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Report on

A SYSTEM THEORY CRITERION FOR POSITIVE REAL MATRICES

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ABSTRACT

The concept of a Positive real function is a vast and important topic in Network theory. Positive real functions have been studied extensively in the continuous-time case in the context of network synthesis. But due to some applications of Positive real function in the control systems theory, it seems possible that this concept can be employed in control systems investigation. The main aim of this paper is to develop a system theoretic criterion of a matrix, which is formulated in terms of the parameters of a control system realization of matrix.

Contents

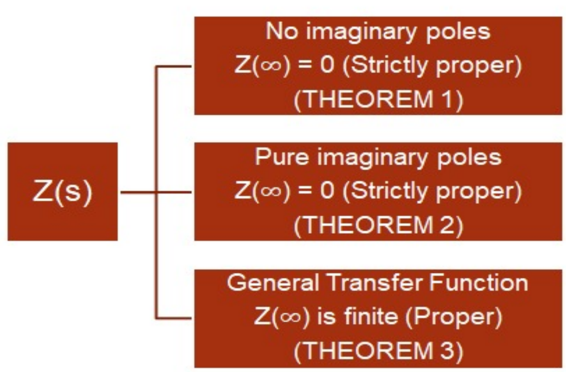
1 Introduction

1.1 Motivation

The motivation of this paper would be construct a link between Network theory and Control theory through the concept of Positive real functions. Popov developed a criterion for the stability of a feedback system [1]. This was the reason behind the idea of this paper where we will use control theory concepts to define Positive real matrices.

1.2 Workflow

Before defining the final Theorem which will act as a criterion for a generalized positive real matrices, we will derive a couple of theorems for the subspace of the Positive real function. These theorems in turn will help in constructing the final theorem. These theorems will act as special cases of the final theorem. Initially, we will consider matrices which doesn't have any imaginary poles and which are zero at (s = ∞) (This means strictly proper transfer function). Then the matrices with only imaginary poles and strictly proper transfer function. Finally the general case where it is proper function and imaginary poles permitted. These theorems are derived with the help of several preliminary concepts such as Minimal Realizations, Lyapunov's Direct method, Spectral Factorization, etc. The concepts are stated as Lemmas at the beginning and further more Lemmas are derived according to our need to derive the theorems.



2 Literature Review

2.1 Preliminary Concepts

2.1.1 Definition of Positive Real Transfer matrices:

An n x n matrix Z(s) of functions of a complex variable s is called Positive Real if the following conditions are satisfied:

1. Z(s) has elements that are analytic for Re[s] $>$ 0
2. Z*(s) = Z(s*), Re[s] $>$ 0, and
3. Z'(s*) + Z(s) is positive semi-definite for Re[s] $>$ 0

=> Here the 3rd condition is the sufficient condition to prove the positive real condition. Physically this condition means that the for all the left hand plane of s, The function should be mapped to the 1st and 4th quadrant in Nyquist plane.

2.1.2 Realizations:

If M(s) is an m x n matrix of rational functions, with M(∞) = 0, then [F,G,H] is termed a realization of M if

$$M(s) = H'(sI - F)^{-1}G$$

where the state space model is given by:

$$\dot{x} = Fx + Gu$$
$$y = H'x$$

=> This realization can easily be derived by taking Laplace transform on the state space model. As there exist infinitely many state space representation for a particular model we use minimal realization. A realization is minimal if it is both controllable and observable. If the realization is not minimal then we apply pole zero cancellation to get the minimal realization.

2.1.3 Lemma 1:

It states the if $[F_1, G_1, H_1]$ *and* $[F_2, G_2, H_2]$ *are two minimal realizations of M(s). Then there exists a non-singular* T *such that:*

$$F_2 = TF_1T^{-1} \tag{1}$$

$$G_2 = TG_1 \tag{2}$$

$$H_2 = (T')^{-1}H_1 \tag{3}$$

=>

- Here the 1st postulate is nothing but the commutative law between F and T if F_1 and F_2 are similar to F
- This Lemma is based on the concept of Similarity Transform derived using Kalman Decomposition for minimal realization.
- Here T is given by $[T_{CO}T_{CO'}T_{C'O}T_{C'O'}]$ where T_{CO} stands for the transformation for the subspace which is controllable and observable. Similarly other transforms can also derived.
- This Lemma is used further to develop the minimal realization of factorization which we will study later.

