exact with respect to the exotic t-structure on $\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ and the perverse t-structure on $\mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{IW}})$.*

(2) *In addition, the functor $\mathcal{F}_{\mathcal{IW}}$ restricts to an equivalence of abelian categories*

$$^p H^0(\mathcal{F}_{\mathcal{IW}})\colon \mathrm{ExCoh}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) \xrightarrow{\sim} \mathcal{P}(\mathrm{Hk}_{\mathcal{IW}}). \tag{10.5}$$

We have already seen in §6 that the simple objects of $\mathcal{P}(\mathrm{Hk}_{\mathcal{IW}})$ denoted $\mathrm{IC}^{\mathcal{IW}}_\nu$ are in bijection with $\mathbb{X}_\bullet$. On the coherent side, the space $\mathrm{Hom}(\Delta^{\mathrm{ex}}_\nu, \nabla^{\mathrm{ex}}_\nu)$ is one-dimensional and the image of Δ^{ex}_ν under any non-zero map is a simple object in $\mathrm{ExCoh}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$. We denote this simple object by L^{ex}_ν.

Lemma 10.3.   (1) *The realization functor*

$$\mathcal{D}^b(\mathrm{ExCoh}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])) \to \mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$$

is an equivalence of ∞-categories.

(2) *For any $\nu \in \mathbb{X}^+_\bullet$, there are isomorphisms*

$$\nabla^{\mathrm{ex}}_\nu \simeq \mathcal{O}(\nu), \tag{10.6}$$

$$\Delta^{\mathrm{ex}}_{-\nu} \simeq \mathcal{O}(-\nu) \tag{10.7}$$

(3) *For any $\nu \in \mathbb{X}_\bullet$, there are isomorphisms*

$$\nabla^{\mathrm{ex}}_\nu|_{\tilde{\mathcal{O}}_{\mathrm{r}}} \simeq \mathcal{O}(\nu^+)|_{\tilde{\mathcal{O}}_{\mathrm{r}}}, \tag{10.8}$$

$$\Delta^{\mathrm{ex}}_\nu|_{\mathcal{O}_{\mathrm{r}}} \simeq \mathcal{O}(\nu^-)|_{\mathcal{O}_{\mathrm{r}}}, \tag{10.9}$$

where $\mathcal{O}_{\mathrm{r}} \subset \hat{\mathcal{N}}$ is the regular orbit, and ν^+ (resp. ν^-) is the dominant (resp. anti-dominant) W_{fin}-conjugate of ν.

(4) *Assume G is of type A. Then, for any $\nu \in \mathbb{X}_\bullet$, there is an isomorphism*

$$\mathcal{F}_{\mathcal{IW}}(L^{\mathrm{ex}}_\nu) \cong \mathrm{IC}^{\mathcal{IW}}_\nu. \tag{10.10}$$

Proof. The first three properties appear in [AR, Corollary 7.1.6, Lemmas 7.2.1 and 7.2.2]. We will explain how they follow from Proposition 10.1 for type A groups and prove the last claim. It is well-known that $\mathcal{D}^b(\mathcal{P}(\mathrm{Hk}_{\mathcal{IW}})) \cong \mathcal{D}_{\mathrm{cons}}(\mathrm{Hk}_{\mathcal{IW}})$. Assertion (1) then follows from the equivalence (9.1). The first isomorphism in assertion (2) can be easily deduced from (6.7) and Proposition 10.1, and the second isomorphism follows analogously. The statement (3) follows from a standard induction argument on the length of the minimal element $w \in W_{\mathrm{fin}}$ such that $\nu = w\nu^+$(resp. $\nu = w\nu^-$) using (2) and [AR, Proposition 7.1.4]. Assertion (4) follows directly from Proposition 10.1 and Corollary 10.2. $\square$

Corollary 10.4. *For any $A \in \mathrm{ExCoh}(\hat{\mathcal{N}}_{\mathrm{Spr}})$, $A|_{\mathcal{O}_{\mathrm{r}}}$ is a $\hat{G}$-equivariant vector bundle on $\mathcal{O}_{\mathrm{r}}$.*

Proof. The result follows from $\hat{G}/Z_{\hat{G}}(n_0) \simeq \mathcal{O}_{\mathrm{r}}$ and Lemma 10.3. $\square$

Proposition 10.5. *For $\nu \in \mathbb{X}_\bullet$, we have*

$$L^{\mathrm{ex}}_\nu|_{\mathcal{O}_{\mathrm{r}}} \simeq \begin{cases} \mathcal{O} & \text{if } \ell(w_\nu) = 0 \\ 0 & \text{otherwise} \end{cases} \tag{10.11}$$

