

# CS 561 Artificial Intelligence

## Lecture # 6-8

### Probabilistic Reasoning Over Time

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# Outline

- Time and Uncertainty
- Inference in Temporal Models
  - Filtering and Prediction
  - Smoothing
  - Most Likely explanation
- Hidden Markov Models
- Kalman filters
- Dynamic Bayesian Networks
  - Particle filters

# Time and Uncertainty

- The Bayesian networks we have seen so far describe **static scenarios** — each random variable has a single fixed value in a single problem instance.
- Now we will consider the problem of describing probabilistic **environments that evolve over time** —
  - Examples: target tracking, condition monitoring, making sense of spoken or written sequence of words, ...
- World is viewed as a series of snapshots, or time slices, each of which contains a set of random variables, some observable and some not
  - same subset of variables is observable at each time slice

# Time and Uncertainty

- **Assumptions:** discrete time steps, state sequence starts at  $t = 0$  and evidence starts arriving at  $t = 1$ .
- **Notations**
  - $\mathbf{X}_t$  : set of unobservable state variables at time  $t$
  - $\mathbf{E}_t$  : set of observable evidence variables at time  $t$
  - $\mathbf{E}_t = \mathbf{e}_t$  : observation at time  $t$  for some set of values
  - $\mathbf{X}_{a:b}$  : the set of variables from  $\mathbf{X}_a$  to  $\mathbf{X}_b$
- **Transition Model:**
  - describes the probability distribution of the variables at time  $t$ , given the state of the world at past times (previous values)
- **Sensor Model** (or Observation Model):
  - describes the probability of each percept at time  $t$ , given the current state of the world

# Transition and Sensor Models

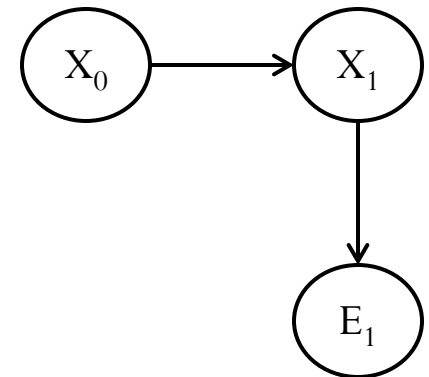
- **Transition model:**  $\mathbf{P}(\mathbf{X}_t | \mathbf{X}_{0:t-1})$
- Markov Assumption:
  - $\mathbf{X}_t$  depends on only finite fixed number of previous states
  - First-order Markov Process:  $\mathbf{P}(\mathbf{X}_t | \mathbf{X}_{0:t-1}) = \mathbf{P}(\mathbf{X}_t | \mathbf{X}_{t-1})$
  - Second-order Markov Process:  $\mathbf{P}(\mathbf{X}_t | \mathbf{X}_{0:t-1}) = \mathbf{P}(\mathbf{X}_t | \mathbf{X}_{t-2}, \mathbf{X}_{t-1}, )$
- **Sensor Model :**
  - $\mathbf{E}_t$  could depend on previous variables as well as the current state variables, but we make sensor Markov assumption (any state should be capable of providing a precise sensor reading of itself)
  - $\mathbf{P}(\mathbf{E}_t | \mathbf{X}_{0:t}, \mathbf{E}_{0:t-1}) = \mathbf{P}(\mathbf{E}_t | \mathbf{X}_t)$
- Assume **stationary process**: transition model  $\mathbf{P}(\mathbf{X}_t | \mathbf{X}_{t-1})$  and sensor model  $\mathbf{P}(\mathbf{E}_t | \mathbf{X}_t)$  are the same for all  $t$

# Complete Joint Distribution

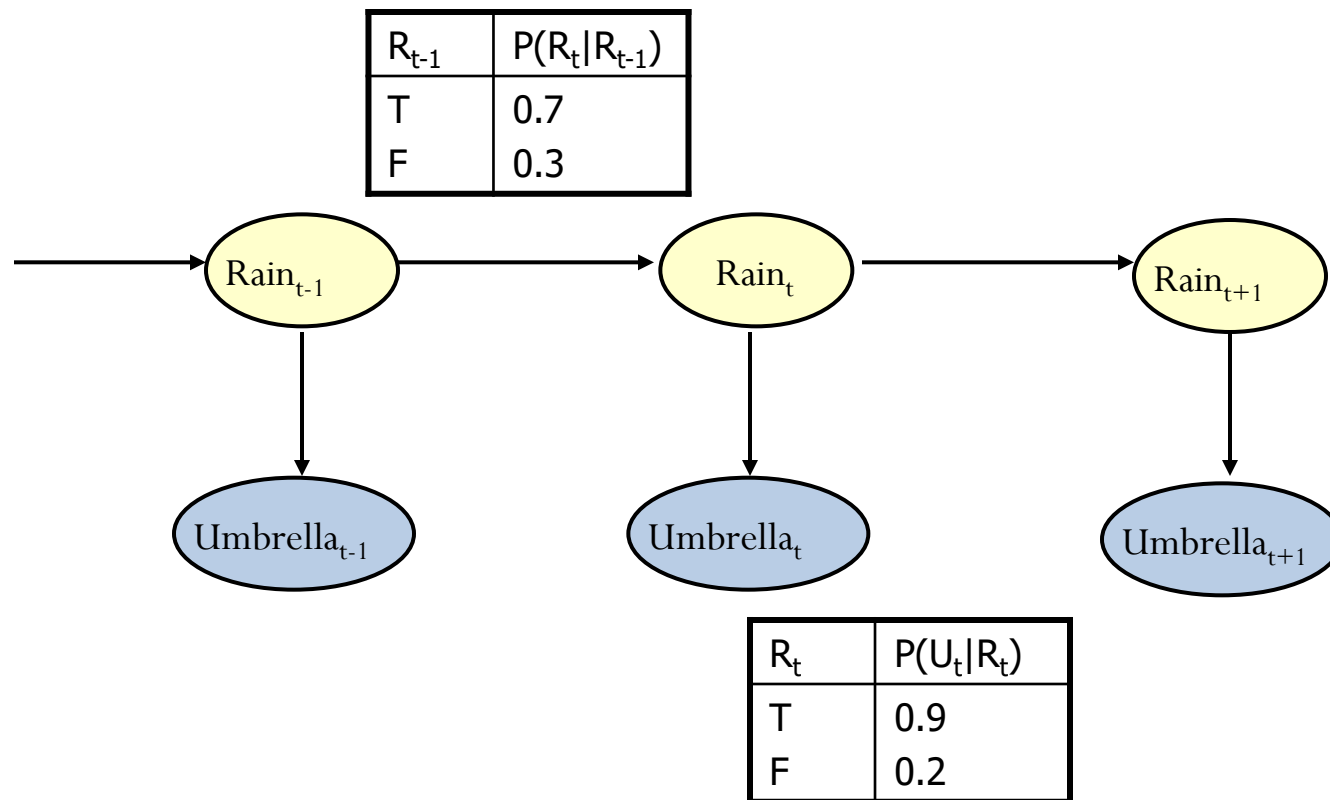
- Given:
  - Transition model:  $\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-1})$
  - Sensor model:  $\mathbf{P}(\mathbf{E}_t|\mathbf{X}_t)$
  - Prior probability distribution at time  $t = 0$  :  $\mathbf{P}(\mathbf{X}_0)$
- Then we can specify complete joint distribution:

$$\mathbf{P}(\mathbf{X}_{0:t}, \mathbf{E}_{1:t}) = \mathbf{P}(\mathbf{X}_0) \prod_{i=1}^t \mathbf{P}(\mathbf{X}_i|\mathbf{X}_{i-1})\mathbf{P}(\mathbf{E}_i|\mathbf{X}_i)$$

$$\begin{aligned}\mathbf{P}(\mathbf{X}_0, \mathbf{X}_1, \mathbf{E}_1) &= \mathbf{P}(\mathbf{E}_1|\mathbf{X}_1, \mathbf{X}_0) \mathbf{P}(\mathbf{X}_1, \mathbf{X}_0) \\ &= \mathbf{P}(\mathbf{E}_1|\mathbf{X}_1)\mathbf{P}(\mathbf{X}_1|\mathbf{X}_0) \mathbf{P}(\mathbf{X}_0) \\ &= \mathbf{P}(\mathbf{X}_0) \prod_{i=1}^1 \mathbf{P}(\mathbf{X}_i|\mathbf{X}_{i-1})\mathbf{P}(\mathbf{E}_i|\mathbf{X}_i)\end{aligned}$$



# Example:



# Inference in Temporal Models

- **Filtering (state estimation)**  $\mathbf{P}(\mathbf{X}_t | \mathbf{e}_{1:t})$ 
  - computing current belief state (the posterior distribution over the most recent state) given all evidence to date;
- **Prediction**  $\mathbf{P}(\mathbf{X}_{t+k} | \mathbf{e}_{1:t})$  for  $k > 0$ 
  - computing the posterior distribution over the future state, given all evidence to date
- **Smoothing**  $\mathbf{P}(\mathbf{X}_k | \mathbf{e}_{1:t})$  for  $0 \leq k < t$ 
  - computing the posterior distribution over a past state, given all the evidence up to the present.
- **Most likely explanation:**  $\operatorname{argmax}_{\mathbf{x}_{1:t}} \mathbf{P}(\mathbf{x}_{1:t} | \mathbf{e}_{1:t})$ 
  - given sequence of observations, find sequence of states that is most likely to have generated those observations.



