Simple Sorting Algorithms

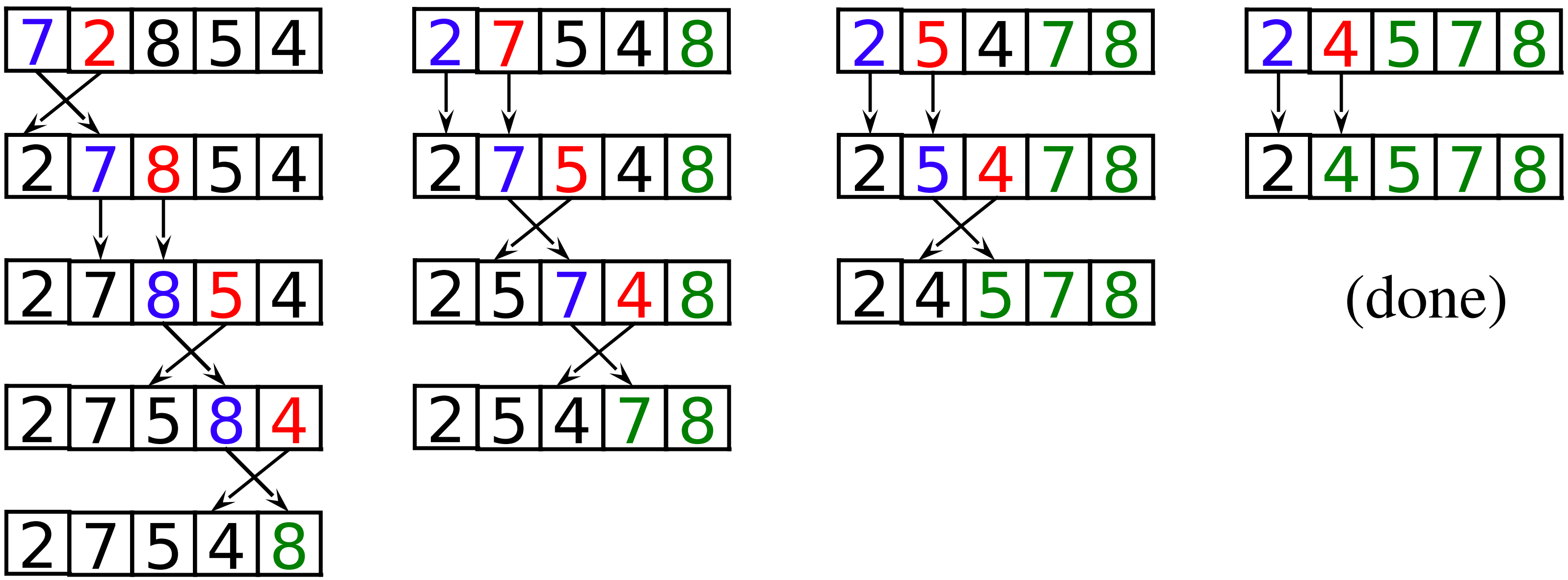
Outline

- We are going to look at three simple sorting techniques: Bubble Sort, Selection Sort, and Insertion Sort
- We are going to develop the notion of a loop invariant
- We will write code for Bubble Sort and Selection Sort, then derive their loop invariants
- We will *start* with a loop invariant for Insertion Sort, and derive the code for it
- We will analyze the running time for each of the above

Bubble Sort

- Compare each element (except the last one) with its neighbor to the right
  - If they are out of order, swap them
  - This puts the largest element at the very end
  - The last element is now in the correct and final place
- Compare each element (except the last *two*) with its neighbor to the right
  - If they are out of order, swap them
  - This puts the second largest element next to last
  - The last two elements are now in their correct and final places
- Compare each element (except the last *three*) with its neighbor to the right
  - Continue as above until you have no unsorted elements on the left

Example of Bubble Sort



Code for Bubble Sort

```
public static void bubbleSort(int[] a) {
   int outer, inner;
   for (outer = a.length - 1; outer > 0; outer--) {  // counting down
      for (inner = 0; inner < outer; inner++) {        // bubbling up
         if (a[inner] > a[inner + 1]) {  // if out of order...
            int temp = a[inner];          // ...then swap
            a[inner] = a[inner + 1];
            a[inner + 1] = temp;
         }
      }
   }
}
```

