ON THE CONNECTEDNESS OF p-ADIC PERIOD DOMAINS.

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Abstract. We prove that all p-adic period domains (and their non-minuscule analogues) are geometrically connected. This answers a question of Hartl [Har13] and has interesting consequences to the geometry of Shimura and local Shimura varieties.

Contents

1. Introduction

Period domains and their geometric properties are recurring themes in analytic geometry when studying Shimura varieties and their p-adic analogues called local Shimura varieties. In the p-adic analytic case, their study starts with a question of Grothendieck [Gro71], and the relevant objects are defined to be the image of the Grothendieck–Messing period morphism, which are analytic open subsets of a generalized flag variety. These domains have been studied since then, for example in the works of Drinfeld [Dri76], Gross–Hopkins [HG94], Rapoport–Zink [RZ96], Hartl [Har08], Scholze–Weinstein [SW13, SW20], Chen–Fargues–Shen [CFS21] among many. The purpose of this article is to prove that p-adic period domains are connected. Our results generalize a key lemma in [Che14] and answers [Har13, Conjecture 6.5]. It is also a key ingredient for p-adic uniformization of Newton strata on Shimura varieties.

Let (G, b, μ) be a p-adic shtuka datum, i.e. G is a reductive group over $\mathbb{Q}_p$, b is an element of the Kottwitz set $B(G) = G(\breve{\mathbb{Q}}_p)/\mathrm{ad}_\varphi(G(\breve{\mathbb{Q}}_p))$ in the sense of [Kot85], and $\mu \in \mathrm{Hom}(\mathbb{G}_m, G_{\bar{\mathbb{Q}}_p})/\mathrm{ad}(G(\bar{\mathbb{Q}}_p))$ is a geometric conjugacy class of cocharacters. We also require that $b \in B(G, \mu)$. Let E over $\mathbb{Q}_p$ be the reflex field of μ, i.e. the finite field extension over which the conjugacy class of μ is defined. We let $\mathbb{C}_p$ denote a completed algebraic closure of

$\mathbb{Q}_p$ and we let $\breve{E} \subseteq \mathbb{C}_p$ denote the compositum of E and $\breve{\mathbb{Q}}_p$ in $\mathbb{C}_p$, let Γ denote the Galois group of $\mathbb{Q}_p$. Given a characteristic p perfectoid space S, one can construct functorially a G-bundle over the relative Fargues–Fontaine curve $X_{FF,S}$ which we denote by $\mathcal{E}_b$. Attached to (G, μ) one can define a spatial diamond $\mathrm{Gr}_{G,\mu}$ over $\mathrm{Spd}\, \breve{E}$ that parametrizes B^+_{dR}-lattices with G-structure bounded by μ in the Bruhat order [SW20, §§19-22]. Moreover, using Beauville–Laszlo descent one can identify $\mathrm{Gr}_{G,\mu}$ with the moduli space of G-bundle modifications of $\mathcal{E}_b$

$$\mathrm{Gr}_{G,\mu}(S) = \{(\mathcal{E}, f) \mid f : \mathcal{E} \dashrightarrow \mathcal{E}_b,\, \mathrm{rel}(f) \leq \mu\}/\cong$$

whose relative position is bounded by μ. This gives the Beauville–Laszlo uniformization map

$$\mathcal{BL}_b : \mathrm{Gr}_{G,\mu} \to \mathrm{Bun}_G,$$

where Bun_G denotes the small v-stack of G-bundles on the Fargues–Fontaine curve [FS21]. By [SW20, Corollary 22.5.1, Proposition 24.1.2], the b-admissible locus, $\mathrm{Gr}^b_{G,\mu} := \mathcal{BL}_b^{-1}(\mathrm{Bun}^1_G)$, is non-empty and open in $\mathrm{Gr}_{G,\mu}$[1].

When μ is minuscule and G is quasi-split we have an identification $\mathrm{Gr}_{G,\mu} = (G/P_\mu)^\diamondsuit$, where P_μ is the parabolic subgroup defined by μ. In this case, $\mathrm{Gr}_{G,\mu}$ is (the diamond attached to) a generalized flag variety. Moreover, we also have a formula:

$$\mathrm{Gr}^b_{G,\mu} = \pi_{\mathrm{GM}}(\mathcal{M}^\diamondsuit_{(G,b,\mu)})$$

Where $\mathcal{M}_{(G,b,\mu)}$ is the local Shimura variety attached to (G, b, μ) and π_{GM} is the Grothendieck–Messing period morphism [RV14, SW20]. Our main theorem is the following:

Theorem 1.1. *The map $\mathrm{Gr}^b_{G,\mu} \to \mathrm{Spd}\, \breve{E}$ has connected geometric fibers. Moreover, $\mathrm{Gr}^b_{G,\mu} \subset \mathrm{Gr}_{G,\mu}$ is geometrically dense as spaces over $\mathrm{Spd}\, \breve{E}$.*

Let us put Theorem 1.1 in context. In [Kis17] Kisin uses in an essential way the connected components of affine Deligne–Lusztig varieties (ADLV) to study integral models of Shimura varieties making significant progress towards the Langlands–Rapoport conjecture [LR87]. On the other hand, in [Che14] Chen uses the connected components of ADLV to derive her main results on connected components of local Shimura varieties (LSV). These two works motivated Chen, Kisin and Viehmann [CKV15] to compute in great generality the connected components of ADLV building on previous work of Viehmann [Vie08]. Since then, several authors have pushed the strategy of [CKV15] to compute connected components of ADLV deriving as corollaries interesting results on the geometry of integral models of Shimura varieties (see [Nie18, Theorem 1.1], [HZ20, Theorem 0.1], [Ham20, Theorem 1.1(3)], [Nie21, Theorem 0.2]).