# Filtering and Prediction

- Given the result of filtering up to time  $t$ , the result for  $t + 1$  is computed from the new evidence i.e.
  - recursive estimation is used to compute  $\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1})$  as a function of  $\mathbf{e}_{t+1}$  and  $\mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$

$$\begin{aligned}
 \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t+1}) &= \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t}, \mathbf{e}_{t+1}) \\
 &= \alpha \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}, \mathbf{e}_{1:t}) \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t}) \\
 &= \alpha \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t}) \\
 &= \alpha \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \sum_{x_t} P(\mathbf{X}_{t+1} | \mathbf{x}_t, \mathbf{e}_{1:t}) \mathbf{P}(\mathbf{x}_t | \mathbf{e}_{1:t}) \\
 &= \alpha \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \sum_{x_t} P(\mathbf{X}_{t+1} | \mathbf{x}_t) \mathbf{P}(\mathbf{x}_t | \mathbf{e}_{1:t})
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t}) &= \frac{\mathbf{P}(\mathbf{X}_{t+1}, \mathbf{e}_{1:t})}{\mathbf{P}(\mathbf{e}_{1:t})} \\
 &= \frac{\sum_{x_t} \mathbf{P}(\mathbf{X}_{t+1}, \mathbf{x}_t, \mathbf{e}_{1:t})}{\mathbf{P}(\mathbf{e}_{1:t})}
 \end{aligned}$$

from transition model

from current state  
distribution



# Filtering and Prediction

$$\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t+1}) = \alpha \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \sum_{x_t} P(\mathbf{X}_{t+1} | \mathbf{x}_t) \mathbf{P}(\mathbf{x}_t | \mathbf{e}_{1:t})$$

Update belief as new observation arrives

Propagate belief

- This leads to a recursive definition:
  - $\mathbf{f}_{1:t+1} = \alpha \text{ FORWARD}(\mathbf{f}_{1:t}, \mathbf{e}_{t+1})$
  - where  $\mathbf{f}_{1:t}$  is  $\mathbf{P}(\mathbf{X}_t | \mathbf{e}_{1:t})$  can be considered as “message” propagated along the sequence, modified by each transition and updated by each new observation, the process begins with  $\mathbf{f}_{1:0} = \mathbf{P}(\mathbf{X}_0)$

- **Example:** Compute  $\mathbf{P}(R_2 | u_{1:2})$ 
  - day 0, no observations  $\mathbf{P}(R_0) = \langle 0.5, 0.5 \rangle$  prior belief
  - day 1,  $U_1 = \text{true}$ , prediction from  $t = 0$  to  $t = 1$  is

$$\begin{aligned} \mathbf{P}(R_1) &= \sum_{r_0} \mathbf{P}(R_1 | r_0) P(r_0) \\ &= \langle 0.7, 0.3 \rangle \times 0.5 + \langle 0.3, 0.7 \rangle \times 0.5 = \langle 0.5, 0.5 \rangle \end{aligned}$$

$R_{t-1}$	$P(R_t   R_{t-1})$
T	0.7
F	0.3

$R_t$	$P(U_t   R_t)$
T	0.9
F	0.2

# Filtering and Prediction

- Then update step simply multiplies by the probability of the evidence for  $t = 1$  and normalizes

$$\mathbf{P}(R_1|u_1) = \alpha \mathbf{P}(u_1|R_1) \sum_{r_0} \mathbf{P}(R_1|r_0)P(r_0)$$

$$\begin{aligned} \mathbf{P}(R_1|u_1) &= \alpha \mathbf{P}(u_1|R_1) \mathbf{P}(R_1) = \alpha \langle 0.9, 0.2 \rangle \langle 0.5, 0.5 \rangle \\ &= \alpha \langle 0.45, 0.1 \rangle \approx \langle 0.818, 0.182 \rangle \end{aligned}$$

$R_{t-1}$	$P(R_t R_{t-1})$
T	0.7
F	0.3

- day 2,  $U_2 = \text{true}$ , the prediction from  $t = 1$  to  $t = 2$  is

$$\mathbf{P}(R_2|u_1) = \sum_{r_1} \mathbf{P}(R_2|r_1)P(r_1|u_1)$$

$$= \langle 0.7, 0.3 \rangle \times 0.818 + \langle 0.3, 0.7 \rangle \times 0.182 \approx \langle 0.627, 0.373 \rangle$$

$R_t$	$P(U_t R_t)$
T	0.9
F	0.2

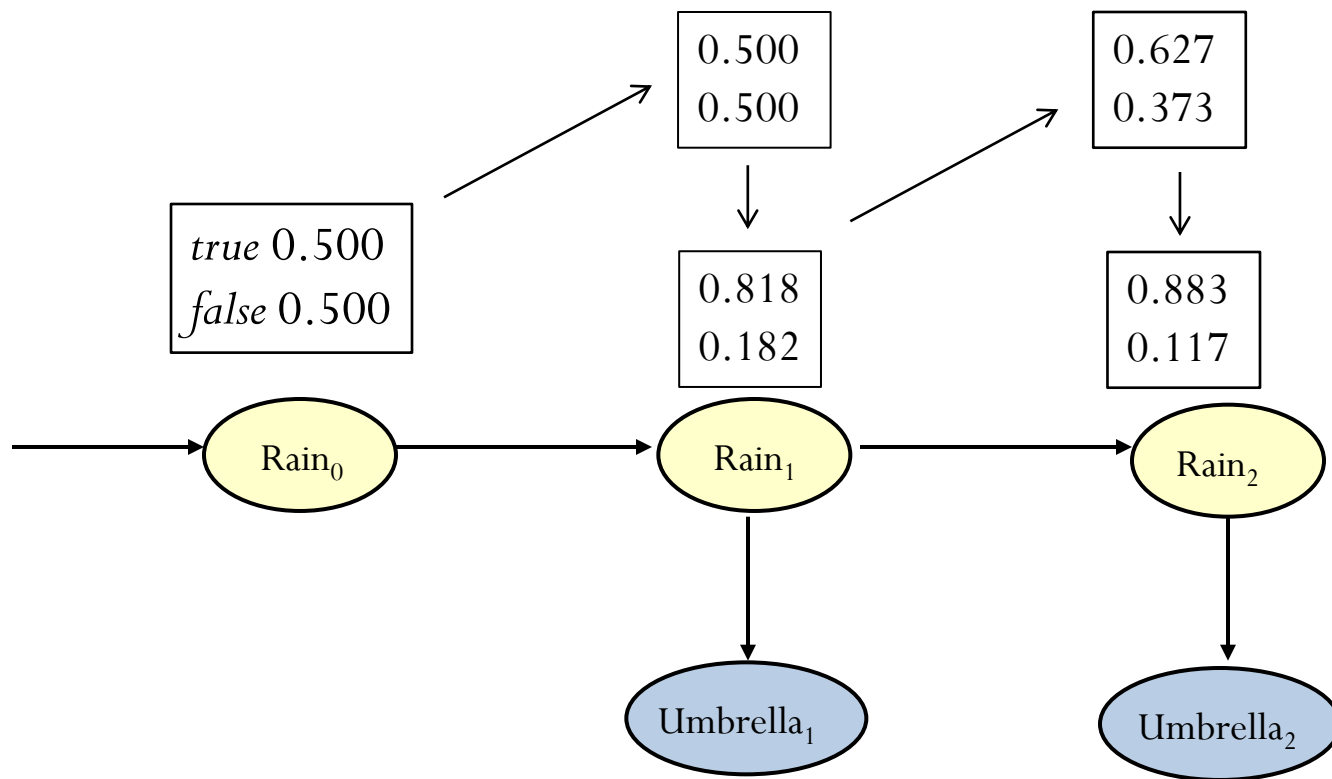
- Update it with evidence for  $t = 2$

$$\mathbf{P}(R_2|u_1, u_2) = \alpha \mathbf{P}(u_2|R_2) \sum_{r_1} \mathbf{P}(R_2|r_1)P(r_1|u_1)$$

$$\begin{aligned} \mathbf{P}(R_2|u_1, u_2) &= \alpha \mathbf{P}(u_2|R_2) \mathbf{P}(R_2|u_1) = \alpha \langle 0.9, 0.2 \rangle \langle 0.627, 0.373 \rangle \\ &= \alpha \langle 0.565, 0.075 \rangle \approx \langle 0.883, 0.117 \rangle \end{aligned}$$

# Filtering and Prediction

- the probability of rain increases from day 1 to day 2 because it persists



# Filtering and Prediction

- Prediction is filtering without the addition of new evidence:

$$\mathbf{P}(\mathbf{X}_{t+k+1} | \mathbf{e}_{1:t}) = \sum_{\mathbf{x}_{t+k}} \mathbf{P}(\mathbf{X}_{t+k+1} | \mathbf{x}_{t+k}) P(\mathbf{x}_{t+k} | \mathbf{e}_{1:t})$$

- Note that this computation involves only the transition model and not the sensor model.

# Smoothing

- Compute  $\mathbf{P}(\mathbf{X}_k | \mathbf{e}_{1:t})$ ,  $0 \leq k < t$ : distribution over past states given evidence up to the present

$$\begin{aligned}
 \mathbf{P}(\mathbf{X}_k | \mathbf{e}_{1:t}) &= \mathbf{P}(\mathbf{X}_k | \mathbf{e}_{1:k}, \mathbf{e}_{k+1:t}) \\
 &= \alpha \mathbf{P}(\mathbf{X}_k | \mathbf{e}_{1:k}) \mathbf{P}(\mathbf{e}_{k+1:t} | \mathbf{X}_k, \mathbf{e}_{1:k}) \text{ (using Bayes' rule)} \\
 &= \alpha \mathbf{P}(\mathbf{X}_k | \mathbf{e}_{1:k}) \mathbf{P}(\mathbf{e}_{k+1:t} | \mathbf{X}_k) \text{ (using conditional independence)} \\
 &= \alpha \mathbf{f}_{1:k} \times \mathbf{b}_{k+1:t} \text{ ("} \times \text{" pointwise multiplication of vectors)}
 \end{aligned}$$

$\mathbf{b}_{k+1:t}$

$$\begin{aligned}
 \mathbf{P}(\mathbf{e}_{k+1:t} | \mathbf{X}_k) &= \sum_{\mathbf{x}_{k+1}} \mathbf{P}(\mathbf{e}_{k+1:t} | \mathbf{X}_k, \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} | \mathbf{X}_k) \\
 &= \sum_{\mathbf{x}_{k+1}} \mathbf{P}(\mathbf{e}_{k+1:t} | \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} | \mathbf{X}_k) \\
 &= \sum_{\mathbf{x}_{k+1}} \mathbf{P}(\mathbf{e}_{k+1}, \mathbf{e}_{k+2:t} | \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} | \mathbf{X}_k) \\
 &= \sum_{\mathbf{x}_{k+1}} \mathbf{P}(\mathbf{e}_{k+1}, \mathbf{e}_{k+2:t} | \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} | \mathbf{X}_k) \\
 &= \sum_{\mathbf{x}_{k+1}} \mathbf{P}(\mathbf{e}_{k+1} | \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{e}_{k+2:t} | \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} | \mathbf{X}_k)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{P}(\mathbf{e}_{k+1:t} | \mathbf{X}_k) &= \frac{\mathbf{P}(\mathbf{e}_{k+1:t}, \mathbf{X}_k)}{\mathbf{P}(\mathbf{X}_k)} \\
 &= \frac{\sum_{\mathbf{x}_{k+1}} \mathbf{P}(\mathbf{e}_{k+1:t}, \mathbf{X}_k, \mathbf{x}_{k+1})}{\mathbf{P}(\mathbf{X}_k)}
 \end{aligned}$$

