ME627 Assignment 1

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January 27, 2021

1 Question 1

$$\ddot{x} + x + x^3 = 0; x(0) = A; \dot{x} = 0;$$

Multiplying both sides by $\dot{x}$ and then integrating

$$\ddot{x}\dot{x} + x\dot{x} + x^3\dot{x} = 0;$$
$$\implies \frac{\dot{x}^2}{2} + \frac{x^2}{2} + \frac{x^2}{4} = E;$$

Where E is the integration constant. We can evaluate it using the initial conditions.

$$E = \frac{A^2}{2} + \frac{A^4}{4};$$
$$\therefore \frac{\dot{x}^2}{2} + \frac{x^2}{2} + \frac{x^2}{4} = \frac{A^2}{2} + \frac{A^4}{4};$$
$$\implies \dot{x}^2 = \frac{1}{2}(A^4 + 2A^2 - x^4 - 2x^2;)$$
$$\implies \dot{x} = \frac{1}{\sqrt{2}}\sqrt{(A^4 + 2A^2) - (x^4 + 2x^2)};$$
$$\implies \frac{dx}{\sqrt{(A^4 + 2A^2) - (x^4 + 2x^2)}} = \frac{1}{\sqrt{2}}dt;$$

The following equation is then integrated using Maple

$$x = Au; dx = Adu;$$
$$\int_0^1 \frac{A}{\sqrt{(A^4 + 2A^2) - (x^4 + 2x^2)}}\, du = \frac{1}{\sqrt{2}}\frac{T}{4}$$

T = Time period.

Hence, the solution from Maple:

$$T = 2\pi(1 - \frac{3}{8}A^2)$$

And, for $A \ll 1, T \approx 2\pi$

The following page contains the maple code for problem 1.

$restart : Digits := 16;$

$$Digits := 16 \quad \textbf{(1)}$$

$$f1 := \frac{A}{\mathrm{sqrt}\left(\left(2 \cdot A^2 + A^4\right) - \left(2 \cdot A^2 . u^2 + A^4 . u^4\right)\right)}$$

$$f1 := \frac{A}{\sqrt{2\,A^2 + A^4 - 2\,A^2 \cdot u^2 - A^4 \cdot u^4}} \quad \textbf{(2)}$$

$\#integrate(f1, u = 0..1)$
$f2 := convert(series(f1, A = 0, 4), polynom)$ assuming $A > 0$

$$f2 := \frac{1}{\sqrt{-2\,u^2 + 2}} - \frac{\left(-u^4 + 1\right) A^2}{2\left(-2\,u^2 + 2\right)^{3/2}} \quad \textbf{(3)}$$

$integrate(f2, u = 0..1)$

$$-\frac{3\,A^2 \sqrt{2}\,\pi}{32} + \frac{\sqrt{2}\,\pi}{4} \quad \textbf{(4)}$$

$eq := \% = \dfrac{T}{\mathrm{sqrt}(2) \cdot 4}$

$$eq := -\frac{3\,A^2 \sqrt{2}\,\pi}{32} + \frac{\sqrt{2}\,\pi}{4} = \frac{T\sqrt{2}}{8} \quad \textbf{(5)}$$

$Tsol := solve(\%, T)$

$$Tsol := -\frac{3}{4}\,A^2\,\pi + 2\,\pi \quad \textbf{(6)}$$

$F := subs(A = 0.001, f1); integrate(F, u = 0..1)$

$$F := \frac{0.001}{\sqrt{2.000001\ 10^{-6} - 2\,\left(1.\ 10^{-6}\right) \cdot u^2 - \left(1.\ 10^{-12}\right) \cdot u^4}}$$

$$1.110720318019563 \quad \textbf{(7)}$$

$evalf(\%) = \dfrac{T}{\mathrm{sqrt}(2) \cdot 4}; solve(\%, T)$

$$1.110720318019563 = \frac{\sqrt{2}\ T}{8}$$

$$6.283182950986493 \quad \textbf{(8)}$$

$evalf(subs(A = 0.001, Tsol))$

$$6.283182950985096 \quad \textbf{(9)}$$

$evalf(2 \cdot \mathrm{Pi})$

$$6.283185307179586 \quad \textbf{(10)}$$

2 Question 2

Now, the damping term is 0. Hence, the time period should be much larger.

