Assignment 03

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We want to define an implicit function given a set of points.

Intuition

Given a point cloud, we want an implicit function that defines a surface describing the points.

- Point set may have some incomplete information
- Point set may have noise
- Point set may be dense in some areas, sparse in others

We assume we know point set and can find the normal for each point.

Normal to a point is a vector rooted at that point, pointing out of the body whose surface we are trying to construct.

We try to search for a function whose gradient we take to be the normals given to us.

The surface described by the zero-set of this function will mostly be tangential to each point in the point set, hence we will get a very smooth surface. It will be globally fitted.

One significant benefit of this sort of surface reconstruction is that even if there is noise, surface generated will be relatively smooth.

Mathematical background

The input data S is a set of samples $s \in S$, each consisting of a point s.p and an inward-facing normal $s.\vec{N}$, assumed to lie on or near the surface M of an unknown model M. Our goal is to reconstruct a watertight, triangulated approximation to the surface by approximating the indicator function of the model and extracting the isosurface.

We define a function with value less than zero outside the model and greater than zero inside the model. Then, we extract the zero-set.

We reconstruct the surface of the model by solving for the indicator function of the shape.

$$\chi_m(p) = \begin{cases} 1 & \text{if } p \in M \\ 0 & \text{if } p \notin M \end{cases}$$

The key challenge is to accurately compute the indicator function from the samples. In this section, we derive a relationship between the gradient of the indicator function and an integral of the surface normal field. We then approximate this surface integral by a summation over the given oriented point samples. Finally, we reconstruct the indicator function from this gradient field as a Poisson problem.

Because the indicator function is a piecewise constant function, explicit computation of its gradient field would result in a vector field with unbounded values at the surface boundary. To avoid this, we convolve the indicator function with a smoothing filter and consider the gradient field of the smoothed function.

The following lemma formalizes the relationship between the gradient of the smoothed indicator function and the surface normal field.

Lemma: Given a solid M with boundary δM, let χ_m denote the indicator function of M, $\vec{N}_{\delta M}(p)$ be the inward surface normal at $p \in$ δM, $\hat{F}(q)$ be a smoothing filter, and $\hat{F}_p(q) = \hat{F}(q-p)$ its translation to the point p. The gradient of the smoothed indicator function is equal to the vector field obtained by smoothing the surface normal field:

$$\nabla(\chi_m * \hat{F})(q_0) = \int_{\delta M} \hat{F}_p(q_0)\vec{N}_{\delta M}(p)dp \tag{1}$$

We can prove the above lemma by partial differentiation of both the sides with respect to x, y & z at q_0. Thus, the normal field $\vec{N}$ is gradient of *smoothed* χ_m.

Of course, we cannot evaluate the surface integral since we do not yet know the surface geometry. However, the input set of oriented points provides precisely enough information to approximate the integral with a discrete summation. Specifically, using the point set S to partition δM into distinct patches $P_s \subset \delta M$, we can approximate the integral over a patch P_s by the value at point sample s.p, scaled by the area of the patch:

$$\nabla(\chi_m * \hat{F})(q) = \sum_{s \in S} \int_{P_s} \hat{F}_p(q) \vec{N}_{\delta M}(p) dp \tag{2}$$

$$\approx \sum_{s \in S} |P_s| \hat{F}_{s.p}(q) s.\vec{N} \equiv \vec{V}(q) \tag{3}$$

Having formed a vector field $\vec{V}$, we want to solve for the function $\hat{\chi}$ such that $\nabla\hat{\chi} = \nabla\vec{V}$. However, $\vec{V}$ is generally not integrable (i.e. it is not curlfree), so an exact solution does not generally exist. To find the best least-squares approximate solution, we apply the divergence operator to form the Poisson equation.

$$\Delta\hat{\chi} = \nabla.\vec{V}$$

The problem of computing the indicator function thus reduces to inverting the gradient operator, i.e. finding the scalar function χ whose gradient best approximates a vector field.If we apply the divergence operator, this variational problem transforms into a standard Poisson problem: compute the scalar function χ whose Laplacian (divergence of gradient) equals the divergence of the vector field $\vec{V}$ defined by the samples.

$$\Delta\hat{\chi} = \nabla.\nabla\chi = \nabla.\vec{V} \tag{4}$$

For any poisson equation of the form $\nabla^2 f(x_i) = \nabla.g_i$ subject to some boundary conditions, we can represent f as following:

$$f(x) = \sum_{j=1}^{k} a_j B_j(||x - c_j||)$$

We need to solve a linear system for the coefficients a_j & compute second derivatives of B_j to solve the poisson equation.

How it is working

We first present our reconstruction algorithm under the assumption that the point samples are uniformly distributed over the model surface. We define a space of functions with high resolution near the surface of the model and coarser resolution away from it, express the vector field $\vec{V}$ as a linear sum of functions in this space, set up and solve the Poisson equation, and extract an isosurface of the resulting indicator function.

The implementation goes on as following:

- Set Octree
- Compute Vector Field
    - Define the function space
    - Splat the samples
- Compute Indicator Function
    - Compute Divergence
    - Solve Poisson Equation
- Extract Isosurface