Analysis of Bubble Sort

- ```
  for (outer = a.length - 1; outer > 0; outer--) {
     for (inner = 0; inner < outer; inner++) {
        if (a[inner] > a[inner + 1]) {
           // code for swap omitted
  }  }  }
  ```
- Let `n = a.length =` size of the array
- The outer loop is executed `n-1` times (call it `n`, that's close enough)
- Each time the outer loop is executed, the inner loop is executed
  - Inner loop executes `n-1` times at first, linearly dropping to just once
  - On average, inner loop executes about `n/2` times for each execution of the outer loop
  - In the inner loop, the comparison is always done (constant time), the swap might be done (also constant time)
- Result is $n * n/2 * k$, that is, $O(n^2/2 + k) = O(n^2)$
  ```

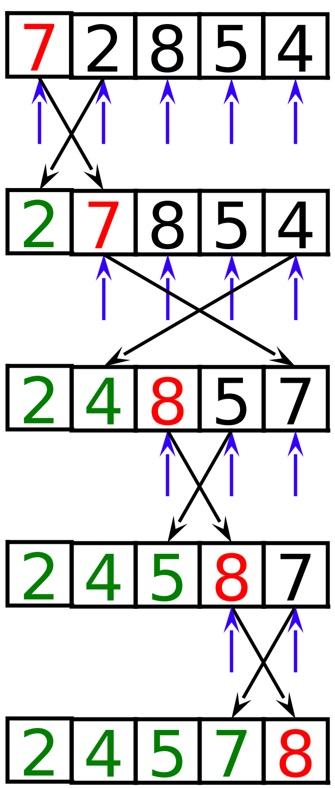
Loop invariants

- You run a loop in order to change things
- Oddly enough, what is usually most important in understanding a loop is finding an invariant: that is, *a condition that doesn't change*
- In Bubble Sort, we put the largest elements at the end, and once we put them there, we don't move them again
  - The variable `outer` starts at the last index in the array and decreases to `0`
  - Our invariant is: Every element to the right of `outer` is in the correct place
  - That is, `for all j > outer, if i < j, then a[i] <= a[j]`
  - When this is combined with the loop exit test, `outer == 0`, we know that *all* elements of the array are in the correct place

Another sort: Selection Sort

- Given an array of length `n`,
  - Search elements `0` through `n-1` and select the smallest
    - Swap it with the element in location `0`
  - Search elements `1` through `n-1` and select the smallest
    - Swap it with the element in location `1`
  - Search elements `2` through `n-1` and select the smallest
    - Swap it with the element in location `2`
  - Search elements `3` through `n-1` and select the smallest
    - Swap it with the element in location `3`
  - Continue in this fashion until there’s nothing left to search

Example and analysis of Selection Sort



- The Selection Sort might swap an array element with itself--this is harmless, and not worth checking for
- Analysis:
  - The outer loop executes n-1 times
  - The inner loop executes about n/2 times on average (from n to 2 times)
  - Work done in the inner loop is constant (swap two array elements)
  - Time required is roughly (n-1)*(n/2)
  - You should recognize this as $O(n^2)$

Code for Selection Sort

