AA/EE547: Wi 22

Module 2: Stability

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References. Chapter 8 and 9, **[JH]**; Chapter 4 and Chapter 7, **[C&D]**. review your notes on norms including (induced) matrix norms from [510]

Contents

1 M2-RL1: Introduction to Stability

You may recall from an undergrad controls or signals class input-output stability. In our study of stability, we will start with state related stability concepts (stability in the sense of Lyapunov).

Intuition. Let's consider the following examples.

a. Continuous Time. Recall that the solution to

$$\dot{x} = -\lambda x$$

is

$$x(t) = x_0 e^{-\lambda t},$$

and if $\lambda > 0$ solution decays to zero, otherwise it blows up. This λ is an 'eigenvalue' for this scalar system, and its sign can be used to characterize a notion of stability.

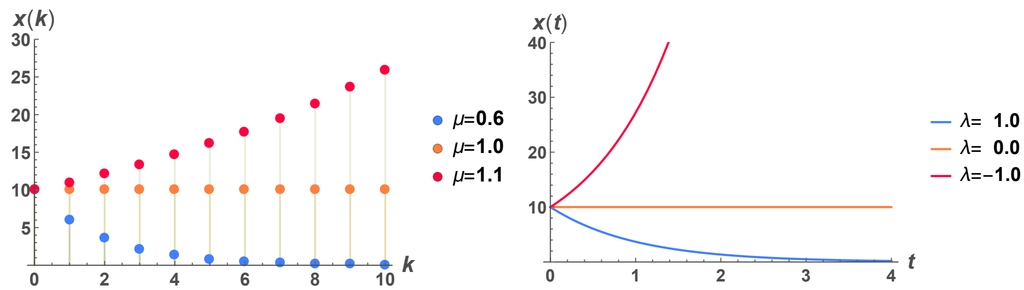
b. Discrete Time. Recall that the solution to

$$x_{k+1} = \mu x_k$$

is

$$x_k = \mu^k x_0$$

and if $|\mu| < 1$, then the solution decays to zero and otherwise, it blows up.



(a) discrete time (b) continuous time

Examples.

1. **Linear systems.** Consider the linear system $\dot{x} = Ax$.
   - If A is non-singular then $x^* = 0$ is the unique equilibrium.
   - If A is singular, then the null space defines a continuum of equilibria.
2. **Logistic Growth Model.** In population dynamics, the logistic growth model is common:

$$\dot{x} = f(x) = r\left(1 - \frac{x}{K}\right)x,\ \ r > 0,\ \ K > 0$$

   The equilibria are determined by solving

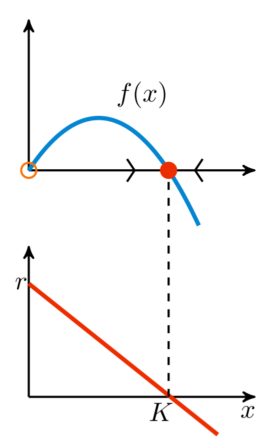
$$f(x) = r\left(1 - \frac{x}{K}\right)x = 0$$

   Clearly

$$x^* = 0,\ \ x^* = K$$

   are equilibrium. The state variable $x > 0$ denotes the population and K is the carrying capacity.When $x \in \mathbb{R}$ is scalar, stability can be determined from the sign of $f(x)$ around the equilibrium. In this example, $f(x) > 0$ for all $x \in (0, K)$ and $f(x) < 0$ for all $x > K$. Thus

   - $x = 0$ is an unstable equilibrium
   - $x = K$ is stable (in fact asymptotically so)



Notice that the second example is a **nonlinear system**. Local stability properties of an equilibrium x^* for a nonlinear system can be determined by linerizing the dynamics (i.e., the vector field $f(x)$) about the point x^*: i.e.,

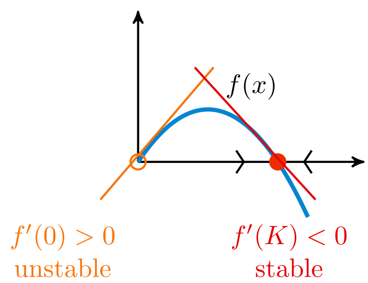
$$f(x^* + \delta\tilde{x}) = \underbrace{f(x^*)}_{=0} + \underbrace{Df\big|_{x=x^*}}_{=:A}\delta\tilde{x} + O(\delta^2)$$

so that the linearized model is given by

$$\dot{\tilde{x}} = A\tilde{x}$$

Stability of this system implies stability of the equilibrium x^* for the nonlinear dynamics.