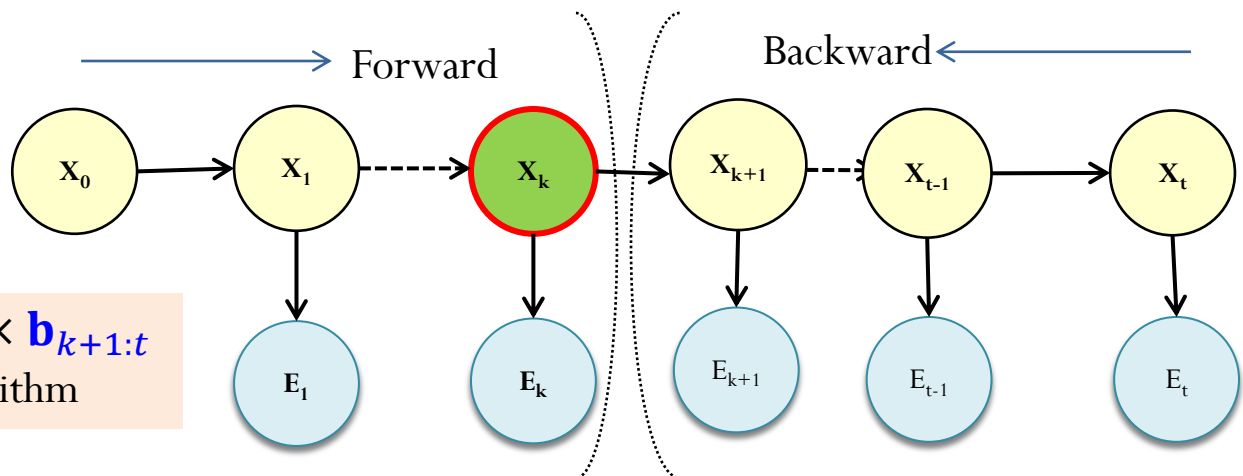
Recursively computed

$\mathbf{b}_{k+2:t}$

Apply conditional independence of  $\mathbf{e}_{k+1}$  and  $\mathbf{e}_{k+2:t}$  given  $\mathbf{X}_{k+1}$

# Smoothing

- This leads to a recursive definition
  - $\mathbf{b}_{k+1:t} = \alpha \text{BACKWARD}(\mathbf{b}_{k+2:t}, \mathbf{e}_{k+1:t})$  where BACKWARD implements the update  $\sum_{\mathbf{x}_{k+1}} \mathbf{P}(\mathbf{e}_{k+1} | \mathbf{x}_{k+1}) \mathbf{b}_{k+2:t} \mathbf{P}(\mathbf{x}_{k+1} | \mathbf{X}_k)$
  - time and space needed for each update are constant and independent of  $t$
  - Backward phase is initialised with  $\mathbf{b}_{t+1:t} = \mathbf{P}(\mathbf{e}_{t+1:t} | \mathbf{X}_t) = \mathbf{P}(\_ | \mathbf{X}_t) = \mathbf{1}$  (vector of 1s)



$$\mathbf{P}(\mathbf{X}_k | \mathbf{e}_{1:t}) = \alpha \mathbf{f}_{1:k} \times \mathbf{b}_{k+1:t}$$

Forward-backward algorithm

# Smoothing

- Example: Estimate the probability of rain at time  $k = 1$ , given umbrella observations on days 1 and 2.

$$\mathbf{P}(\mathbf{X}_k | \mathbf{e}_{1:t}) = \alpha \mathbf{P}(\mathbf{X}_k | \mathbf{e}_{1:k}) \mathbf{P}(\mathbf{e}_{k+1:t} | \mathbf{X}_k)$$

$$\mathbf{P}(R_1 | u_1, u_2) = \alpha \mathbf{P}(R_1 | u_1) \mathbf{P}(u_2 | R_1)$$

$$\mathbf{P}(R_1 | u_1) = \langle 0.818, 0.182 \rangle ,$$

already computed using filtering

$$\mathbf{P}(\mathbf{e}_{k+1:t} | \mathbf{X}_k) = \sum_{\mathbf{x}_{k+1}} \mathbf{P}(\mathbf{e}_{k+1} | \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{e}_{k+2:t} | \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} | \mathbf{X}_k)$$

$$\begin{aligned} \mathbf{P}(u_2 | R_1) &= \sum_{r_2} P(u_2 | r_2) P(e_{3:2} | r_2) \mathbf{P}(r_2 | R_1) \\ &= (0.9 \times 1 \times \langle 0.7, 0.3 \rangle) + (0.2 \times 1 \times \langle 0.3, 0.7 \rangle) \\ &= \langle 0.69, 0.41 \rangle \end{aligned}$$

$$\mathbf{P}(R_1 | u_1, u_2) = \alpha \langle 0.818, 0.182 \rangle \times \langle 0.69, 0.41 \rangle \approx \langle 0.883, 0.117 \rangle$$

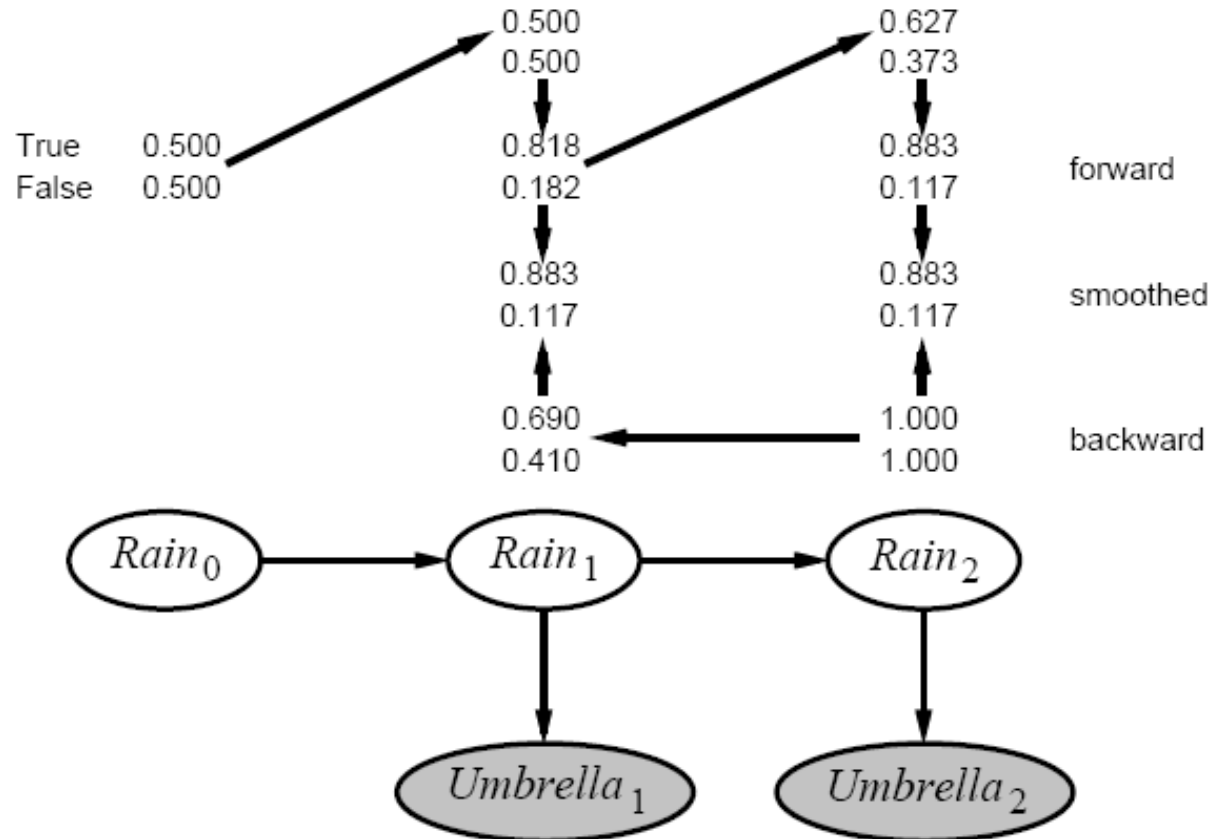
$R_{t-1}$	$P(R_t   R_{t-1})$
T	0.7
F	0.3

$R_t$	$P(U_t   R_t)$
T	0.9
F	0.2

Smoothed estimate for rain on day 1 is more than filtered estimate



# Smoothing Example



# Most likely Explanation

- Suppose that  $[true, true, false, true, true]$  is the observed umbrella sequence for the first five days. What is the weather sequence most likely to explain this?
  - there are  $2^5$  possible weather sequences – which is the most likely
- Most likely sequence  $\neq$  sequence of most likely states
  - the latter can be obtained by a combination of smoothing and filtering, the former cannot!
- Most likely path to each  $\mathbf{x}_{t+1}$  = most likely path to some  $\mathbf{x}_t$  plus one more step (recursive definition, due to the Markov property)

$$\begin{aligned}
 & \max_{\mathbf{x}_1 \dots \mathbf{x}_t} \mathbf{P}(\mathbf{x}_1, \dots, \mathbf{x}_t, X_{t+1} | \mathbf{e}_{1:t+1}) \quad \text{m}_{1:t+1} \\
 &= \alpha \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \max_{\mathbf{x}_t} (\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{x}_t) \max_{\mathbf{x}_1 \dots \mathbf{x}_{t-1}} P(\mathbf{x}_1, \dots, \mathbf{x}_{t-1}, \mathbf{x}_t | \mathbf{e}_{1:t})) \quad \text{m}_{1:t}
 \end{aligned}$$