$$\ddot{x} + x^3 = 0; x(0) = A; \dot{x} = 0;$$

Similarly, performing the calculations, the integral equation we obtain is this:

$$x = Au; dx = Adu;$$

$$\int_0^1 \frac{A}{\sqrt{A^4 - x^4}} du = \frac{1}{\sqrt{2}} \frac{T}{4}$$

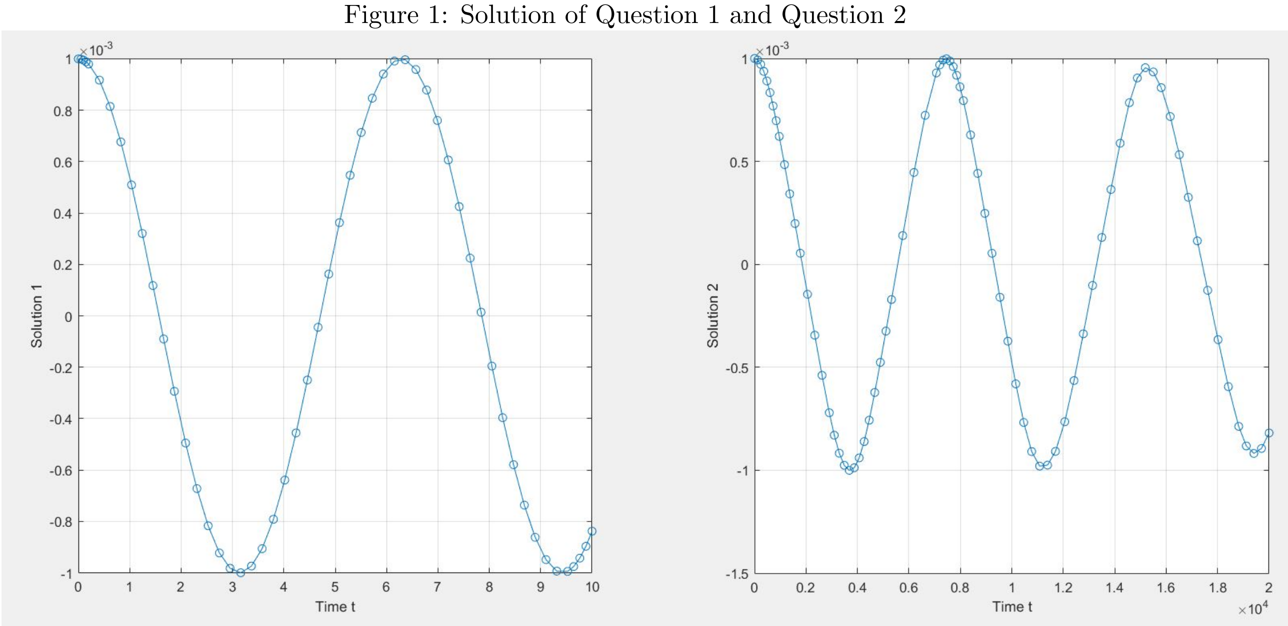
Solving in Maple, we get:

$$T = \frac{7.4163}{A}$$

Since, A is in the denominator, with A≪1, we will obviously have a very large time period T.

Say, A = 0.001. Using maple, we obtain the following answers
Question 1: 6.2832, Question 2: 7416.2988
We can also plot the solutions of the above problems in Matlab.

Figure 1: Solution of Question 1 and Question 2



It can be clearly seen from the order of time scales, that time period of the second is much bigger than the first.

The following page contains the maple code for problem 2.

$restart : Digits := 16$

$$Digits := 16 \tag{1}$$

$fl := \dfrac{A}{\mathrm{sqrt}(A^4 - A^4 . u^4)}$

$$fl := \frac{A}{\sqrt{A^4 - A^4 \cdot u^4}} \tag{2}$$

$\#integrate(fl, u = 0 .. 1)$

$f2 := convert(series(fl, A = 0, 4), polynom)$ assuming $A > 0$

$$f2 := \frac{1}{\sqrt{-u^4 + 1}\, A} \tag{3}$$

$integrate(f2, u = 0 .. 1)$

$$\frac{\mathrm{B}\left(\frac{1}{4}, \frac{1}{2}\right)}{4\, A} \tag{4}$$

$$f2 \tag{5}$$

$eq := \% = \dfrac{T}{\mathrm{sqrt}(2) \cdot 4}$

$$eq := \frac{\mathrm{B}\left(\frac{1}{4}, \frac{1}{2}\right)}{4\, A} = \frac{T\sqrt{2}}{8} \tag{6}$$

