The Uncertainty Principle Approach to the Balian-Low Theorem

A Project Report Submitted for the Course

MA699 Project

by

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CERTIFICATE

This is to certify that the work contained in this report entitled **"The Uncertainty Principle Approach to the Balian-Low Theorem"** submitted by **Purushottam Priyam Rathaur** (**Roll No: 222123039**) to Department of Mathematics, Indian Institute of Technology Guwahati towards the requirement of the course **MA699 Project** has been carried out by him under my supervision.

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ABSTRACT

The aim of the project is to study The Uncertainty Principle Approach to Balian-Low theorem to higher dimensions, focusing on symplectic lattices in $\mathbb{R}^{2d}$ and for Riesz bases to higher dimension.

Contents

List of notations

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$\mathbb{R}$: The set of all real numbers.

$\mathbb{R}^n$: The n-dimensional real Euclidean space.

$\mathcal{C}_c(\mathbb{R}^n)$: The set of all real-valued continuous functions on $\mathbb{R}^n$ having compact support.

$\mathcal{C}_0(\mathbb{R}^n)$: The set of all real-valued continuous functions on $\mathbb{R}^n$ vanishing at infinity.

$\mathcal{M}(\mathbb{R}^n)$: The class of all finite Borel measures on $\mathbb{R}^n$.

$L^p(\mathbb{R}^n)$: The space of all measurable functions whose p-th power is integrable, where $1 \leq p < \infty$.

$\hat{f}$: The Fourier transform of f.

Λ : The lattice, a discrete subgroup of $\mathbb{R}^n$.

$\mathcal{G}(g, \Lambda)$: A Gabor system forming an orthonormal or Riesz basis for $L^2(\mathbb{R})$.

$GL(n, \mathbb{R})$: The set of all invertible matrices of size $n \times n$ with real entries.

$\mathcal{S}(\mathbb{R})$: The Schwartz space

$\{g_{m,n}\}$: Exact frame for $L^2(\mathbb{R})$.

$\Re(z)$: Real part of z ,where z ∈C.

Chapter 1

Position and Momentum Operator and Uncertainty Princuple

1.1 Position Operator and Momentum Operator

Let f be a function belongs to Schwartz space $\mathcal{S}(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$.
Then the position operator X_j and momentum operator P_j are defined as:

$$(X_j f)(x) = x_j f(x) \quad \text{for } j = 1, 2, 3, \ldots, d$$

$$(P_j f)(x) = \frac{1}{2\pi i} \frac{\partial f}{\partial x_j} \quad \text{for } j = 1, 2, 3, \ldots, d$$

where $x = (x_1, x_2, \ldots, x_d)$ is a point in $\mathbb{R}^d$.

$$\|X_j f\|_{L^2(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |X_j(f(x))|^2 \, dx \right)^{1/2}$$

$$\|P_j f\|_{L^2(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |P_j(f(x))|^2 \, dx \right)^{1/2}$$

1.2 Translation and Modulation operator

For $x, \omega \in \mathbb{R}^d$, we define

Translation operator: $T_x f(t) = f(t-x)$

Modulation operator : $M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)$.

These both operators are unitary and we denote the corresponding time–frequency shift by

$$\boxed{\pi(z) f(t) = M_\omega T_x f(t) = e^{2\pi i \omega \cdot t} f(t-x)}$$

where $z = (x, \omega) \in \mathbb{R}^{2d}$ for a point in the time–frequency plane $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$,

1.3 Theorem Uncertainty Thoerem

1.3.1 The classical uncertainty principle inequality

If $f \in L^2(\mathbb{R})$ and $x_0, \gamma_0 \in \mathbb{R}$, then

$$\|f\|_2^2 \leq 4\pi \|(x-x_0) f(x)\|_2 \|(\gamma-\gamma_0) \hat{f}(\gamma)\|_2, \tag{1.1}$$

where $\|f\|_2^2 = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$.

