Quantum Tunneling through a Single Potential Barrier: An Analytical Approach

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Problem Setup

Suppose the potential $V(x)$ is given by:

$$V(x) = \begin{cases} V_0, & 0 \leq x \leq L \\ 0, & \text{otherwise} \end{cases}$$

We analyze the case for a particle with energy $E > 0$ incident from the left. This paper focuses on the quantum tunneling, so we assume the particle's energy is less than the barrier height: $0 < E < V_0$.

Time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

Wave Function Solutions in Each Region

Region I $(x < 0, V = 0)$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_I}{dx^2} = E\psi_I \implies \frac{d^2\psi_I}{dx^2} + k_1^2\psi_I = 0$$

Define $k_1^2 = \frac{2mE}{\hbar^2}$

$$\psi_I(x) = Ae^{ik_1x} + Be^{-ik_1x}$$

Region II $(0 \leq x \leq L, V = V_0)$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{II}}{dx^2} + V_0\psi_{II} = E\psi_{II} \implies \frac{d^2\psi_{II}}{dx^2} - k_2^2\psi_{II} = 0$$

Define $k_2^2 = \frac{2m(V_0-E)}{\hbar^2}$

$$\psi_{II}(x) = Ce^{k_2x} + De^{-k_2x}$$

Region III $(x > L, V = 0)$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{III}}{dx^2} = E\psi_{III} \implies \frac{d^2\psi_{III}}{dx^2} + k_1^2\psi_{III} = 0$$

Define $k_1^2 = \frac{2mE}{\hbar^2}$

$$\psi_{III}(x) = Fe^{ik_1 x}$$

From the boundary conditions

We apply continuity for $\psi(x)$ and $\frac{d\psi}{dx}$ at both $x = 0$ and $x = L$. This gives four conditions:

$$\psi_I(0) = \psi_{II}(0) \implies A + B = C + D \tag{1}$$

$$\psi_{II}(L) = \psi_{III}(L) \implies Ce^{k_2 L} + De^{-k_2 L} = Fe^{ik_1 L} \tag{2}$$

$$\left.\frac{d\psi_I}{dx}\right|_{x=0} = \left.\frac{d\psi_{II}}{dx}\right|_{x=0} \implies ik_1(A - B) = k_2(C - D) \tag{3}$$

$$\left.\frac{d\psi_{II}}{dx}\right|_{x=L} = \left.\frac{d\psi_{III}}{dx}\right|_{x=L} \implies k_2(Ce^{k_2 L} - De^{-k_2 L}) = ik_1 Fe^{ik_1 L} \tag{4}$$

Solving the system

From (2) and (4):

$$(2) + \frac{(4)}{k_2} \implies 2Ce^{k_2 L} = \left(1 + \frac{ik_1}{k_2}\right) Fe^{ik_1 L}$$

$$C = \left(1 + \frac{ik_1}{k_2}\right) \frac{F}{2} e^{ik_1 L - k_2 L} \tag{5}$$

$$(2) - \frac{(4)}{k_2} \implies 2De^{-k_2 L} = \left(1 - \frac{ik_1}{k_2}\right) Fe^{ik_1 L}$$

$$D = \left(1 - \frac{ik_1}{k_2}\right) \frac{F}{2} e^{ik_1 L + k_2 L} \tag{6}$$

From (1) and (3):

$$(1) + \frac{(3)}{ik_1} \implies 2A = \left(1 - \frac{ik_2}{k_1}\right) C + \left(1 + \frac{ik_2}{k_1}\right) D \tag{7}$$

Note: Define $\gamma = \frac{k_1^2 - k_2^2}{k_1 k_2}$. Then:

$$\left(1 - \frac{ik_2}{k_1}\right)\left(1 + \frac{ik_1}{k_2}\right) = 2 + i\left(\frac{k_1^2 - k_2^2}{k_1 k_2}\right) = 2 + i\gamma$$

$$\left(1 + \frac{ik_2}{k_1}\right)\left(1 - \frac{ik_1}{k_2}\right) = 2 - i\left(\frac{k_1^2 - k_2^2}{k_1 k_2}\right) = 2 - i\gamma$$

