Removing Blur & Multiplicative Noise Using Convex Optimization

Pushen Wang, Jimmy Ye

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1 Abstract

A common problem in signal processing is recovering the true signal from a noise-corrupted signal. We consider a specific domain of this general problem, multiplicative noise and blurring in images. Both are common in images due to sensor imperfections, shadows, motion, poor focus, etc. In particular, we would like to find a model which captures properties of a less noisy image and is amenable to computationally efficient methods of convex optimization. Since it is usually impossible to recover the true image, the models and approaches are evaluated by how well they recover significant details like edges, whether they introduce additional artifacts, and their efficiency in terms of time complexity.

2 Introduction

Suppose that an imaging system captures an image which is distorted by blurring and multiplicative noise. This process could be described as:

$$u_0 = (Au)\eta \tag{1}$$

where u is the "true" image, A is a blurring operator and η is a multiplicative noise. The problem is to restore the original image u from the degraded image u_0, and the recovering process is called de-blurring and de-noising.

Different constraints may be placed on blurring operator A and multiplicative noise η in different applications. For example, η follows a Gamma distribution in synthetic aperture radar(SAR), and follows a Rayleigh distribution in ultrasound imaging, and when A is the identity matrix, this problem will degrade into multiplicative removal only.

Historically, it was common to use least-squares, i.e. L_2 norm, as the optimization objective, however, it has been found that the total variation (TV) norm, which is essentially L_1 norm of the derivative, is better [**?**]. We consider how to formulate multiplicative noise and blurring, using the TV objective, as a computationally efficient optimization problem.

3 Basic Formulation

We first consider additive noise as a simpler problem before extending the optimization problem to multiplicative noise and blurring. As in [**?**], we denote the pixel values of a noisy image, or the observed intensity function, $u_0(x, y)$, the desired denoised image as $u(x, y)$, and the additive noise as n. Then we have

$$u_0(x, y) = u(x, y) + n(x, y) \tag{2}$$

The total variation norm for a function with domain $[a, b]$ is defined as

$$V(f) = \sup_P \sum_{i=1}^{n_p - 1} \big| f(x_{i+1}) - f(x_i) \big| \tag{3}$$

where P is the set of all partitions $p = \{x_0, ..., x_{n_p}\}$ of the domain.

This definition can be extended to an n-dimensional Ω:

$$V(f, \Omega) = \sup\{\int_\Omega f(x)(\nabla \cdot \phi(x))\,\mathrm{d}x : \phi \in C_c^1(\Omega), \|\phi\|_{L^\infty(\Omega)} \leq 1\} \tag{4}$$

where $C_c^1(\Omega)$ denotes the space of continuously differentiable functions with compact support (i.e. in $\Omega \subset \mathbb{R}^n$, f is nonzero only on a bounded set) and the norm used is supremum on sets of nonzero measure.

The following theorem gives a more manageable definition:

$$V(f) = \int_\Omega \big\|\nabla f(x)\big\|_2 dx \tag{5}$$

Now we have our optimization problem [**?**]:

$$\text{minimize} \int_\Omega \sqrt{u_x^2 + u_y^2}\,\mathrm{d}x\,\mathrm{d}y \tag{6}$$

$$\text{subject to} \int_\Omega (u - u_0)\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{7}$$

$$\int_\Omega \frac{1}{2}(u - u_0)^2\,\mathrm{d}x\,\mathrm{d}y = \sigma^2 \tag{8}$$

Here, u_x and u_y are the partial derivatives of u, and the constraints enforce our simplifying assumption that the additive noise follows a Gaussian distribution with mean zero and standard deviation σ. Note that $\big\|\nabla f(x)\big\|_2 = \sqrt{u_x^2 + u_y^2}$.

However, this problem is optimizing over a function of u, which is itself a function. We can apply the Euler-Lagrange equations to obtain a PDE, a form more amenable to numerical solutions, which when satisfied gives us the desired u which optimizes TV [**?**]:

$$0 = \frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{u_x^2 + u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{u_x^2 + u_y^2}}\right) - v_1 - v_2(u - u_0) \text{ on } \Omega, \text{with} \tag{9}$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \text{the boundary of } \Omega \tag{10}$$

We solve numerically by introducing an additional time parameter, optimizing $u(x, y, t)$ [**?**]:

$$u_t = \frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{u_x^2 + u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{u_x^2 + u_y^2}}\right) - v(u - u_0) \text{ for } t > 0 \tag{11}$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \text{the boundary of } \Omega \tag{12}$$

where $u(x, y, 0)$ is given. Note that the mean constraint is satisfied so long as $u(x, y, 0)$ has the same mean as u_0.

We compute $v(t)$ by multiplying both sides of the previous equation by $(u - u_0)$ and integrating by parts [?]:

$$v = -\frac{1}{2\sigma^2} \int \left[\sqrt{u_x^2 + u_y^2} - \left(\frac{(u_0)_x u_x}{\sqrt{u_x^2 + u_y^2}} + \frac{(u_0)_y u_y}{\sqrt{u_x^2 + u_y^2}} \right) \right] \mathrm{d}x \, \mathrm{d}y \tag{13}$$

This is essentially Rosen's gradient projection method [?], a variation of gradient descent that takes at each step, the projection of $-\nabla f$ onto the feasible set, instead of $-\nabla f$ itself, as the increment, and is therefore applicable to constrained optimization problems.