Coming back to the nonlinear example of the logistic growth dynamics, we have that



1.1 Continuous Time

Recall that for a given linear system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t)\\ y(t) &= C(t)x(t) + D(t)u(t)\end{aligned}$$

the zero input response is given by

$$x(t) = \Phi(t, t_0)x_0$$

where $\Phi(t, t_0)$ is the state transition matrix and $x(t_0) = x_0$. Note that

$$x_0 = 0 \Longrightarrow x(t) = 0 \ \forall t$$

The points $x_e = 0$ is called the *equilibrium point*.

Definition 1 (Stable Equilibrium)**.** The following are characterizations of stability (in the sense of Lyapunov).

a. **Marginally Stable**: Consider the equilibrium point $x_e = 0$.

$$x_e \text{ is stable} \iff \forall\ x_0 \in \mathbb{R}^n,\ \forall t_0 \in \mathbb{R}^n,\ t \mapsto x(t) = \Phi(t, t_0)x_0 \text{ is bounded } \forall\ t \geq t_0.$$

Note: the effect of initial conditions does not grow unbounded with time (but it may grow temporarily during a transient phase).

b. **Asymptotic Stability**. Consider the equilibrium point $x_e = 0$.

$$x_e = 0 \text{ is asymptotically stable} \iff x_0 = 0 \text{ is stable and } x(t) = \Phi(t, t_0)x_0 \longrightarrow 0 \text{ as } t \to \infty.$$

Note: the effect of initial conditions eventually disappears with time.

c. **Exponential Stability**. Consider the equilibrium point $x_e = 0$.

$$x_e = 0 \text{ is exponentially stable} \iff \exists M, \alpha > 0 :\ \|x(t_0)\| \leq M \exp(-\alpha(t - t_0))\|x_0\|$$

We say an equilibrium point or the system is **unstable** if it is not marginally stable in the sense of Lyapunov. For such systems, the effect of initial conditions (may) grow over time (depending on the specific initial conditions).

Theorem 2 (Asymptotic Stability of Linear CT Systems)**.** The following claim holds:

$$x = 0 \quad \text{is asymptotically stable} \iff \Phi(t,0) \to 0 \quad \text{as } t \to \infty.$$

Proof. ($\Longleftarrow$) Observe that

$$x(t) = \Phi(t,t_0)x_0 = \Phi(t,0)\Phi(0,t_0)x_0$$

since $\Phi(t,0) \to 0$ as $t \to \infty$ then $\|\Phi(t,0)\| \to 0$ as $t \to \infty$ and

$$\|x(t)\| \leq \|\Phi(t,0)\|\|\Phi(0,t_0)\|\|x_0\|$$

Thus $x(t) \to 0$ as $t \to \infty$.
($\Longrightarrow$) We argue by contradiction: assume that $t \to \Phi(t,0)$ does not tend to zero as $t \to \infty$, i.e. $\exists\ i,j$ such that[1]

$$\Phi_{ij}(t,0) \not\to 0 \ \text{ as } \ t \to \infty$$

Choose

$$x_0 = \left.\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} \quad \text{where 1in the } j\text{-th spot}$$

Thus, we have

$$x_i(t) = \Phi_{ij}(t,0) \not\to 0 \ \text{ as } \ t \to \infty$$

contradicting the asymptotic stability of zero. □

A stability concept that is equivalent to exponential stability for LTV systems is **uniform asymptotic stability**.

