LONDON NUMBER THEORY STUDY GROUP ON FARGUES-SCHOLZE'S GEOMETRISATION OF THE LOCAL LANGLANDS CORRESPONDENCE, PART I

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Introduction

The goal of this study group is to go through the first four chapters of the upcoming paper [5] of Fargues-Scholze on Fargues's geometric local Langlands program.

Let E be a local field whose residue field has characteristic p, and let G be a connected reductive group over E. Roughly speaking, the local Langlands correspondence predicts the existence of a relation between certain (infinite dimensional) representations of $G(E)$ - the automorphic side - and representations of the Weil group of E - the Galois side. The easiest example $G = GL_1$ boils down to local class field theory for E.

The discovery of the Fargues-Fontaine curve [1] paved the way for a new approach to the study of the local Langlands correspondence, using ideas from geometric Langlands. More precisely, the formalism of diamonds developed by Scholze [7] allows to put in family, in a suitable sense, the construction of Fargues and Fontaine, and to define the stack Bun_G of G-bundles over the Fargues-Fontaine curve. A key observation is that the automorphism group of the trivial G-bundle is not the algebraic group G, but the locally profinite group $G(E)$. This translates into the fact that the locus of geometrically fibrewise trivial G-bundles inside Bun_G is isomorphic to the classifying stack of $G(E)$, hence l-adic sheaves on it are related to the automorphic side of the local Langlands correspondence. Fargues's conjecture [2] gives a geometric enhancement of the correspondence, in which representations of $G(E)$ are replaced by (perverse) l-adic sheaves on Bun_G.

The aim of this study group is to define the stack Bun_G and study some of its geometric properties - let us remark that to state Fargues's conjecture one needs to make sense of $D_{et}(\mathrm{Bun}_G, \bar{\mathbb{Q}}_l)$, a highly non-trivial task in itself. We will start by introducing the Fargues-Fontaine curve, and we will devote the first part of the seminar to the proof of the classification theorem of vector bundles on it and to the study of families of vector bundles. In particular, we will learn the new proofs given in [5] of the classification theorem (first proved in [1]) and of the results of Kedlaya-Liu on the Harder-Narasimhan filtration in families [6]. We will then define the stack Bun_G, study it via the Beauville-Laszlo uniformisation and describe its Harder-Narasimhan strata. We shall subsequently discuss some topics in the geometry of diamonds which are needed for a deeper study of Bun_G: we will introduce universally locally acyclic sheaves and study several notions of smoothness in the perfectoid world. One of the main theorems we will prove states that Bun_G is a (cohomologically and formally) smooth Artin v-stack. Finally, we will study the category $D_{et}(\mathrm{Bun}_G; \Lambda)$ for a ring Λ killed by an integer prime to p.

References. We will mainly focus on [5, Chapters I-IV]. Other references with useful material on the Fargues-Fontaine curve and Fargues's conjecture include [4, 10, 11]. It may be worthwile to point out that the whole story is already interesting in the case $G = GL_1$; see [3].

Talks

Talk 1: Motivation and overview & Adic spaces, perfectoid spaces and diamonds. October 7th - Speaker: Matteo Tamiozzo

In the first part of the talk, we will review the local Langlands program, Fargues's geometrization conjecture and the ultimate results of Fargues-Scholze. We will also give an overview of the contents of each talk.

In the second part of the talk, we will recall some background material in p-adic geometry. Recall the notions of perfectoid space and the tilting equivalence, introduce diamonds and describe the diamond attached to an analytic adic space. Introduce the pro-étale topology and the v-topology, discuss totally disconnected spaces and state [7, Lemma 7.18]. Finally, define closed immersions, separated, proper and partially proper morphisms of v-sheaves. A good reference for this introductory material is [10, Chapter 2].

Talk 2: The Fargues-Fontaine curve. October 14th - Speaker: Chris Birkbeck

The aim of this talk is to introduce the Fargues-Fontaine curve as an adic space and as a diamond.