- identical to filtering except  $\mathbf{f}_{1:t}$  replaced by this term and summation is replaced by maximization over  $\mathbf{X}_t$

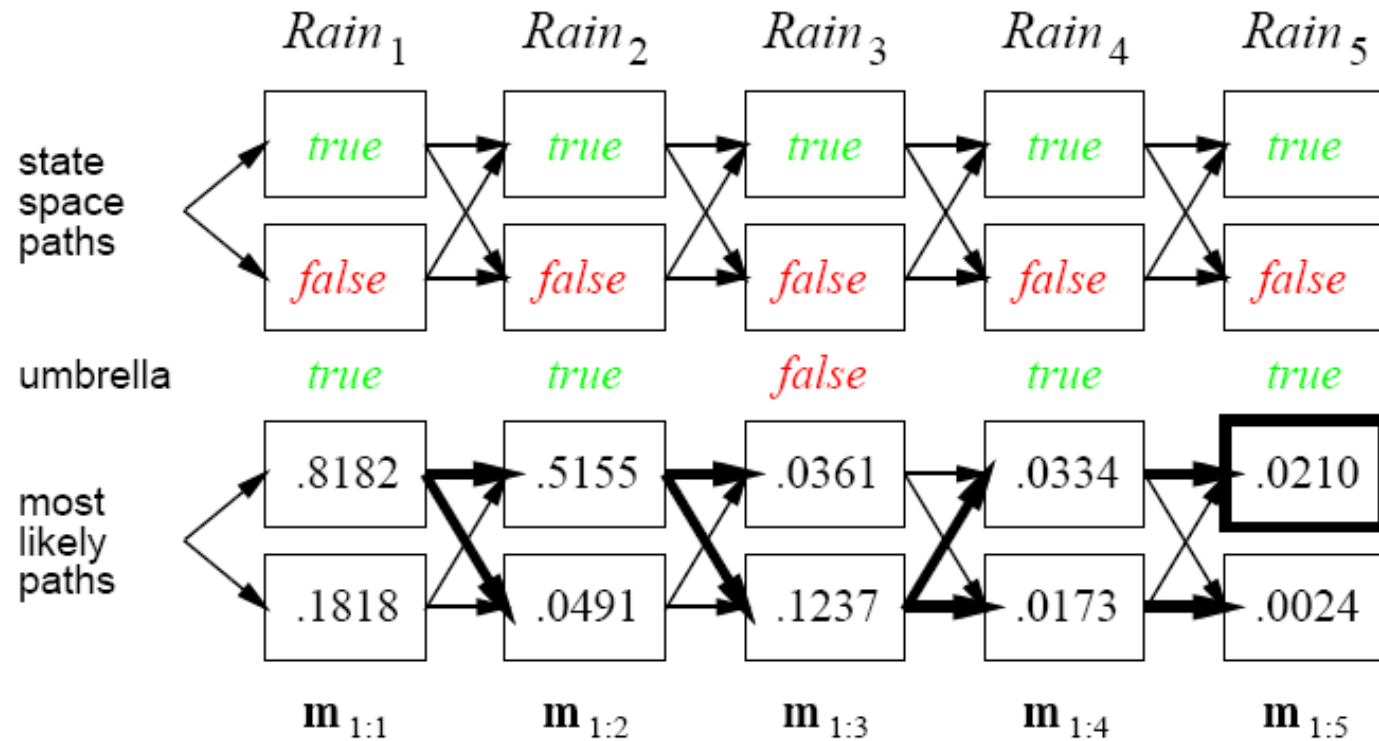
# Most likely Explanation

- computing most likely sequence is similar to filtering
  - runs forward along the sequence, computing the  $\mathbf{m}$  messages at each time step where  $\mathbf{m}$  is given as

$\mathbf{m}_{1:t} = \max_{\mathbf{x}_1 \dots \mathbf{x}_{t-1}} P(\mathbf{x}_1, \dots, \mathbf{x}_{t-1}, \mathbf{X}_t | \mathbf{e}_{1:t})$  i.e. the probabilities of the most likely path to each state  $\mathbf{X}_t$

- at the end this procedure will give the probability for the most likely sequence reaching each of the final states:
  - we can just select the most likely sequence over all
- in order to identify the actual sequence (as opposed to just computing the probability) the algorithm will also need to record, for each state, the best state that leads to it.

# Viterbi algorithm



Viterbi algorithm : time and space complexity  $O(t)$ ,  $t$  is the length of sequence

# Hidden Markov Models

- HMM is a temporal probabilistic model in which the state of the process is described by a single discrete random variable

$\mathbf{X}_t$  is a single, discrete variable (usually  $\mathbf{E}_t$  is too)

Domain of  $\mathbf{X}_t$  is  $\{1, \dots, S\}$

Transition matrix  $\mathbf{T}_{ij} = P(X_t = j | X_{t-1} = i)$ , e.g.,  $\begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$

Sensor matrix  $\mathbf{O}_t$  for each time step, diagonal elements  $P(e_t | X_t = i)$

e.g., with  $U_1 = \text{true}$ ,  $\mathbf{O}_1 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.2 \end{pmatrix}$

Forward and backward messages as column vectors:

$$\mathbf{f}_{1:t+1} = \alpha \mathbf{O}_{t+1} \mathbf{T}^\top \mathbf{f}_{1:t}$$

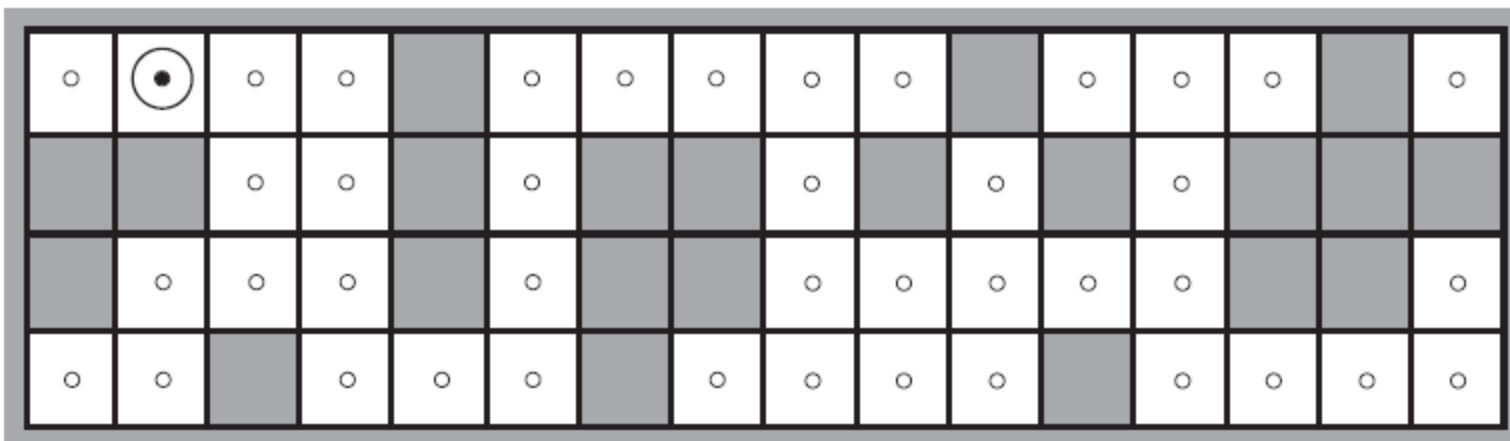
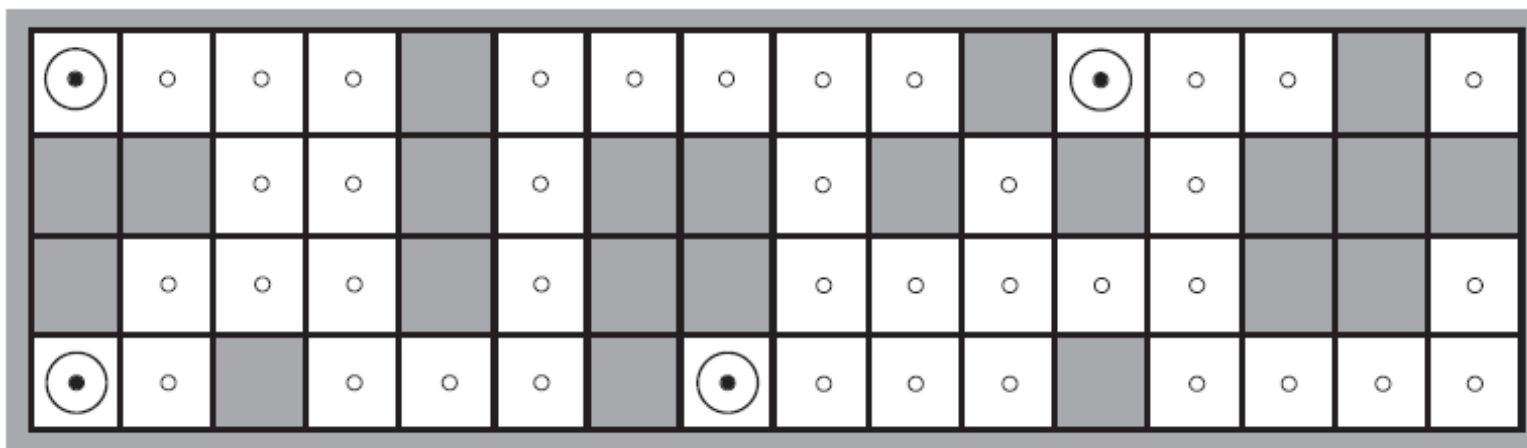
$$\mathbf{b}_{k+1:t} = \mathbf{T} \mathbf{O}_{k+1} \mathbf{b}_{k+2:t}$$

$$P(\mathbf{X}_{t+1} | e_{1:t+1}) = \alpha P(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \sum_{x_t} P(\mathbf{X}_{t+1} | \mathbf{X}_t) P(\mathbf{x}_t | \mathbf{e}_{1:t})$$

