Algebra y Geometría Analítica Lineal

Guía UTN -Resueltos

POTENCIA DE MATRICES DIAGONALIZABLES

2-

a-

MATRIZ INVERSA

$$A^{-1} = \frac{Adj(A)}{Det(A)} = \frac{M_{Cof}^{T}(A)}{Det(A)}$$

$$P^{-1} = \frac{\begin{pmatrix} 6 & 0 \\ -1 & 1 \end{pmatrix}^{T}}{6} = \begin{pmatrix} \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{6}} \\ \mathbf{0} & \frac{\mathbf{1}}{\mathbf{6}} \end{pmatrix}$$

b-

MATRIZ SEMEJANTE

$$A\ es\ Semejante\ a\ B \Leftrightarrow \exists P / P^{-1}AP = B$$

$$P^{-1}AP = \begin{pmatrix} 1 & -\frac{1}{6} \\ 0 & \frac{1}{6} \end{pmatrix} \begin{pmatrix} -4 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} -4 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} \mathbf{-4} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} \end{pmatrix} = \boldsymbol{B}$$

POTENCIA DE MATRIZ SEMEJANTE - DIAGONAL

$$[A]_{ij}^{n} = a_{ii}^{n}$$

$$B^{8} = \begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix}^{8} = \begin{pmatrix} \mathbf{65536} & \mathbf{0} \\ \mathbf{0} & \mathbf{2562} \end{pmatrix}$$

OBSERVEN

$$Si\ B\ Es\ Semejante\ a\ A$$
$$y\ P = M_{BB}(T)\ Autoespacial \Rightarrow B\ Diagonal\ de\ Autovalores$$

3-

a-

$$P^{-1} = \frac{\begin{pmatrix}3 & -1\\ -2 & -1\end{pmatrix}^T}{-5} = \begin{pmatrix} -\frac{3}{5} & \frac{2}{5}\\ \frac{1}{5} & \frac{1}{5}\end{pmatrix}$$

b-

Por tratarse de una Matriz Semejante se conserva la Matriz P:

$$B^5 = P^{-1}D^5P = \begin{pmatrix}-\frac{3}{5} & \frac{2}{5}\\ \frac{1}{5} & \frac{1}{5}\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1024\end{pmatrix}\begin{pmatrix}-1 & 2\\ 1 & 3\end{pmatrix} = \begin{pmatrix}\frac{3}{5} & \frac{2048}{5}\\ -\frac{1}{5} & \frac{1024}{5}\end{pmatrix}\begin{pmatrix}-1 & 2\\ 1 & 3\end{pmatrix} =$$

$$\begin{pmatrix}409 & 1230\\ 205 & 614\end{pmatrix}$$

4-

a-

Por tratarse de una matriz de 3x3 con 3 AUTOVALORES distintos, podemos asegurar que la matriz es DIAGONALIZABLE, Y podemos usar:

$$P^{-1}D^3P = A^3 \ Siendo\ D = M_{BB}\ (T)$$

b-

En este caso tenemos 3 autovalores iguales con multiplicidad algebraica 3, sin embargo, es imposible que, para mantener la indeterminación del sistema homogéneo, tengamos indeterminación para un sistema de 3x3 con autoespacio de 3 para que coincida el geométrico con el algebraico. Por lo tanto, se debe hacer la MULTIPLICACIÓN.

5-

a-

Sabiendo que:

$$(x,y)\begin{pmatrix}a & \frac{b}{2}\\ \frac{b}{2} & c\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = -d \Rightarrow (x,y)\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 3$$

Busquemos los Autovalores de A para hallar la Matriz P:

$$\left|\begin{pmatrix}1-\lambda & 2\\ 2 & 1-\lambda\end{pmatrix}\right| = (1-\lambda)^2 - 4 = 0 \Rightarrow \boldsymbol{\lambda = -1 \vee \lambda = 3}$$

