SEMINAR PROGRAM, SUMMER 2023 “HOMOLOGICAL AND FROBENIUS METHODS IN COMMUTATIVE ALGEBRA”

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Abstract. As a disclaimer, the program that follows is still a draft and was written in a rush. It serves the purpose of permitting the prospective students to judge the contents and structure of the seminar by themselves in order to decide their participation. A complete program will eventually become available before the summer semester starts.

1. Preliminary meeting

The preliminary meeting will take place on the 6th of February at 14pm at SR 1C, in order to give an overview of the contents and distribute the talks at the end.

2. Introduction

The first part of the seminar will focus on homological algebra, which refers to an area of commutative algebra that mostly deals with the behavior of the derived functors Tor of the tensor product $- \otimes_R -$ and Ext of the homomorphism modules $\mathrm{Hom}(-,-)$. In other words, we study (co)homological invariants of modules M over commutative rings with unit R that indicate the failure of $N \mapsto N \otimes_R M$, resp. $N \mapsto \mathrm{Hom}_R(N, M)$, to preserve short exact sequences.

We will also study the notion of depth of an R-module M, coming from the length of regular sequences in M, i.e. elements $x_i \in M$ which are non-zero-divisors in $M/(x_1, \dots, x_{i-1})M$, and relate these locally to the behavior of the Ext functors. We will study the notion of Cohen–Macaulay and Gorenstein rings and modules and see examples of those. These should be thought of as the class of “smooth objects” in homological algebra, since they admit a satisfactory theory of duality.

In the second part, we will consider exclusively rings over the prime field $\mathbb{F}_p$ of characteristic p, and study the behavior of the Frobenius map $\varphi\colon R \to R$ with respect to the previously introduced concepts. In particular, we will cover Kunz’s theorem that characterizes regular characteristic p rings as those whose Frobenius is flat. We will investigate the concept of φ-split and φ-regular rings, and see what type of singularities they possess. This will culminate with a proof of Fedder’s criterion, that allows one to explicitly compute whether a ring is φ-split or φ-regular. Depending on time and the number of participants, we might also see some global applications to projective geometry.

3. Prerequisites and sources

The only prerequisite for this seminar is having frequented a lecture in Commutative Algebra. We will basically assume the contents of [AM18] which includes the notions of prime ideals, Krull dimension, flatness, integral extensions, completions, etc. We also expect familiarity with categorical language. We will not assume a background in algebraic geometry, but it would naturally be a very helpful tool to get a better intuition behind certain concepts.

During Talks 1-2, the main reference that we will follow is [Wei95]. This is a classical book on homological algebra, which will guide us through the more technical notions of projective and injective resolutions, left and right functors, and the corresponding functors Tor and Ext derived from $\otimes_R^L$ and Hom_R. Talks 3-7 of the seminar follow [Eis13], another classical book in commutative algebra, dealing with regular sequences, depth, Koszul complexes, Cohen–Macaulay, Gorenstein rings, projective and global dimension, and coherent duality. Additionally, there is also the original EGAIV volume [Gro64, Gro65, Gro66, Gro67]: it is written in French (there's no translation) and it uses scheme theory, but it is the most comprehensive source out there, which means it should be used by the interested student to gather extra information, or check for further proofs/corollaries. Talks 8-11 and 13 will follow Karen Smith's course notes [Smi19] which are reasonably elementary and mostly written in the language of rings and not schemes. Additionally, we might refer sometimes to [BK05], which deals more with algebraic varieties, and [MP21], which works with rings, but is more advanced. Talk 12 can be based on any preferred book on algebraic geometry. Please note that talks 12-13 (which depend on participation and interest) deal with the theory of φ-singularities for projective varieties.

4. Talks description

Talk 1: Resolutions and derived functors. Introduce the notion and prove the existence of projective and injective resolutions for a finitely presented R-module M over a Noetherian ring following [Wei95, Sections 2.2-2.3]. Explain the notion of left and right derived functors as in [Wei95, Sections 2.4-2.5].

Talk 2: The functors Tor and Ext. Define the right derived functor Tor of the tensor product as in [Wei95, Sections 3.1-3.2], as well as the left derived functor Ext of the Hom functor, see [Wei95, Sections 3.3-3.4]. Give various examples of how to calculate these functors.

Talk 3: Derivations and the Jacobi criterion. Introduce the module of differentials $\Omega_{S/R}$ for a ring homomorphism $R \to S$ along with its main properties, following [Eis13, Section 16]. State and prove the Jacobian criterion, see [Eis13, Theorem 16.19].

Talk 4: Regular sequences and the Koszul complex. Define the notion of a regular sequence x_i of an R-module M, and the Koszul complex, following [Eis13, Section 17].

Talk 5: Depth and Cohen–Macaulay modules. Define the notion of depth of an R-module M, where R is a local ring. Define what it means for M to be a Cohen–Macaulay module, and interpret it in terms of the Ext groups, following [Eis13, Section 18].

Talk 6: Global dimension and Auslander–Buchsbaum. Explain the notion of projective dimension of an R-module M and the global dimension of a noetherian ring R, see [Eis13, Subsection 19.1]. Give the Auslander–Buchsbaum characterization of regular and Cohen–Macaulay rings in terms of homological invariants, see [Eis13, Subsection 19.3].

Talk 7: Duality and Gorenstein rings. Discuss coherent duality for Cohen–Macaulay rings and define Gorenstein rings, following [Eis13, Section 21].

Talk 8: Frobenius and Kunz's theorem. Given a characteristic p ring, define the Frobenius homomorphism and prove its basic properties. Prove Kunz's theorem characterizing regular rings, see [Smi19, Chapter 2].

Talk 9: Frobenius splitting. Define what it means for a ring R to be Frobenius split. Give some examples and counterexamples of such rings, following [Smi19, Sections 3.1-3.3]

Talk 10: Frobenius regularity. Introduce the notion of Frobenius regularity and show that such ring is always normal, following [Smi97, Section 3.4] Give an example of a φ-split ring which is not φ-regular.

Talk 11: Fedder's criterion. Prove Fedder's criterion on φ-split and φ-regular ring, see [Smi97, Chapter 4].

Talk 12: Projective algebraic geometry. Explain the basic construction of the projective variety attached to a graded ring. Define line bundles and the Picard group.

Talk 13: Global splittings and cohomology vanishing. Explain the notion of φ-split and φ-regular projective varieties, and show that φ-split varieties satisfy certain cohomological vanishing for ample line bundles, as in [Smi97, Sections 3.5-3.6]. Mention the Mehta–Ramanathan critetion, see [BK05, Proposition 1.3.11].

References

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