$Tsol := evalf(solve(\%, T))$

$$Tsol := \frac{7.416298709205488}{A} \tag{7}$$

$F := subs(A = 0.001, fl); integrate(F, u = 0 .. 1)$

$$F := \frac{0.001}{\sqrt{1.\ 10^{-12} - (1.\ 10^{-12}) \cdot u^4}}$$

$$1311.028777146060 \tag{8}$$

$evalf(\%) = \dfrac{T}{\mathrm{sqrt}(2) \cdot 4}; solve(\%, T)$

$$1311.028777146060 = \frac{T\sqrt{2}}{8}$$

$$7416.298709205488 \tag{9}$$

$evalf(subs(A = 0.001, Tsol))$

$$7416.298709205488 \tag{10}$$

3 Question 3

$$A_{ij} = sin(i)cos(j)$$

The matlab code is given below:

```
clc;
clear;

A = zeros(5,5); % initializing the A matrix

for i = 1:5
    for j=1:5
        A(i,j) = sin(i)*cos(j); % substituting the values
    end
end


[v,d] = eig(A);
% columns of v are the eigenvectors and the digonal elements of d are the eigwevalues
```

The eigenvalues were: 0.1592, 0, 0, 0, 0
And the corresponding eigenvectors:

$$\begin{bmatrix} 0.4821 + 0.0000i \\ 0.5209 + 0.0000i \\ 0.0808 + 0.0000i \\ -0.4336 + 0.0000i \\ -0.5493 + 0.0000i \end{bmatrix}, \begin{bmatrix} 0.3367 + 0.0000i \\ 0.5598 + 0.0000i \\ 0.0869 + 0.0000i \\ -0.4660 + 0.0000i \\ -0.5904 + 0.0000i \end{bmatrix}, \begin{bmatrix} 0.2761 + -0.0756i \\ 0.6832 + 0.0000i \\ 0.0613 + -0.0393i \\ -0.4681 + -0.1123i \\ -0.3883 + -0.2518i \end{bmatrix}, \begin{bmatrix} 0.2761 + 0.0756i \\ 0.6832 + 0.0000i \\ 0.0613 + 0.0393i \\ -0.4681 + 0.1123i \\ -0.3883 + 0.2518i \end{bmatrix}, \begin{bmatrix} 0.3072 + 0.0000i \\ 0.3958 + 0.0000i \\ -0.0050 + 0.0000i \\ -0.3365 + 0.0000i \\ -0.7973 + 0.0000i \end{bmatrix}$$

4 Question 4

$$\dot{x_1} = -sin(x_1)(1 + x_2)$$
$$\dot{x_2} = x_1 + x_2$$

$$\dot{x_1} = 0 \implies x_1^* = n\pi \text{ or } x_2^* = -1$$
$$\dot{x_2} = 0 \implies x_1^* = -x_2^*$$

(x_1^*, x_2^*) are the equilibrium points. We can choose the following set of points for the further stability analysis around them:

$$(1, -1), (0, 0), (\pi, -\pi)$$

The Jacobian matrix D is:

$$D = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}, \text{where, } \dot{x_1} = f_1(x_1, x_2) \text{ and } \dot{x_2} = f_2(x_1, x_2)$$

$$\therefore D = \begin{bmatrix} -cos(x1)(x2 + 1) & -sin(x1) \\ 1 & 1 \end{bmatrix}$$

For $(x_1^*, x_2^*) = (1, -1)$,

$$D = \begin{bmatrix} 0 & -0.8415 \\ 1.0000 & 1.0000 \end{bmatrix}$$

Eigenvalues: 0.5000 + 0.7691i, 0.5000 - 0.7691i. Corresponding Eigenvectors:

$$\begin{bmatrix} -0.3685 + 0.5667i \\ 0.7369 + 0.0000i \end{bmatrix}, \begin{bmatrix} -0.3685 - 0.5667i \\ 0.7369 + 0.0000i \end{bmatrix}$$

For $(x_1^*, x_2^*) = (0, 0)$,

$$D = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

Eigenvalues: 1, -1. Corresponding Eigenvectors:

$$\begin{bmatrix} 0 \\ 1.0000 \end{bmatrix}, \begin{bmatrix} 0.8944 \\ -0.4472 \end{bmatrix}$$