Moreover, there is equality if and only if

$$f(x) = C e^{2\pi i x \gamma_0} e^{-s(x-x_0)^2}, \tag{1.2}$$

for $C \in \mathbb{C}$ and $s > 0$. Here, $|| \cdot ||_2$ designates the L^2 norm, and the Fourier transform $\hat{f}(\gamma)$ of f is formally defined as

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \gamma} \, dx. \tag{1.3}$$

Proof : We prove it for $f \in \mathcal{S}(\mathbb{R})$. For sake of convenience we may assume $x_0 = \gamma_0 = 0$

$$\int_{\mathbb{R}} (xf(x))\overline{f'(x)}\, dx = x|f(x)|^2\Big|_{x=-\infty}^{x=\infty} - \int_{\mathbb{R}} \left(|f(x)|^2 + xf'(x)\overline{f(x)}\right)\, dx.$$

As $f \in \mathcal{S}(\mathbb{R})$ then , $\lim_{x\to\pm\infty} x|f(x)|^2 = 0$, hence equation can be re-arranged as ,

$$\int_{\mathbb{R}} |f(x)|^2\, dx = -\int_{\mathbb{R}} xf(x)\overline{f'(x)}\, dx - \int_{\mathbb{R}} x\overline{f(x)}f'(x)\, dx = -2\Re\left(\int_{\mathbb{R}} xf(x)\overline{f'(x)}\, dx\right),$$

as $x \in \mathbb{R}$.

$$\left|-2\Re\left(\int_{\mathbb{R}} xf(x)\overline{f'(x)}\, dx\right)\right| \leq 2\left|\int_{\mathbb{R}} xf(x)\overline{f'(x)}\, dx\right|.$$

Now, we use the Cauchy-Schwarz inequality to obtain

$$\left|\int_{\mathbb{R}} -xf(x)\overline{f'(x)}\, dx\right| = |\langle xf, f'\rangle| \leq \|-xf\|_2\|f'\|_2 = \left(\int_{\mathbb{R}} x^2|f(x)|^2\, dx\right)^{1/2}\left(\int_{\mathbb{R}} |f'(x)|^2\, dx\right)^{1/2}.$$

$$\implies \int_{\mathbb{R}} |f(x)|^2\, dx \leq 2\left(\int_{\mathbb{R}} x^2|f(x)|^2\, dx\right)^{1/2}\left(\int_{\mathbb{R}} |f'(x)|^2\, dx\right)^{1/2}.$$

By Plancherel's theorem and the Fourier transform of derivatives, we have

$$\left(\int_{\mathbb{R}} |f'(x)|^2\, dx\right)^{1/2} = \left(\int_{\mathbb{R}} |(2\pi i\gamma)\hat{f}(\gamma)|^2\, d\gamma\right)^{1/2}.$$

Combining this with the previous lines, we obtain

$$\frac{\|f\|_2^2}{4\pi} \leq \|xf\|_2\|\gamma\hat{f}\|_2$$

$$\implies \|f\|_2^2 \leq 4\pi\|xf\|_2\|\gamma\hat{f}\|_2$$

And the equality holds only when $\boxed{-x\bar{f(x)} = kf'(x)}$ where $k \geq 0$. We can express $f(x)$ as

$$\begin{aligned} f(x) &= u(x) + iv(x) \\ \implies -xu(x) &= ku'(x), \\ \text{and } xv(x) &= kv'(x) \end{aligned}$$

Solving these differential equations, we get

$$\begin{aligned} -xu(x) &= ku'(x) \\ -xdx &= \frac{k}{u(x)}du(x) \\ -x^2/2k + c_1 &= \ln(u(x)) \\ u(x) &= e^{c_1} \cdot e^{-x^2/2k} \\ u(x) &= Ae^{-x^2/2k} \\ \implies u(x) &= Ae^{-sx^2} \end{aligned}$$

$$\begin{aligned} xv(x) &= kv'(x) \\ \frac{x^2}{2k} + c_2 &= \ln(v(x)) \\ v(x) &= e^{c_2} \cdot e^{\frac{x^2}{2k}} \\ v(x) &= Be^{\frac{x^2}{2k}} \implies v(x) = Be^{sx^2} \end{aligned}$$

We have ,

$$\begin{aligned} f(x) &= u(x) + iv(x) \\ &\implies f(x) = Ae^{-sx^2} + Be^{sx^2} \end{aligned}$$

and as f is in Schwartz space, $B = 0$.
Hence $f(x) = Ae^{-sx^2}$.

