Functional Analysis

Closed Graph Theorem

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Nov 11th , 2023

Definition and Theorem that required in proof of Closed Graph Theorem

Banach space: A Normed Linear Space X is called a **Banach space** if it is complete with respect to the given norm $\|\cdot\|_X$. In other words, every Cauchy sequence in X converges to a limit in X.

Open Map :
Let X and Y be topological spaces. A function $f : X \to Y$ is called an **open map** if, for every open set U in X, the image $f(U)$ is an open set in Y. In other words, f preserves the openness of sets.

Thorem

Theorem (Bounded Inverse Theorem)

Let X and Y be Banach spaces, and $T : X \to Y$ be a bounded linear operator. If T is bijective, then its inverse $T^{-1} : Y \to X$ is also a bounded linear operator.

Theorem (A)

Let X and Y be NLS , and $T : X \to Y$ be a linear operator. $Ker(T)$ is Null space of T . Define $\bar{T} : X/ker(T) \to Y$ by $\bar{T}([x]) = Tx$ for $x \in X$. Then T is an open map if and only if $\bar{T}$ is an open map.

The Closed Graph Theorem

Theorem (The Closed Graph Theorem)

Let X and Y be Banach spaces, and $T : X \to Y$ be a linear transformation such that the graph of T, denoted by $G(T) = \{(x, Tx) \in X \times Y \mid x \in X\} \subseteq X \times Y$, is closed. Then T is continuous.

Proof:

X , Y : Banach spaces $\Rightarrow X \times Y$ is a Banach space,
$G(T)$ is closed $\Rightarrow G(T)$ is also a Banach space.
Define map $P : G(T) \to X$
by $P(x, Tx) = x$.
To show that P is a bijection, linear, and bounded

Continued..

One-One: Suppose $P(x_1, Tx_1) = P(x_2, Tx_2)$.

$$\Rightarrow \quad x_1 = x_2,$$
$$\Rightarrow (x_1, Tx_1) = (x_2, Tx_2).$$

Therefore, P is One-One.

Onto: *For* $x \in X$

we have $(x, Tx) \in G(T)$

s.t. $P(x, Tx) = x$

hence P *is onto*

Continued.

Linear: Let (x_1, Tx_1) and (x_2, Tx_2) be in $G(T)$ and α in K.

$$\begin{aligned}
P((x_1, Tx_1) + (x_2 + Tx_2)) &= P(x_1 + x_2, T(x_1 + x_2)) \\
&= x_1 + x_2 \\
&= P(x_1, T(x_1)) + P(x_2, T(x_2)) \\
P(\alpha(x_1, T(x_1))) &= P(\alpha x_1, \alpha T(x_1)) \\
&= P(\alpha x_1, T(\alpha x_1)) \\
&= \alpha x_1 \\
&= \alpha P(x_1, T(x_1))
\end{aligned}$$

Hence P is linear.

Bdd: $$\begin{aligned}\|P(x, T(x))\|_X &= \|x\|_X \\ &\leq \max\{\|x\|_X, \|Tx\|_Y\} \\ &= \|(x, T(x))\|_*\end{aligned}$$

Hence, P is bounded. $\|\cdot\|_*$ **denote the norm defined on** $X \times Y$.

Continued...

So, P is linear, bijective, and bounded. Hence, by the Bounded Inverse Theorem, $P^{-1} : X \to G(T)$ is continuous.
Now, define $Q : G(T) \to Y$ by $Q(x, Tx) = Tx$.
Linear: Let (x_1, Tx_1) and (x_2, Tx_2) be in $G(T)$ and α in K.

$$\begin{aligned}
Q((x_1, Tx_1) + (x_2, Tx_2)) &= Q(x_1 + x_2, T(x_1 + x_2)) \\
&= Tx_1 + Tx_2 \\
&= Q((x_1, T(x_1))) + Q((x_2, T(x_2))) \\
Q(\alpha(x_1, Tx_1)) &= Q(\alpha x_1, \alpha T(x_1)) \\
&= T(\alpha x_1) \\
&= \alpha x_1 \\
&= \alpha Q((x_1, T(x_1)))
\end{aligned}$$

Hence Q is linear.

Continued...

Bdd: $\|Q((x, T(x))\|_Y = \|T(x)\|_Y$
$$\leq \max\{\|x\|_X, \|T(x)\|_Y\}$$
$$= \|(x, T(x))\|_*$$
Hence, Q *is* **bounded**, and Q is **continuous** as Q is **linear**.