Definition 3 (Uniform Asymptotic Stability)**.** We say that the zero solution of $\dot{x} = A(t)x(t)$ on $t \geq 0$ is uniformly asymptotically stable if and only if

a. $t \mapsto \Phi(t,t_0)$ is bounded on $t \geq t_0$ uniformly in $t_0 \in \mathbb{R}_+$, i.e.,

$$\exists\ k < \infty: \ \forall t_0 \in \mathbb{R}_+,\ \|\Phi(t,t_0)\| \leq k,\ \forall t \geq t_0$$

b. $t \mapsto \Phi(t,t_0)$ tends to zero as $t \to \infty$ uniformly in $t_0 \in \mathbb{R}_+$, i.e.,

$$\forall \varepsilon > 0,\ \exists T(\varepsilon) > 0: \ \forall t_0 \in \mathbb{R}_+, \|\Phi(t,t_0)\| \leq \varepsilon,\ \forall t \geq t_0 + T(\varepsilon) \tag{1}$$

In the LTI case, i.e. $\dot{x} = Ax$,

$$\text{asymptotic stability} \iff \text{exponential stability}$$

Indeed, $\Phi(t,t_0) = \exp(A(t-t_0))$ depends only on the elapsed time $t - t_0$, so that the zero solution of $\dot{x} = Ax$ is asymptotically stable if and only if the zero solution is uniformly asymptotically stable if and only if the zero solution is exponentially stable.

Notation. Recall that $\exists!$ means "there exists a unique".

[1]The notation Φ_{ij} denotes the (i,j) entry of the matrix Φ.

Proposition 4 (LTV Exponential Stability)**.**

$$A(\cdot) \text{ uniformly asymptotically stable } \iff A(\cdot) \text{ exponentially stable}$$

Proof. ($\Longleftarrow$) Suppose that $A(\cdot)$ is exponentially stable. Then, items a. and b. of Definition 3 hold with $k = m$ and $T(\varepsilon) > 0$ such that $\exp(-\alpha T(\varepsilon)) \leq \varepsilon m^{-1}$. Hence, we have that $A(\cdot)$ is uniformly asymptotically stable.

($\Longrightarrow$) Suppose that $A(\cdot)$ is uniformly asymptotically stable. Given any $T > 0$, we have that

$$\forall t \geq t_0,\ \exists!\ n \in \mathbb{N},\ \exists!\ s \in [0, T):\ t - t_0 = nT + s$$

Let $t_0 \in \mathbb{R}_+$ be arbitrary but fixed and for b., pick some $T(\varepsilon) > 0$ for $\varepsilon = 1/2$. Then, by (1), we have that

$$\|\Phi(s + t_0 + T, t_0)\| \leq 1/2 \quad \forall\ s \geq 0.$$

By the properties of the state transition matrix (cf Module 1), we know that

$$\Phi(s + t_0 + 2T, t_0) = \Phi(s + t_0 + 2T, t_0 + T)\Phi(t_0 + T, t_0)$$

This in turn implies that

$$\|\Phi(s + t_0 + 2T, t_0)\| \leq \|\Phi(s + t_0 + 2T, t_0 + T)\|\|\Phi(t_0 + T, t_0)\| \leq 2^{-2},$$

we have that (by induction),

$$\forall s \geq 0, \forall n = 1, 2, \ldots,\ \|\Phi(s + t_0 + nT, t_0)\| \leq 2^{-n} \tag{$**$}$$

Pick $\alpha > 0$ s.t. $\exp(\alpha T) = 2$. Then,

$$\forall s \in [0, T), 1 \leq 2\exp(-\alpha s) \implies \forall s \in [0, T),\ 2^{-n} \leq 2\exp(-\alpha(s + nT))$$

Combining this with the upper bound on the norm of the state transition matrix in ($**$), we have that

$$\forall s \in [0, T), \forall n = 1, 2, \ldots,\ \|\Phi(s + t_0 + nT, t_0)\| \leq 2\exp(-\alpha(s + nT))$$

Now, using a. and the fact that $1 \leq 2\exp(-\alpha s)$, we have that

$$\forall s \in [0, T),\ \|\Phi(s + t_0, t_0)\| \leq k \leq 2k\exp(-\alpha s)$$

so that

$$\forall t \geq t_0,\ \|\Phi(t, t_0)\| \leq 2k\exp(-\alpha(t - t_0))$$

$\square$

1.2 Stability of Discrete Time Linear Systems

Consider

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C_k x_k + D_k u_k \end{aligned}$$

The zero state is again the equilibrium for this system—i.e., $x_e = 0$.