2.1.4 Lemma 2:

If the elements of $M_1(.)$ *and* $M_2(.)$ *have no poles in common then,*

$$\delta[M_1 + M_2] = 2\delta[M_2] \tag{4}$$

This Lemma is discussed together with Lemma 3 and Lemma 4.

2.1.5 Lemma 3:

If the elements of $M_1(.)$ *and* $M_2(.)$ *have no poles in common then,*

$$\delta[M_1M_2] <= \delta[M_1] + \delta[M_2] \tag{5}$$

2.1.6 Lemma 4:

If $M_1(.)$ is n x r and $M_2(.)$ is r x n and they have no poles in common, rank $M_1(s_0) = r$ at any pole s_0 of an element of M_2 and rank $M_2(s_0) = r$ at any pole s_0 of an element of M_1 then,

$$\delta[M_1 M_2] = \delta[M_1] + \delta[M_2] \tag{6}$$

=> Here $\delta[M]$ is the dimension the minimal realization(i.e. minimal F matrix) and is termed as the degree of M(s). The above stated Lemmas are the postulates of the degree of M(s). These properties can be verified if we assume this degree as McMillan degree and using Smith- McMillan decomposition. A given matrix is in SM form if

$$M(s) = diag\left[\frac{\epsilon_1(s)}{\psi_1(s)}, \frac{\epsilon_2(s)}{\psi_2(s)}, ..., \frac{\epsilon_r(s)}{\psi_r(s)}, 0, ...0\right]$$

Let us take $M(s) = \frac{P(s)}{D(s)}$ where D(s) is LCM of all the denominators of transfer function matrix and P(s) is just a polynomial Here $\epsilon_i(s) = det|_i(P(s))$ i.e. i'th determinant of P(s) and $\delta_r = D_G(s)$

We take $M_1 = A_1 T_1 B_1$ and $M_2 = A_2 T_2 B_2$ where A_1, B_2 are n x n poly. matrices and B_1, A_2 are r x r matrices. Hence to convert $M_1 and M_2$ into SM form we take T_1 and T_2 similar to the M(s) SM equation.

We define $\hat{T}_1$ and $\hat{T}_2$ similar to T_1 and T_2 but with appending zeros. Hence by the properties of degree of McMillan we have

$$\begin{aligned}
\delta[M_1 M_2] &= \delta[T_1 B_1 A_2 T_2] \\
&= \delta[\hat{T}_1 B_1 A_2 \hat{T}_2] \\
&= \delta[\hat{T}_1] + \delta[\hat{T}_2] \\
&= \delta[M_1] + \delta[M_2]
\end{aligned}$$

2.1.7 Lemma 5:

Let F be a p x p matrix with eigenvalues all possessing negative real part. Then to each p x q matrix L (q arbitrary) there corresponds a unique symmetric non negative definite solution P to the equation.

$$PF + F'P = -LL' \tag{7}$$

COROLLARY: The only matrices which commute with

$$\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix} \tag{8}$$

are of the form

$$\begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \tag{9}$$

where T_1, T_2' commute with F
=> This Lemma is the straight application of Lyapunov's Direct Method where instead of Q we take LL' This Lemma can be proved using Lyapunov's stability criterion i.e. by proving $\dot{\mathrm{V}}$ is always negative semi-definite for any positive definite $V = x^T P x$.

Here we are concerned with the Corollary which will use alongside the eq(1) to prove many of the topics following.

Proof of Corollary : Suppose

$$\begin{bmatrix} T_1 & S \\ R & T_2 \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix} \begin{bmatrix} T_1 & S \\ R & T_2 \end{bmatrix}$$

$$\begin{bmatrix} T_1F & -SF' \\ RF & -T_2F' \end{bmatrix} = \begin{bmatrix} FT_1 & FS \\ -F'R & -F'T_2 \end{bmatrix}$$

Then,

$$FT_1 = T_1F, \tag{10a}$$

$$F'T_2 = T_2F', \tag{10b}$$

$$FS + SF' = 0(Lemma5), \tag{10c}$$

$$RF + F'R = 0(Lemma5) \tag{10d}$$

2.1.8 Lemma 6:

Let the n x n matrix Z(s) be positive real, and suppose that Z(s) + Z'(s) has rank r almost everywhere. Then there exists an r x n matrix W(s) such that:

$$Y(s) = Z(s) + Z'(-s) = W'(-s)W(s) \tag{11}$$

and

1. W has elements which are analytic for Re[s]>0, and for Re[s]>=0 if Z(s) has elements which are analytic for Re[s]>=0
2. rank[W] = r for Re[s] > 0
3. W is unique save for multiplication on the left by an arbitrary orthogonal matrix.

