Kalman Filter

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State-space model

For a state-space model where y_t are observed and x_t are the latent states over time.

$$y_t = y_{t-1} + e_t, e_t \sim N(0, \sigma^2)$$
$$x_t = x_{t-1} + a_t, a_t \sim N(0, \tau^2), t > 1$$

$\mathbf{x_1 | y_1}$

$$x_1 \sim N(0, K^2),$$

K large

$$P(x_1|y_1) \propto P(y_1|x_1)P(x_1)$$
$$\propto \exp\left(\frac{-1}{2\sigma^2}(y_1 - x_1)^2\right)\exp\left(\frac{-x_1^2}{2K^2}\right)$$
$$x_1|y_1 \sim N\left(\left(\frac{1}{\sigma^2} + \frac{1}{K^2}\right)^{-1}\frac{y_1}{\sigma^2}, \left(\frac{1}{\sigma^2} + \frac{1}{K^2}\right)^{-1}\right)$$

$\mathbf{x_{t-1} | y_{t-1}, ..., y_1}, t > 1$

$$x(t|t-1) := E(x_{t-1}|y_{t-1}, ..., y_1)$$
$$s(t|t-1) := Var(x_{t-1}|y_{t-1}, ..., y_1) + \tau^2$$
$$x_t|y_{t-1}, ..., y_1 \sim N(x(t|t-1), s(t|t-1))$$
$$P(x_t|y_t, ..., y_1) \propto P(y_t|x_t)P(x_t|y_{t-1}, ..., y_1)$$
$$\propto \exp\left(\frac{-1}{2\sigma^2}(y_t - x_t)^2\right)\exp\left(\frac{-1}{s(t|t-1)}(x_1 - x(t|t-1))^2\right)$$
$$x_t|y_t, ..., y_1 \sim N\left(\left(\frac{1}{\sigma^2} + \frac{1}{s(t|t-1)}\right)^{-1}\left(\frac{y_1}{\sigma^2} + \frac{x(t|t-1)}{s(t|t-1)}\right), \left(\frac{1}{\sigma^2} + \frac{1}{s(t|t-1)}\right)^{-1}\right)$$

Kalman filter for $\mathbf{t > 1}$

$$\epsilon_1 := y_1 - E(y_1) = y_1$$
$$R_1 = Var(y_1) := K^2 + \sigma^2$$
$$\epsilon_t := y_t - E(y_t|y_{t-1}, ..., y_1) = x_t - x(t|t-1) + e_t$$
$$R_t := Var(e_t) = s(t|t-1) + \sigma^2$$

Likelihood of $\mathbf{y_t, ..., y_1}$

$$\begin{aligned} P(y_t, ..., y_1|\theta) &= P(y_1|\theta)...P(y_{t-1}, ..., y_1|\theta) \\ &= \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi R_t(\theta)}} \exp\left(\frac{-\epsilon_t(\theta)^2}{2R_t(\theta)}\right) \end{aligned}$$