Recall from Module 1 that the solution solution and output are given by

$$\begin{aligned} x_k &= \Phi(k, k_0)x_0 + \sum_{\ell=k_0}^{k-1} \Phi(k, \ell + 1)B_\ell u_\ell \\ y_k &= C_k\Phi(k, k_0)x_0 + C_k\left(\sum_{\ell=k_0}^{k-1} \Phi(k, \ell + 1)B_\ell u_\ell\right) + D_k u_k \end{aligned}$$

Asymptotic stability can be characterized informally by the statement that every solution of $x_{k+1} = A_k x_k$ tends to zero as $k \to \infty$.

Definition 5 (Asymptotic Stability)**.** Consider $x_k \equiv 0$ (i.e., the zero solution of $x_{k+1} = A_k x_k$). The zero solution is asymptotically stable if and only if for all $x_0 \in \mathbb{R}^n$, for all $k_0 \in \mathbb{N}$,

a. $k \mapsto x_k = \Phi(k, k_0)x_0$ is bounded on $k \geq k_0$
b. $k \mapsto x_k = \Phi(k, k_0)x_0 \to 0$ as $k \to \infty$.

Note: any solution x_k on any $[k_0, k]$ is a finite set and hence, b. $\implies$ a. Due to linearity, we get the following theorem.

Theorem 6 (Asymptotic Stability of Linear DT Systems)**.** Let $\det(A_k) \neq 0$ for all $k \in \mathbb{N}$. The zero solution of $x_{k+1} = A_k x_k$ on $k \geq 0$ is asymptotically stable if and only if $\Phi(k, 0) \to 0$ as $k \to \infty$.

Practice Problem. Try to prove this theorem by following the same proof structure as Theorem 2. Hint: not that since $\det(A_k) \neq 0$, $\forall k \geq k_0 \geq 0$, $\Phi(k, k_0) = \Phi(k, 0)\Phi(0, k_0)$.

Analogous to the continuous time case, exponential stability is a property of the system $x_{k+1} = A_k x_k$ which if possessed, guarantees that every solution of the system is bounded by a decaying exponential depending on the elapsed time $k - k_0$. Indeed, we have the following formal definition.

Definition 7 (DT Exponential Stability)**.** The zero solution of $x_{k+1} = A_k x_k$ on $k \geq 0$ is exponentially stable if and only if $\exists\ \rho \in [0, 1)$ and $m > 0$ such that for all $k_0 \in \mathbb{N}$,

$$\|\Phi(k, k_0)\| \leq m\rho^{k-k_0},\ \forall\ k \geq k_0$$

where the matrix norm is arbitrary.[2]

The following are important observations:

- The constants $\rho \in [0, 1)$ and $m > 0$ are fixed, meaning that they are independent of k_0. Further, the constant $\alpha \geq 0$ such that $\rho = \exp(-\alpha)$ is the exponential decay rate.
- An equivalent statement: the zero solution is exponentially stable if and only if

$$\exists \rho \in [0, 1),\ m > 0 :\ \forall (x_0, k_0) \in \mathbb{R}^n \times \mathbb{N},\ \|x_k\| \leq m\|x_0\|\rho^{k-k_0},\ \forall k \geq k_0$$

We say that the zero solution is uniformly asymptotically stable if and only if

a. $k \mapsto x_k = \Phi(k, k_0)x_0$ on $k \geq k_0$ is uniformly bounded (bound is independent of k_0)
b. $k \mapsto x_k = \Phi(k, k_0)x_0 \to 0$ uniformly as $k \to \infty$.