Now, Chen proves and uses Theorem 1.1 for period domains that arise from Rapoport–Zink data as a key stepping stone to derive the main results

[1]We warn the reader that in some literature $\mathrm{Gr}^b_{G,\mu}$ denotes $\mathcal{BL}_1^{-1}(\mathrm{Bun}^b_G)$ instead.

in her work. This is where the connected components of ADLV enter in her argument. In [GLX22], the first author together with Lim and Xu show that Chen's reasoning can be reversed, and use Theorem 1.1 to compute the connected components of ADLV and the connected components of LSV ([GLX22, Theorem 1.14, Theorem 1.2.(3)]).

Let the notation be as in [GLX22], our work has the following corollary.

Corollary 1.2. *If (b, μ) is HN-irreducible, then the Kottwitz map is bijective*

$$\kappa_G : \pi_0(X_\mu^{\mathcal{K}_p}(b)) \to c_{b,\mu}\pi_1(G)_I^\varphi. \tag{1.1}$$

Proof. It follows from [GLX22, Theorem 1.2] and Theorem 1.1. □

Our work together with [GLX22] finishes the problem of computing connected components of ADLV in mixed characteristic.

Let us sketch the proof of our main theorem in the case where G is quasi-split. Fix a Borel $B \subseteq G$. When b is basic Theorem 1.1 can be proved directly, and it is a result of Hansen–Weinstein, so we assume that b is not basic. Let $P \subseteq G$ be the parabolic generated by B and the centralizer of ν_b.

To prove that a space is connected it suffices to prove that a dense subset of it is connected, this allows us to replace $\mathrm{Gr}_{G,\mu}$ by the dense open subset $L^+P \cdot \xi^\mu$. Now, by Beauville–Laszlo descent, $L^+P \cdot \xi^\mu$ gets identified with the space of modifications of $\mathcal{E}_b^P$, where $\mathcal{E}_b^P$ is the Harder–Narasimhan P-reduction of $\mathcal{E}_b$. Moreover, on this open subset we have a factorization:

$$\mathcal{BL}_b : L^+P \cdot \xi^\mu \xrightarrow{\mathcal{BL}_{P,b}} \mathrm{Bun}_P \to \mathrm{Bun}_G.$$

Recall the following general fact. Let X be a connected locally spatial diamond that is smooth and partially proper over $\mathrm{Spa}\,\mathbb{C}_p$. Suppose that X is endowed with a filtration by an open immersion $j : U \to X$ and complementary closed immersion $i : Z \to X$. For U to be connected, it suffices that $\dim(Z) < \dim(X)$ [Han21, Corollary 4.11].

In our case $X = L^+P \cdot \xi^\mu$ and $U = L^+P \cdot \xi^\mu \cap \mathcal{BL}_b^{-1}(\mathrm{Bun}_G^1)$. An important observation is that the non-empty fibers of $\mathcal{BL}_{P,b}$ are $\mathrm{Aut}_{\mathrm{Fil}}(\mathcal{E}_p)$-torsors, see Lemma 3.3. In particular, they have the same dimension. Also, $\mathcal{BL}_{P,b}$ factors through one connected component $\mathrm{Bun}_P^\kappa \subseteq \mathrm{Bun}_P$ determined by $\mu - \nu_b$.

Let $Y = \mathrm{Bun}_P^\kappa \setminus \mathcal{BL}_b^{-1}(\mathrm{Bun}_G^1)$. The second key point is that $\dim(Y) < \dim(\mathrm{Bun}_P^\kappa)$. To prove this, we study the following diagram:

$$\begin{tikzcd} \mathrm{Bun}_P^{b_M} \arrow[r] \arrow[d] & \mathrm{Bun}_P \arrow[r] \arrow[d] & \mathrm{Bun}_G \\ \mathrm{Bun}_M^{b_M} \arrow[r] & \mathrm{Bun}_M & \end{tikzcd} \tag{1.2}$$

where M is the Levi quotient of P, $b_M \in B(M)$ and the square is Cartesian. When b_M is basic and ν_{b_M} is G-dominant, $\mathrm{Bun}_P^{b_M} \to \mathrm{Bun}_G$ is smooth and dimensions are easy to understand. On the other hand, the case when b_M is basic and ν_{b_M} is a non-negative sum of positive coroots, i.e. $\nu_{b_M} \in \mathbb{Q}_{\geq 0}\Phi_G^+$, can be understood inductively from the case where ν_{b_M} is G-dominant. This is where $b \in B(G, \mu)$ is important. Indeed, in this case $\mu^\diamond - \nu_b$ is in $\mathbb{Q}_{\geq 0}\Phi_G^+$ and the relevant $b_M \in B(M)_{\text{basic}}$ satisfies that ν_{b_M} is also in $\mathbb{Q}_{\geq 0}\Phi_G^+$.

We now explain the organization of this article. We start §2 with some cohomological considerations that allow us to work with the notion of dimension in a meaningful way. Then, we make some preparations explaining the combinatorics involving the induction process that reduces the $\nu_{b_M} \in \mathbb{Q}_{\geq 0}\Phi_G^+$ case to the G-dominant case. Afterwards, we bound dimensions of Newton strata that arise from the diagram 1.2. Finally, §3 is dedicated to proving Theorem 1.1.

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2. Bounding dimensions of Newton strata.