Forward-backward algorithm needs time  $O(S^2t)$  and space  $O(St)$

# HMM Example

- Robot localization



(b) Possible locations of robot After  $E_1 = \text{NSW}$ ,  $E_2 = \text{NS}$

# Kalman Filter

- State estimation in **linear dynamically systems(LDS)**, a Bayesian model (Dynamic Bayesian Network) similar to HMM but
  - state space of hidden variables is continuous
  - hidden and observed variables have a Gaussian distribution (i.e. dependencies are linear Gaussian)
- LDS represents a system of one or more real-valued variables that evolve linearly over time, with some Gaussian noise
- LDs are often used to model the dynamics of moving objects and to track their current positions given noisy measurements

# Kalman Filter

- Kalman Filter model assumes that the state of a system at a time  $t + 1$  evolved from the prior state at time  $t$  according to

$$\mathbf{X}_{t+1} = \mathbf{F}\mathbf{X}_t + \mathbf{B}\mathbf{u}_{t+1} + \mathbf{w}_{t+1}$$

- where  $\mathbf{X}_{t+1}$  is the state vector containing the terms of interest (e.g. position, velocity, heading at time  $t+1$ ),
- $\mathbf{u}_{t+1}$  vector of control inputs (steering angle, braking force)
- $\mathbf{F}$  is the state transition matrix,
- $\mathbf{B}$  is the control input matrix that applies the effect of each input parameter in the vector  $\mathbf{u}_{t+1}$  on the state vector (e.g. applies the effect of brake force on the system velocity and position)
- $\mathbf{w}_{t+1}$  noise vector containing noise terms for each parameter in the state vector-multivariate normal distribution with zero mean and  $\mathbf{Q}_t$  covariance matrix

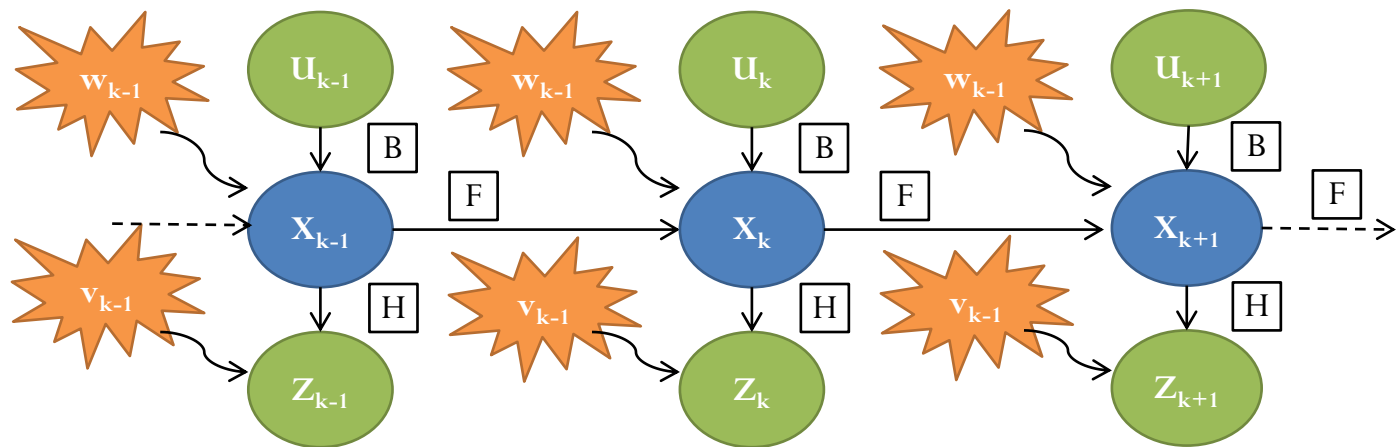


# Kalman Filter

- System measurements (observations) can be made with the following model

$$\mathbf{Z}_{t+1} = \mathbf{H}\mathbf{X}_{t+1} + \mathbf{v}_{t+1}$$

- Where  $\mathbf{Z}_{t+1}$  vector of measurements (observations),  $\mathbf{H}$  is the transformation matrix that maps the state vector parameters into the measurement domain (sensor model),  $\mathbf{v}_{t+1}$  is the vector containing the measurement noise terms for each observation in the measurement vector- assumed to be zero mean Gaussian with covariance  $\mathbf{R}_t$

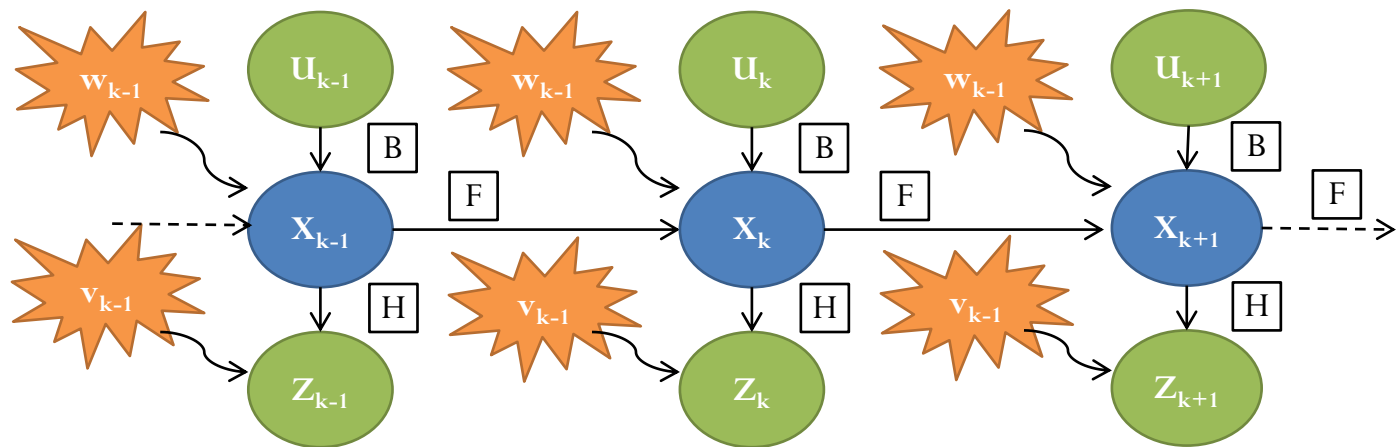


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# Kalman Filter

- Example: 1-D Tracking

$$\mathbf{X}_{t+1} = \begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix} \quad \text{State vector: position and velocity}$$

Control input: driver may apply braking or accelerating input to the system, which I considered here as a function of an applied force  $f_t$  and mass of vehicle  $m$ .

$$\mathbf{u}_t = \frac{f_t}{m}$$

Position and velocity of the vehicle at  $t+1$  can be given as :

$$x_{t+1} = x_t + \dot{x}_t \Delta t + \frac{f_t (\Delta t)^2}{2m} \quad \dot{x}_{t+1} = \dot{x}_t + \frac{f_t \Delta t}{m}$$

The above linear equations can be written in matrix form:

$$\begin{bmatrix} x_{t+1} \\ \dot{x}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} + \begin{bmatrix} \frac{(\Delta t)^2}{2} \\ \Delta t \end{bmatrix} \frac{f_t}{m}$$

$$\mathbf{X}_{t+1} = \mathbf{F}\mathbf{X}_t + \mathbf{B}\mathbf{u}_{t+1}$$



Image source: <http://www.iconcox.com/>

Using kalman filter we can estimate the state  $\hat{x}_{t+1}$  by combining models of the system and noisy measurements of certain parameters or linear functions of parameters.

Estimates are provided by probability density functions rather than discrete values.

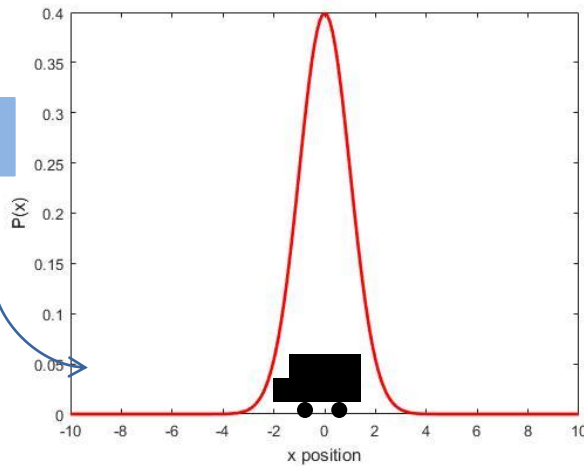
# Kalman Filter

- allows recursive estimation of  $\hat{\mathbf{x}}_{t+1}$  by combining prior knowledge, predictions from systems models, and noisy measurements
- Involves two stages:
  - time update (prediction)
  - measurement update (correction/update)
- **1-D tracking Example:** Estimate the position of the vehicle when information is available from two sources(i) predictions based on the last known position and velocity of the vehicle (ii) measurement from some positioning system.
  - These two are combined to give the best possible estimate of the position of the vehicle.

