$$G_r(t+1,\omega) \;=\; e^{-2\pi i\omega}\, G_r(t,\omega)\,\delta_r(t,\omega)$$

for every $t$ and $\omega$.

Now, we have that

$$\begin{aligned} |G_r(1,\omega)|\, e^{i\varphi_r(1,\omega)} = G_r(1,\omega) \;&=\; e^{-2\pi i\omega}\, G_r(0,\omega)\,\delta_r(0,\omega) \\ &= |G_r(0,\omega)\,\delta_r(0,\omega)|\, e^{-2\pi i\omega + i\theta_r(0,\omega)+i\varphi_r(0,\omega)} \\ &=\; |G_r(1,\omega)|\, e^{-2\pi i\omega + i\theta_r(0,\omega)+i\varphi_r(0,\omega)}. \end{aligned}$$

Hence for each $r > 0$ there is an integer $l_r$ such that for all $\omega \in [0,1]$,

$$\varphi_r(1,\omega) \;=\; -2\pi\omega + \theta_r(0,\omega) + \varphi_r(0,\omega) + 2\pi l_r.$$

Finally, we have

$$\begin{aligned} 0 \;&=\; \big(\varphi_r(0,0)-\varphi_r(1,0)\big) \;+\; \big(\varphi_r(1,0)-\varphi_r(1,1)\big) \\ &\qquad +\; \big(\varphi_r(1,1)-\varphi_r(0,1)\big) \;+\; \big(\varphi_r(0,1)-\varphi_r(0,0)\big) \\ &=\; (-\theta_r(0,0) - 2\pi l_r) \;+\; (-2\pi k_r) \;+\; (-2\pi + \theta_r(0,1) + 2\pi l_r) \;+\; (2\pi k_r) \\ &=\; -2\pi + \theta_r(0,1) - \theta_r(0,0). \end{aligned}$$

Letting $r \to 0$, we find that $0 = -2\pi$, a contradiction. $\square$

## 7. The Uncertainty Principle Approach to the BLT

We shall present a proof of the BLT due to Daubechies and Janssen [DJ93], based on the operator theory associated with the Classical Uncertainty Principle Inequality and inspired by the elegant proof of the BLT for orthornormal bases by Battle [Bat88].

We shall use the basic properties of frames, Gabor systems, and the Zak transform discussed in §2.1, §2.2, and §3.1. In particular, we may consider only Gabor systems with $a = b = 1$. In this case, $\{g_{m,n}\}$ is a frame for $L^2(\mathbf{R})$ if and only if it is an exact frame (Theorem 3.1d). Therefore, the dual frame $\{\tilde{g}_{m,n}\}$ is biorthogonal to $\{g_{m,n}\}$, that is, $\langle g_{m,n}, \tilde{g}_{m',n'}\rangle = \delta_{m,m'}\delta_{n,n'}$. Also, $Zg_{m,n}(t,\omega) = e_m(t)\, e_n(\omega) Zg(t,\omega)$.

### 7.1. Uncertainty Principles

The Classical Uncertainty Principle Inequality was stated in Theorem 1.2. We prove it for $f \in \mathcal{S}(\mathbf{R})$:

$$\begin{aligned} \|f\|_2^2 \;&=\; \int t\,\big(|f|^2\big)'(t)\,dt \;\le\; \int |t|\,\big(2\,|\overline{f(t)}\,f'(t)|\big)\,dt \\ &\le\; 2\,\|tf(t)\|_2\,\|f'\|_2 \\ &=\; 4\pi\,\|tf(t)\|_2\,\|\gamma\hat{f}(\gamma)\|_2. \end{aligned}$$

A standard closure argument (for example, [Ben94, Remark 7.31]), which we also use in Lemma 7.2, extends the inequality to all $f \in L^2(\mathbf{R})$. There is a theory of weighted and local uncertainty principles emanating from Theorem 1.2 and its Hilbert space analogue, Theorem 7.1. This theory is the subject of [Ben94, Section 7.8], [Ben96, Chapter 6].

The Classical Uncertainty Principle Inequality can be formulated in general Hilbert spaces. Given a Hilbert space $H$ with inner product $\langle f, g\rangle$ and norm $\|f\| = \langle f, f\rangle^{1/2}$ and given operators $A$, $B$ mapping $H$ into $H$ (or mapping domains $D(A)$, $D(B) \subseteq H$ into $H$), we define the *commutator* of $A$ and $B$ to be

$$[A, B] \;=\; AB - BA.$$

If $A$ is *selfadjoint* (i.e., $\langle Af, f\rangle = \langle f, Af\rangle$ for all $f \in H$), then the *expectation* of $A$ at $f \in D(A)$ is $E_f(A) = \langle Af, f\rangle$ and the *variance* of $A$ at $f \in D(A^2)$ is $\sigma_f^2(A) = E_f(A^2) - (E_f(A))^2$.