For $(x_1^*, x_2^*) = (\pi, -\pi)$,

$$D = \begin{bmatrix} -2.1416 & 0 \\ 1.0000 & 1.0000 \end{bmatrix}$$

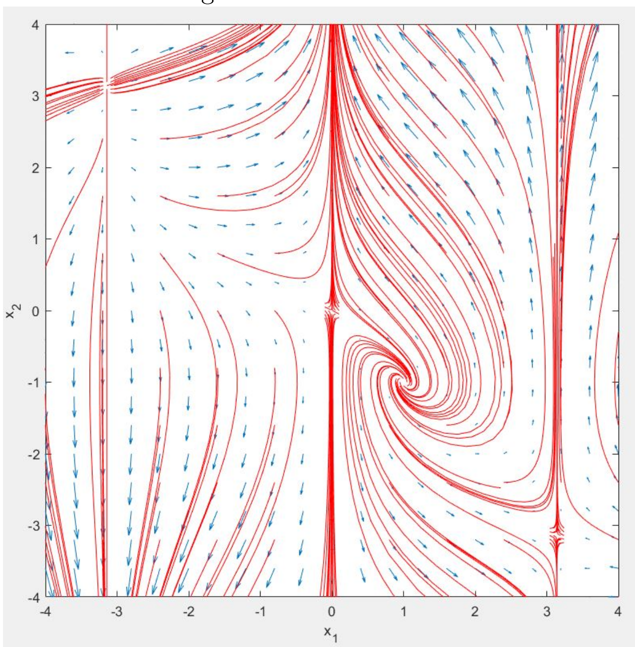
Eigenvalues: 1.0000, -2.1416. Corresponding Eigenvectors:

$$\begin{bmatrix} 0 \\ 1.0000 \end{bmatrix}, \begin{bmatrix} 0.9529 \\ -0.3033 \end{bmatrix}$$

Further, from the eigenvalues, we can see that all of the points are unstable.

Phase portrait in the bounded region: $x_1 \in [-4, 4]$ and $x_2 \in [-4, 4]$ is shown below. 4 fixed points can be seen in this region: $(-\pi, \pi), (0, 0), (1, -1)$ and $(\pi, -\pi)$

Figure 2: Phase Portrait



The matlab code is given below:

```
clc;
clear;

syms x1 x2

f1 = -sin(x1)*(1+x2); % x_1_dot = f1
f2 = x1 + x2; % x_2_dot = f2

S = solve([f1==0, f2==0], [x1 x2]);
x_1 = S.x1;
x_2 = S.x2;

q_star = [x_1,x_2; pi,-pi]; % some equilibrium points

D = [diff(f1,x1), diff(f1,x2); diff(f2,x1), diff(f2,x2)]; % Jacobian matrix
for i=1:3
    [x1, x2] = deal(q_star(i,1), q_star(i,2));
    D_i(:,:,i) = double(subs(D)); % Calculating jacobian Matrix at each of the equilibrium points
end

i=3;
[v,d] = eig(D_i(:,:,i)); % Eigenvalues and Eigenvectors at those points

%% PHASE PLOT

f = @(t,x) [-sin(x(1))*(1+x(2)); x(1) + x(2)];

spacing = 0.4;
[X,Y] = meshgrid(-4:spacing:4);

U = zeros(size(X));
V = zeros(size(X));

t=0;
for i = 1:numel(X)
    xdot = f(t,[X(i); Y(i)]);
    U(i) = xdot(1);
    V(i) = xdot(2);
end

quiver(X,Y,U,V,1); % quiver directions showing the direction of phase plot
xlabel('x_1')
ylabel('x_2')
axis square;
xlim([-4 4]);
ylim([-4 4]);

hold on

% some standard lines showing the phase plot
for x10 = -4:0.8:4
    for x20 = -4:0.8:4
        tim_end = 4;
```

```
        [ts,xs] = ode45(f,[0,tim_end],[x10;x20]);
        plot(xs(:,1),xs(:,2),'r');
    end
end

% some lines showing the phase plot in the region near the fixed points
e = 0.1;
for xq = [-pi, 0, 1, pi]
    for x10 = -e:e/2:e
        for x20 = -e:e/2:e
            tim_end = 10;
            [ts,xs] = ode45(f,[0,tim_end],[x10+xq;x20-xq]);
            plot(xs(:,1),xs(:,2),'r');
        end
    end
end
```