Quantum Tunneling through a Single Potential Barrier: Equivalence of the Transfer Matrix and Analytical Solutions

Rati Kaewkam

11 November 2025

Problem Setup

Suppose the potential $V(x)$ is given by:

$$V(x) = \begin{cases} V_0, & 0 \leq x \leq L \\ 0, & \text{otherwise} \end{cases}$$

We analyze the case for a particle with energy $E > 0$ incident from the left. This paper focuses on the quantum tunneling, so we assume the particle's energy is less than the barrier height: $0 < E < V_0$.

The Time-independent Schrödinger equation is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

Wave Function Solutions in Each Region

We define $k_1^2 = \frac{2mE}{\hbar^2}$ and $k_2^2 = \frac{2m(V_0-E)}{\hbar^2}$.

- **Region I ($x < 0, V = 0$):**

$$\psi_I(x) = Ae^{ik_1x} + Be^{-ik_1x}$$

- **Region II ($0 \leq x \leq L, V = V_0$):**

$$\psi_{II}(x) = Ce^{k_2x} + De^{-k_2x}$$

- **Region III ($x > L, V = 0$):**

$$\psi_{III}(x) = Fe^{ik_1x}$$

Boundary Conditions

We apply continuity for $\psi(x)$ and $\frac{d\psi}{dx}$ at both $x = 0$ and $x = L$. This gives four conditions:

$$\psi_I(0) = \psi_{II}(0) \implies A + B = C + D \tag{1}$$

$$\psi_{II}(L) = \psi_{III}(L) \implies Ce^{k_2 L} + De^{-k_2 L} = Fe^{ik_1 L} \tag{2}$$

$$\left.\frac{d\psi_I}{dx}\right|_{x=0} = \left.\frac{d\psi_{II}}{dx}\right|_{x=0} \implies ik_1(A - B) = k_2(C - D) \tag{3}$$

$$\left.\frac{d\psi_{II}}{dx}\right|_{x=L} = \left.\frac{d\psi_{III}}{dx}\right|_{x=L} \implies k_2(Ce^{k_2 L} - De^{-k_2 L}) = ik_1 Fe^{ik_1 L} \tag{4}$$

Method 1: Analytical Approach (Solving the System)

Solving the system of four equations leads to expressions for the coefficients. By solving (2) and (4) for C and D:

$$C = \left(1 + \frac{ik_1}{k_2}\right)\frac{F}{2}e^{ik_1 L - k_2 L} \tag{5}$$

$$D = \left(1 - \frac{ik_1}{k_2}\right)\frac{F}{2}e^{ik_1 L + k_2 L} \tag{6}$$

And by solving (1) and (3) for A and B:

$$A = \left(1 - \frac{ik_2}{k_1}\right)\frac{C}{2} + \left(1 + \frac{ik_2}{k_1}\right)\frac{D}{2} \tag{7}$$

$$B = \left(1 + \frac{ik_2}{k_1}\right)\frac{C}{2} + \left(1 - \frac{ik_2}{k_1}\right)\frac{D}{2} \tag{8}$$

Substituting (5) and (6) into (7) and solving for the transmission amplitude F/A, we find the transmission probability $T = \left|\frac{F}{A}\right|^2$. This yields the well-known analytical result:

$$T = \frac{1}{\cosh^2(k_2 L) + \frac{\gamma^2}{4}\sinh^2(k_2 L)}$$

where $\gamma = \frac{k_1^2 - k_2^2}{k_1 k_2}$.

Method 2: Transfer Matrix Approach

We can rewrite the equations from the boundary conditions in matrix form. Let $\begin{pmatrix} A \\ B \end{pmatrix} = \hat{M}\begin{pmatrix} F \\ 0 \end{pmatrix}$. We can find $\hat{M}$ by relating the coefficients across each boundary.

We define two operators, $\hat{M}_\alpha$ (for the boundary at $x = 0$) and $\hat{M}_\beta$ (for the boundary at $x = L$), along with scalar factors λ_1 and λ_2.

- **Scalars:**

$$\lambda_1 = \frac{1}{2} \quad \text{and} \quad \lambda_2 = \frac{e^{ik_1 L}}{2}$$

- **Operator $\hat{M}_\alpha$ (from $x = 0$):**

$$\hat{M}_\alpha = \begin{pmatrix} 1 - \frac{ik_2}{k_1} & 1 + \frac{ik_2}{k_1} \\ 1 + \frac{ik_2}{k_1} & 1 - \frac{ik_2}{k_1} \end{pmatrix}$$

- **Operator $\hat{M}_\beta$ (from $x = L$):**

$$\hat{M}_\beta = \begin{pmatrix} \left(1 + \frac{ik_1}{k_2}\right) e^{-k_2 L} & 0 \\ \left(1 - \frac{ik_1}{k_2}\right) e^{k_2 L} & 0 \end{pmatrix}$$

The total relationship is given by:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \lambda_1 \lambda_2 \hat{M}_\alpha \hat{M}_\beta \begin{pmatrix} F \\ 0 \end{pmatrix} = \lambda_1 \lambda_2 \hat{O} \begin{pmatrix} F \\ 0 \end{pmatrix}$$

where $\hat{O} = \hat{M}_\alpha \hat{M}_\beta$.

Operator $\hat{O}$ Elements

We only need the O_{11} element to find A:

$$A = \lambda_1 \lambda_2 O_{11} F$$

Calculating $O_{11} = (M_{\alpha,11})(M_{\beta,11}) + (M_{\alpha,12})(M_{\beta,21})$:

$$O_{11} = \left(1 - \frac{ik_2}{k_1}\right)\left(1 + \frac{ik_1}{k_2}\right) e^{-k_2 L} + \left(1 + \frac{ik_2}{k_1}\right)\left(1 - \frac{ik_1}{k_2}\right) e^{k_2 L}$$

Using the identities (where $\gamma = \frac{k_1^2 - k_2^2}{k_1 k_2}$):

- $\left(1 - \frac{ik_2}{k_1}\right)\left(1 + \frac{ik_1}{k_2}\right) = 2 + i\gamma$
- $\left(1 + \frac{ik_2}{k_1}\right)\left(1 - \frac{ik_1}{k_2}\right) = 2 - i\gamma$

We get:

$$O_{11} = (2 + i\gamma) e^{-k_2 L} + (2 - i\gamma) e^{k_2 L}$$

Now, substituting this back into the equation for A:

$$A = \left(\frac{1}{2}\right)\left(\frac{e^{ik_1 L}}{2}\right)\left[(2 + i\gamma) e^{-k_2 L} + (2 - i\gamma) e^{k_2 L}\right] F$$

$$A = \frac{F e^{ik_1 L}}{4}\left[(2 + i\gamma) e^{-k_2 L} + (2 - i\gamma) e^{k_2 L}\right]$$

This equation is identical to the one derived from the analytical approach (Page 3 of the previous document).

Conclusion: Equivalence

Since $A = \lambda_1 \lambda_2 O_{11} F$, the transmission amplitude is

$$\frac{F}{A} = \frac{1}{\lambda_1 \lambda_2 O_{11}}$$

Calculating $T = \left|\frac{F}{A}\right|^2$ from this expression yields the exact same result:

$$T = \frac{1}{\cosh^2(k_2 L) + \frac{\gamma^2}{4}\sinh^2(k_2 L)}$$

This demonstrates that the Transfer Matrix method and the standard Analytical Solution are equivalent for this problem.