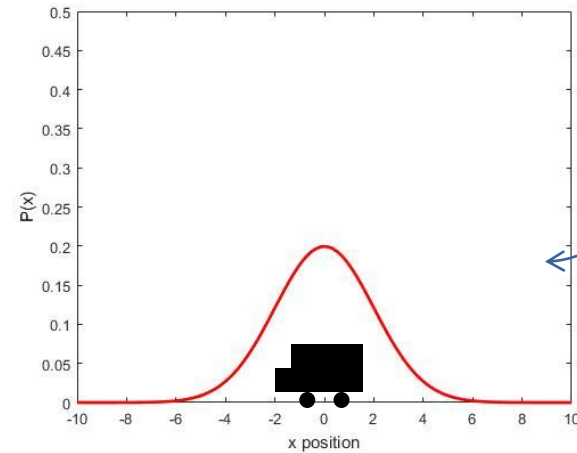
# Kalman Filter

- 1-D tracking Example:

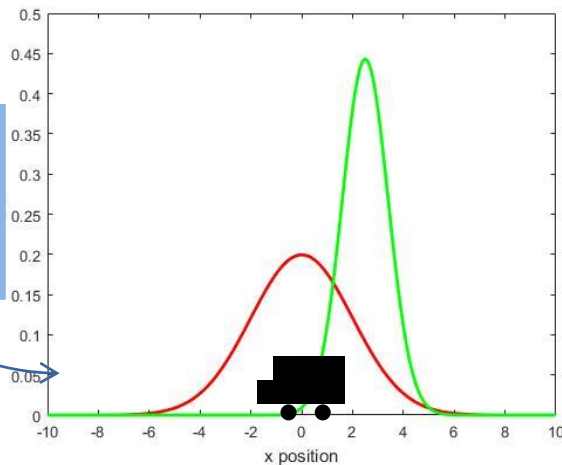
Prior belief



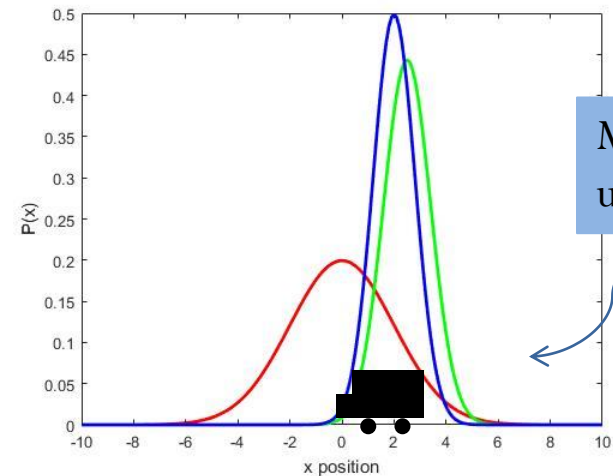
Time update



Measurement  
/observation  
available



Measurement  
update



# Kalman Filter

$$P(x_0) = \alpha e^{-\frac{1}{2}\left(\frac{(x_0-\mu_0)^2}{\sigma_0^2}\right)}$$

Gaussian Prior

$$P(x_{t+1}|x_t) = \alpha e^{-\frac{1}{2}\left(\frac{(x_{t+1}-x_t)^2}{\sigma_x^2}\right)}$$

$$P(z_t|x_t) = \alpha e^{-\frac{1}{2}\left(\frac{(z_t-x_t)^2}{\sigma_z^2}\right)}$$

Sensor model assumes Gaussian noise with variance  $\sigma_z^2$

Given the prior, the one step predicted distribution can be given as :

$$P(x_1) = \int_{-\infty}^{\infty} P(x_1|x_0)P(x_0)dx_0 = \alpha e^{-\frac{1}{2}\left(\frac{(x_1-\mu_0)^2}{\sigma_0^2+\sigma_x^2}\right)}$$

Product of two Gaussian Functions is another Gaussian function

Now, when  $z_1$  is available do the correction or measurement update step:

$$P(x_1|z_1) = \alpha P(z_1|x_1)P(x_1) = \alpha e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2}\right)}$$

$\mu_1$  or  $\mu_{t+1}$  is in slide 32



# Updating Gaussian distributions

Prediction step: if  $\mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$  is Gaussian, then prediction

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}) = \int_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t)P(\mathbf{x}_t|\mathbf{e}_{1:t}) d\mathbf{x}_t$$

is Gaussian. If  $\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})$  is Gaussian, then the updated distribution

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = \alpha \mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})$$

is Gaussian

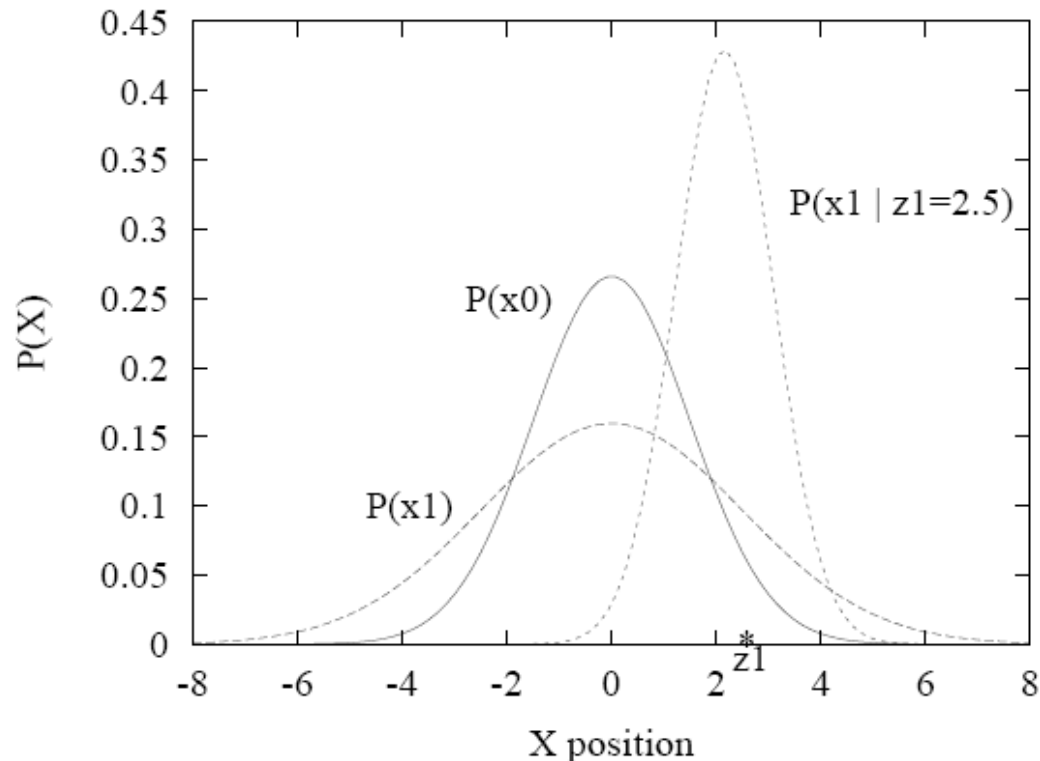
Hence  $\mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$  is multivariate Gaussian  $N(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$  for all  $t$

# Simple 1-D example

Gaussian random walk on  $X$ -axis, s.d.  $\sigma_x$ , sensor s.d.  $\sigma_z$

$$\mu_{t+1} = \frac{(\sigma_t^2 + \sigma_x^2)z_{t+1} + \sigma_z^2\mu_t}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2}$$

$$\sigma_{t+1}^2 = \frac{(\sigma_t^2 + \sigma_x^2)\sigma_z^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2}$$





# General Kalman update

Transition and sensor models:

$$\begin{aligned}P(\mathbf{x}_{t+1}|\mathbf{x}_t) &= N(\mathbf{F}\mathbf{x}_t, \Sigma_x)(\mathbf{x}_{t+1}) \\P(\mathbf{z}_t|\mathbf{x}_t) &= N(\mathbf{H}\mathbf{x}_t, \Sigma_z)(\mathbf{z}_t)\end{aligned}$$

$\mathbf{F}$  is the matrix for the transition;  $\Sigma_x$  the transition noise covariance  
 $\mathbf{H}$  is the matrix for the sensors;  $\Sigma_z$  the sensor noise covariance

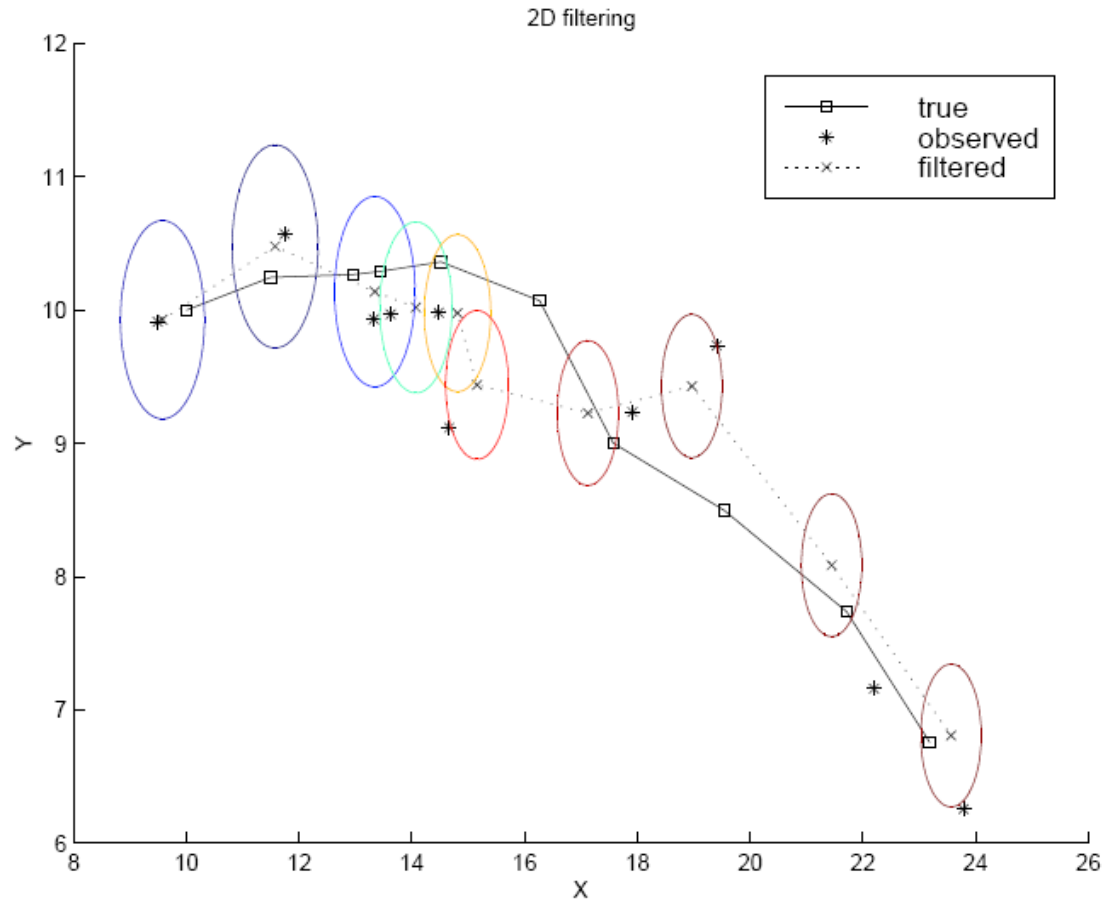
Filter computes the following update:

$$\begin{aligned}\mu_{t+1} &= \mathbf{F}\mu_t + \mathbf{K}_{t+1}(\mathbf{z}_{t+1} - \mathbf{H}\mathbf{F}\mu_t) \\ \Sigma_{t+1} &= (\mathbf{I} - \mathbf{K}_{t+1})(\mathbf{F}\Sigma_t\mathbf{F}^\top + \Sigma_x)\end{aligned}$$