```
public static void selectionSort(int[] a) {
    int outer, inner, min;
    for (outer = 0; outer < a.length - 1; outer++) { // outer counts down
        min = outer;
        for (inner = outer + 1; inner < a.length; inner++) {
            if (a[inner] < a[min]) {
                min = inner;
            }
            // Invariant: for all i, if outer <= i <= inner, then a[min] <= a[i]
        }
        // a[min] is least among a[outer]..a[a.length - 1]
        int temp = a[outer];
        a[outer] = a[min];
        a[min] = temp;
        // Invariant: for all i <= outer, if i < j then a[i] <= a[j]
    }
}
```

Invariants for Selection Sort

- For the inner loop:
  - This loop searches through the array, incrementing `inner` from its initial value of `outer+1` up to `a.length-1`
  - As the loop proceeds, `min` is set to the index of the smallest number found so far
  - Our invariant is:
    `for all i such that outer <= i <= inner, a[min] <= a[i]`
- For the outer (enclosing) loop:
  - The loop counts up from `outer = 0`
  - Each time through the loop, the minimum remaining value is put in `a[outer]`
  - Our invariant is:
    `for all i <= outer, if i < j then a[i] <= a[j]`

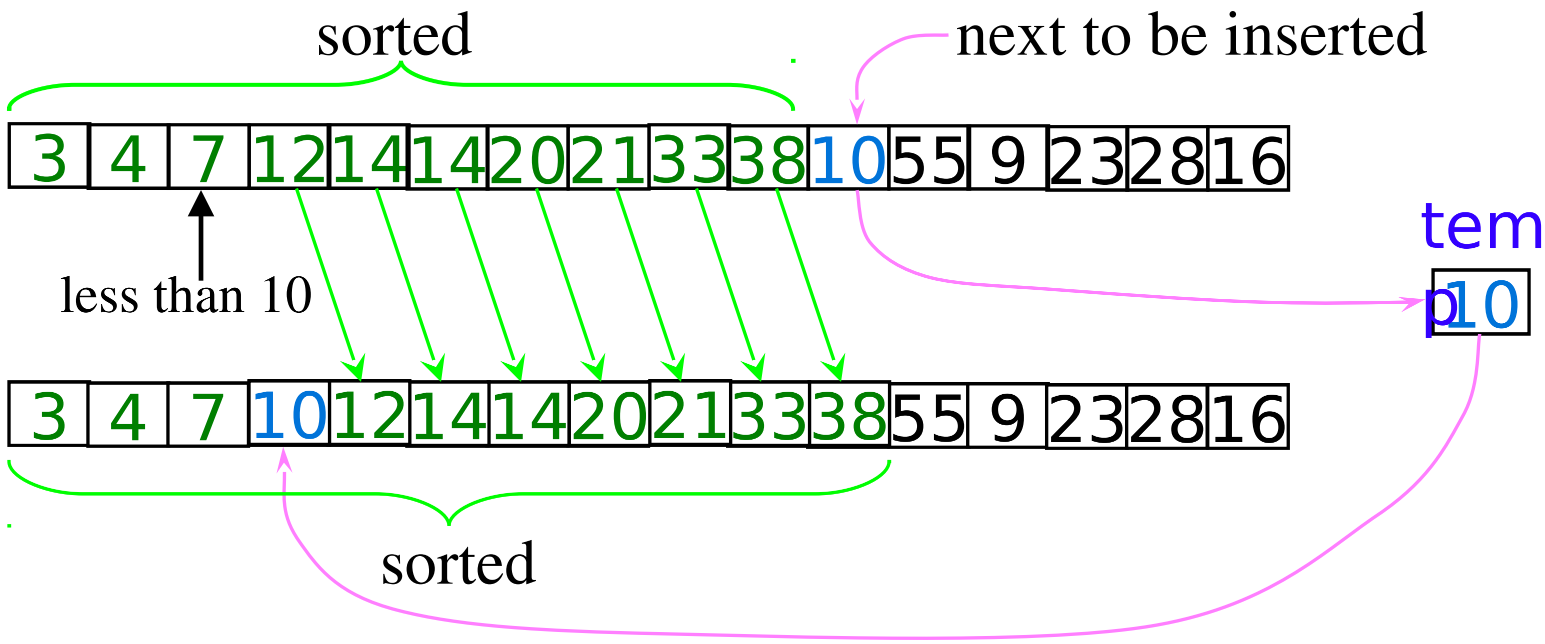
Summary so far

- We've looked at code for Bubble Sort and Selection Sort
  - We've figured out their loop invariants
  - We've figured out their running time
- Next, we are going to start with a loop invariant that ought to result in a sort method
  - We will derive the sort method
  - We will figure out its running time

Insertion sort

- The outer loop of insertion sort is:
  `for (outer = 1; outer < a.length; outer++) {...}`
- The invariant is that all the elements to the left of `outer` are sorted *with respect to one another*
  - For all i < outer, j < outer, if i < j then a[i] <= a[j]
  - This does *not* mean they are all in their final correct place; the remaining array elements may need to be inserted
  - When we increase `outer`, `a[outer-1]` becomes to its left; we must keep the invariant true by inserting `a[outer-1]` into its proper place
  - This means:
    - Finding the element's proper place
    - Making room for the inserted element (by shifting over other elements)
    - Inserting the element

One step of insertion sort



Code for Insertion Sort

```
public static void insertionSort(int[] array) {
    int inner, outer;

    for (outer = 1; outer < array.length; outer++) {
        int temp = array[outer];
        inner = outer;
        while (inner > 0 && array[inner - 1] >= temp) {
            array[inner] = array[inner - 1];
            inner--;
        }
        array[inner] = temp;
        // Invariant: For all i < outer, j < outer, if i < j then a[i] <=
a[j]
    }
}
```

Analysis of insertion sort

- We run once through the outer loop, inserting each of n elements; this is a factor of n
- On average, there are $n/2$ elements already sorted
  - The inner loop looks at (and moves) half of these
  - This gives a second factor of $n/4$
- Hence, the time required for an insertion sort of an array of n elements is proportional to $n^2/4$
- Discarding constants, we find that insertion sort is $O(n^2)$

Summary

- Bubble Sort, Selection Sort, and Insertion Sort are all $O(n^2)$
- As we will see later, we can do much better than this with somewhat more complicated sorting algorithms
- Within $O(n^2)$,
  - Bubble Sort is very slow, and should probably never be used for anything
  - Selection Sort is intermediate in speed
  - Insertion Sort is usually the fastest of the three--in fact, for small arrays (say, 10 or 15 elements), insertion sort is faster than more complicated sorting algorithms
- Selection Sort and Insertion Sort are “good enough” for small arrays
- Use of a Bubble Sort tends to elicit derision from your colleagues

The End