SDS 385: Exercises 1 - Preliminaries

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Problem 1

(A)

$$\hat{\beta} = \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{N} \frac{w_i}{2} \left(y_i - x_i^T \beta\right)^2 \tag{1}$$
$$= \arg\min_{\beta \in \mathbb{R}^p} \frac{1}{2}(Y - X\beta)^T W (Y - X\beta) \tag{2}$$

$$\frac{1}{2}(Y - X\beta)^T W (Y - X\beta) = \frac{1}{2}(Y^T - \beta^T X^T) W (Y - X\beta) \tag{3}$$
$$= \frac{1}{2}(Y^T W - \beta^T X^T W)(Y - X\beta) \tag{4}$$
$$= \frac{1}{2}(Y^T W Y - \beta^T X^T W Y - Y^T W X\beta + \beta^T X^T W X\beta) \tag{5}$$
$$= \frac{1}{2}(Y^T W Y - 2(X\beta)^T W Y + \beta^T X^T W X\beta) \tag{6}$$
$$= \frac{1}{2} Y^T W Y - (X\beta)^T W Y + \frac{1}{2}\beta^T X^T W X\beta, \tag{7}$$

because

$$\beta^T X^T W Y = (X\beta)^T W Y, \tag{8}$$

and

$$Y^T W X\beta = (Y^T W X\beta)^T \because Y^T W X\beta \in \mathbb{R}^1 \tag{9}$$
$$(Y^T W X\beta)^T = (W X\beta)^T Y = (X\beta)^T W^T Y = (X\beta)^T W Y. \tag{10}$$

We want to minimize the objective function from Eqn. (7), so we take the gradient with respect to β and set it equal to zero. For each of the three terms, their are respective gradients with respect to β are

(i)

$$\frac{\partial}{\partial \beta} \frac{1}{2} Y^T W Y = 0 \tag{11}$$

(ii)

$$\frac{\partial}{\partial \beta} - (X\beta)^T W Y = -X^T W Y \tag{12}$$

(iii)

$$\frac{\partial}{\partial \beta} \frac{1}{2}\beta^T X^T W X\beta = \frac{1}{2}\beta^T (X^T W X + (X^T W X)^T) \tag{13}$$
$$= X^T W X\beta. \tag{14}$$

Summing these terms and equaling them to zero yields

$$X^T W X\beta - X^T W Y = 0 \tag{15}$$
$$\therefore (X^T W X)\hat{\beta} = X^T W Y \tag{16}$$

(B)

(C)

(D)

Problem 2

(A)

(B)

(C)

```r
##########################################################
######### Created by Spencer Woody on 23 Aug 2016 #########
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library(Matrix)
library(microbenchmark)

# No. 1 pt C

# No. 1 pt D

N <- 2000
P <-  500

X <- matrix(rnorm(N * P), nrow = N)
mask <- matrix(rbinom(N * P, 1, 0.05), nrow = N)
X <- mask * X

# END
```