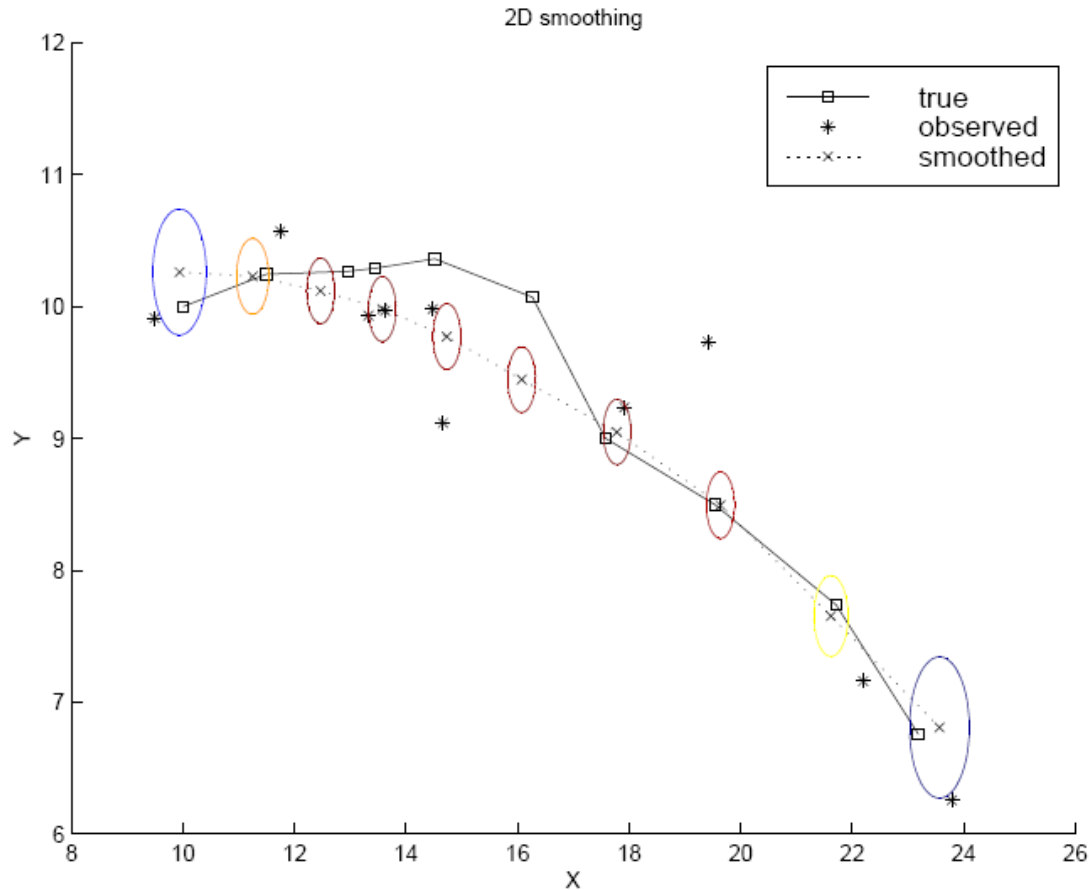
where  $\mathbf{K}_{t+1} = (\mathbf{F}\Sigma_t\mathbf{F}^\top + \Sigma_x)\mathbf{H}^\top(\mathbf{H}(\mathbf{F}\Sigma_t\mathbf{F}^\top + \Sigma_x)\mathbf{H}^\top + \Sigma_z)^{-1}$   
is the **Kalman gain matrix**

$\Sigma_t$  and  $\mathbf{K}_t$  are independent of observation sequence, so compute offline

# 2-D tracking example: Filtering



# 2-D tracking example: smoothing



# Dynamic Bayesian Networks

- **Dynamic Bayesian Networks (DBN)**
  - Each slice of DBN can have any number of state  $\mathbf{X}_t$  and evidence variables  $\mathbf{E}_t$ , variables and their links same from slice to slice
  - First order Markov process – each variable can have parents only in its own slice or the immediately preceding slice
- Vehicle localization task: track current location using data obtained from possibly faulty sensor.
- System state can be encoded using:
  - Location, Velocity, Weather, Failure (failure status of the sensor), and Obs (the current observation ): **one such set of variables for every point t**

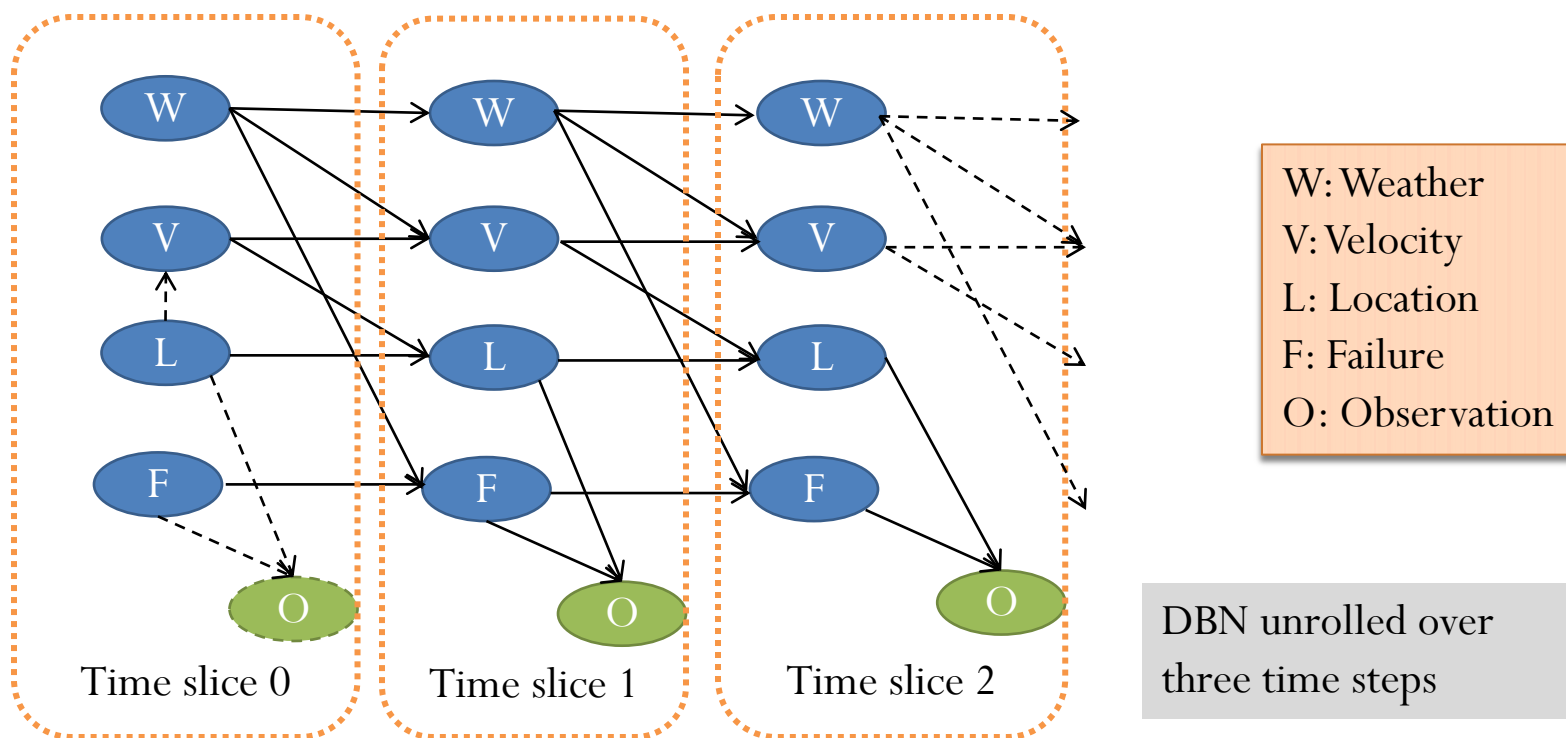


Image source: <http://www.iconcox.com/>

Given a sequence of observations about the car, where is it now?  
Where is it likely to be in 10 minutes?  
Did it stop at red light?

# Dynamic Bayesian Networks

- current observation depends on vehicle's location (and the map, which is not explicitly modeled), and on the error status of the sensor
- bad weather makes the sensor more likely to fail
- vehicle's location depends on the previous position and velocity



# Dynamic Bayesian Networks

- **HMM- simplest DBN** with a single state variable and a single observation variable
- **Discrete DBN- HMM** with all the state variables in the DBN as a single state variable whose values are all possible tuples of values of the individual state variables
- When the state of a complex system is decomposed into its constituent variables, it can take advantage of sparseness in the temporal probability model
  - Sparse dependencies  $\rightarrow$  exponentially fewer parameters
  - Example: 20 state variables, three parents each
  - DBN has  $20 \times 2^3 = 160$  probabilities and corresponding HMM has  $2^{20}$  states and so  $2^{40}$  probabilities in transition matrix
- Every Kalman Filter model is a DBN, but not all DBNs are KFs; real world requires non-Gaussian posteriors

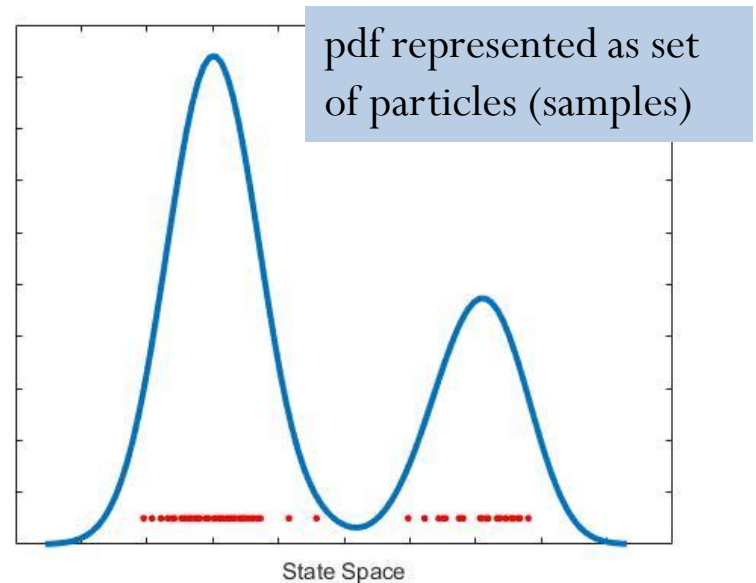
# Inference in DBNs

- DBNs are Bayesian networks – inference algorithms for Bayesian networks can be used by unrolling
- **Unrolling the DBN**- given a sequence of observations, replicate slices until the network accommodates the observations
- Recursive computations (e.g. in filtering and smoothing) require constant time update (independent of  $t$ ), however, the “constant” time is exponential in the number of state variables.
- Approximate inference preferred over exact inference due to computational costs in networks with large number of variables.