Thus, by taking the appropriate numerical approximations of u_t and $v(t)$, we have a completely specified numerical algorithm for optimizing u in the case of Gaussian-distributed additive noise.

4 Multiplicative Noise & Deblurring

We can account for multiplicative noise and blurring (formulation is given by (**??**)) by modifying our constraints, which then affects the last terms of u_t, but the core idea, including the optimization algorithm, remains the same.

The new constraints [?]:

$$\int_\Omega \left(\frac{u_0(x, y)}{Au(x, y)} \right) \mathrm{d}x \, \mathrm{d}y = 1 \tag{14}$$

$$\int_\Omega \left(\frac{u_0}{Au} - 1 \right)^2 \mathrm{d}x \, \mathrm{d}y = \sigma^2 \tag{15}$$

which are still under the assumption that the noise η is Gaussian-distributed, but since it is multiplicative, it now has mean 1. Note also the introduction of the blurring term A, a linear convolution operator (transforming matrices / 2D functions rather than vectors / 1D functions).

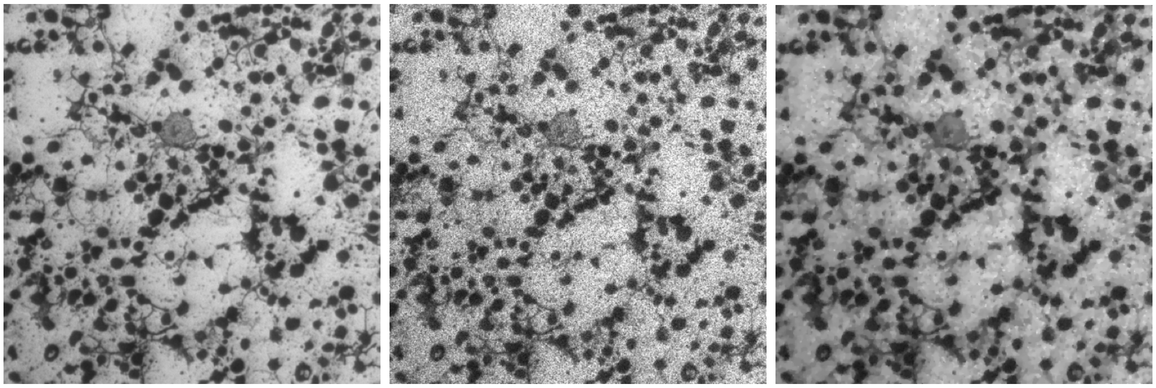
The new u_t [?]:

$$u_t = \frac{\partial}{\partial x} \left(\frac{u_x}{\sqrt{u_x^2 + u_y^2}} \right) + \frac{\partial}{\partial y} \left(\frac{u_y}{\sqrt{u_x^2 + u_y^2}} \right) - v_1 A^* \left(\frac{u_0}{(Au)^2} \right) \left(\frac{u_0}{Au} \right) - v_2 A^* \frac{u_0}{(Au)^2} \tag{16}$$

where A^* is the adjoint operator of A (intuitively, a sort of dual operator which generalizes the idea of the complex conjugate of a complex number, or the conjugate transpose of a matrix).

We have shown that it is possible, with minimal additions and modifications, to extend our numerically solvable additive noise model to a model which simultaneously captures multiplicative noise and convolutional blurring.

Figure **??** is the result of a Gaussian blur with $\sigma^2 = 2$ and multiplicative noise with $\sigma = 0.2$ applied to the original Figure **??**, and Figure **??** shows the result of the total variation based algorithm in [?]:



(a) Original image (b) Noisy image (c) Denoised image

Figure 1: Experiment results given in [?]

5 Advanced Techniques

There are some additional improvements we can make to the quality of our denoised images as well as the efficiency of our algorithm. In particular, we can find a convex approximation to our nonconvex problem, which guarantees a good global optimum, rather than simply assuming local near-convexity to achieve good local optimums. We can then apply a different optimization algorithm, the alternating direction method of multipliers (ADMM), which outputs better images and is generally at least an order of magnitude faster [?].

The basic idea given in [?] is to transform the multiplicative noise to additive noise through a logarithm:

$$\log u_0 = \log Au + \log \eta$$

And then similar to the previous approach, based on total variation, this problem can be formulated as:

$$\begin{aligned} &\text{minimize } ||\nabla u||_1 && (17)\\ &\text{subject to } u \in \mathbb{R}^n && (18)\\ &\qquad ||\log u_0 - \log Au||_1 \leq \alpha, && (19) \end{aligned}$$

where n is the number of pixels, α is the trade-off between the fit to f and the amount of regularization, and $||\cdot||_1$ is L_1-norm.

References

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[3] Fan Wang and Michael K Ng. A fast minimization method for blur and multiplicative noise removal. *International Journal of Computer Mathematics*, 90(1):48–61, 2013.