2.1. **Dimension for stacky maps.** In the following sections we bound the dimensions of certain Artin v-stacks. Since we do not intend to develop foundations, we will work with an ad-hoc notion of dimension. Let $f : X \to Y$ be a *fine* morphism of Artin v-stacks [GHW22, Definition 1.3] and let $n \in \mathbb{N}$. Let $S \to Y$ be a map with S a spatial diamond, let $f_S : X_S \to S$ denote the base change, and let $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(X_S, \mathbb{F}_\ell)$.

Definition 2.1. We say that the *ℓ-cohomological dimension of f is bounded by n*, which we abbreviate as $\dim_\ell(f) \leq n$ if: for all $S \to Y$ and $\mathcal{F}$ as above:

$$f_{S,!}\mathcal{F} \in D^{\leq 2n}_{\acute{e}t}(S, \mathbb{F}_\ell), \tag{2.1}$$

and we write $\dim_\ell(X) \leq n$ when $Y = *$.

Convention 2.2. From now on we will only consider maps of Artin v-stacks that are fine and we will not include this adjective in our statements.

Actually, the stacky morphisms used in this article are all obtained as compositions of smooth maps and locally closed immersions which are all fine morphisms.

Lemma 2.3. *Let $f : X \to Y$ and $g : Y \to Z$ be map of Artin v-stacks such that $\dim_\ell(f) \leq n$ and $\dim_\ell(g) \leq m$. Then $\dim_\ell(g \circ f) \leq m + n$.*

Proof. Let $S \to Z$ be a map and denote by X_S and Y_S the base changes. Let $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(X_S, \mathbb{F}_\ell)$. Observe that $f_{S,!}\mathcal{F}[2n] \in D^{\leq 0}_{\text{ét}}(Y_S, \mathbb{F}_\ell)$, which implies that $g_{!,S} f_{S,!}\mathcal{F}[2n] \in D^{\leq 2m}(S, \mathbb{F}_\ell)$. It follows that $\dim_\ell(g \circ f) \leq n + m$. □

Lemma 2.4. *Let $f : X \to Y$ be a map of Artin v-stacks. Suppose that for any $s : \mathrm{Spa}(C, C^+) \to X$ the fibers satisfy $\dim_\ell(X_s) \leq n$. Then $\dim_\ell(f) \leq n$.*

Proof. This follows from [Sch17, Theorem 1.9.(2)], [GHW22, Theorem 1.4.(4)], since $\mathcal{F} \in D^{\leq 2n}_{\text{ét}}(S, \mathbb{F}_\ell)$ can be checked on geometric point. □

Lemma 2.5. *Let $f : X \to Y$ be a surjective ℓ-cohomologically smooth map of Artin v-stacks with constant ℓ-dimension d. Let $g : Y \to Z$ be a map of Artin v-stacks. Then $\dim_\ell(g) \leq n$ if and only if $\dim_\ell(g \circ f) \leq n + d$.*

Proof. To bound $\dim_\ell(g \circ f)$ it suffices by Lemma 2.3 to prove $\dim_\ell(f) \leq d$. It suffices to prove that $\mathrm{RHom}(f_{S,!}\mathcal{F}, \mathcal{G}) = 0$ for every map $S \to Y$, every object $\mathcal{G} \in D^{\geq 2d+1}_{\text{ét}}(S, \mathbb{F}_\ell)$ and every object $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(X, \mathbb{F}_\ell)$. By adjunction, we may prove $\mathrm{RHom}(\mathcal{F}, f^!_S\mathcal{G}) = 0$ instead. Now, by ℓ-cohomological smoothness $f^!\mathcal{G} = f^*\mathcal{G} \otimes f^!\mathbb{F}_\ell$ and $f^!\mathbb{F}_\ell$ is an invertible object in $D_{\text{ét}}(X, \mathbb{F}_\ell)$ concentrated in degree $-2d$. In particular, $f^!_S\mathcal{G} \in D^{\geq 1}_{\text{ét}}(X, \mathbb{F}_\ell)$ while $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(X_S, \mathbb{F}_\ell)$.

To prove $\dim_\ell(g) \leq n$, let $S \to Z$ a map with S a spatial diamond, let $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(Y_S, \mathbb{F}_\ell)$ and let $\mathcal{G} \in D^{\geq 2n+1}_{\text{ét}}(S, \mathbb{F}_\ell)$. As above, it suffices to prove:

$$\mathrm{RHom}(\mathcal{F}, g^!_S\mathcal{G}) = 0 \tag{2.2}$$

In other words, we wish to prove that $g^!_S\mathcal{G} \in D^{\geq 1}_{\text{ét}}(Y_S, \mathbb{F}_\ell)$, for all $\mathcal{G} \in D^{\geq 2n+1}_{\text{ét}}(S, \mathbb{F}_\ell)$. This can be verified on geometric points so we may show

$$f^*_S g^!_S\mathcal{G} \in D^{\geq 1}_{\text{ét}}(X_S, \mathbb{F}_\ell) \tag{2.3}$$

instead, since f_S is surjective. By smoothness, $f^!_S\mathbb{F}_\ell \in D^{-2d}_{\text{ét}}(X_S, \mathbb{F}_\ell)$ is an invertible object and $f^*_S g^!_S\mathcal{G} = f^!_S g^!_S\mathcal{G} \otimes (f^!_S\mathbb{F}_\ell)^{-1}$. Since, by assumption $f^!_S g^!_S\mathcal{G} \in D^{\geq 1-2d}_{\text{ét}}(X_S, \mathbb{F}_\ell)$, we can verify that 2.3 holds. □