### *7.1. Theorem*

*Given selfadjoint (but not necessarily continuous) operators $A$, $B$ on a Hilbert space $H$, if $f \in D(A^2) \cap D(B^2) \cap D(i[A, B])$ and $\|f\| = 1$, then*

$$E_f(i[A, B])^2 \;\le\; 4\,\sigma_f^2(A)\,\sigma_f^2(B).$$

The proof of Theorem 7.1 is not difficult (for example, [Ben94, Theorem 7.32]), and we obtain the Classical Uncertainty Principle Inequality as a corollary, as follows. Define the *position operator* $P$ and *momentum operator* $M$ by

$$Pf(t) \;=\; t\,f(t) \quad \text{and} \quad Mf \;=\; (P\hat{f})^{\vee} \;=\; (\gamma\,\hat{f}(\gamma))^{\vee},$$

when these make sense. Both $P$ and $M$ are selfadjoint and if $f \in \mathcal{S}(\mathbf{R})$, then $(f')^{\wedge} = 2\pi i P\hat{f}$, $f' = 2\pi i Mf$, and $[P, M]f = -\frac{1}{2\pi i} f$. By Theorem 7.1, we have

$$E_f(-\tfrac{1}{2\pi} I)^2 \;\le\; \sigma_f^2(P)\,\sigma_f^2(M),$$

where $I$ is the identity operator. On the other hand, $E_f(I) = \langle If, f\rangle = \|f\|_2^2$, and

$$\sigma_f^2(P) \;=\; \|Pf\|_2^2 - \langle Pf, f\rangle^2 \;\le\; \|Pf\|_2^2 \;=\; \|tf(t)\|_2^2,$$

and

$$\sigma_f^2(M) \;=\; \|Mf\|_2^2 - \langle Mf, f\rangle^2 \;\le\; \|Mf\|_2^2 \;=\; \|\gamma\,\hat{f}(\gamma)\|_2^2,$$

since both $\langle Pf, f\rangle = \int t\,|f(t)|^2\,dt$ and $\langle Mf, f\rangle = \int \gamma\,|\hat{f}(\gamma)|^2\,d\gamma$ are real, hence have nonnegative squares. The Classical Uncertainty Principle Inequality follows immediately.

## 7.2. The Weak BLT

The fact that $[P, M] = -\frac{1}{2\pi i} I$ forms the core of the uncertainty principle approach to proving the BLT. We state this fact in the following form.

### 7.2. *Lemma*

*If* $f, g \in L^2(\mathbf{R})$ *satisfy* $Pf, Pg \in L^2(\mathbf{R})$ *and* $P\hat{f}, P\hat{g} \in L^2(\hat{\mathbf{R}})$*, then*

$$\langle Pf, Mg\rangle - \langle Mf, Pg\rangle \;=\; \frac{1}{2\pi i}\langle f, g\rangle. \tag{7.1}$$

**Proof.** $Mf$ and $Mg$ are well defined since $P\hat{f}, P\hat{g} \in L^2(\hat{\mathbf{R}})$. By standard techniques we can find $\varphi_k, \psi_k \in \mathcal{S}(\mathbf{R})$ such that $\varphi_k \to f$, $P\varphi_k \to Pf$, $M\varphi_k \to Mf$, and $\psi_k \to g$, $P\psi_k \to Pg$, $M\psi_k \to Mg$, all in $L^2$-norm. Since $\varphi_k, \psi_k \in \mathcal{S}(\mathbf{R})$ and $P$, $M$ are selfadjoint, we have

$$\begin{aligned}
\langle P\varphi_k, M\psi_k\rangle - \langle M\varphi_k, P\psi_k\rangle \;&=\; \langle MP\varphi_k, \psi_k\rangle - \langle PM\varphi_k, \psi_k\rangle \\
&=\; -\langle [P, M]\varphi_k, \psi_k\rangle \\
&=\; \frac{1}{2\pi i}\langle \varphi_k, \psi_k\rangle.
\end{aligned}$$

However, the inner product is continuous, so $\langle \varphi_k, \psi_k\rangle \to \langle f, g\rangle$, $\langle P\varphi_k, M\psi_k\rangle \to \langle Pf, Mg\rangle$, and $\langle M\varphi_k, P\psi_k\rangle \to \langle Mf, Pg\rangle$. Therefore (7.1) holds. □

Next, we compute the commutators of $P$ and $M$ with the translation and modulation operators $\tau_n$ and $e_m$ defined in (2.3).