Busquemos los Autoespacios Asociados a los Autovalores:

$$\begin{cases} Si\ \lambda = -1 \Rightarrow x = -y \Rightarrow S_{\lambda=-1} = \{(-1,1)\} \\ Si\ \lambda = -3 \Rightarrow 2x = 2yS_{\lambda=-1} \Rightarrow S_{\lambda=3} = \{(1,1)\} \end{cases}$$

La matriz P:

$$\begin{cases} P = (S_{\lambda_1}\ S_{\lambda_2}) = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} \\ \mathbf{P_{\perp}} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} \end{cases} = M(Id)_{BE} = P_{\perp} \wedge P_{\perp}^{-1} = P_{\perp}^{T}$$

La ecuación en Base B en canónico:

$$\begin{cases} (x,y)M(T)_{BB'}\begin{pmatrix}x\\ y\end{pmatrix} = (x,y)A\begin{pmatrix}x\\ y\end{pmatrix} \wedge\ A = A^T (SIMETRICA) \\ M(Id)_{BE}\begin{pmatrix}x'\\ y'\end{pmatrix} = P_{\perp}\begin{pmatrix}x'\\ y'\end{pmatrix} = \begin{pmatrix}x\\ y\end{pmatrix} \Rightarrow (x,y) = (x',y')P_{\perp}^{T} \quad \Rightarrow \\ \mathbf{D} = \begin{pmatrix}-1 & 0\\ 0 & 3\end{pmatrix} \end{cases}$$

$$\Rightarrow (x',y')P_{\perp}^{T}AP_{\perp}\begin{pmatrix}x'\\ y'\end{pmatrix} = (x',y')D\begin{pmatrix}x'\\ y'\end{pmatrix} = (x',y')\begin{pmatrix}-1 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix} = 3$$

$$\mathbf{y'^2 - \frac{x'^2}{3} = 1}$$

$$[V]_{B'} = (0,1) \Rightarrow P_{\perp}[V]_{B'} = [C]_E \Rightarrow V = (0,1)\ es\ \left(\frac{1}{\sqrt{2}};\frac{1}{\sqrt{2}}\right)$$

b-

Sabiendo que:

$$(x,y)\begin{pmatrix}a & \frac{b}{2}\\ \frac{b}{2} & c\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = -d \Rightarrow (x,y)\begin{pmatrix}5 & 3\\ 3 & 5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 8$$

Busquemos los Autovalores de A para hallar la Matriz P:

$$\left|\begin{pmatrix}5-\lambda & 3\\ 3 & 5-\lambda\end{pmatrix}\right| = (5-\lambda)^2 - 9 = 0 \Rightarrow \boldsymbol{\lambda = 2 \vee \lambda = 8}$$

Busquemos los Autoespacios Asociados a los Autovalores:

$$\begin{cases} Si\ \lambda = 2 \Rightarrow x = -y \Rightarrow S_{\lambda=2} = \{(-1,1)\} \\ Si\ \lambda = 8 \Rightarrow 2x = 2yS_{\lambda=-1} \Rightarrow S_{\lambda=8} = \{(1,1)\} \end{cases}$$

La matriz P:

$$\begin{cases} P = (S_{\lambda_1}\ S_{\lambda_2}) = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} \\ \mathbf{P_{\perp}} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} \end{cases} = M(Id)_{BE} = P_{\perp} \wedge P_{\perp}^{-1} = P_{\perp}^{T}$$

La ecuación en Base B en canónico:

$$\begin{cases} (x,y)M(T)_{BB'}\begin{pmatrix}x\\y\end{pmatrix} = (x,y)A\begin{pmatrix}x\\y\end{pmatrix} \wedge \ A = A^T (SIMETRICA) \\ M(Id)_{BE}\begin{pmatrix}x'\\y'\end{pmatrix} = P_\perp \begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \Rightarrow (x,y) = (x',y')P_\perp^T \quad \Rightarrow \\ \mathbf{D} = \begin{pmatrix}\mathbf{2} & \mathbf{0}\\ \mathbf{0} & \mathbf{8}\end{pmatrix} \end{cases}$$