Proof. This is [AR, Proposition 7.2.4] and we sketch it here for type A groups. If $\ell(w_\nu) = 0$, then $\nabla_\nu^{\mathcal{IW}} \cong \mathrm{IC}_\nu^{\mathcal{IW}}$, and we conclude the proof by Lemma 10.3. In general, there exists a unique $\mu \in \mathbb{X}_\bullet$ such that $\ell(w_\mu) = 0$ and $\mathrm{Fl}_{\mathcal{I},w_\nu}$ and $\mathrm{Fl}_{\mathcal{I},w_\mu}$ belong to the same connected component. Then the proof of Lemma 6.5 and Proposition 6.6 imply that $\mathrm{IC}_{w_\mu}^{\mathcal{IW}}$ is a composition factor of $\nabla_\mu^{\mathcal{IW}}$. Then Lemma 10.3 yields that L_μ^{ex} is a composition factor of ∇_ν^{ex}. Combining Lemma 3.9, Lemma 10.3, Corollary 10.4 and the previous discussion, both L_μ^{ex} and ∇_ν^{ex} restricts to an equivariant $\hat{G}$-line bundle on $\tilde{\mathcal{O}}_{\mathrm{r}}$. In particular, as a composition factor of ∇_ν^{ex}, L_ν^{ex} restricts to 0. $\qquad\square$

Denote by $\mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])_{\mathrm{nr}}$ the full subcategory of perfect complexes supported on the complement of $\mathcal{O}_{\mathrm{r}}$.

Lemma 10.6. *The category $\mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])_{\mathrm{nr}}$ is generated by $\{L_\nu^{\mathrm{ex}}|\nu \in \mathbb{X}_\bullet, \ell(w_\nu) > 0\}$ under cones and shifts.*

Proof. This is [AR, Lemma 7.2.7] and we could also prove it for type A groups via Corollary 10.4 and Proposition 10.5. $\qquad\square$

For the rest of this section, we apply the previous discussion to study the relation between $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$ and its full subcategory $\mathcal{P}_0^c(\mathrm{Hk}_{\mathcal{I}})$, culminating in the proof that they coincide in type A and so do $H \subset Z_{\hat{G}}(n_0)$ as promised in Remark 8.4. Recall we define the functor

$$\Pi^0 : \mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}_0^c(\mathrm{Hk}_{\mathcal{I}}) \tag{10.12}$$

in §8. By definition, it factors through the anti-spherical category and we will denote by Π_{as}^0 the resulting functor $\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$. Theorem 9.1 and Corollary 9.2 show that

$$\mathcal{F}^{\mathrm{as}} := \mathcal{D}^b(\Pi_{\mathrm{as}}^0) \circ \mathcal{F} : \mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) \to \mathcal{D}^b(\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}})) \tag{10.13}$$

is an equivalence for type A groups G.

Proposition 10.7. *Assume G is of type A. Then, there exists a unique t-exact equivalence of ∞-categories*

$$\mathcal{F}^{\mathrm{r}} : \mathrm{Perf}([\hat{G}\backslash\mathcal{O}_{\mathrm{r}}]) \to \mathcal{D}^b(\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})), \tag{10.14}$$

fitting into the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) & \xrightarrow{\mathcal{F}^{\mathrm{as}}} & \mathcal{D}^b(\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}})) \\
\downarrow & & \downarrow {\scriptstyle \mathcal{D}^b(\Pi_{\mathrm{as}}^0)} \\
\mathrm{Perf}([\hat{G}\backslash\mathcal{O}_{\mathrm{r}}]) & \xrightarrow{\mathcal{F}^{\mathrm{r}}} & \mathcal{D}^b(\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})),
\end{array}
\tag{10.15}
$$

where the left vertical arrow is induced by restriction.

Proof. The proof follows from the idea of [AR, Proposition 7.2.6]. We sketch the proof here and refer to *loc.cit* for details. We first observe that $\mathcal{D}^b(\Pi_{\mathrm{as}}^0) \circ \mathcal{F}^{\mathrm{as}}(L_\nu^{\mathrm{ex}}) = \Pi^0(\mathrm{IC}_{w_\nu})$ by Lemma 10.3 and Corollary 9.2. Then it follows from Lemma 10.6 that $\mathcal{D}^b(\Pi_{\mathrm{as}}^0) \circ \mathcal{F}^{\mathrm{as}}$ restricts to zero on $\mathrm{Perf}(\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}})_{\mathrm{nr}}$. Note that $\mathrm{Perf}(\hat{G}\backslash\mathcal{O}_{\mathrm{r}})$ is the quotient in Cat_∞ of $\mathrm{Perf}(\hat{\mathcal{N}}_{\mathrm{Spr}})$ by the non-regular full subcategory (idempotent completions are not necessary as the Springer variety and the regular orbit are smooth). On the other hand, as noticed before this proposition, the functor

$$\mathcal{D}^b(\Pi_{\mathrm{as}}^0) : \mathcal{D}^b(\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}})) \to \mathcal{D}^b(\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})) \tag{10.16}$$

is a quotient map in Cat_∞ with kernel given by the full subcategory generated by $\Pi^{\mathrm{as}}(\mathrm{IC}_w)$ with $\ell(w) > 0$. Thus $\mathcal{F}^{\mathrm{r}}$ is an equivalence.