Define the space $\mathcal{Y}_S$, prove that it is an analytic adic space and describe the attached diamond [9, Proposition 11.2.1]; state [9, Proposition 11.3.1] and sketch the proof. Introduce the analogous object with $\mathcal{O}_E$ in place of $\mathbf{Z}_p$, define the FF curve and the space Div^1 [5, I.1.2] (see also [9, Figure 12.1]). Show that affinoid connected opens are PID (see [5, Cor. 1.1.10]). Introduce relative de Rham period rings as in [9, p. 138] and link them to completed local rings in the case of a point. Introduce the vector bundles $\mathcal{O}(\lambda)$ on the FF curve, explaining their relation to isocrystals, see [5, p. 13]. State the classification theorem [5, Theorem I.2.13]. Finally, define the algebraic FF curve as in before [5, Prop. 1.2.8] (see also [1, pp. 6-7] for motivation), prove that proposition and state GAGA.

Talk 3: Banach-Colmez spaces. October 21st - Speaker: Ashwin Iyengar

This talk introduces vector bundles on the Fargues-Fontaine curve and studies the associated Banach-Colmez spaces.

Define Banach-Colmez spaces, explain the relation between $\mathcal{O}(1)$ and Lubin-Tate formal groups and prove [5, Corollary I.2.4]. Finally state [5, Proposition I.2.5], describing Banach-Colmez spaces attached to the vector bundles $\mathcal{O}(\lambda)$ (ignoring cohomological smoothness). You can describe the proof in some of the easier cases (see also [9, Section 15.2]); it is important that you treat the case of $\mathrm{BC}(\mathcal{O}(-1)[1])$, see also [5, Ex. 1.3.12]. Finally, establish [5, Prop. 1.2.15] on projectivized BC spaces (admitting Theorem 1.2.6, to be discussed in the next talk).

Talk 4: Families of vector bundles. October 28th - Speaker: TBA

In this talk we finish the proof of classification of vector bundles on the FF curve and prove semi-continuity results for families.

State the ampleness result [5, Theorem I.2.6] and sketch the proof. Mention without proof that the theorem formally implies algebraisation and GAGA for the FF curve [5, Proposition I.2.7]. Then cover [5, Section I.2.4], describing the Harder-Narasimhan filtration and prove [5, Theorem I.2.13], classifying vector bundles on the FF curve; in the proof of [5, Lemma I.2.14] you can ignore the equal characteristic case. Finally prove the existence of the Harder-Narasimhan filtration in families and upper semicontinuity of the Harder-Narasimhan polygon [5, Theorem I.2.18].

Talk 5: Bun_G, the B^+_{dR}-affine Grassmannian and Beauville-Laszlo uniformisation. November 4th - Speaker: TBA

In this talk we define the stack of G-bundles on the FF curve, and prove that it is uniformised by the B^+_{dR}-affine Grassmannian; this uniformisation will be one of the key tools to study the geometry of Bun_G later on.

Define Bun_G and explain why it is a (small) v-stack [5, Section II.1]. Then introduce the B_{dR}^+-affine Grassmannian, Schubert cells [9, Sections 19.1, 19.2] and discuss [9, Proposition 19.2.3]; then state [9, Theorem 19.2.4, Corollary 19.3.4]. Define the Beauville-Laszlo morphism and prove [5, Proposition II.3.1], admitting Theorem II.2.4, Lemma II.2.6 (to be proved in the next talk). Discuss as much as time permits Fargues's theorem [4, Théorème 7.1], which is the key ingredient in the proof of the previous proposition. Finally, prove lemma II.3.5.

Talk 6: Stratas of Bun_G. NOVEMBER 11TH - SPEAKER: TBA

The aim of this talk is to describe Harder-Narasimhan strata on Bun_G.

Start by describing the topological space $|\,\mathrm{Bun}_G\,|$ as in [5, Section II.2]. Then describe the Harder-Narasimhan stratification for general G-bundles [5, Section II.2.2], and prove that the Kottwitz invariant is locally constant [5, Theorem II.2.7] (only discuss the first proof). Then describe the locus of fibrewise trivial bundles [5, II.2.3]. Determine the automorphism group in the semistable case as in [5, II.4] and then refine the treatement to more general non-semistable strata [5, II.5].

Talk 7: Six functor formalism for diamonds. NOVEMBER 18TH - SPEAKER: TBA

In this talk we are going to explain the main contributions of [7] to the étale cohomology of diamonds (or more generally v-stacks).