### 7.3. *Lemma*

**(a)** $[e_m\tau_n, P] = e_m\tau_n P - Pe_m\tau_n = -n\, e_m\tau_n$.

**(b)** $[e_m\tau_n, M] = e_m\tau_n M - Me_m\tau_n = -m\, e_m\tau_n$.

**Proof.** As the two parts are similar, we prove only part a. We compute

$$\begin{aligned}
(e_m\tau_n Pf)(t) \;-\; (Pe_m\tau_n f)(t) \;&=\; e_m(t)\,(\tau_n Pf)(t) \;-\; t\,(e_m\tau_n f)(t) \\
&=\; e_m(t)\,(Pf)(t-n) \;-\; t\,e_m(t)\,(\tau_n f)(t) \\
&=\; e_m(t)\,(t-n)\,f(t-n) \;-\; t\,e_m(t)\,f(t-n) \\
&=\; -n\,e_m(t)\,f(t-n) \\
&=\; -n\,e_m(t)\,(\tau_n f)(t) \\
&=\; -n\,(e_m\tau_n f)(t). \qquad \square
\end{aligned}$$

We can now prove a weak version of the BLT.

### 7.4. Theorem (Weak BLT)

*Assume $g \in L^2(\mathbf{R})$ is such that $\{g_{m,n}\}$ is an exact frame for $L^2(\mathbf{R})$. Then we cannot have all of $Pg,\ P\tilde{g} \in L^2(\mathbf{R})$ and $P\hat{g},\ P\hat{\tilde{g}} \in L^2(\hat{\mathbf{R}})$, that is, we must have*

$$\|tg(t)\|_2\ \|\gamma\hat{g}(\gamma)\|_2\ \|t\tilde{g}(t)\|_2\ \|\gamma\hat{\tilde{g}}(\gamma)\|_2 \ = \ +\infty.$$

**Proof.** Assume all four functions were elements of $L^2$. Note that

$$\forall\, f, h \in L^2(\mathbf{R}), \qquad \langle f, h_{m,n}\rangle \ = \ \langle f_{-m,-n}, h\rangle.$$

Also, by Lemma 7.3a,

$$\forall\, f \in L^2(\mathbf{R}), \qquad P(f_{m,n}) \ = \ (Pf)_{m,n} \ + \ n\, f_{m,n}.$$

Since $P$ is selfadjoint and $\{g_{m,n}\}$ is biorthonormal to its dual frame $\{\tilde{g}_{m,n}\}$, we can therefore compute

$$\begin{aligned}
\langle Pg, \tilde{g}_{m,n}\rangle \ &= \ \langle g, P(\tilde{g}_{m,n})\rangle \ = \ \langle g, (P\tilde{g})_{m,n})\rangle \ + \ n\,\langle g, \tilde{g}_{m,n})\rangle \\
&= \ \langle g_{-m,-n}, P\tilde{g}\rangle \ + \ n\,\delta_{m,0}\,\delta_{n,0} \\
&= \ \langle g_{-m,-n}, P\tilde{g}\rangle.
\end{aligned}$$

Now, by the $L^2$-inversion formula, both $Mg$ and $M\tilde{g}$ exist and are in $L^2(\mathbf{R})$, so by Lemma 7.3b we similarly obtain

$$\langle g_{m,n}, M\tilde{g}\rangle \ = \ \langle M(g_{m,n}), \tilde{g}\rangle \ = \ \langle (Mg)_{m,n}, \tilde{g}\rangle \ + \ m\,\langle g_{m,n}, \tilde{g}\rangle \ = \ \langle Mg, \tilde{g}_{-m,-n}\rangle.$$