Theorem 8 (Equivalence of Asymptotic and Exponenital Stability)**.**

$$A(\cdot) \text{ is uniformly asymptotically stable } \iff A(\cdot) \text{ is exponentially stable}$$

Practice Problem. Prove this theorem. Hint: use the same method as in the CT case. In the necessity direction use a. and b. with $2\varepsilon = 1$ and pick $\rho \in [0, 1)$ such that $2\rho^K > 1$ so that $\|\Phi(k, k_0)\| \leq 2\ell\rho^{k-k_0}$ for all $k \geq k_0$.

[2] Recall from [510] that finite dimensional norms are equivalent.

2 M2-RL2: Spectral Conditions for Stability of LTI Systems

In the case of LTI systems, characterization of stability reduces to analyzing spectral conditions. For this we need to remind ourselves of some of the results from [510] on functions of a matrix.

Recall from [510] that we have the following result about representing any analytic function[3] as a sum of polynomials.

Let $A \in \mathbb{C}^{n\times n}$ and let $\sigma(A)$ denote the spectrum of A (containing distinct eigenvalues of A) with $p = |\sigma(A)|$. The minimal polynomial of A is given as above by

$$\psi_A(\lambda) = \prod_{i=1}^{p} (\lambda - \lambda_i)^{m_i}$$

Facts about functions of a matrix. Let $f(s)$ be any function of s analytic on the spectrum of A and $q(s)$ be a polynomial such that

$$f^{(k)}(\lambda_\ell) = q^{(k)}(\lambda_\ell)$$

for $0 \le k \le m_\ell - 1$ and $1 \le \ell \le p$. Then

$$f(A) = q(A)$$

where p is the number of distinct roots of the characteristic polynomial $\chi_A(s)$ and

$$m_k = \min\{\mu \ge 1 : \mathcal{N}((A - \lambda_k I)^\mu) = \mathcal{N}((A - \lambda_k I)^{\mu+1})\}$$

i.e., m_k is the ascent of $A - I\lambda_k$.

In fact, if $m = \sum_{i=1}^{p} m_i$ then

$$q(s) = a_1 s^{m-1} + a_2 s^{m-2} + \cdots + a_m s^p$$

where $a_1, a_2, \ldots, a_n$ are functions of

$$(f(\lambda_1), f^{(1)}(\lambda_1), f^{(2)}(\lambda_1), \ldots, f^{(m_1)}(\lambda_1), f(\lambda_2), \ldots)$$

and hence

$$f(A) = a_1 A^{m-1} + \cdots + a_m A^0 = \sum_{\ell=1}^{p} \sum_{k=0}^{m_\ell - 1} q_{k,\ell}(A) f^{(k)}(\lambda_\ell)$$

where $q_{k,\ell}$'s are polynomials independent of f.

This leads to the following theorem.

Theorem 9 (General Form of $f(A)$)**.** Let $A \in \mathbb{C}^{n\times n}$ have a minimal polynomial ψ_A given by

$$\psi_A(s) = \prod_{k=1}^{p} (s - \lambda_k)^{m_k}$$

Let the domain Δ contain $\sigma(A)$, then for any analytic function $f : \Delta \to \mathbb{C}$. we have

$$f(A) = \sum_{k=1}^{p} \sum_{\ell=0}^{m_k - 1} f^{(\ell)}(\lambda_k) q_{k,\ell}(A)$$

where $q_{k,\ell}$'s are polynomials independent of f.

[3] An *analytic function* is a function that is locally given by a convergent power series—i.e., f is real analytic on an open set U in the real line if for any $x_0 \in U$ one can write

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where $a_n \in \mathbb{R}$ for each n and the series is convergent to $f(x)$ for $x$in a neighborhood of x_0.