1.4 Commutator of operators

The commutator of two operators A and B defined on a Hilbert space H is denoted as $[A, B]$ and defined as

$$\boxed{[A, B] = AB - BA}$$

1.4.1 Commatator of Momentum operator

$$\boxed{[X_j, P_j] = \frac{i}{2\pi} I}$$

Proof:

$$\begin{aligned}
(X_jP_j - P_jX_j)f(x) &= X_jP_jf(x) - P_jX_jf(x) \\
&= X_j\left(\frac{1}{2\pi i}\frac{\partial f}{\partial x_j}\right) - P_j(x_jf(x)) \\
&= \frac{1}{2\pi i}x_j\frac{\partial f}{\partial x_j} - \frac{1}{2\pi i}x_j\frac{\partial f}{\partial x} - \frac{f(x)}{2\pi i} \\
&= \frac{i}{2\pi}f(x).
\end{aligned}$$

Hence we conclude that

$$\boxed{X_jP_j - P_jX_j = \frac{i}{2\pi} I}$$

1.5 Balian-Low Theorem on lattice in $\mathbb{R}$

1.5.1 The Weak BLT

The fact that $[X, P] = -\frac{1}{2\pi i} I$ forms the core of the uncertainty principle approach to proving the BLT. We state this fact in the following form.

Lemma 1 : *If $f, g \in L^2(\mathbb{R})$ satisfy $Xf, Xg \in L^2(\mathbb{R})$ and $X\hat{f}, X\hat{g} \in L^2(\widehat{\mathbb{R}})$, then*

$$\langle Xf, Pg\rangle - \langle Pf, Xg\rangle = \frac{1}{2\pi i}\langle f, g\rangle. \tag{1.4}$$

Proof. Pf and Pg are well defined since $X\hat{f}, X\hat{g} \in L^2(\widehat{\mathbb{R}})$. As we know that $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$,so that we can find $\varphi_k, \psi_k \in \mathcal{S}(\mathbb{R})$ such that $\varphi_k \to f, X\varphi_k \to$

$Xf, P\varphi_k \to Pf$, and $\psi_k \to g, X\psi_k \to Xg$, $P\psi_k \to Pg$, all in L^2-norm. Since $\varphi_k, \psi_k \in \mathcal{S}(\mathbb{R})$ and X, P are self-adjoint, we have

$$\begin{aligned}\langle X\varphi_k, P\psi_k\rangle - \langle P\varphi_k, X\psi_k\rangle &= \langle XP\varphi_k, \psi_k\rangle - \langle PX\varphi_k, \psi_k\rangle \\ &= -\langle [X,P]\varphi_k, \psi_k\rangle \\ &= \frac{1}{2\pi i}\langle \varphi_k, \psi_k\rangle.\end{aligned}$$

However, the inner product is continuous, so $\langle \varphi_k, \psi_k\rangle \to \langle f, g\rangle$, $\langle X\varphi_k, P\psi_k\rangle \to \langle Xf, Pg\rangle$, and $\langle P\varphi_k, X\psi_k\rangle \to \langle Pf, Xg\rangle$. Therefore (1.4) holds. $\square$

Next, we compute the commutators of X and P with the translation and modulation operators T_x and M_ω defined in section 1.2 , Here we are taking x and $\omega \in \mathbb{Z}$.