Since $f = \sum \langle f, g_{m,n}\rangle\, \tilde{g}_{m,n} = \sum \langle f, \tilde{g}_{m,n}\rangle\, g_{m,n}$ for every $f \in L^2(\mathbf{R})$, we therefore have

$$\begin{aligned}
\langle Pg, M\tilde{g}\rangle \ &= \ \sum_{m,n} \langle Pg, \tilde{g}_{m,n}\rangle\, \langle g_{m,n}, M\tilde{g}\rangle \\
&= \ \sum_{m,n} \langle g_{-m,-n}, P\tilde{g}\rangle\, \langle Mg, \tilde{g}_{-m,-n}\rangle \\
&= \ \sum_{m,n} \langle Mg, \tilde{g}_{m,n}\rangle\, \langle g_{m,n}, P\tilde{g}\rangle \\
&= \ \langle Mg, P\tilde{g}\rangle.
\end{aligned}$$

Therefore, by biorthonormality and Lemma 7.2,

$$1 \ = \ \langle g, \tilde{g}\rangle \ = \ 2\pi i \Big( \langle Pg, M\tilde{g}\rangle - \langle Mg, P\tilde{g}\rangle \Big) \ = \ 0,$$

a contradiction. $\square$

### 7.3. Equivalence of the weak BLT and the BLT

We can give several special cases illustrating the relationship between the weak BLT and the usual BLT (Theorem 1.1).

**7.5. Example.** a. If the Gabor frame $\{g_{m,n}\}$ is actually an orthonormal basis, then $\tilde{g} = g$ and the equivalence is clear. This is precisely Battle's proof of the BLT [Bat88].

b. If $g$ generates a tight exact frame with bounds $A = B$, then $\tilde{g} = S^{-1}g = A^{-1}g$, and the equivalence is again clear. However, any tight exact frame is a multiple of an orthonormal basis.

c. If $\operatorname{supp}(g)$ is contained in an interval of length 1, then the frame operator $S$ is $Sf = f \cdot \lambda$, where $\lambda(t) = \sum |g(t-n)|^2$ (for example, [HW89]). Any Gabor frame must have $A \le \lambda(t) \le B$ a.e. (Theorem 3.5), so $S$ is multiplication by an essentially constant function. Therefore, $\tilde{g} = S^{-1}\tilde{g} = g/\lambda$, and thus $Pg \in L^2(\mathbf{R})$ if and only if $P\tilde{g} \in L^2(\mathbf{R})$. Similarly, if $\operatorname{supp}(\hat{g})$ is contained in an interval of length 1, then $(Sf)^\wedge = \hat{f} \cdot \Lambda$, where $\Lambda$ is the essentially constant function $\sum |\hat{g}(\gamma - m)|^2$, and so $P\hat{g} \in L^2(\hat{\mathbf{R}})$ if and only if $P\hat{\tilde{g}} \in L^2(\hat{\mathbf{R}})$. □

The BLT will follow from the weak BLT if we can prove that

$$Pg \in L^2(\mathbf{R}) \;\Leftrightarrow\; P\tilde{g} \in L^2(\mathbf{R}) \quad \text{and} \qquad P\hat{g} \in L^2(\hat{\mathbf{R}}) \;\Leftrightarrow\; P\hat{\tilde{g}} \in L^2(\hat{\mathbf{R}}), \tag{7.2}$$

whenever $\{g_{m,n}\}$ is an exact frame. We verify (7.2) in Theorem 7.7. First, however, we compute the Zak transform of the dual function $\tilde{g}$.

### *7.6. Proposition*

*If $g \in L^2(\mathbf{R})$ and $\{g_{m,n}\}$ is a frame, then*

$$Z\tilde{g} \;=\; 1/\overline{Zg}.$$

**Proof.** If $\{g_{m,n}\}$ is a frame, then $0 < A \le |Zg|^2 \le B < \infty$ a.e. on $Q$. Therefore, $h = Z^{-1}(1/\overline{Zg}) \in L^2(\mathbf{R})$. Given $m, n \in \mathbf{Z}$ we then compute

$$\begin{aligned}
\langle h, g_{m,n}\rangle \;&=\; \langle Zh, Zg_{m,n}\rangle \;=\; \langle 1/\overline{Zg}, e_m(t)e_n(\omega)Zg\rangle \\
&=\; \langle 1, e_m(t)e_n(\omega)\rangle \\
&=\; \delta_{m,0}\,\delta_{n,0} \\
&=\; \langle \tilde{g}, g_{m,n}\rangle.
\end{aligned}$$

Since $\{g_{m,n}\}$ is complete in $L^2(\mathbf{R})$ and $h, \tilde{g} \in L^2(\mathbf{R})$, it follows that $h = \tilde{g}$. □

The following theorem is due to Daubechies and Janssen [DJ93].

### *7.7. Theorem*

*If $\{g_{m,n}\}$ is an exact frame, then* (7.2) *holds.*