=>

- The proof is derived again using Smith McMillan decomposition. It is present in D.C Youla's "On the factorization of Rational Matrices" paper. It is based on Spectral Factorization.
- This Lemma is one of the important one which we will be using because this converts the parallel network into series network and there by making it easy to derive the proofs.
- This factorization is applied widely in communication theory but due the above property we will use it here.

2.2 Additional Concepts Derived

2.2.1 Lemma 7:

Let Z and W be related as in Lemma 6, [F,G,H] be a minimal realization for Z(s) and [A,K,L] be a minimal realization of W(s). Then the matrices A and F are similar.
=> *Proof :* WKT Z(s) and Z'(-s) are

$$\dot{x}_1 = Fx_1 + Gu_1$$
$$y_1 = H'x_1$$
$$and$$
$$\dot{x}_2 = -F'x_1 + Hu_2$$
$$y_2 = -G'x_2$$

Hence

$$\{F_1, G_1, H_1\} = \left\{ \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ H \end{bmatrix}, \begin{bmatrix} H \\ -G \end{bmatrix} \right\} \tag{12}$$

=> Here $[F_1, G_1, H_1]$ is minimal as $\delta[Z(s) + Z'(s)] = 2\delta[Z(s)]$

Similarly if we W'(-s).W(s) can be wriiten as :

$$\dot{x}_1 = -A'x_1 + Lu_1$$
$$y_1 = -K'x_1$$

and input this y_1 to W(s) so we get:

$$\dot{x}_2 = Ax_2 + Ky_1$$
$$y_2 = L'x_2$$

We have

$$\{F_2, G_2, H_2\} = \left\{ \begin{bmatrix} A & 0 \\ LL" & -A' \end{bmatrix}, \begin{bmatrix} K \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -K \end{bmatrix} \right\} \tag{13}$$

Here $[F_2, G_2, H_2]$ is minimal as $\delta[W'(-s)W(s)] = 2\delta[Z(s)]$

We now apply similarity transform from Lemma 1 on the above realization with T = $\begin{bmatrix} I & 0 \\ P & I \end{bmatrix}$ where P is given by the Lemma 5.
We get :

$$\{F_3, G_3, H_3\} = \left\{ \begin{bmatrix} A & 0 \\ 0 & -A' \end{bmatrix}, \begin{bmatrix} K \\ PK \end{bmatrix}, \begin{bmatrix} PK \\ -K \end{bmatrix} \right\} \tag{14}$$

COROLLARY A: Let Z(s) have a minimal realization [F,G,H] and let Z and W be related as in Lemma 6. Then there exists matrices K,L such that W has a minimal Realization [F,K,L]

COROLLARY B: Two minimal realizations of $Y(s) = Z(s) + Z'(-s) = W'(-s)W(s)$ are given by :

$$\{F_1, G_1, H_1\} = \left\{ \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ H \end{bmatrix}, \begin{bmatrix} H \\ -G \end{bmatrix} \right\} \tag{15}$$

and

$$\{F_3, G_3, H_3\} = \left\{ \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} K \\ PK \end{bmatrix}, \begin{bmatrix} PK \\ -K \end{bmatrix} \right\} \tag{16}$$

where P is given by Lemma 5
=> The corollary A can be derived using Lemma 1 which means that F and A are similar. Similarly corollary B is represented. In the paper there is slight mistake in representing the G_3 matrix. Here I have used the correct form.
Our main motivation would be relate the minimal realizations of Z(s) and W(s) as much as possible. Hence to do so we formulate another Lemma.