Lemma 2 : $$\begin{aligned}[M_\omega T_x, X] &= M_\omega T_x X - X M_\omega T_x = -x M_\omega T_x \\ [M_\omega T_x, P] &= M_\omega T_x P - P M_\omega T_x = -\omega M_\omega T_x.\end{aligned} \tag{1.5}$$

Proof. As the two parts are similar, we prove only part a. We compute

$$\begin{aligned}(M_\omega T_x X f)(t) - (X M_\omega T_x f)(t) &= M_\omega(t)(T_x X f)(t) - t(M_\omega T_x f)(t) \\ &= M_\omega(t)(Xf)(t-x) - tM_\omega(t)(T_x f)(t) \\ &= M_\omega(t)(t-x)f(t-x) - tM_\omega(t)f(t-x) \\ &= -xM_\omega(t)f(t-x) \\ &= -xM_\omega(t)(T_x f)(t) \\ &= -x(M_\omega T_x f)(t).\end{aligned}$$

$\square$

We can now prove a weak version of the BLT.

1.5.2 Weak BLT

Theorem (Weak BLT): Assume $g \in L^2(\mathbb{R})$ is such that $\{g_{m,n}\}$ is an exact frame for $L^2(\mathbb{R})$. Then we cannot have all of $Xg, X\tilde{g} \in L^2(\mathbb{R})$ and $X\hat{g}, X\hat{\tilde{g}} \in L^2(\hat{\mathbb{R}})$, that is, we must have

$$\|tg(t)\|_2 \|\gamma\hat{g}(\gamma)\|_2 \|t\tilde{g}(t)\|_2 \|\gamma\hat{\tilde{g}}(\gamma)\|_2 = +\infty.$$

Proof. Assume all four functions were elements of L^2. Note that

$$\forall f, h \in L^2(\mathbb{R}), \langle f, h_{m,n}\rangle = \langle f_{-m,-n}, h\rangle.$$

Also, by Lemma 2.a,

$$\forall f \in L^2(\mathbb{R}), X(f_{m,n}) = (Xf)_{m,n} + nf_{m,n}.$$

Since X is selfadjoint and $\{g_{m,n}\}$ is biorthonormal to its dual frame $\{\tilde{g}_{m,n}\}$, we can therefore compute

$$\begin{aligned}\langle Xg, \tilde{g}_{m,n}\rangle = \langle g, X(\tilde{g}_{m,n})\rangle &= \langle g, (X\tilde{g})_{m,n}\rangle + n\langle g, \tilde{g}_{m,n}\rangle \\ &= \langle g_{-m,-n}, X\tilde{g}\rangle + n\delta_{m,0}\delta_{n,0} \\ &= \langle g_{-m,-n}, X\tilde{g}\rangle.\end{aligned}$$

Now, by the L^2-inversion formula, both Pg and $P\tilde{g}$ exist and are in $L^2(\mathbb{R})$, so by Lemma 2.b we similarly obtain

$$\begin{aligned}\langle g_{m,n}, P\tilde{g}\rangle = \langle P(g_{m,n}), \tilde{g}\rangle &= \langle (Pg)_{m,n}, \tilde{g}\rangle + m\langle g_{m,n}, \tilde{g}\rangle \\ &= \langle Pg, \tilde{g}_{-m,-n}\rangle.\end{aligned}$$

Since $f = \sum\langle f, g_{m,n}\rangle\tilde{g}_{m,n} = \sum\langle f, \tilde{g}_{m,n}\rangle g_{m,n}$ for every $f \in L^2(\mathbb{R})$, we therefore have

$$\begin{aligned}
\langle Xg, P\tilde{g}\rangle &= \sum_{m,n} \langle Xg, \tilde{g}_{m,n}\rangle \langle g_{m,n}, P\tilde{g}\rangle \\
&= \sum_{m,n} \langle g_{-m,-n}, X\tilde{g}\rangle \langle Pg, \tilde{g}_{-m,-n}\rangle \\
&= \sum_{m,n} \langle Pg, \tilde{g}_{m,n}\rangle \langle g_{m,n}, X\tilde{g}\rangle \\
&= \langle Pg, X\tilde{g}\rangle.
\end{aligned}$$

Therefore, by biorthonormality and Lemma 1,

$$\begin{aligned}
1 &= \langle g, \tilde{g}\rangle \\
&= 2\pi i \left(\langle Xg, P\tilde{g}\rangle - \langle Pg, X\tilde{g}\rangle\right) \\
&= 0,
\end{aligned}$$

a contradiction. $\square$

for proof of this$\langle g, \tilde{g}\rangle = 1$ see the Book on Frame thoery Titled : An Introduction to frames and Riesz Bases by Ole Christensen .