$Q \circ P^{-1} : X \to Y$ is a composition of two continuous functions, and as the composition of two continuous functions is again continuous, we conclude that
$Q \circ P^{-1}$ is a continuous function.
Moreover:
$Q \circ P^{-1}(x) = Q(P^{-1}(x))$
$=$ Q((x,T(x))) $=$ T(x)
$\Rightarrow Q \circ P^{-1} = T$
$\Rightarrow T$ is **continuous** *as* $Q \circ P^{-1}$ is **continuous**.

CLOSED GRAPH THEOREM IMPLIES OPEN MAPPING THEOREM

Theorem (Open Mapping Theorem)

Suppose X and Y are Banach spaces, and $T : X \to Y$ is a surjective (onto) bounded linear operator. Then, T is an open map, meaning that for any open set U in X, the image $T(U)$ is an open set in Y.

Proof :

CASE I: T is ONE-ONE :

T is one-one $\Rightarrow$ T^{-1} exists

By the Bounded Inverse Thoerem T^{-1} is bounded , i.e. continuous

.

Hence for any open set U in X $(T^{-1})^{-1}(U)$ is open .

$\Rightarrow$T(U) *is open ,i.e T is an open map.*

CASE II : T is not ONE-ONE:

Define $\bar{T} : X/\ker(T) \to Y$ by $\bar{T}([x]) = Tx$ for x $\in X$.

Linear:

let $[x_1], [x_2] \in X/\ker(T)$, and let $a \in \mathbb{K}$. Then,

$$\begin{aligned}\bar{T}(a[x_1] + [x_2]) &= \bar{T}([ax_1 + x_2]) \\ &= T(ax_1 + x_2) \\ &= aTx_1 + Tx_2 \\ &= a\bar{T}([x_1]) + \bar{T}([x_2]).\end{aligned}$$

Hence, $\bar{T}$ is linear.

Continued...

One-One :
Assume $\bar{T}([x_1]) = \bar{T}([x_2])$ for some $[x_1], [x_2] \in X/\ker(T)$.
$\Rightarrow \mathrm{T}\mathrm{x}_1 = Tx_2$
$\Rightarrow \mathrm{x}_1 - x_2 \in \ker(T)$
which means $[x_1] = [x_2]$.
Therefore, $\bar{T}$ is One-One.

Onto:
Let $y \in Y$. Since T is surjective, there exists $x \in X$ such that $Tx = y$.
Consider $[x] \in X/\ker(T)$.
Then, $\bar{T}([x]) = Tx = y$.
Thus, $\bar{T}$ is surjective.

Therefore, $\bar{T}$ is a linear, injective, and surjective map.

Continued...

Hence Inverse on $\bar{T}$ exists , $\bar{T}^{-1} : Y \to X/\ker(T)$
For any $y_1, y_2 \in Y$, let $[x_1] = \bar{T}^{-1}(y_1)$ and $[x_2] = \bar{T}^{-1}(y_2)$.
Let,

$$\bar{T}^{-1}(y_1 + y_2) = [x]$$

$$\Rightarrow [x_1] + [x_2] = \bar{T}^{-1}(y_1) + \bar{T}^{-1}(y_2)$$

$$\Rightarrow \bar{T}([x_1] + [x_2]) = \bar{T}(\bar{T}^{-1}(y_1) + \bar{T}^{-1}(y_2))$$

$$\Rightarrow \bar{T}([x_1] + [x_2]) = y_1 + y_2$$

$$\Rightarrow [x_1] + [x_2] = \bar{T}^{-1}(y_1 + y_2)$$

$$\Rightarrow \bar{T}^{-1}(y_1 + y_2) = \bar{T}^{-1}(y_1) +^{-1}(y_2).$$

Continued...

For any $y \in Y$ and scalar α,
let $[x] = \bar{T}^{-1}(y)$. Then,

$$\alpha[x] = \alpha \bar{T}^{-1}(y)$$

$$\Rightarrow [\alpha x] = \alpha \bar{T}^{-1}(y)$$

$$\Rightarrow \bar{T}([\alpha x]) = \alpha \bar{T}(\bar{T}^{-1}(y))$$

$$\Rightarrow \bar{T}([\alpha x]) = \alpha y$$

$$\Rightarrow \alpha[x] = \bar{T}^{-1}\alpha y.$$

Hence $\bar{T}^{-1}$ is Linear .
Therefore, $\bar{T}^{-1}$ is a linear, injective, and surjective map.

Continued...

Also Graph of $\bar{T}^{-1}$ is closed .
Hence by Closed Graph Theorem , $\bar{T}^{-1}$ is bounded , that is $\bar{T}$ is Open map . Hence by THEOREM(A) . T is open Map.

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This conclude the proof.
Thank You