2.2.2 Lemma 8:

Let Z(s) have a minimal realization [F,G,H] and let Z and W be related by Lemma 6. Then there exists a matrix such that [F,G,] is a minimal realiza-

tion for W => Proof : WKT W(s) has min. realization [F,K,L] from above corollary. Hence we apply similarity transform (Lemma 1) on the above result such that the T must commute with $\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}$. After applying the transform we get

$$T \begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} K \\ PK \end{bmatrix}$$

By Lemma 5, we get $T_1 G = K$ and $\hat{L} = (T_1)' L$

2.3 Theorems

2.3.1 Theorem 1 :

Let Z(.) be a matrix of rational functions such that $Z(\infty) = 0$ *and Z has poles only in Re[s]<0. Let [F,G,H] be a minimal realization of Z. Then Z(.) is positive real if and only if there exist a symmetric positive definite matrix P and a matrix L such that*

$$PF + F'P = -LL' \tag{17}$$

$$PG = H \tag{18}$$

=> *Sufficiency Proof:* We recall the definition of positive real function. WKT only 3rd condition needs to be verified.

$$\begin{aligned} Z'(s^*) + Z(s) &= G'(s^*I - F')^{-1}H + H'(sI - F)^{-1}G \\ &= G'\{(s^*I - F')^{-1}P + P(sI - F)^{-1}\}G && \text{(by substituti} \\ &= G'(s^*I - F')^{-1}\{Ps + s^*P - PF - F'P\}(sI - F)^{-1}G \\ &= G'(s^*I - F')^{-1}P(sI - F)^{-1}G(s + s^*) \\ &+ G'(s^*I - F')^{-1}LL(sI - F)^{-1}G && \text{(by substitutin} \end{aligned}$$

=> Here $(s + s^*)$ is equivalent to 2Re[s] which is always positive and the 1st term is of the form $A'PA$ where P is always positive hence even this product is always positive. The last is nothing but of the form A^TA so even this is positive, hence $Z'(s^*) + Z(s)$ is positive semi-definite for Re[s]>0.

Necessity Proof: Let W(s) and Z(s) be related as in Lemma 6, and [F,G,L] be minimal realization of W(s. From the above Lemmas WKT

$$\left\{ \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ H \end{bmatrix}, \begin{bmatrix} H \\ -G \end{bmatrix} \right\} \tag{19}$$

and

$$\left\{ \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ PG \end{bmatrix}, \begin{bmatrix} PG \\ -G \end{bmatrix} \right\} \tag{20}$$

are two minimal realizations of Y(s) from Z(s) perspective and W(s) perspective.

Similar to what we did in Lemma 8, here also we will apply similarity transform above with T commuting $\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}$ we get

$$T \begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} G \\ PG \end{bmatrix}$$

and

$$(T^{-1})' \begin{bmatrix} H \\ -G \end{bmatrix} = \begin{bmatrix} PG \\ -G \end{bmatrix}$$

By corollary of Lemma 5 we get, $T_1 G = G$ and $(T_1^{-1})' H = PG$

Next we take controllability matrix i.e.

$$\begin{aligned}
[G, FG, ...] &= [T_1 G, F T_1 G, ...] \quad \text{(substituting } G = T_1 G) \\
&= [T_1 G, T_1 F G, ...] \quad \text{(By taking the commutative law or (1))} \\
&= T_1 [G, FG, ...]
\end{aligned}$$

Hence $T = I$, and $PG = H$

2.3.2 Theorem 2:

Let a positive real Z(s) have all pure imaginary poles with $Z(\infty) = 0$ *and let [F,G,H] be a minimal realization for Z. Then there exists a symmetric positive definite P such that:*

$$PF + F'P = 0, \tag{21}$$

$$PG = H. \tag{22}$$

=> *Proof:* Here $P^* = (T')^{-1}PT^{-1}$ means that if P is a part of any one minimal realization then it is also applicable for all the minimal realization by the similarity theorem. Here we consider

$$Z(s) = \sum_i \frac{A_i s + B_i}{s^2 + \omega_i^2} \tag{23}$$

if we consider $[F_i, G_i, H_i]$ and the minimal realization of any element and P_i from the above equation, with the help of direct sum techniques we can write as :

$$F = \oplus_i F_i$$

$$P = \oplus_i P_i$$

where $\oplus$ denotes direct sum.