Substitute (5) and (6) into the equation for $2A$ (eq. 7):

$$2A = \left[\left(1 - \frac{ik_2}{k_1}\right)\left(1 + \frac{ik_1}{k_2}\right)\frac{F}{2}e^{ik_1L-k_2L} + \left(1 + \frac{ik_2}{k_1}\right)\left(1 - \frac{ik_1}{k_2}\right)\frac{F}{2}e^{ik_1L+k_2L}\right]$$
$$2A = \frac{F}{2}e^{ik_1L}\left[(2+i\gamma)e^{-k_2L} + (2-i\gamma)e^{k_2L}\right]$$
$$2A = \frac{F}{2}e^{ik_1L}\left[2(e^{k_2L}+e^{-k_2L}) - i\gamma(e^{k_2L}-e^{-k_2L})\right]$$
$$2A = Fe^{ik_1L}\left[2\cosh(k_2L) - i\gamma\sinh(k_2L)\right]$$

Now, we find the transmission amplitude $\frac{F}{A}$:

$$\frac{F}{A} = \frac{2e^{-ik_1L}}{2\cosh(k_2L) - i\gamma\sinh(k_2L)} = \frac{e^{-ik_1L}}{\cosh(k_2L) - \frac{i\gamma}{2}\sinh(k_2L)}$$

The Transmission Probability: $T = \left|\frac{F}{A}\right|^2 = \left(\frac{F}{A}\right)\left(\frac{F}{A}\right)^*$

$$T = \left(\frac{e^{-ik_1L}}{\cosh(k_2L) - \frac{i\gamma}{2}\sinh(k_2L)}\right)\left(\frac{e^{+ik_1L}}{\cosh(k_2L) + \frac{i\gamma}{2}\sinh(k_2L)}\right)$$
$$T = \frac{1}{\cosh^2(k_2L) + \left(\frac{\gamma}{2}\right)^2\sinh^2(k_2L)}$$

Approximation for $k_2L \gg 1$

If $k_2L \gg 1$, then $\cosh(k_2L) \approx \frac{e^{k_2L}}{2}$ and $\sinh(k_2L) \approx \frac{e^{k_2L}}{2}$. The denominator becomes:

$$\cosh^2(k_2L) + \left(\frac{\gamma}{2}\right)^2\sinh^2(k_2L) \approx \left(\frac{e^{k_2L}}{2}\right)^2 + \left(\frac{\gamma}{2}\right)^2\left(\frac{e^{k_2L}}{2}\right)^2 = \frac{e^{2k_2L}}{4}\left(1 + \frac{\gamma^2}{4}\right)$$

So,

$$T \approx \frac{1}{\frac{e^{2k_2L}}{4}\left(1+\frac{\gamma^2}{4}\right)} = \frac{4e^{-2k_2L}}{1+\frac{\gamma^2}{4}}$$

We simplify the $1 + \frac{\gamma^2}{4}$ term:

$$1 + \frac{\gamma^2}{4} = 1 + \frac{1}{4}\left(\frac{k_1^2 - k_2^2}{k_1k_2}\right)^2 = \frac{4k_1^2k_2^2 + (k_1^2-k_2^2)^2}{4k_1^2k_2^2}$$
$$= \frac{4k_1^2k_2^2 + k_1^4 - 2k_1^2k_2^2 + k_2^4}{4k_1^2k_2^2} = \frac{(k_1^2+k_2^2)^2}{4k_1^2k_2^2}$$

Substitute $k_1^2 = \frac{2mE}{\hbar^2}$ and $k_2^2 = \frac{2m(V_0-E)}{\hbar^2}$:

$$
\begin{aligned}
k_1^2 + k_2^2 &= \frac{2mV_0}{\hbar^2} \\
4k_1^2 k_2^2 &= 4\left(\frac{2mE}{\hbar^2}\right)\left(\frac{2m(V_0-E)}{\hbar^2}\right) \\
1 + \frac{\gamma^2}{4} &= \frac{(\frac{2mV_0}{\hbar^2})^2}{4(\frac{2mE}{\hbar^2})(\frac{2m(V_0-E)}{\hbar^2})} = \frac{V_0^2}{4E(V_0-E)}
\end{aligned}
$$

Finally, substituting this back into the expression for T:

$$T \approx \frac{4e^{-2k_2L}}{\frac{V_0^2}{4E(V_0-E)}} = \frac{16E(V_0-E)}{V_0^2} e^{-2k_2L}$$

$$T \approx \frac{16E}{V_0}\left(1 - \frac{E}{V_0}\right) e^{-2k_2L}$$