Lemma 2.6. *Let $f : X \to Y$ be a map of Artin v-stacks. Let $i : Z \to X$ be a closed immersion and let $j : U \to X$ denote the complementary open immersion. Suppose that $\dim_\ell(i \circ f) \leq n$ and that $\dim_\ell(j \circ f) \leq n$, then $\dim_\ell(f) \leq n$. Conversely if $\dim_\ell(f) \leq n$ then $\dim_\ell(i \circ f) \leq n$ and $\dim_\ell(j \circ f) \leq n$.*

Proof. Notice that the fibers of j and i are 0-dimensional. By Lemma 2.3 the second claim follows. For the first claim, let $\mathcal{F} \in D^{\leq 0}_{\text{ét}}(X, \mathbb{F}_\ell)$, and consider the following distinguished triangle:

$$f_! j_! j^*\mathcal{F} \to f_!\mathcal{F} \to f_! i_* i^*\mathcal{F} \to f_! j_! j^*\mathcal{F}[1] \tag{2.4}$$

We may pass to geometric fibers, where one of the terms vanish. □

2.2. **Averages of coweights.** Let G be a quasi-split reductive group over $\mathbb{Q}_p$ and let $T \subset B \subset G$ be a pair consisting of a maximal torus that is maximally $\mathbb{Q}_p$-split and a Borel both defined over $\mathbb{Q}_p$. Let Φ_G be the absolute root system of G with respect to T and Δ_G the basis of positive simple absolute roots with respect to B. We let $X_*(T)$ denote the set of geometric cocharacters and denote by $X_*(T)_{\mathbb{Q}}$ and $X_*(T)_{\mathbb{R}}$ the resulting rational vector space. We use the symbol M to denote a standard Levi of G defined over $\mathbb{Q}_p$, and by Δ_M the induced base of positive simple roots.

Definition 2.7. We say that $\nu \in X_*(T)_{\mathbb{Q}}$ is *M-dominant* (resp. *M-central*) if $\langle \alpha, \nu \rangle \geq 0$ (resp. $\langle \alpha, \nu \rangle = 0$) for all $\alpha \in \Delta_M$ and denote by $X_*(T)_{\mathbb{Q}}^{+_M}$ the convex set of M-dominant vectors in $X_*(T)_{\mathbb{Q}}$.

Following [Sch22], we now define the so called *M-average* of ν:

$$\mathrm{av}_M(\nu) = \frac{1}{|W_M|} \sum_{w \in W_M} w\nu \tag{2.5}$$

where W_M denotes the absolute Weyl group of M.

Lemma 2.8. *The M-average $\mathrm{av}_M(\nu)$ is the unique M-central $\mu \in X_*(T)_{\mathbb{Q}}$ whose difference $\mu - \nu$ is spanned by $\Delta_M^\vee$.*

Proof. Notice that $\mathrm{av}_M(\nu)$ is W_M-invariant by definition. Also, a vector is W_M-invariant if and only if it is M-central. □

It also follows that $\langle 2\rho_G - 2\rho_M, \nu \rangle = \langle 2\rho_G - 2\rho_M, \mathrm{av}_M(\nu) \rangle$. We study how averaging interacts with the notion of positivity presented below.

Definition 2.9. We say that $\nu \in X_*(T)_{\mathbb{Q}}$ is *non-negative* if it belongs to the convex hull of $X_*(Z_G)_{\mathbb{Q}}$ and $\mathbb{Q}_{\geq 0}\alpha^\vee$, where Z_G is the center of G and α runs over Δ_G. The convex set of non-negative vectors is denoted by $X_*(T)_{\mathbb{Q}}^{\geq 0}$.

Our definition above corresponds to the inequality $\nu_{\mathrm{ad}} \geq 0$ in the usual Bruhat order of $X_*(T_{\mathrm{ad}})$, where T_{ad} denotes the image of T in the adjoint group G_{ad} of G. A dominant vector is necessarily non-negative, but the converse rarely ever holds. In the following, we note that averaging preserves non-negativity, compare with [Sch22, Lemma 3.1].

Proposition 2.10. *The function av_M preserves $X_*(T)_{\mathbb{R}}^{\geq 0}$.*

Proof. It suffices to see that it preserves $X_*(Z_G)_{\mathbb{Q}}$ and $\mathbb{Q}_{\geq 0}\alpha^\vee$. This is clear for M-central coweights, so it suffices to consider $\mathrm{av}_M(\alpha^\vee)$ for $\alpha \in \Delta_G \setminus \Delta_M$. But then $w\alpha^\vee$ is a positive coroot for all $w \in W_M$, thereby finishing the proof. □

Remark 2.11. If $G = \mathrm{GL}_n$, we may interpret ν as a polygon and its non-negativity as meaning the polygon never crosses the straight line connecting its extremities. The vector $\mathrm{av}_M(\nu)$ corresponds to connecting vertices according to a partition of n. In this case, it is visually clear that this partial average polygon lies above the total average polygon, since we started with a non-negative one.