Then Z(s) can be written in a simpler form:

$$Z(s) = \frac{As + B}{s^2 + \omega_0^2}$$

Z(s) can be written using complex vectors as :

$$Z(s) = \frac{xx^{*\prime}}{s - j\omega_0} + \frac{x^* x'}{s + j\omega_0}$$

After adding and subtracting $xx', x^*x^{*\prime}$ and substituting

$$y_1 = \frac{x + x^*}{\sqrt{2}}, y_2 = j\frac{x - x^*}{\sqrt{2}}$$

we get Z(s) as :

$$Z(s) = [y_1 y_2] \frac{1}{s^2 + \omega_0^2} \begin{bmatrix} s & \omega_0 \\ -\omega_0 & s \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} \tag{24}$$

This is in the canonical form of Z(s) where $s^2 + \omega_0^2$ is determinant and with the middle matrix it is $(sI - F)^{-1}$ matrix. The other matrices are H' and G. Hence we get the minimal representation as

$$[F, G, H, P] = \left\{ \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix}, \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}, \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

2.3.3 Theorem 3:

Let Z(.) be a matrix of rational transfer functions such that $Z(\infty)$ is finite and Z has poles which lie in Re[s]<0 or are simple imaginary poles. Let $[F, G, H, Z(\infty)]$ be a minimal realization of Z. Then Z(.) is positive real if and only if there exist a symmetric positive definite P and matrices W_0 and L such that:

$$PF + F'P = -LL' \tag{25}$$

$$PG = H - LW_0 \tag{26}$$

$$W_0' W_0 = Z'(\infty) + Z(\infty) \tag{27}$$

=> Here W_0 and $Z(\infty)$ are nothing the feed-forward matrix of their respective state space representation *Sufficiency Proof:* Similar to Theorem 1 we consider:

$$Z'(s^*) + Z(s) = Z'(\infty) + Z(\infty) + G'(s^*I - F')^{-1}H + H'(sI - F)^{-1}G$$

(by substitutin

(by substitutin

$$+ G'(s^*I - F')^{-1}P(sI - F)^{-1}G(s + s^*)$$

Similar to theorem we can see that this equation is positive definite. Necessity Proof: $[F, G, L, W_0]$ be minimal realization of Y(s). Consider only for real poles then, from theorem 1 by replacing the equation of PG we get :

$$[F_1, G_1, H_1, W_0'W_0] = \left\{ \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ H \end{bmatrix}, \begin{bmatrix} H \\ -G \end{bmatrix}, W_0'W_0 \right\} \tag{28}$$

and

$$[F_3, G_3, H_3, W_0'W_0] = \left\{ \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ PG + LW_0 \end{bmatrix}, \begin{bmatrix} PG + LW_0 \\ -G \end{bmatrix}, W_0'W_0 \right\} \tag{29}$$

Now let us relax the condition on poles

$$Z(s) = Z_1(s) + Z_2(s) \tag{30}$$

Then selecting the condition from theorem2 and substituting

$$P = P_1 + P_2 \tag{31}$$

$$F = F_1 + F_2 \tag{32}$$

$$G' = [G_1', G_2'], \tag{33}$$

$$H' = [H_1', H_2'], \tag{34}$$

$$L' = [0, L_2'], \tag{35}$$

$$\tag{36}$$

Thus (23),(24),(25) can be verified.

3 Conclusion

1. This Criterion later came to be known as Positive Real(PR) Lemma. It provides a conceptual link between the Network theory and Control Theory.

2. This criterion solved the problem negative storage function problem, which was solved using only Nyquist criterion.

3. The one drawback of this criterion is that it is valid under a state space co-ordinate transformation, as they have merely been established for a particular class of F(of the form F1 + F2)

4 Future Scope

1. It was the basic idea behind the development of Kalman-Yakubovich-Popov lemma.

$$PA + A'P = -LL' - \epsilon P$$
$$PB = C' - L'W$$
$$W'W = D + D^T$$

2. This criterion solved the problem negative storage function problem, which was solved using only Nyquist criterion.

3. A scientist named Khalil worked on the above Lemma and verified it to a no. of systems. One of his work is there in the references.

4. The one drawback of this criterion is that it is valid under a state space co-ordinate transformation, as they have merely been established for a

particular class of F(of the form F1 + F2)

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