Recalling our derivation of functions of matrices from [510], we can show that

$$\exp(tA) = \sum_{k=1}^{p} \sum_{\ell=0}^{m_k-1} t^{\ell} \exp(\lambda_k t) p_{k\ell}(A)$$

This gives rise to the following stability condition:

Proposition 10 (Continuous Time)**.** Consider the differential equation $\dot{x} = Ax$, $x(0) = x_0$. From the above expression:

$$\{\exp(At) \to 0 \ \text{ as } \ t \to \infty\} \iff \{\forall \lambda_k \in \sigma(A),\ \mathrm{Re}(\lambda_k) < 0\}$$

and

$$\{t \mapsto \exp(At) \text{ is bounded on } \mathbb{R}_+\} \iff \left\{ \begin{array}{ll} \forall \lambda_k \in \sigma(A), & \mathrm{Re}(\lambda_k) < 0 \ \& \\ m_k = 1 \text{ when} & \mathrm{Re}(\lambda_k) = 0 \end{array} \right\}$$

We have a similar situation for discrete time systems:

$$\forall\ \nu \in \mathbb{N},\ A^{\nu} = \sum_{k=1}^{p} \sum_{\ell=1}^{m_k-1} \nu(\nu-1)\cdots(\nu-\ell+1)\lambda_k^{\nu-\ell} p_{k\ell}(A)$$

The above gives rise to the following stability condition:

Proposition 11 (Discrete Time)**.** Consider the discrete time system $x(k+1) = Ax(k)$, $k \in \mathbb{N}$, with $x(0) = x_0$. Then for $k \in \mathbb{N}$, $x(k) = A^k x_0$. From the above equation, we have that

$$\left\{A^k \to 0 \text{ as } k \to \infty\right\} \iff \{\forall\ \lambda_i \in \sigma(A),\ |\lambda_i| < 1\}$$

and

$$\left\{k \to A^k \text{ is bounded on } \ \mathbb{N}_+\right\} \iff \left\{ \begin{array}{ll} \forall \lambda_i \in \sigma(A), & |\lambda_i| \leq 1 \ \& \\ m_i = 1 \ \text{ when} & |\lambda_i| = 1 \end{array} \right\}$$

2.1 LTI: Asymptotic Stability is Equivalent to Exponential Stability

Coming back to the CT case, we will show that asymptotic stability is equivalent to exponentially stable using the function of matrix expansion.

Let's prove the claim in Proposition 10:

Claim 1.

$$\dot{x} = Ax \text{ is exponentially stable } \iff \sigma(A) \subset \mathbb{C}_-^{\circ}$$

Proof. The state transition matrix for an LTI system is

$$\Phi(t, t_0) = \exp(A(t - t_0))$$

And from above we know that

$$\exp(At) = \sum_{k=1}^{p} \pi_k(t) \exp(\lambda_k t),\ \{\lambda_k\}_1^p = \sigma(A)$$

where π_k are some matrix polynomials in t. Hence, by taking matrix norms, we have that

$$\|\exp(At)\| \leq \sum_{k=1}^{p} \|\pi_k(t)\| \exp(\mathrm{Re}(\lambda_k)t) \leq \sum_{k=1}^{p} p_k(t) \exp(\mathrm{Re}(\lambda_k)t) \leq p(t)\exp(-\mu t)$$

where $p_k(t)$ are polynomials such that $\|\pi_k(t)\| \leq p_k(t)$, $p(t) = \sum_{k=1}^{p} p_k(t) \geq 0$ and

$$\mu = -\max\{\mathrm{Re}(\lambda) :\ \lambda \in \sigma(A)\}$$

Since a polynomial is growing slower than any growing exponential we have

$$\forall \varepsilon > 0,\ \exists\ m(\varepsilon) > 0 :\ 0 \leq |p(t)| \leq m \exp(\varepsilon t),\ \forall t \geq 0$$

Hence combing this with the above bound on $\|\exp(At)\|$, we have that

$$\forall \varepsilon > 0\ \exists\ m(\varepsilon) > 0 :\ \|\exp(At)\| \leq m \exp(-(\mu - \varepsilon)t)\ \forall t \geq 0$$

Then, if $\sigma(A) \subset \mathbb{C}_-^\circ$, by the above $\mu > 0$. Hence picking $\varepsilon \in (0, \mu)$ we have that

$$\|\Phi(t, t_0)\| \leq m \exp(-\alpha(t - t_0))$$

with $\alpha = \mu - \varepsilon > 0$. On the other hand if $\sigma(A)$ is not included in $\mathbb{C}_-^\circ$, then by the polynomial expansion for $\exp(At)$, $\exp(At)$ does not tend to the zero matrix as $t \to \infty$ and the zero solution is not exponentially stable.

