Parent-offspring pair between samples probability

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Offspring born before time of first sample

If there is a parent-offspring pair (POP) at time t_1, then there is a POP between time t_1 and a later time t_2 for each animal that survives to time t_2. Note that for simplicity if both survive we are counting two POPs rather than distinguishing this case from when just one survives. The probability of each animal surviving is $\phi^{t_2-t_1}$, where ϕ is the probability that an animal survives from time t to time $t+1$. So the expected number of POPs between two such times, for which the offspring is born at or before the first time, is

$$E(PO_{t_1})2\phi^{t_2-t_1} = 2E(N_{t_1})\frac{\phi(\lambda-\phi)}{\lambda-\phi^2}2\phi^{t_2-t_1}$$

$$= 4E(N_{t_2})\frac{\phi(\lambda-\phi)}{\lambda-\phi^2}\left(\frac{\phi}{\lambda}\right)^{t_2-t_1},$$

where $E(PO_{t_1})$ is the expected number of POPs at time t_1 as derived on pages 38-39 of Dissertation.pdf.

Offspring born after time of first sample

If there is a parent-offspring pair where one animal is alive at time t_1 and the other is alive at a later time t_2, then the parent must be alive at time t_1. The offspring may be born after the first time.

The expected number of animals born each year is

$$E(B_t) = \left(1-\frac{\phi}{\lambda}\right)E(N_t),$$

as explained in HSP-probability.pdf, where N_t is the population at time t, and

$$\lambda = \frac{E(N_{t+1})}{E(N_t)}$$

If an animal is born between time t_1 and time $t_1+\alpha+1$, where α is the age of sexual maturity for its species, then its parents must have been alive at time t_1 in order that they be mature and breed the year before the offspring is born. Therefore there is a POP between times t_1 and t_2 for every animal born between time t_1 and time $t_1+\alpha+1$ and surviving to time t_2.

The expected number of animals that were alive at time t_1, and survived until time t, is

$$\left(\frac{\phi}{\lambda}\right)^{t-t_1}E(N_t)$$

The expected number of animals that are mature at time t is the expected number that were alive at time $t-\alpha$, and survived until time t, that is

$$\left(\frac{\phi}{\lambda}\right)^{\alpha}E(N_t)$$

For animals born at time t, where $t > t_1 + \alpha + 1$, the probability that each of their parents was alive at time t_1 is thus

$$\frac{\left(\frac{\phi}{\lambda}\right)^{t-1-t_1} E(N_{t-1})}{\left(\frac{\phi}{\lambda}\right)^{\alpha} E(N_{t-1})} = \left(\frac{\phi}{\lambda}\right)^{t-1-t_1-\alpha},$$

since we know that their parents survived until time $t-1$ and were mature then.

So the expected number of POPs between times t_1 and t_2, in which the offspring was born at a time t between times t_1 and t_2, and survives to time t_2, is

$$2\left(1-\frac{\phi}{\lambda}\right) E(N_t)\phi^{t_2-t},$$

when, $t > t_1 + \alpha + 1$, and

$$2\left(1-\frac{\phi}{\lambda}\right) E(N_t)\phi^{t_2-t}\left(\frac{\phi}{\lambda}\right)^{t-1-t_1-\alpha},$$

when $t \leq t_1 + \alpha + 1$, where the factor of two is due to the offspring having two parents. That is

$$2\left(1-\frac{\phi}{\lambda}\right) E(N_t)\phi^{t_2-t}\left(\frac{\phi}{\lambda}\right)^{max\{0,t-1-t_1-\alpha\}}.$$

Summing over the possible years gives the expected total number of such pairs

$$\sum_{t_1<t\leq t_2} 2\left(1-\frac{\phi}{\lambda}\right) E(N_t)\phi^{t_2-t}\left(\frac{\phi}{\lambda}\right)^{max\{0,t-1-t_1-\alpha\}}$$

$$= 2\left(1-\frac{\phi}{\lambda}\right) E(N_{t_2}) \sum_{t_1<t\leq t_2} \left(\frac{\phi}{\lambda}\right)^{t_2-t}\left(\frac{\phi}{\lambda}\right)^{max\{0,t-1-t_1-\alpha\}}$$

$$= 2\left(1-\frac{\phi}{\lambda}\right) E(N_{t_2}) \sum_{t_1<t\leq t_2} \left(\frac{\phi}{\lambda}\right)^{max\{t_2-t,t_2-1-t_1-\alpha\}}$$

$$= 2\left(1-\frac{\phi}{\lambda}\right) E(N_{t_2}) \left\{\left(\frac{\phi}{\lambda}\right)^{max\{0,t_2-t_1-1-\alpha\}} + ... + \left(\frac{\phi}{\lambda}\right)^{max\{t_2-t_1-1,t_2-t_1-1-\alpha\}}\right\}.$$

Adding the expected number of POPs in which the offspring is born before the time of the first sample, and dividing by the expected total number of pairs of animals between times t_1 and t_2, which can be approximated by $E(N_{t_1})E(N_{t_2})$, gives the probability of a POP between samples, as

$$P(POP_{t_1,t_2}) = \frac{4}{E(N_{t_1})}\frac{\phi(\lambda-\phi)}{\lambda-\phi^2}\left(\frac{\phi}{\lambda}\right)^{t_2-t_1} +$$

$$\frac{2}{E(N_{t_1})}\left(1-\frac{\phi}{\lambda}\right)\left\{\left(\frac{\phi}{\lambda}\right)^{max\{0,t_2-t_1-1-\alpha\}} + ... + \left(\frac{\phi}{\lambda}\right)^{max\{t_2-t_1-1,t_2-t_1-1-\alpha\}}\right\}.$$