$$\Rightarrow (x',y')P_\perp^T A P_\perp \begin{pmatrix}x'\\y'\end{pmatrix} = (x',y')D\begin{pmatrix}x'\\y'\end{pmatrix} = (x',y')\begin{pmatrix}2 & 0\\ 0 & 8\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix} = 8$$

$$\frac{\boldsymbol{x'^2}}{\mathbf{4}} + \frac{\boldsymbol{y'^2}}{\mathbf{1}} = \mathbf{1}$$

$$[V]_{B'} = (0,1) \Rightarrow P_\perp [V]_{B'} = [V]_E \Rightarrow V = (2,0)\ es\ \left(\sqrt{2}; -\sqrt{2}\right)$$

c-

Sabiendo que:

$$(x,y)\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = -d \Rightarrow (x,y)\begin{pmatrix}9 & 12\\ 12 & 16\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 25$$

Busquemos los Autovalores de A para hallar la Matriz P:

$$\left|\begin{pmatrix}9-\lambda & -12\\ -12 & 16-\lambda\end{pmatrix}\right| =)(9-\lambda)(16-\lambda) - 12^2 = 0 \Rightarrow \boldsymbol{\lambda = 0 \vee \lambda = 25}$$

Busquemos los Autoespacios Asociados a los Autovalores:

$$\begin{cases} Si\ \lambda = 0 \Rightarrow 9x = 12y \Rightarrow S_{\lambda=0} = \{(4,3)\} \\ Si\ \lambda = 16 \Rightarrow -16x = 12yS_{\lambda=-1} \Rightarrow S_{\lambda=3} = \{(-3,4)\} \end{cases}$$

La matriz P:

$$\begin{cases} P = (S_{\lambda_1}\ S_{\lambda_2}) = \begin{pmatrix}4 & -3\\ 3 & 4\end{pmatrix} \\ \mathbf{P}_\perp = \begin{pmatrix}\frac{\mathbf{4}}{\mathbf{5}} & \frac{\mathbf{3}}{\mathbf{5}} \\ -\frac{\mathbf{3}}{\mathbf{5}} & \frac{\mathbf{4}}{\mathbf{5}}\end{pmatrix} \end{cases} = M(Id)_{BE} = P_\perp \wedge P_\perp^{-1} = P_\perp^T$$

La ecuación en Base B en canónico:

$$\begin{cases} (x,y)M(T)_{BB'}\begin{pmatrix}x\\y\end{pmatrix} = (x,y)A\begin{pmatrix}x\\y\end{pmatrix} \wedge \ A = A^T (SIMETRICA) \\ M(Id)_{BE}\begin{pmatrix}x'\\y'\end{pmatrix} = P_\perp \begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \Rightarrow (x,y) = (x',y')P_\perp^T \quad \Rightarrow \\ \mathbf{D} = \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{25}\end{pmatrix} \end{cases}$$

$$\Rightarrow (x',y')P_\perp^T A P_\perp \begin{pmatrix}x'\\y'\end{pmatrix} = (x',y')D\begin{pmatrix}x'\\y'\end{pmatrix} = (x',y')\begin{pmatrix}0 & 0\\ 0 & 25\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix} = 25$$

$$\boldsymbol{y'^2 = 1}$$

d-

Sabiendo que:

$$(x,y)\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + (d\ e)\begin{pmatrix}x\\y\end{pmatrix} = -d \Rightarrow (x,y)\begin{pmatrix}9 & 12\\ 12 & 16\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + (0\ 1)\begin{pmatrix}x\\y\end{pmatrix} = 1$$

Busquemos los Autovalores de A para hallar la Matriz P:

$$\left|\begin{pmatrix}9-\lambda & -12\\ -12 & 16-\lambda\end{pmatrix}\right| =)(9-\lambda)(16-\lambda) - 12^2 = 0 \Rightarrow \boldsymbol{\lambda = 0 \vee \lambda = 25}$$