Chapter 2

Lattice , Gabor system and Modulation Space

2.1 Definition

2.1.1 Lattice in $\mathbb{R}^d$

A subset Λ of $\mathbb{R}^d$ is said to be a lattice if $\exists$ a matrix $A \in GL(n, \mathbb{R})$ such that $\Lambda = A\mathbb{Z}^d$,
i.e $\qquad \Lambda = \{Ax : x \in \mathbb{Z}^d\}$.

Notes

Λ is discrete subgroup of $\mathbb{R}^d$ satisfying following conditions :

(i). Λ is closed under addition and subtraction .

(ii) $\exists\ \epsilon > 0$ such that if $x \neq y \in \Lambda$ then $|x - y| \geq \epsilon$

2.1.2 Riesz Basis

Riesz basis in HILBERT Space : A collection of vectors $\{x_k\}$ in a Hilbert space H is a Riesz basis for H if it is image of an orthonormal basis for H under an invertible linear transformation.

2.1.3 Frame:

A sequence $\{x_k\}$ in a Hilbert space H is a frame if $\exists\ A, B \geq 0$, such that for all $x \in H$

$$A\|x\|^2 \leq \sum_k |\langle x, x_k\rangle|^2 \leq B\|x\|^2 \tag{2.1}$$

Note

(i) If $\{x_k\}$ is an orthonormal set in Hilbert space H, then
$\sum_k |\langle x, x_k\rangle|^2 \leq \|x\|^2$

(ii) If $\{x_k\}$ is an orthonormal basis in Hilbert space H, then
$\|x\|^2 = \sum_k |\langle x, x_k\rangle|^2$

Theorem:
Suppose $\{x_n\}$ be a sequence in a Hilbert space H. Then the following are equivalent:

1. $\{x_n\}$ is frame with bounds A and B
2. $S(x) = \sum_n \langle x, x_n\rangle x_n$ is bounded linear operator with $AI \leq S \leq BI$,
   $S(x)$ is called frame operator.

Corollary :

1. S is an invertible operator and $B^{-1}I \leq S^{-1} \leq A^{-1}$.
2. $\{S^{-1}x_n\}$ is a frame with bounds B^{-1} and A^{-1} , called the dual frame of $\{x_n\}$.
3. $\forall x \in H$, $x = \sum \langle x, S^{-1}x_n\rangle x_n = \sum \langle x, x_n\rangle S^{-1}x_n$

2.1.4 Symplectic lattices and operators

In time–frequency analysis, compositions of the symmetric time–frequency shifts $M_{\omega/2}T_xM_{\omega/2}$ often occur, and the symplectic form $[\cdot,\cdot]$ defined by

$$\boxed{[(x_1,\omega_1),(x_2,\omega_2)] = x_2\cdot\omega_1 - x_1\cdot\omega_2} \quad (x_1,\omega_1),(x_2,\omega_2) \in \mathbb{R}^{2d}$$

The symplectic group $\mathrm{Sp}(d)$ is the group of all matrices $M \in \mathrm{GL}(2d, \mathbb{R})$ that leave the symplectic form $[\cdot, \cdot]$ invariant, i.e., $M \in \mathrm{Sp}(d)$ satisfies

$$[Mx, My] = [x, y] \quad \text{for all} \quad x, y \in \mathbb{R}^{2d}.$$

As a consequence of the Stone–von Neumann theorem, a symplectic transformation $M \in \mathrm{Sp}(d)$ corresponds to a unitary symplectic operator $\mu(M)$ on $L^2(\mathbb{R}^d)$ which satisfies

$$\pi(Mz) = \mu(M)\pi(z)\mu(M)^{-1} \quad \text{for all} \quad z \in \mathbb{R}^{2d}.$$

Definition 2: A lattice $\Lambda \subseteq \mathbb{R}^{2d}$ is a symplectic lattice if $\Lambda = \alpha M \mathbb{Z}^{2d}$ for some $\alpha \in \mathbb{R} \setminus \{0\}$ and $M \in \mathrm{Sp}(d)$.