□

2.2 Application to Numerical Integration

Suppose we are given $\dot{x} = Ax$, $x(0) = x_0$, $A \in \mathbb{C}^{n\times n}$, $x \in \mathbb{C}^n$. Call $t \mapsto x(t)$ the exact solution $x(t) = \exp(At)x_0$. Note that $t \mapsto x(t)$ is analytic in t. Call $(\xi_0, \xi_1, \dots)$ the sequence of computed values.

There are many numerical integration schemes. We will focus on two first order schemes: forward and backward Euler. These two schemes connect to stability very nicely.

Scheme 1: Forward Euler Method. For small $h > 0$, we have for any $t_k \in \mathbb{R}_+$,

$$\begin{aligned} x(t_k + h) &= x(t_k) + h\dot{x}(t_k) + \mathrm{O}(h^2) \\ &= x(t_k) + hAx(t_k) + \mathrm{O}(h^2) \end{aligned}$$

In other words, we have approximately

$$x(t_k + h) \simeq (I + hA)x(t_k)$$

So if we perform repeatedly this step starting at $t_0 = 0$, we have the computed sequence $\{\xi_i\}_0^\infty$ by

$$\xi_m = (I + hA)^m x_0,\ m = 0, 1, 2, \dots$$

From the spectral mapping theorem and the above equation for ξ_m we have the following.

Example. Consider

$$\dot{x}(t) = \lambda x(t)$$

with $\lambda \in \mathbb{C}$. Then the equation is stable if $\mathrm{Re}(\lambda) \leq 0$. In this case the system is exponentially decaying

$$\lim_{t\to\infty} x(t) = 0$$

When is the numerical solution x_i also decaying—i.e., when is $\lim_{i\to\infty} x_i = 0$? First, the **forward Euler** numerical solution is given by the following recursion where h is the step-size:

$$x_{k+1} = x_k + h\lambda x_k$$

Iterating the recursion, we have that

$$x_{k+1} = (1 + h\lambda)^{k+1} x_0$$

The solution is decaying (stable) if $|1 + h\lambda| \le 1$. That is, for any step-size choice $h > 0$ such that

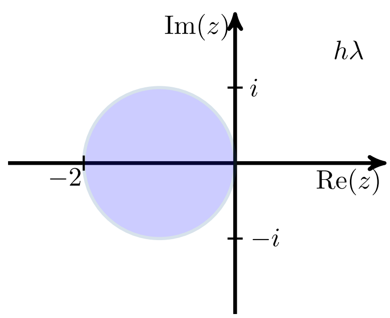
$$(1 + h\mathrm{Re}(\lambda))^2 + \mathrm{Im}(\lambda)^2 \le 1$$

$$\begin{aligned} 1 + 2h\mathrm{Re}(\lambda)) + h^2\mathrm{Re}(\lambda))^2 + h^2\mathrm{Im}(\lambda)^2 \le 1 &\iff 2h\mathrm{Re}(\lambda)) + h^2(\mathrm{Re}(\lambda))^2 + \mathrm{Im}(\lambda)^2) \le 0 \\ &\iff h(2\mathrm{Re}(\lambda) + h|\lambda|^2) \le 0 \\ &\iff h \le -2\frac{\mathrm{Re}(\lambda)}{|\lambda|^2} \\ &\iff 0 \le h \le \frac{2|\mathrm{Re}(\lambda)|}{|\lambda|^2}, \quad \text{when system is stable—i.e., } \lambda < 0. \end{aligned}$$

Another way to think about this is letting $z = \lambda h$ so that we have the following constraint:

$$1 + 2x + x^2 \le 1$$

which gives us the circle shown in the figure below (right).