Start by giving an overview of the six functor formalism in the case of schemes, as well as of the main reduction arguments behind the constructions of [7]: passing to geometric points $\mathrm{Spa}(C, C^+)$ by localizing and taking limits; cohomology of the disk for base change theorems; and the adjoint functor theorem of ∞-categories. Discuss the embedding of categories $D(Y_{\mathrm{et}}) \subset D(Y_v)$ and the difference between the left side and $D_{\mathrm{et}}(Y)$, see [7, §14]. Mention some of the base change results of [7, §16]. Define the four (non-shriek) functors as in [7, §17]. Introduce the canonical compactification of [7, §18] and explain proper base change of [7, §19]. Discuss constructibility, dimension and finally define $Rf_!$, see [7, §§20-22]. Define at last $Rf^!$ and cohomological smoothness as in [7, §23] putting emphasis on the ball example of [7, §24]. Mention Verdier duality, see [7, §25]

Talk 8: Universally locally acyclic sheaves. NOVEMBER 25TH - SPEAKER: TBA

In this talk we are going to study ULA sheaves, which play a decisive role in verifying smoothness and geometric Satake later on.

Define ULA sheaves (see [5, Def. III.2.1]) and relate them to locally constant constructible sheaves (see [5, Props. III.2.6, III.2.10]). Show that ULA sheaves behave as expected along proper pushforward and smooth pullback, see [5, §III.2.2]. Show that ULA sheaves imply base change for and can be characterized in terms of Verdier duality (see [5, Prop. III.2.15, III.2.19, Cor. III.2.22]). If time permits, formulate ULA sheaves à la Lu-Zheng (see [5, III.2.3.3]). Prove [5, Prop. III.2.33] for locally spatial diamonds and [5, Cor. III.2.34].

Talk 9: Smoothness in the perfectoid world. DECEMBER 2ND - SPEAKER: TBA

This talk will discuss and relate the several notions of smoothness (formal, sous-perfectoid and cohomological) in the realm of v-stacks.

Define formally smooth morphisms of v-stacks, see [5, Def. III.3.1], give their basic properties and examples, and prove that Bun_G and $\mathrm{BC}([\mathcal{E}_1 \to \mathcal{E}_0])$ are formally smooth over the respective bases (see [5, Props. III.3.7, III.3.8]). Then define and go over the main features of smooth maps of sous-perfectoid spaces, see [5, §III.4.1], skipping most of the classical proofs. Define the space of sections $\mathcal{M}_Z$, state the Jacobian criterion [5, Thm. III.4.2] and explain the case $Z = \mathbf{P}^n$, see [5, Prop. III.4.20].

Talk 10: Bun_G as a cohomologically smooth Artin v-stack. DECEMBER 9TH - SPEAKER: TBA

In this talk, we will define Artin v-stacks, descend some of the previously seen notions and show that Bun_G is cohomologically smooth. Define Artin stacks, see [5, III.3.1], highlighting the various separatedness issues and quotients by group actions. Define cohomological smoothness for Artin v-stacks and explain how the notions of previous talks extend (see [5, III.1.1.2]). Use Beauville-Laszlo uniformization to show Bun_G is cohomologically smooth, see [5, Thm. III.1.18] and go through the arguments of the proof carefully.

Talk 11: M_Z^{sm} is cohomologically smooth. December 16th - Speaker: TBA

Here we prove the Jacobian criterion of [5, §III.4]. Recall its statement and prove formal smoothness of $\mathcal{M}_Z^{\mathrm{sm}}$, see [5, III.4.3]. Prove that the constant sheaf $\mathbf{F}_l$ is universally locally acyclic, summarizing approximation arguments, see [5, III.4.4], and finally show that $Rf^!\mathbf{F}_l$ is invertible as in [5, III.4.5].

We will try to fit two additional lectures towards the end of the seminar, which use the material discussed so far to describe $D_{\mathrm{et}}(\mathrm{Bun}_G)$.

Talk 12: $D_{\mathrm{et}}(\mathrm{Bun}_G)$ and local charts.

Talk 13: Dictionary between $D_{\mathrm{et}}(\mathrm{Bun}_G)$ and smooth representations.

References

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