Note: If M is symplectic, then $|\det(M)| = 1$, so $\mathrm{vol}(\alpha M \mathbb{Z}^{2d}) = |\alpha|$. Since $\mathrm{Sp}(1) = \mathrm{SL}(2, \mathbb{R})$, every lattice in $\mathbb{R}^2$ is a symplectic lattice. However, this is not the case when $d > 1$.

2.2 Time-frequency shifts and Gabor systems

For $x, \omega \in \mathbb{R}^d$, we define $T_x f(t) = f(t - x)$ and $M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)$ to be the unitary operators of translation and modulation. We denote the corresponding time–frequency shift by

$$\boxed{\pi(z) f(t) = M_\omega T_x f(t) = e^{2\pi i \omega \cdot t} f(t - x)}$$

Where $z = (x, \omega) \in \mathbb{R}^{2d}$ for a point in the time–frequency plane $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$,

2.2.1 Gabor system :

Given a function $\mathbf{g} \in L^2(\mathbb{R}^d)$, called a window function, and a lattice Λ in the time-frequency plane $\mathbb{R}^{2d}$, the corresponding Gabor system is

$$\mathcal{G}(\mathbf{g}, \Lambda) = \{\pi(\lambda)\mathbf{g} \,|\, \lambda \in \Lambda\}.$$

If $\mathcal{G}(\mathbf{g}, \Lambda)$ is a frame for its closed span $H = \text{span}\{\pi(\lambda)\mathbf{g}\}_{\lambda\in\Lambda}$ in $L^2(\mathbb{R}^d)$, i.e., there exist $A, B > 0$ such that

$$\forall f \in H, \quad A\|f\|_2^2 \leq \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)\mathbf{g}\rangle|^2 \leq B\|f\|_2^2 \tag{2.2}$$

then the associated Gabor frame operator is

$$S_{\mathbf{g},\Lambda} f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\mathbf{g}\rangle \pi(\lambda)\mathbf{g}. \tag{2.3}$$

This is a positive, invertible operator of H onto itself. The canonical dual window is $\gamma = S_{\mathbf{g},\Lambda}^{-1}\mathbf{g} \in H$, and the canonical dual frame is the Gabor system $G(\gamma, \Lambda)$. We have the frame expansions

$$\forall f \in H, \quad f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\gamma\rangle \pi(\lambda)\mathbf{g} = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\mathbf{g}\rangle \pi(\lambda)\gamma. \tag{2.4}$$

2.3 Modulation Space

Modulation spaces are a family of Banach spaces defined by the behavior of the short-time Fourier transform with respect to a test function from the Schwartz space.

Modulation spaces are defined as follows. For $1 \leq p \leq \infty$, a non-negative function $m(x, \omega)$ on $\mathbb{R}^{2d}$ and a test function $g \in \mathcal{S}(\mathbb{R}^d)$, the modulation space $M_m^p(\mathbb{R}^d)$ is defined by

$$M_m^p(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \ : \ \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \omega)|^p m(x, \omega)^p dx \right) d\omega \right)^{1/p} < \infty \right\}.$$

In the above equation, $V_g f$ denotes the short-time Fourier transform of f with respect to g evaluated at (x, ω), namely

$$V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t - x)} e^{-2\pi i t\cdot\omega} dt = \langle f, M_w T_x g\rangle$$

In other words, $f \in M_m^p(\mathbb{R}^d)$ is equivalent to $V_g f \in L_m^p(\mathbb{R}^{2d})$. The space $M_m^p(\mathbb{R}^d)$ is the same, independent of the test function $g \in \mathcal{S}(\mathbb{R}^d)$ chosen.