As a corollary, we get the following technical result that is relevant in the next subsection:

Lemma 2.12. *Let $\nu \in X_*(T)^{\geq 0}_{\mathbb{Q}}$ be invariant under Γ and M-central. There is a sequence of standard Levi subgroups $M = M_0 \subset \cdots \subset M_i \subset \cdots \subset M_k = G$ defined over $\mathbb{Q}_p$ and also of Γ-invariant vectors $\nu = \nu_0, \dots, \nu_i, \dots \nu_k = \mathrm{av}_G(\nu)$ in $X_*(T)^{\geq 0}_{\mathbb{Q}}$ such that the following properties hold*

(1) $\nu_j = \mathrm{av}_{M_j}(\nu_i)$ *for* $j \geq i$.
(2) ν_i *is* M_{i+1}*-dominant.*

Proof. Suppose $\langle \alpha, \nu \rangle \leq 0$ for all $\alpha \in \Delta_G \setminus \Delta_M$. Since $\langle \alpha, \nu \rangle = 0$ for $\alpha \in \Delta_M$ by hypothesis, we also get $\langle \rho_G, \nu \rangle \leq 0$. On the other hand, the convex hull of $X_*(Z_G)_{\mathbb{Q}}$ and $\mathbb{Q}_{\geq 0}\alpha^\vee$ for all $\alpha \in \Delta_G$ pairs non-negatively with the strictly dominant weight ρ_G, and it vanishes exactly on G-central elements. Therefore, the only possibility would be $M = G$, in which case $k = 0$.

Otherwise, there exists some $\alpha \in \Delta_G \setminus \Delta_M$ such that $\langle \alpha, \nu \rangle > 0$. By Γ-invariance, this holds for its entire Γ-orbit. Now let L be the standard Levi defined over $\mathbb{Q}_p$ with $\Delta_L = \Delta_M \cup \Gamma\alpha$ and consider $\mathrm{av}_L(\nu)$. By Proposition 2.10 $\mathrm{av}_L(\nu)$ is non-negative and L-central, which finishes the proof of the lemma by induction on the cardinality of $\Delta_G \setminus \Delta_M$. $\square$

2.3. **Newton strata in Bun_P.** We let $\mathbb{M}$ denote the set of standard Levi subgroups of G containing T. Let $B(\mathbb{M})$ denote the set of pairs $\{(M, b_M)\}$ where $M \in \mathbb{M}$ and $b_M \in B(M)$. For all $M \in \mathbb{M}$, we have $B(M) \subset B(\mathbb{M})$.

Fix $b \in B(\mathbb{M})$ with $b = (M, b_M)$ and let $P = MB$ denote the standard parabolic containing B and with standard Levi M. We let $\nu_b \in (X_*(T) \otimes \mathbb{Q})^\Gamma$ denote the M-dominant Newton point of b_M. For $M \subseteq L$ we let $P_L := P \cap L$ and define $\mathrm{Bun}^b_{P_L}$ by the following diagram with Cartesian square:

$$\begin{tikzcd} \mathrm{Bun}^b_{P_L} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_L} \arrow[r] \arrow[d] & \mathrm{Bun}_L \\ \mathrm{Bun}^{b_M}_M \arrow[r] & \mathrm{Bun}_M & \end{tikzcd} \tag{2.6}$$

Theorem 2.13 (Hamann)**.** *The map of Artin v-stacks $\mathrm{Bun}_{P_L} \to \mathrm{Bun}_M$ is ℓ-cohomologically smooth. In particular, $\mathrm{Bun}^b_{P_L} \to \mathrm{Bun}^{b_M}_M$ is ℓ-cohomologically smooth. Moreover, the later map is of relative ℓ-dimension $\langle 2\rho_L - 2\rho_M, \nu_b \rangle$.*

Proof. This follows from [Ham22, Proposition 3.16, Proposition 4.7]. $\square$

When $M \subseteq L$ we let $i_L(b)$ denote the pair (L, b_M) and $a_L(b)$ denote the pair (L, b_L) where b_L is the unique basic element in $B(L)$ with the same image image under the Kottwitz map, i.e. with $\kappa_L(b_L) = \kappa_L(b_M)$. One verifies that $\mathrm{av}_L(\nu_b) = \nu_{a_L(b)}$, and consequently:

$$\langle 2\rho_L - 2\rho_M, \nu_b \rangle = \langle 2\rho_L - 2\rho_M, \nu_{a_L(b)} \rangle.$$

Definition 2.14. We say that $b \in B(\mathbb{M})$ is *basic* if $b_M \in B(M)$ is basic. We say that $b \in B(\mathbb{M})$ is *dominant* if ν_b is G-dominant. We say that $b \in B(\mathbb{M})$ is *non-negative* if ν_b lies in the monoid generated by $X_*(Z_G)^\Gamma_\mathbb{Q}$ and $\mathbb{Q}_{\geq 0}\alpha^\vee$ for every $\alpha \in \Delta_G$.

Notice that if b is basic and dominant then $\mathrm{Bun}^b_P = \mathcal{M}_b$, the Fargues–Scholze chart attached to $i_G(b) \in B(G)$ [FS21, §V.3]. Also, if b is basic and anti-dominant $\mathrm{Bun}^b_P \to \mathrm{Bun}_G$ induces an isomorphism $\mathrm{Bun}^b_P \cong \mathrm{Bun}^b_G$.

Let $g \in B(\mathbb{M})$ with $g = (L_g, g_L)$ and $M \subseteq L_g$. Let $L_g \subseteq L$ and let

$$\mathrm{Bun}^{(b,g)}_{P_L} := \Delta^{-1}_{M,L_g}(\mathrm{Bun}^{b_M}_M \times \mathrm{Bun}^{g_L}_{L_g}) \subseteq \mathrm{Bun}_{P_L} \tag{2.7}$$

Here $\Delta_{M,L_g} : \mathrm{Bun}_{P_L} \to \mathrm{Bun}_M \times \mathrm{Bun}_{L_g}$ is the composition of $\mathrm{Bun}_{P_L} \to \mathrm{Bun}_{P_{L_g}}$ and the diagonal map $\mathrm{Bun}_{P_{L_g}} \to \mathrm{Bun}_M \times \mathrm{Bun}_{L_g}$.

