# Linear algebra notes

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## Linear Algebra and Its Applications by Gilbert Strang (1976)

### Preface

- Teaching of LA has become too abstract.

### 1 Gaussian Elimination

### 1.1 Introduction

- Central problem of LA is solving simultaneous linear equations
- Start with n by n systems as simplest and most important
- Two algorithms taught in high school, elimination and Cramer's rule
- In elimination multiples of one equation are subtracted from the others to produce smaller systems until only one variable remains, and then values are back-substituted
- Cramer's rule gives solutions in terms of ratios of n by n determinants
- Cramer's rule is a disaster, and elimination always used
- Elimination has four deeper aspects
  - Interpretation as LU factorization of coefficient matrix essential for theory
  - Identification of problems due to disordered equations, or zero or infinitely many solutions
  - Rough number of operations
  - Sensitivity to round off error

### 1.2 Example of Gaussian Elimination

- The leading coefficient of the equation that is being subtracted from the others is called the pivot
- The technique finds a unique solution iff none of the pivots are zero
- Ignoring the right-hand side of the equations, subtracting one equation from the another requires n operations, one to find the multiple required, and n - 1 to find the remaining coefficients (counting multiplication-subtraction as a single operation)
- There are n - 1 equations to subtract from, so each elimination step takes n(n - 1) operations
- Each step produces a system with one fewer unknowns, so altogether the elimination process takes

$$P = n(n-1) + ... + 1(1-1)$$

$$= \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k$$

$$= \frac{1}{3}n\left(n + \frac{1}{2}\right)(n+1) - \frac{1}{2}n(n+1)$$

$$= \frac{n^3 - n}{3}$$

steps

- The closed form for $\sum_{k=1}^{n} k^2$ is easy to prove, but apparently not derivable by any perfectly general method
- When n is large the number of steps is well-approximated by $n^3/3$
- Back-substitution takes one step for the first variable, two for the second, and so on, giving

$$Q = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \approx \frac{n^2}{2}$$

steps

### 1.3 Matrix notation and matrix multiplication

- A linear system of n equations in m unknowns can be represented by

$$A\mathbf{x} = \mathbf{a_1}x_1 + ... + \mathbf{a_m}x_m = \mathbf{b},$$

where

$$A = [\mathbf{a_1}...\mathbf{a_m}],$$

$$\mathbf{a_1}, ..., \mathbf{a_m}, \mathbf{b} \in \mathbb{R}^n,$$

and

$$\mathbf{x} = [x_1...x_m]^T.$$

- Matrix multiplication:
  - The matrix product EA is the matrix satisfying $(EA)\mathbf{x} = E(A\mathbf{x})$ for all $\mathbf{x}$
  - Each column of EA is a linear combination of the columns of E, given by the corresponding column of A
  - Associativity - $(AB)C = A(BC)$
  - Distributivity - $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$
  - Non-commutativity - $AB \neq BA$
- The identity matrix I is the n by n matrix with ones on the main diagonal and zeros elsewhere such that $IA = AI = A$ for all matrices A

### 1.4 Gaussian elimination = Triangular factorization