Forward Euler

Fact 12. Suppose $\sigma(A) \subset \mathbb{C}_-^\circ$ (equivalently, the origin is exponentially stable). Let h_0 be the largest positive h such that

$$\max_i |1 + h\lambda_i| = 1$$

Under these conditions,

1. The iterates ξ_m are such that $\{\xi_m\}_0^\infty \to 0$ exponentially for all ξ_0 if and only if $h \in (0, h_0)$.
2. If $h > h_0$, then for almost all x_0, the sequence of computed values $\{\xi_k\}_0^\infty$ is such that $\{\|\xi_m\}_0^\infty$ grows exponentially.

Interpretation. Even if $\sigma(A) \subset \mathbb{C}_-^\circ$ (and hence the exact solution $x(t) \to 0$ exponentially), for $h > h_0$, for almost all x_0, the sequence of computed vectors $\{\|\xi_m\|\}_0^\infty$ blows up. It is for this reason that in practice we often prefer the backward Euler method.

Scheme 2: Backward Euler. For small $h > 0$, we have that for any $t_k \in \mathbb{R}$,

$$\begin{aligned} x(t_k) &= x(t_k + h) - h\dot{x}(t_k + h) + \mathrm{O}(h^2) \\ &= x(t_k + h) - hAx(t_k + h) + \mathrm{O}(h^2) \end{aligned}$$

Thus we have approximately

$$x(t_k + h) \simeq (I - hA)^{-1} x(t_k)$$

So if we perform repeatedly this step, starting from $t_0 = 0$, we get the computed sequence $\{\xi_i\}_0^\infty$ given by

$$\xi_m = (I - hA)^{-m} x_0, \;\; m = 0, 1, 2, \ldots$$

Now, the spectrum of $(I - hA)^{-1}$ is $\{(1 - h\lambda_i)^{-1}\}_{i=1}^{\sigma}$. Hence by the above expression for ξ_m, we have

$$\begin{aligned} &\xi_m \to 0 \text{ as } m \to \infty \\ \iff\;& \forall \lambda_i \in \sigma(A), \;\; |(1 - h\lambda_i)^{-1}| < 1 \\ \iff\;& \forall \lambda_i \in \sigma(A), \;\; |1 - h\lambda_i| > 1 \end{aligned}$$

Note that if $\mathrm{Re}(\lambda_i) < 0$, then $|1 - h\lambda_i| > 1$, since $h > 0$. Thus we have shown the following result.

Example. Consider

$$\dot{x}(t) = \lambda x(t)$$

with $\lambda \in \mathbb{C}$. Then the equation is stable if $\mathrm{Re}(\lambda) \leq 0$. In this case the system is exponentially decaying

$$\lim_{t \to \infty} x(t) = 0$$

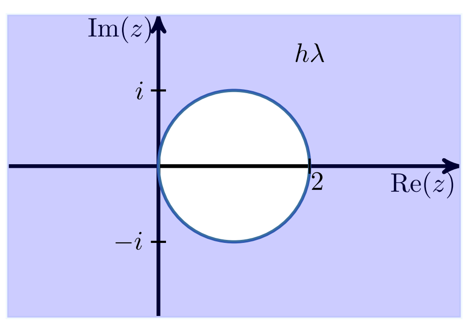
When is the numerically solution x_i also decaying, $\lim_{i \to \infty} x_i = 0$?

$$x_{i+1} = x_i + h\lambda x_{i+1} \iff x_{i+1} = \left(\frac{1}{1 - h\lambda} \right)^{i+1} x_0$$

The solution is decaying (stable) if $|1 + h\lambda| \geq 1$. Indeed, letting $z = h\lambda$ the above constraint is basically saying

$$1 + 2z + z^2 \geq 1$$

which is illustrated in the figure below.



Backward Euler

Fact 13. If $\sigma(A) \subset \mathbb{C}_-^\circ$, then for all $h > 0$, for all $x_0 \in \mathbb{C}^n$, the computed sequence $\{\xi_m\}_0^\infty$ obtained via backward Euler goes to zero exponentially.

This is very important in practice, because if h is unfortunately chosen too large the computed sequence may lose accuracy but at least it will never blow up!

3 M2-RL2: Lyapunov Stability