Proposition 2.15. *If $b \in B(\mathbb{M})$ is basic and non-negative, then Bun^b_P contains an open subspace $\mathcal{T}_b \subset \mathrm{Bun}^b_P$ such that $f_b : \mathcal{T}_b \to \mathrm{Bun}_G$ is ℓ-cohomologically smooth of relative dimension $\langle 2\rho_G - 2\rho_M, \nu_b\rangle$. Moreover, f_b factors through $\mathrm{Bun}^{a_G(b)}_G$ and $\dim_\ell(\mathrm{Bun}^b_P \setminus \mathcal{T}_b) < \langle 2\rho_G - 2\rho_M, \nu_b\rangle$.*

Proof. We do this by induction on the cardinality of $\Delta_G \setminus \Delta_M$. If b is basic and dominant then $\mathrm{Bun}^b_P \to \mathrm{Bun}_G$ is smooth by [FS21, Theorem V.3.7] and with notation as in eq. (2.7) $\mathcal{T}_b = \mathrm{Bun}^{(b,a_G(b))}_P$ satisfies the desired properties. We may choose $L = M_1$ as in the statement of Lemma 2.12. We let $Q \subset G$ denote the parabolic generated by L and B and we let $P_L = L \cap P$. We have the following commutative diagram with Cartesian squares:

$$\begin{tikzcd}
\mathrm{Bun}^b_P \arrow[r] \arrow[d] & \mathrm{Bun}_P \arrow[r] \arrow[d] & \mathrm{Bun}_Q \arrow[r] \arrow[d] & \mathrm{Bun}_G \\
\mathrm{Bun}^b_{P_L} \arrow[r] \arrow[d] & \mathrm{Bun}_{P_L} \arrow[r] \arrow[d] & \mathrm{Bun}_L & \\
\mathrm{Bun}^{b_M}_M \arrow[r] & \mathrm{Bun}_M & &
\end{tikzcd} \tag{2.8}$$

After pullback by $\mathrm{Bun}^{b_L}_L \to \mathrm{Bun}_L$, and by induction, we get a commutative diagram in which $\mathcal{T}_b$ is defined so that all squares are Cartesian:

$$\begin{tikzcd}
\mathcal{T}_b \arrow[r] \arrow[d] & \mathrm{Bun}^{(b,a_L(b))}_P \arrow[r] \arrow[d] & \mathrm{Bun}^b_P \arrow[r] \arrow[d] & \mathrm{Bun}^b_{P_L} \arrow[d] \\
\mathcal{T}_{a_L(b)} \arrow[r] & \mathrm{Bun}^{a_L(b)}_Q \arrow[r] & \mathrm{Bun}_Q \arrow[r] & \mathrm{Bun}_L
\end{tikzcd} \tag{2.9}$$

By induction, the map $\mathcal{T}_{a_L(b)} \to \mathrm{Bun}_G$ is ℓ-cohomologically smooth and $\mathcal{T}_b \to \mathcal{T}_{a_L(b)}$ is also ℓ-cohomologically smooth, so the same holds for their composition. The claim on dimensions follow since $\mathrm{Bun}_G \to *$ is ℓ-smooth of dimension 0 and $\mathrm{Bun}^b_P \to *$ is ℓ-smooth of dimension $\langle 2\rho_G - 2\rho_M, \nu_b\rangle$.

For the second claim, let $g \in B(L)$ be in the image of $\mathrm{Bun}_{P_L}^b$. We get a smooth map $\mathrm{Bun}_P^{(b,g)} \to \mathrm{Bun}_Q^g$ of ℓ-dimension $\langle 2\rho_L - 2\rho_M, \nu_b\rangle$. By Theorem 2.13, the map $\mathrm{Bun}_Q^g \to \mathrm{Bun}_L^g$ is smooth and it has ℓ-dimension $\langle 2\rho_G - 2\rho_L, \nu_g\rangle$. Now, $\langle 2\rho_G - 2\rho_L, \nu_g\rangle = \langle 2\rho_G - 2\rho_L, \nu_b\rangle$, since $\kappa_L(b) = \kappa_L(g)$. In particular, $\mathrm{Bun}_P^{(b,g)} \to \mathrm{Bun}_Q^g \to \mathrm{Bun}_L^g$ is smooth of relative dimension $\langle 2\rho_G - 2\rho_M, \nu_b\rangle$. Now, when $g \neq a_L(b)$, $\dim_\ell(\mathrm{Bun}_L^g) < 0$, so that by Lemma 2.3 and Lemma 2.6 $\dim_\ell(\mathrm{Bun}_P^{(b,g)}) < \langle 2\rho_G - 2\rho_M, \nu_b\rangle$ and $\dim_\ell(\mathrm{Bun}_P^b \setminus \mathrm{Bun}_P^{(b,a_L(b))}) < \langle 2\rho_G - 2\rho_M, \nu_b\rangle$. By induction, $\dim_\ell(\mathrm{Bun}_P^{(b,a_L(b))} \setminus \mathcal{T}_b) < \langle 2\rho_G - 2\rho_M, \nu_b\rangle$ since $\mathrm{Bun}_P^{(b,a_L(b))} \setminus \mathcal{T}_b \to \mathrm{Bun}_Q^{a_L(b)} \setminus \mathcal{T}_{a_L(b)}$ is smooth. By Lemma 2.6, $\dim_\ell(\mathrm{Bun}_P^b \setminus \mathcal{T}_b) < \langle 2\rho_G - 2\rho_M, \nu_b\rangle$, since we have an open and closed decomposition $\mathrm{Bun}_P^b \setminus \mathcal{T}_b = (\mathrm{Bun}_P^{(b,a_L(b))} \setminus \mathcal{T}_b) \cup (\mathrm{Bun}_P^b \setminus \mathrm{Bun}_P^{(b,a_L(b))})$. □