# Approximate inference in DBNs

- **Particle Filter**

- **key idea:** at any given time, belief represented with a collection of particles (samples) drawn from the distribution
- We need to estimate the state i.e  $P(\mathbf{X}_t | \mathbf{e}_{1:t})$  where  $\mathbf{e}_{1:t}$  is the observation sequence.
  - Each particle contains one set of values for the state variables
  - We want to draw the samples from the posterior  $P(\mathbf{X}_t | \mathbf{e}_{1:t})$ , but we don't have explicit representation of the posterior
  - But we have prior belief, sample from prior and then weight each sample based on observations made.





# Particle Filtering

## Particle filter

**Input:** **e**-the incoming evidence,  
N-the number of samples to be  
maintained, a DBN with prior  
 $\mathbf{P}(\mathbf{X}_0)$ , transition model  $\mathbf{P}(\mathbf{X}_1|\mathbf{X}_0)$ ,  
sensor model  $\mathbf{P}(\mathbf{E}_1|\mathbf{X}_1)$ , **S** vector of  
samples of size N, **W**-vector of  
weights of size N

**Output:** a set of samples for the  
next time step

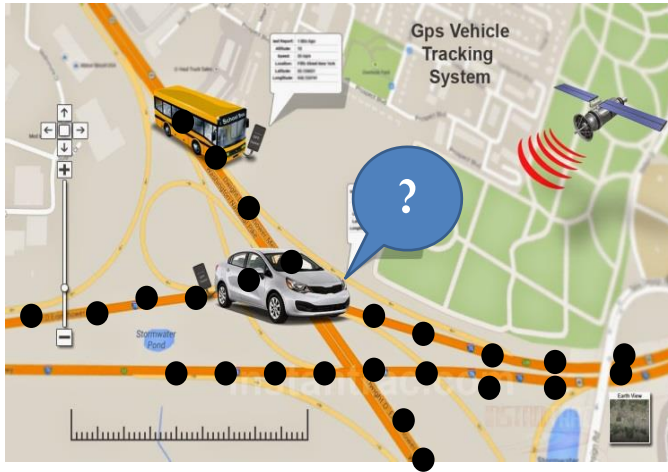
```
Initialize S by drawing samples from  $\mathbf{P}(\mathbf{X}_0)$  and  $\mathbf{W} \leftarrow 1$ 
for i = 1 to N do
     $\mathbf{S}[i] \leftarrow$  sample from  $\mathbf{P}(\mathbf{X}_1|\mathbf{X}_0 = \mathbf{S}[i])$ 
     $\mathbf{W}[i] \leftarrow \mathbf{P}(\mathbf{e}|\mathbf{X}_1=\mathbf{S}[i])$ 
for j = 1 to N do
     $\text{temp\_S}[j] \leftarrow$  sample from S with probability  $\mathbf{W}[i]$ 
S  $\leftarrow$  temp_S
W  $\leftarrow 1$ 
return S
```

Start with sampling from initial pdf  
 $\mathbf{P}(\mathbf{X}_0)$

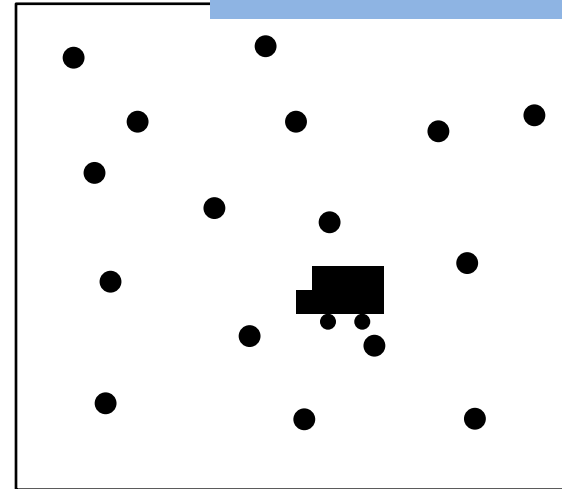
For each time step, repeat the  
following

- time update (prediction)
- measurement update  
(correction/update)
- Resample

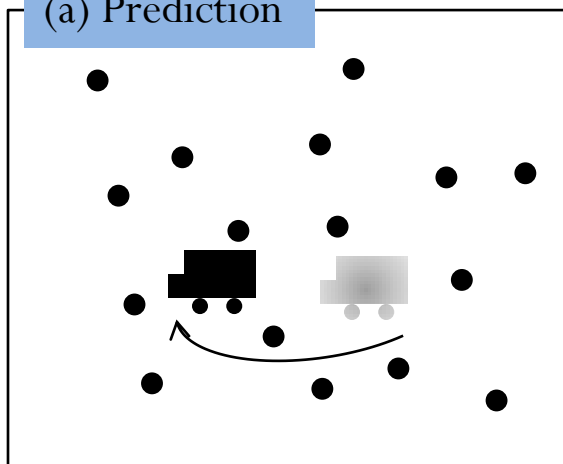
# Particle Filtering



Initial distribution

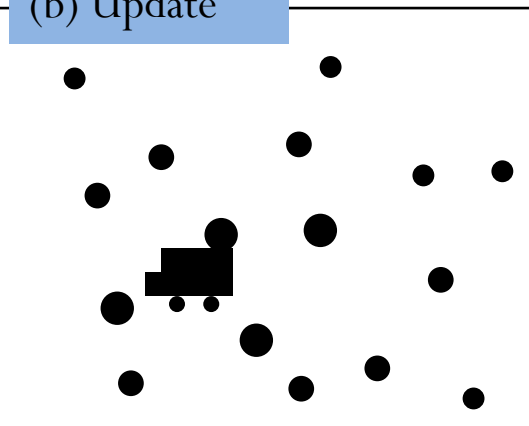


(a) Prediction



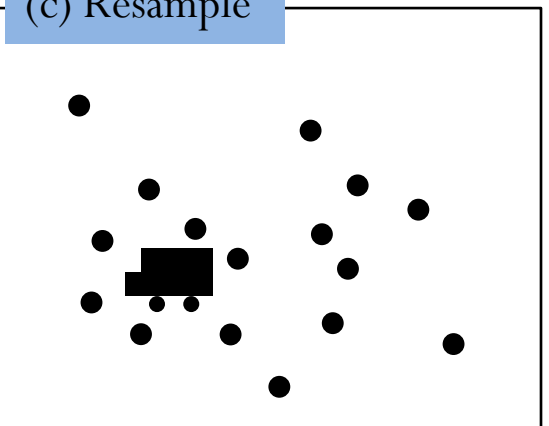
Sample from  $P(x_{t+1}|x_t)$

(b) Update



Weight samples with  
 $P(z_{t+1}|x_{t+1})$

(c) Resample

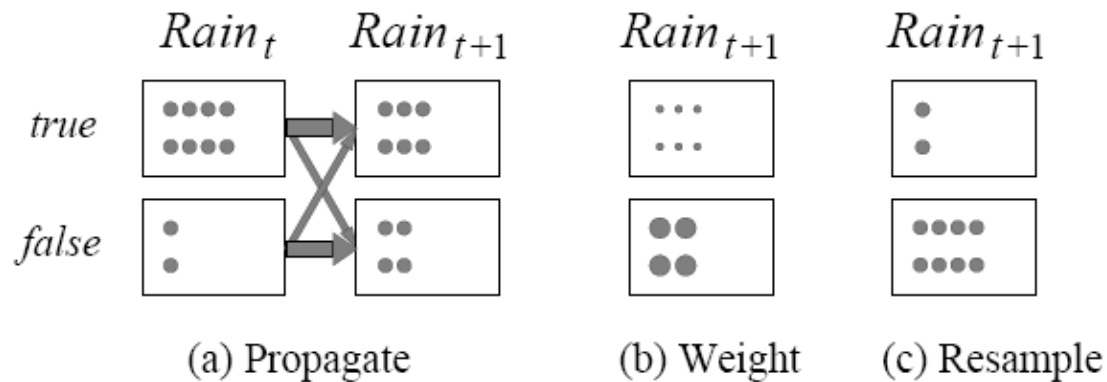


Resample proportional to  
weight

# Particle Filtering

The particle filtering update cycle for the Umbrella DB with  $N = 10$ , showing the sample population of each state.

(a) At time  $t$ , 8 samples indicate *rain* and 2 indicate  $\sim$ *rain*. Each is propagated forward by sampling the next state through the transition model. At  $t+1$ , 6 samples indicate *rain* and 4 indicate  $\sim$ *rain*.



(b)  $\sim$ *umbrella* is observed at  $t+1$ . Each sample is weighted by its likelihood for the observation, as indicated by the size of the circles.

(c) A new set of samples is generated by weighted random selection from the current set, resulting in 2 samples that indicate  $\sim$ *rain* and 8 that indicate *rain*.

# What did we discuss in L6-8?

- What are temporal models and how to represent them?
- How Markov assumption simplifies the transition and observation models?
- How to perform Filtering, Prediction, Smoothing, and Most likely explanation (inference tasks) recursively with constant time per step?
- What are Dynamic Bayesian Networks (DBNs), HMMs, Kalman Filtering for Linear Dynamical systems and how to represent them and how to infer from such models?
- What is Particle filtering and how is it useful to DBNs when exact inference in such networks is intractable?