For our purpose the following special cases of the modulation spaces will be sufficient .

Definition Let $v(z) \geq 1$ be a submultiplicative weight function on $\mathbb{R}^{2d}$ with at most polynomial growth. Then the modulation space M_p^v, where $1 \leq p \leq \infty$, is defined as the subspace of all $f \in \mathcal{S}(\mathbb{R}^d)'$ such that the norm

$$\|f\|_{M_p^v} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p m(x,\omega)^p dx \right) d\omega \right)^{1/p}$$

For the BLT,

L_s^2 denotes the weighted L^2-space with norm

$$\|f(t)\|_{L_s^2}^2 = \int |f(t)|^2 (1+|t|^2)^s \, dt,$$

and H^s denotes the Bessel potential space with norm

$$\|f(\omega)\|_{H^s}^2 = \int |\hat{f}(\omega)|^2 (1+|\omega|^2)^s \, d\omega.$$

Remark .

(a) If $v(x,\omega) = (1+|x|^2)^{s/2}$, then $M_v^2 = L_s^2$.

(b) If $v(x,\omega) = (1+|\omega|^2)^{s/2}$, then $M_v^2 = H^s$.

The weights that we shall use are
$m(x,\omega) = (1+|x|^2+|\omega|^2)^{1/2}$
$m_j(x,\omega) = (1+|x_j|^2+|\omega_j|^2)^{1/2}$, $j = 1,\ldots,d$. Remark implies that $M_m^2 = L_1^2 \cap H^1$.

2.4 Result

Consequently, if $f \in L^2(\mathbb{R}^d)$, then

$$f \in M_{m_j}^2 \iff \|X_j f\|^2 \|P_j f\|^2 < \infty$$

and

$$f \in M_m^2 \iff \left(\int_{\mathbb{R}^d} |x| \, |g(x)|^2 \ dx \right) \left(\int_{\mathbb{R}^d} |\omega| \, |\hat{g}(\omega)|^2 \ d\omega \right) < \infty$$

Chapter 3

The Balian–Low theorem for symplectic lattices in higher dimensions

3.1 The Balian-Low theorem

3.1.1 The weak subspace BLT for arbitrary lattice

Theorem 1 : *Let Λ be a lattice in $\mathbb{R}^{2d}$. If $\mathbf{g} \in L^2(\mathbb{R}^d)$ is such that $\mathcal{G}(\mathbf{g}, \Lambda)$ is a Riesz basis for its closed span $H = \overline{span}\{\pi(\lambda)\mathbf{g}\}_{\lambda\in\Lambda}$ in $L^2(\mathbb{R}^d)$ and the dual window is $\gamma = S^{-1}_{\mathbf{g},\Lambda}\mathbf{g}$, then for each $j = 1, \ldots, d$, one of $X_j\mathbf{g}$, $P_j\mathbf{g}$, $X_j\gamma$, or $P_j\gamma$ cannot lie in H.* In particular, if $G(\mathrm{g}, \Lambda)$ is a Riesz basis for $L^2(\mathbb{R}^d)$, then:

(a) For each $j = 1, \ldots, d$, either $\mathbf{g} \notin M^2_{m_j}$ or $\gamma \notin M^2_{m_j}$.

(b) Either $\mathbf{g} \notin M^2_m$ or $\gamma \notin M^2_m$.

Proof. Assume that $X_jg, P_jg, X_j\gamma, P_j\gamma \in H$. We can compute that for any $(p, q) \in \mathbb{R}^d$ we have

$$\langle X_jg, M_qT_p\gamma\rangle = \langle T_{-p}M_{-q}g, X_j\gamma\rangle \text{ and}$$
$$\langle M_qT_pg, P_j\gamma\rangle = \langle P_jg, T_{-p}M_{-q}\gamma\rangle.$$