By Corollary 10.2, the restriction functor $\mathcal{F}^{\mathrm{as}} : \mathrm{Perf}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) \to \mathcal{D}^b(\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}}))$ is t-exact with respect to the exotic t-structure on the source and the tautological t-structure on the target. Then, to prove $\mathcal{F}^{\mathrm{r}}$ is t-exact, it suffices to show that every simple object in $\mathrm{Coh}([\hat{G}\backslash\mathcal{O}_{\mathrm{r}}])$ is the restriction of an exotic coherent sheaf. The verification of the later assertion can be argued entirely on the coherent side as in [AR, Proposition 7.2.6]. $\qquad\square$

Proposition 10.8. *With notations in Proposition 8.3, we have* $\mathcal{P}_0^c(\mathrm{Hk}_{\mathcal{I}}) = \mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$, *and* $H = Z_{\hat{G}}(n_0)$.

Proof. The proof is completely analogous to [AR, Proposition 7.2.8] by our previous preparations. By construction, the projective objects in $\mathrm{Coh}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ map to $\mathcal{P}_0^c(\mathrm{Hk}_{\mathcal{I}})$ under $\Pi_{\mathrm{as}}^0 \circ \mathcal{F}^{\mathrm{as}}$. Also every coherent sheaf on the regular orbit is a quotient of a projective object in $\mathrm{Coh}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ by [AR, Lemma 7.2.9]. Then it follows from Proposition 10.7 that the first assertion holds. Recall the equivalence Φ^0 from Proposition 8.3 between $\mathrm{Rep}\,H$ and $\mathcal{P}_0(\mathrm{Hk}_{\mathcal{I}})$. On the coherent side, we have an equivalence $\Psi : \mathrm{Coh}(\hat{G}\backslash\mathcal{O}_{\mathrm{r}}) \simeq \mathrm{Rep}(Z_{\hat{G}}(n_0))$ induced by the isomorphism $\mathcal{O}_{\mathrm{r}} \simeq \hat{G}/Z_{\hat{G}}(n_0)$ by the definition of the regular orbit itself. The second statement follows by showing that $\mathrm{For}_H^{Z_{\hat{G}}(n_0)} \circ \Psi$ is equivalent to $\Phi^0 \circ F^{\mathrm{r}}$ and we refer to the end of the proof of [AR, Proposition 7.2.8] for details. $\qquad\square$

11. Equivariant coherent sheaves on the nilpotent cone

Recall the Springer resolution

$$p_{\mathrm{Spr}} : \hat{\mathcal{N}}_{\mathrm{Spr}} = \hat{G} \times^{\hat{B}} \mathrm{Lie}(\hat{U}) \to \hat{\mathcal{N}} \tag{11.1}$$

of the nilpotent cone of the dual group $\hat{G}$, defined over the coefficient field $\Lambda = \bar{\mathbb{Q}}_\ell$. In this section, we study the category $\mathrm{Coh}([\hat{G}\backslash\hat{\mathcal{N}}])$ by establishing a connection with a certain quotient of $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ and proving main results of [Bez09] in the mixed-characteristic setting for type A groups.

Let $\mathcal{P}_{\mathrm{bas}}(\mathrm{Hk}_{\mathcal{I}})$ denote the quotient of $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})$ by the Serre subcategory spanned by the IC sheaves of $\mathrm{Fl}_{\mathcal{I},\leq w}$ for non-minimal w in its W_{fin}-double coset. Recall the anti-spherical category $\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}})$ in Theorem 6.7. The natural functor $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}_{\mathrm{bas}}(\mathrm{Hk}_{\mathcal{I}})$ factors through the quotient

$$\Pi_{\mathrm{bas}}^{\mathrm{as}} : \mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}_{\mathrm{bas}}(\mathrm{Hk}_{\mathcal{I}}). \tag{11.2}$$

In the sequel, we will relate this category to equivariant coherent sheaves on the nilpotent cone.

