Expected numbers of full and half-sibling pairs

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Expected number of full-sibling pairs

Let

- $FSP_{t,b_1,b_2,i}$ be the number of pairs of animals at time t, that include a particular animal i, that is born at time b_1, and an animal born at a later time b_2, that has the same mother and the same father (full-sibling pairs), and
- FSP_{t,b_1,b_2} be the total number of pairs at time t that include any animal born at time b_1, and an animal born at a later time b_2, that has the same mother and the same father.

Then following the logic as the cases for same-mother and same-father pairs we have that the probability that an animal born at time b_1 survives to time t and an animal born at time b_2 forms a same-father pair with it at time t is $\frac{\phi^{2(t-b_1)}}{M_{b_2-1}F_{b_2}}$. Also following the logic through we find that

$$E(FSP_{t,b_1,b_2}) = \frac{E(SMP_{t,b_1,b_2})\phi^{b_2-b_1+1}}{E(F_{b_2})}$$

$$= \frac{2E(N_t)\left(1-\frac{\phi}{\lambda}\right)^2\left(\frac{\lambda}{\phi}\right)^\alpha\left(\frac{\phi^2}{\lambda}\right)^{t-b_1}\phi^{b_2-b_1+1}}{\frac{E(N_{b_2})}{2}\left(\frac{\phi}{\lambda}\right)^\alpha}$$

$$= 2E(N_t)\left(1-\frac{\phi}{\lambda}\right)^2\left(\frac{\lambda}{\phi}\right)^\alpha\left(\frac{\phi^2}{\lambda}\right)^{t-b_1}\phi^{b_2-b_1+1}\frac{2}{E(N_{b_2})}\left(\frac{\lambda}{\phi}\right)^\alpha$$

$$= 2E(N_t)\left(1-\frac{\phi}{\lambda}\right)^2\left(\frac{\lambda}{\phi}\right)^\alpha\left(\frac{\phi^2}{\lambda}\right)^{t-b_1}\frac{2\phi^{b_2-b_1+1}\lambda^{t-b_2}}{E(N_t)}\left(\frac{\lambda}{\phi}\right)^\alpha$$

$$= 4\left(1-\frac{\phi}{\lambda}\right)^2\left(\frac{\lambda}{\phi}\right)^{2\alpha}\left(\frac{\phi^2}{\lambda}\right)^{t-b_1}\phi^{b_2-b_1+1}\lambda^{t-b_2},$$

$$E(FSP_t) = \sum_{b_1<t}\sum_{b_1<b_2\leq t}E(FSP_{t,b_1,b_2})$$

$$= \sum_{b_1<t}\sum_{b_1<b_2\leq t}4\left(1-\frac{\phi}{\lambda}\right)^2\left(\frac{\lambda}{\phi}\right)^{2\alpha}\left(\frac{\phi^2}{\lambda}\right)^{t-b_1}\phi^{b_2-b_1+1}\lambda^{t-b_2}$$

$$= 4\left(1-\frac{\phi}{\lambda}\right)^2\left(\frac{\lambda}{\phi}\right)^{2\alpha}\phi\sum_{b_1<t}\left(\frac{\phi^2}{\lambda}\right)^{t-b_1}\sum_{b_1<b_2\leq t}\phi^{b_2-b_1}\lambda^{t-b_2},$$

$$\sum_{b_1<t}\left(\frac{\phi^2}{\lambda}\right)^{t-b_1}\sum_{b_1<b_2\leq t}\phi^{b_2-b_1}\lambda^{t-b_2} = \frac{\phi^2}{\lambda}\phi + \left(\frac{\phi^2}{\lambda}\right)^2(\phi^2+\phi\lambda) + \left(\frac{\phi^2}{\lambda}\right)^3(\phi^3+\phi^2\lambda+\phi\lambda^2) + ...$$

$$= \frac{1}{1-\frac{\phi^2}{\lambda}\phi}\left\{\frac{\phi^2}{\lambda}\phi + \left(\frac{\phi^2}{\lambda}\right)^2\phi\lambda + \left(\frac{\phi^2}{\lambda}\right)^3\phi\lambda^2 + ...\right\}$$

$$
\begin{aligned}
&= \frac{\phi\lambda}{\lambda-\phi^3}\left\{\frac{\phi^2}{\lambda}+\frac{\phi^4}{\lambda}+\frac{\phi^6}{\lambda}+...\right\} \\
&= \frac{\phi}{\lambda-\phi^3}\left\{\phi^2+\phi^4+\phi^6+...\right\} \\
&= \frac{\phi}{\lambda-\phi^3}\frac{\phi^2}{1-\phi^2} \\
&= \frac{\phi^3}{(\lambda-\phi^3)(1-\phi^2)},
\end{aligned}
$$

$$
\begin{aligned}
E(FSP_t) &= 4\left(1-\frac{\phi}{\lambda}\right)^2\left(\frac{\lambda}{\phi}\right)^{2\alpha}\phi\frac{\phi^3}{(\lambda-\phi^3)(1-\phi^2)} \\
&= 4\left(1-\frac{\phi}{\lambda}\right)^2\left(\frac{\lambda}{\phi}\right)^{2\alpha}\frac{\phi^4}{(\lambda-\phi^3)(1-\phi^2)}
\end{aligned}
$$

Expected number of half-sibling pairs at time t

Let HSP_t be the number of half-sibling pairs at time t. Then

$$HSP_t = SMP_t + SFP_t - 2FSP_t,$$

and

$$
\begin{aligned}
E(HSP_t) &= E(SMP_t + SFP_t - 2FSP_t) \\
&= E(SMP_t) + E(SFP_t) - 2E(FSP_t)
\end{aligned}
$$

The half-sibling pair *probability* is then found by dividing the expected number of half-sibling pairs by the expected total number of pairs, which we approximate with

$$\frac{E(N_t)\left\{E(N_t-1)\right\}}{2}.$$

End of current work

The same process leads to the probability for pairs between samples at times t_1 and t_2, where $t_1 < t_2$, by simply replacing t with t_1 in the first sum, as the older sibling must be born before the first sample, and replacing t with t_2 in the second sum, as the younger sibling can be born anytime before the second sample. The sums may not simplify as nicely then though. I haven't tried this yet.