3. $\mathrm{Gr}_{G,\mu}^b$ IS CONNECTED

Contrary to the previous section we will momentarily not assume that G is quasi-split. Fix C an algebraically closed non-Archimedean field extension of $\breve{E}$ and recall the Beauville–Laszlo map from the introduction

$$\mathcal{BL}_b : \mathrm{Gr}_{G,\mu} \to \mathrm{Bun}_G, \tag{3.1}$$

where we base change the affine Grassmannian to $\mathrm{Spd}\, C$. Observe that $\mathcal{BL}_b$ factors through the unique connected component of Bun_G parametrized by $\mu^\natural - \kappa_G(b) \in \pi_1(G)_\Gamma$. We formulate Theorem 1.1 as follows:

Theorem 3.1. *If $b \in B(G,\mu)$, then $\mathrm{Gr}_{G,\mu}^b$ is dense in $\mathrm{Gr}_{G,\mu}$ and connected.*

Without loss of generality we may assume that G is adjoint. Moreover, we may replace G by its quasi-split inner form G^*, which is now a pure inner form by adjointness of G. In total, we may assume that G is quasi-split, at the expense of having to prove the more general Theorem 3.2 below.

Let us recall the setup. Let $T \subset B \subset G = G^*$ be as in the previous section. We define an element $\mu^\diamond \in X_*(T)_{\mathbb{Q}}^\Gamma$ given by the formula:

$$\mu^\diamond := \frac{1}{[\Gamma : \Gamma_\mu]} \sum_{\gamma \in \Gamma/\Gamma_\mu} \gamma(\mu), \tag{3.2}$$

where Γ_μ denotes the stabilizer of μ for the Γ-action. Notice that $\langle 2\rho_G, \mu^\diamond\rangle = \langle 2\rho_G, \mu\rangle$, because ρ_G is Γ-invariant.

Let $A_Z(G,\mu) \subset B(G)$ be the set of acceptable elements modulo center, i.e. for which $\mu^\diamond - \nu_b$ is non-negative as in Definition 2.14. This is related to the notion of acceptable elements $A(G,\mu)$ of [RV14, Definition 2.3], in the sense that $A_Z(G,\mu)$ equals the pre-image of $A(G_{\mathrm{ad}}, \mu_{\mathrm{ad}})$ along $B(G) \to B(G_{\mathrm{ad}})$.

If $b \in B(M)$, we let $\mathrm{d}_{\mu,b}^M$ denote the unique basic element in $B(M)$ such that $\kappa_M(\mathrm{d}_{\mu,b}^M) = \mu^\natural - \kappa_M(b)$. When $M = G$ we simply write b_μ for $\mathrm{d}_{\mu,b}^G$. Let

$d = \dim_\ell(\mathrm{Gr}_{G,\mu}) = \langle 2\rho_G, \mu\rangle$ and let $\mathrm{Gr}^{(g,b)}_{G,\mu} := \mathcal{BL}_b^{-1}(\mathrm{Bun}^g_G) \subset \mathrm{Gr}_{G,\mu}$. For example, $\mathrm{Gr}^b_{G,\mu} = \mathrm{Gr}^{(1,b)}_{G,\mu}$.

Theorem 3.2. *If $b \in A_Z(G,\mu)$, then $\mathrm{Gr}^{(b_\mu,b)}_{G,\mu}$ is dense in $\mathrm{Gr}_{G,\mu}$ and connected.*

Proof. To prove that $\mathrm{Gr}^{(b_\mu,b)}_{\mu}$ is dense and connected, it suffices to prove that $\dim_\ell(\mathrm{Gr}^{(g,b)}_{\mu}) < d$ for all $g \in B(G)$ with $g \neq b_\mu$. We consider the Schubert cell $\mathrm{Gr}^\circ_{G,\mu} \subset \mathrm{Gr}_{G,\mu}$. Since $\dim_\ell(\mathrm{Gr}_{G,\mu} \setminus \mathrm{Gr}^\circ_{G,\mu}) < d$ it suffices to prove that $\dim_\ell(\mathrm{Gr}^{(g,b)}_{G,\mu} \cap \mathrm{Gr}^\circ_{G,\mu}) < d$. If b is basic, $\mathcal{BL}_b : \mathrm{Gr}^\circ_{G,\mu} \to \mathrm{Bun}_G$ is smooth of relative dimension d [FS21]. In particular, $\dim_\ell(\mathrm{Gr}^{\circ,(g,b)}_{G,\mu}) = d + \dim_\ell(\mathrm{Bun}^g_G)$. Now, b_μ is the unique basic element in the image of $\mathcal{BL}_b$ and for non-basic elements $\dim_\ell(\mathrm{Bun}^g_G) < 0$. This finishes the proof in this case.