Busquemos los Autoespacios Asociados a los Autovalores:

$$\begin{cases} Si\ \lambda = 0 \Rightarrow 9x = 12y \Rightarrow S_{\lambda=0} = \{(4,3)\} \\ Si\ \lambda = 16 \Rightarrow -16x = 12yS_{\lambda=-1} \Rightarrow S_{\lambda=3} = \{(-3,4)\} \end{cases}$$

La matriz P:

$$\begin{cases} P = (S_{\lambda_1}\ S_{\lambda_2}) = \begin{pmatrix}4 & -3\\ 3 & 4\end{pmatrix} \\ \mathbf{P}_\perp = \begin{pmatrix}\frac{\mathbf{4}}{\mathbf{5}} & \frac{\mathbf{3}}{\mathbf{5}} \\ -\frac{\mathbf{3}}{\mathbf{5}} & \frac{\mathbf{4}}{\mathbf{5}}\end{pmatrix} \end{cases} = M(Id)_{BE} = P_\perp \wedge P_\perp^{-1} = P_\perp^T$$

La ecuación en Base B en canónico:

$$\begin{cases} (x,y)M(T)_{BB'}\begin{pmatrix}x\\y\end{pmatrix} = (x,y)A\begin{pmatrix}x\\y\end{pmatrix} \wedge \ A = A^T (SIMETRICA) \\ M(Id)_{BE}\begin{pmatrix}x'\\y'\end{pmatrix} = P_\perp \begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \Rightarrow (x,y) = (x',y')P_\perp^T \quad \Rightarrow \\ \mathbf{D} = \begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{25}\end{pmatrix} \end{cases}$$

$$\Rightarrow (x',y')P_\perp^T A P_\perp \begin{pmatrix}x'\\y'\end{pmatrix} = (x',y')D\begin{pmatrix}x'\\y'\end{pmatrix} =$$

$$(x',y')\begin{pmatrix}0 & 0\\ 0 & 25\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix} + (0\ 1)\begin{pmatrix}\frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5}\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix} =$$

$$25y'^2 + \left(-\frac{3}{5}; \frac{4}{5}\right)\begin{pmatrix}x'\\y'\end{pmatrix}$$

$$\boldsymbol{\left(y + \frac{2}{125}\right)^2 = \frac{3}{125}\left(x + \frac{4}{75}\right)}$$

6-

Busquemos los Autovalores de A para hallar la Matriz P:

$$\left|\begin{pmatrix} 2-\lambda & k \\ k & 1-\lambda \end{pmatrix}\right| = (2-\lambda)(1-\lambda) - k^2 = 0 \wedge \lambda = 0(SINO\ DA\ CÓNICA) \Rightarrow \boldsymbol{k = \pm\sqrt{2}}$$

NO DA COMO LA GUIA, PERO EN MATLAB ASÍ DA DOS RECTAS

7-

Tenemos entonces la Matriz P ortogonal o la Matriz de Cambio de Base:

$$P_{\perp} = M(Id)_{BE} = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}$$

$$\begin{cases} \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Longrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} \\ (x'\ y')D\begin{pmatrix} x' \\ y' \end{pmatrix} = (x\ y)P_{\perp}DP_{\perp}^T\begin{pmatrix} x \\ y \end{pmatrix} = (x\ y)A\begin{pmatrix} x \\ y \end{pmatrix} = 11 \end{cases}$$

Invertimos todo

$$\begin{cases} (x\ y)A\begin{pmatrix} x \\ y \end{pmatrix} = 11 \\ P_{\perp}DP_{\perp}^T = A \end{cases} \quad \begin{cases} (x\ y)A\begin{pmatrix} x \\ y \end{pmatrix} = 11 \\ \begin{pmatrix} 2 & 3 \\ 3 & 10 \end{pmatrix} = A \end{cases} \Rightarrow$$

$$\mathbf{2x^2 + 10y^2 + 6xy = 11}$$