Kalman Filter example

Example 1: Let us consider a simple situation where we are interested in estimating the temperature in a room. Three sensors are present inside the room that gives noisy measurements. We will try to use the sensor measurements from all the three sensors to estimate the current value of the state variable (i.e. temperature at time t). (This is also an example of sensor data fusion).

The system is given as

$$X_{t+1} = F X_t + B u_t + w_{t+1}$$

w_{t+1} is the noise vector with zero mean and Q_t covariance matrix

$$Z_{t+1} = H X_{t+1} + V_{t+1}$$

V_{t+1} is the measurement noise with zero mean and R_t covariance

The problem is to find an estimate $\hat{X}_{t+1}$ of the state vector X_{t+1} from incomplete and noisy measurement vector Z_{t+1}.

Kalman Filter (Estimator)

$$\hat{X}_{t+1} = A X_t + B U_t + K_{t+1} \left(Z_{t+1} - H \hat{X}_t \right)$$

K_{t+1}: filter gain or Kalman Gain

* The filter gain ensures that estimation error e converges to zero.
(Refer to derivation of Kalman Filter)

$$\lim_{t \to \infty} e(t) \to 0$$

Kalman Gain K_{t+1} is given as follows

$$K_{t+1} = \left(A P_t A^T + Q \right) H^T \left(H \left(A P_t A^T + Q \right) H^T + R \right)^{-1}$$

Here P_t is the error Covariance matrix at time t

$$P_{t+1} = \left(I - K_{t+1} \right) \left(A P_t A^T + Q \right)$$

Let us now define our models for the given problem.

The changes in temperature are modeled as a Gaussian process.

$$X_{t+1} = X_t + W_{t+1}$$ with $\mu = 25$, $\sigma = 1.3$

$A = 1$ $\qquad Q = 1.21$

$B = 0$

measurements are given as

$$Z_{t+1} = H X_{t+1} + V_{t+1}$$

Z_{t+1} is a 3×1 vector for 3 sensors

$$H = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad R = \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}$$

Let at $t = 0$, $X_0 = 20$

Now, we have to estimate $\hat{X}_1$ at $t = 1$

<u>first step</u> : time update (prediction)

$$X_{t+1} = X_t + W_{t+1} \quad \text{with } \mu = 0, \; Q = 1.21$$

$$A = 1$$

$$X_1 = X_0 + W_1$$

$$= 20 + 0.5914$$

this value is generated using MATLAB considering noise of 0 mean and 1.21 variance

$$= 20.6$$

<u>step 2</u> : measurement update (update / correction)

$$Z_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [20.6] + \begin{bmatrix} 0.1042 \\ -0.3980 \\ -0.2229 \end{bmatrix}$$

generated with MATLAB code

$[20.7 \quad]$

Now, we can apply the Kalman Filter (Estimator)

$$\hat{x}_{t+1} = A x_t + B u_t + K_{t+1}(z_{t+1} - H \hat{x}_t)$$

$$\hat{x}_1 = [1] 20 + K_{t+1}\left(\begin{bmatrix} 20.7 \\ 20.2 \\ 20.4 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [20] \right)$$

$$K_{t+1} = ([1] P_0 A^T + Q) H^T (H (A P_0 A^T + Q) H^T + R)^{-1}$$

P_0 = initial value of error co-variance matrix is set to large values.

Let $P_0 = 100$

$$P_1 = (I - K_1)(A P_0 A^T + Q)$$

Using the values of P_0, A, H, Q, R we can compute K_1 and P_1 and subsequently $\hat{x}_1$.