Suppose now that b is not basic, let M denote the centralizer of ν_b, let b_M denote the unique element in $B(M)$ mapping to b whose Newton point is G-antidominant. Now, $\mathrm{Bun}^{b_M}_P \cong \mathrm{Bun}^b_G$ by our choice of b_M, and we let $\mathcal{E}^P_b$ denote the unique P-reduction of $\mathcal{E}_b$ determined by the image of $\mathrm{Bun}^{b_M}_P$ in Bun_P. The space of modifications of $\mathcal{E}^P_b$ gets identified with $\mathrm{Gr}_P \subset \mathrm{Gr}_G$. We consider $\mathrm{Gr}^\circ_{P,\mu} := L^+P \cdot \xi^\mu$, the result of intersecting $\mathrm{Gr}_{G,\mu}$ with the connected component of Gr_P attached to the dominant representative μ. We have a smooth map $\mathrm{Gr}^\circ_{P,\mu} \to \mathrm{Gr}^\circ_{M,\mu}$ of relative dimension $\langle 2\rho_G - 2\rho_M, \mu\rangle$. Moreover, we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Gr}^{\mathrm{d}^M_{\mu,b}}_{P,\mu} \arrow[r] \arrow[d] & \mathrm{Gr}^\circ_{P,\mu} \arrow[r] \arrow[d] & \mathrm{Gr}^\circ_\mu \arrow[d, "\mathcal{BL}_b"] \\
\mathrm{Bun}^{\mathrm{d}^M_{\mu,b}}_P \arrow[r] \arrow[d] & \mathrm{Bun}_P \arrow[r] \arrow[d] & \mathrm{Bun}_G \\
\mathrm{Bun}^{\mathrm{d}^M_{\mu,b}}_M \arrow[r] & \mathrm{Bun}_M &
\end{tikzcd} \tag{3.3}$$

Where $\mathrm{Gr}^{\mathrm{d}^M_{\mu,b}}_{P,\mu}$ is defined so that the square in the left-upper corner is Cartesian. In particular, the upper left arrow is an open immersion. Since $\mathrm{d}^M_{\mu,b} \in B(M)$ is basic, we know that

$$\dim_\ell(\mathrm{Gr}^\circ_{P,\mu} \setminus \mathrm{Gr}^{\mathrm{d}^M_{\mu,b}}_{P,\mu}) < d. \tag{3.4}$$

It suffices to prove that

$$\dim_\ell(\mathrm{Gr}^{\mathrm{d}^M_{\mu,b}}_{P,\mu} \cap \mathrm{Gr}^{(g,b)}_{G,\mu}) < d \tag{3.5}$$

for $g \neq b_\mu$. By Proposition 2.15,

$$\dim_\ell(\mathrm{Bun}^{(\mathrm{d}^M_{\mu,b},g)}_P) < \langle 2\rho_G - 2\rho_M, \nu_{\mathrm{d}^M_{\mu,b}}\rangle = \langle 2\rho_G - 2\rho_M, \mathrm{av}_M(\mu^\diamond - \nu_b)\rangle \tag{3.6}$$

By Lemma 3.3, the fibers of $\mathrm{Gr}_{P,\mu}^{\mathrm{d}^M_{\mu,b}} \to \mathrm{Bun}_P^{\mathrm{d}^M_{\mu,b}}$ have all dimension bounded by $\langle 2\rho_G - 2\rho_M, \nu_b\rangle$. Consequently by Lemma 2.4, we get that (3.5) holds. Indeed: $\langle 2\rho_G - 2\rho_M, \mathrm{av}_M(\mu^\diamond - \nu_b)\rangle = \langle 2\rho_G - 2\rho_M, \mu^\diamond - \nu_b\rangle$ and $\langle 2\rho_G - 2\rho_M, \mu^\diamond\rangle = \langle 2\rho_G - 2\rho_M, \mu\rangle$. □

Lemma 3.3. *The geometric fibers of* $\mathrm{Gr}_{P,\mu}^{\mathrm{d}^M_{\mu,b}} \to \mathrm{Bun}_P^{\mathrm{d}^M_{\mu,b}}$ *are either* $\mathrm{Aut}_{\mathrm{Fil}}(\mathcal{E}_b)$*-torsors or empty. Their dimension is* $\langle 2\rho_G - 2\rho_M, \nu_b\rangle$ *in the former case.*

Proof. It suffices to see that the Beauville–Laszlo map $\mathrm{Gr}^\circ_{P,\mu} \to \mathrm{Bun}_P$ has the same geometric fibers. But those are contained in the geometric fibers of $\mathrm{Gr}_P \to \mathrm{Bun}_P$, which are torsors on the left for the group $A^{-1}P(B_e)A$ where $A \in P(B_{\mathrm{dR}})$ is the Beauville–Laszlo glueing data for the P-torsor $\mathcal{E}_b^P$ [SW20, Theorem 13.5.3.(2)]. The stabilizer of the LP-action on ξ^μ is the product of L^+M and $L^{\geq\langle\alpha,\mu\rangle}U_\alpha$ for all absolute roots in $\Delta_G \setminus \Delta_M$, hence contained in L^+P as μ was assumed to be dominant. Therefore, if $p \in P(B_{\mathrm{dR}})$, $x \in \mathrm{Gr}^\circ_{P,\mu}$ are such that $p \cdot x \in \mathrm{Gr}^\circ_{P,\mu}$, then we conclude that necessarily $p \in P(B^+_{\mathrm{dR}})$. This implies that the non-empty geometric fibers of our map form a torsor under the group $P(B^+_{\mathrm{dR}}) \cap A^{-1}P(B_e)A = \mathrm{Aut}_{\mathrm{Fil}}(\mathcal{E}_b)$. By [FS21, Proposition III.5.1] $\dim_\ell(\mathrm{Aut}_{\mathrm{Fil}}(\mathcal{E}_b)) = \langle 2\rho_G - 2\rho_M, \nu_b\rangle$, and we may conclude the same about the non-empty fibers. □

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