Theorem 11.1. *Assume G is of type A. Then, there exists a unique equivalence of ∞-categories:*

$$\mathcal{F}_{\mathrm{bas}} : \mathcal{D}_{\mathrm{coh}}^b([\hat{G}\backslash\hat{\mathcal{N}}]) \to \mathcal{D}^b(\mathcal{P}_{\mathrm{bas}}(\mathrm{Hk}_{\mathcal{I}})), \tag{11.3}$$

making the following diagram commutes

$$\begin{array}{ccc}
\mathcal{D}_{\mathrm{coh}}^b([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) & \xrightarrow{\mathcal{F}_{\mathrm{as}}} & \mathcal{D}^b(\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}})) \\
\downarrow{\scriptstyle Rp_{\mathrm{Spr}*}} & & \downarrow{\scriptstyle \mathcal{D}^b(\Pi_{\mathrm{bas}}^{\mathrm{as}})} \\
\mathcal{D}_{\mathrm{coh}}^b([\hat{G}\backslash\hat{\mathcal{N}}]) & \xrightarrow{\mathcal{F}_{\mathrm{bas}}} & \mathcal{D}^b(\mathcal{P}_{\mathrm{bas}}(\mathrm{Hk}_{\mathcal{I}})),
\end{array} \tag{11.4}$$

where $\mathcal{F}_{\mathrm{as}}$ is the composition of $\mathcal{F}$ with the functor $\mathcal{D}^b(\mathcal{P}(\mathrm{Hk}_{\mathcal{I}})) \to \mathcal{D}^b(\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}}))$ induced by the quotient functor $\mathcal{P}(\mathrm{Hk}_{\mathcal{I}}) \to \mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}})$.

Proof. The proof follows the idea of [Bez09, Theorem 1] and [AR, Theorem 7.3.1] in equicharacteristic and we sketch it here. Recall that $\mathcal{D}_{\mathrm{coh}}^b([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ is the bounded derived category of its abelian heart for the exotic t-structure, see Lemma 10.3. Let $\mathcal{D}$ be the Verdier quotient

of $\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}])$ by the full subcategory spanned by the L^{ex}_ν with $\nu \notin \mathbb{X}^-_\bullet$ under cones and extensions. Then $Rp_{\mathrm{Spr}*}$ factors as the composition of the quotient $\Pi : \mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}_{\mathrm{Spr}}]) \to \mathcal{D}$ and a functor $\alpha : \mathcal{D} \to \mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\mathcal{N}])$ since $Rp_{\mathrm{Spr}*}(L^{\mathrm{ex}}_\nu) = 0$ for any $\nu \notin \mathbb{X}^-_\bullet$, compare with [Bez09, Lemma 1] and [AR, Lemma 7.3.3].

Similarly, let $\mathcal{D}'$ be the Verdier quotient of $\mathcal{D}^b(\mathcal{P}_{\mathrm{as}}(\mathrm{Hk}_{\mathcal{I}}))$ by the full subcategory spanned by the IC sheaves of the form IC_{w_ν} with $\nu \notin \mathbb{X}^-_\bullet$. Then $\mathcal{D}^b(\Pi^{\mathrm{as}}_{\mathrm{bas}})$ factors through $\mathcal{D}'$ via a functor $\alpha' : \mathcal{D}' \to \mathcal{D}^b(\mathcal{P}_{\mathrm{bas}}(\mathrm{Hk}_{\mathcal{I}}))$. We know by [Miy91, Theorem 3.2] that α' is an equivalence. The equivalence $\mathcal{F}_{\mathrm{as}}$ induces an equivalence $\mathcal{D} \simeq \mathcal{D}'$ by Corollary 9.2 and Lemma 10.3. Hence, it suffices to show α is an equivalence. The essential surjectivity follows from [Bez03, Lemma 7] and full faithfulness follows from the abstract [AR, Lemma 7.3.13] together with a few input calculations. □

In unpublished notes, Deligne introduces an analogue of the perverse t-structure [BBDG18] on the derived category of coherent sheaves on a Noetherian scheme with a dualizing complex. This t-structure has been studied and extended by Arinkin–Bezrukavnikov [AB10]. In this subsection, we compare the perverse t-structure on $\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\mathcal{N}])$ with the exotic t-structure transported from the equivalence of Theorem 11.1 and prove [Bez09, Theorem 2, Corollary 1] in our setting.

The following lemma is due to Bezrukavnikov [Bez03].

Lemma 11.2. *The perverse coherent t-structure corresponding to the perversity function $p(O) = \mathrm{codim}(O)/2$ is the unique t-structure which has all $p_{\mathrm{Spr}*}(\mathcal{O}(\nu))$ lie in its heart.*

Proof. This is [Bez03, Corollary 3]. □

Note that we have an exotic t-structure on $\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}])$ inherited from the derived category of the Springer variety in virtue of our realization of the former as a Verdier quotient of the latter in Corollary 11.3.

Corollary 11.3. *The exotic t-structure on $\mathcal{D}^b_{\mathrm{coh}}([\hat{G}\backslash\hat{\mathcal{N}}])$ identifies with the perverse coherent t-structure with perversity function $p(O) = \mathrm{codim}(O)/2$.*

Proof. The statement follows directly from Theorem 11.1 in light of Lemma 11.2. □

References

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