Then, using the frame expansions 2.4, we have that

$$\begin{aligned}
\langle X_j g, P_j \gamma\rangle &= \left\langle \sum_{(p,q)\in\Lambda} \langle X_j g, M_q T_p \gamma\rangle M_q T_p g, P_j \gamma \right\rangle \\
&= \sum_{(p,q)\in\Lambda} \langle T_{-p} M_{-q} g, X_j \gamma\rangle \langle P_j g, T_{-p} M_{-q} \gamma\rangle \\
&= \left\langle P_j g, \sum_{(p,q)\in\Lambda} \langle X_j \gamma, M_q T_p g\rangle M_q T_p \gamma \right\rangle \\
&= \langle P_j g, X_j \gamma\rangle.
\end{aligned}$$

However, the canonical commutation relation $[X_j, P_j] = -\frac{1}{2\pi i} I$ leads to the contradiction

$$\begin{aligned}
1 &= \langle g, \gamma\rangle \\
&= 2\pi i \left(\langle P_j g, X_j \gamma\rangle - \langle X_j g, P_j \gamma\rangle\right) \\
&= 0.
\end{aligned}$$

□

3.1.2 The BLT on non-lattice

The assumption of lattice structure is not essential to the definition of a Gabor frame. In particular, if Λ is any countable sequence of points in $\mathbb{R}^{2d}$, then $G(g, \Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$ if

$$\sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda) g\rangle|^2$$

is an equivalent norm for $L^2(\mathbb{R})$. Unfortunately, if Λ is not a lattice then although a dual frame $\{h_\lambda\}_{\lambda\in\Lambda}$ will exist, it need not be a Gabor frame of the form $G(\gamma, \Lambda)$. However, for the case of a so-called normalized tight frame, including orthonormal bases in particular, the dual frame coincides with the frame. In this case, we can observe that the proof of Theorem 1 requires no structural assumptions on Λ except that it be symmetric about the origin. Hence we have the following.

Theorem 2. Let Λ be a countable sequence in $\mathbb{R}^{2d}$ such that $\Lambda = -\Lambda$. If $g \in L^2(\mathbb{R}^d)$ is such that $G(g, \Lambda)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$, then:

1. $g \notin M^2_{mj}$ for each $j = 1, \ldots, d$, and

2. $g \notin M^2_m$.

Theorem 3. Let Λ be a symplectic lattice in $\mathbb{R}^{2d}$. If $g \in L^2(\mathbb{R}^d)$ is such that $G(g, \Lambda)$ is a Riesz basis for $L^2(\mathbb{R}^d)$, then $g \notin M^1$.

Let (C_0, l^1) denote the Wiener amalgam space

$$(C_0, l^1) = \left\{ f : f \text{ is continuous and } \sum_{k \in \mathbb{Z}^d} \|f \cdot \chi_{Q+k}\|_\infty < \infty \right\},$$

where $Q = [0, 1)^d$. Because M^1 is embedded into (C_0, l^1), we have for the case $\Lambda = \alpha\mathbb{Z}^d \times (1/\alpha)\mathbb{Z}^d$ that Theorem 3 is implied by the following result known as the Amalgam BLT[11].

Theorem 4. If $g \in L^2(\mathbb{R}^d)$ is such that $G(g, \alpha\mathbb{Z}^d \times (1/\alpha)\mathbb{Z}^d)$ is a Riesz basis for $L^2(\mathbb{R}^d)$, then $g, \hat{g} \notin (C_0, l^1)$.

Since M^2_m is not embedded into (C_0, l^1) nor conversely

The proof of Theorem 3 relies on the fact that M^1 is invariant under symplectic operators. It is unknown whether (C_0, l^1) is invariant under such operators, and it is an open question whether the Amalgam BLT extends to more general lattices than $\alpha\mathbb{Z}^d \times (1/\alpha)\mathbb{Z}^d$.

Finally, we observe that some of the most natural lattices in $\mathbb{R}^{2d}$ are the separable lattices. If a separable lattice with unit volume is symplectic, then it is a product lattice. Every lattice in $\mathbb{R}^2$ is symplectic, but this is not the case in $\mathbb{R}^{2d}$ when $d > 1$. It is an open question whether question as to whether the BLT extends to the case of separable, non-product lattices in higher dimensions.

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