Simulating the Ionosphere

Anna Scaife

Cavendish Astrophysics
University of Cambridge.

MCCT SKADS
Summer School on Third Generation Calibration

Outline

Let's start with the **Z–Jones**...

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$$Z = \begin{bmatrix} e^{i\phi_{\rm iono}} & 0 \\ 0 & e^{i\phi_{\rm iono}} \end{bmatrix}$$

$$\phi_{\rm iono} = -25\lambda \mathrm{TEC}$$

$$\mathrm{TEC} = ...$$

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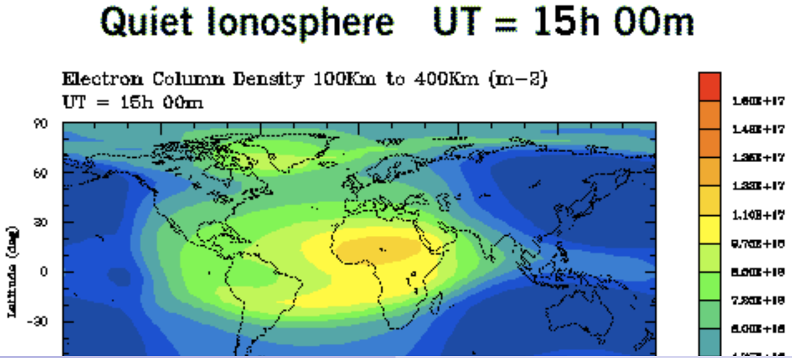
TEC

$$1\ \mathrm{TECU} = 10^{16}\ \mathrm{m}^{-2}$$

$$\mathrm{TEC} = \int_0^{\infty} n_{\mathrm{e}} d\ell + \mathrm{const}$$



- It seems that we should be able to predict the behaviour of the ionosphere...
- ...but, in fact we are limited to interpolation of scattered measurements.
- Counter-intuitively, simulating the ionosphere is in fact an inverse problem.
- Fortunately there are lots of ways of approaching inverse problems.

Interpolation

Let's look at interpolating between measurements...

DEFINITION: Approximating measurements at intermediate scales/positions from scattered measurements.

- We have *sparse* measurements
- i.e. We have *under-sampled* data

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Interpolation schemes

1. Kriging
2. Triangulation based
3. Natural neighbour
4. Splining

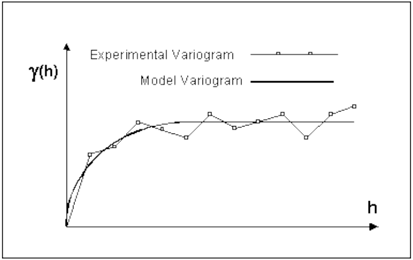
Kriging

Semivariance

$$\gamma(\mathbf{h}) = \frac{1}{2}\left(f(\mathbf{x}) - f(\mathbf{x}+\mathbf{h})\right)^2$$

A typical model:

$$\gamma = \begin{cases} s\left(\frac{3}{2}\left(\frac{h}{a}\right) - \frac{1}{2}\left(\frac{h}{a}\right)^3\right), & 0 \le h \le a \\ s, & h > a \end{cases}$$



Triangulation

Given three measured points we can interpolate to any point within the triangle using:

Delaunay triangulation

$$f(x,y) = \sum_{\mathrm{i}=1}^{3} \phi_{\mathrm{i}}(x,y) f_{\mathrm{i}}$$

- $\phi_{\mathrm{i}}(\mathbf{x})$ is our basis function
- In a simple case we can use linear equations:

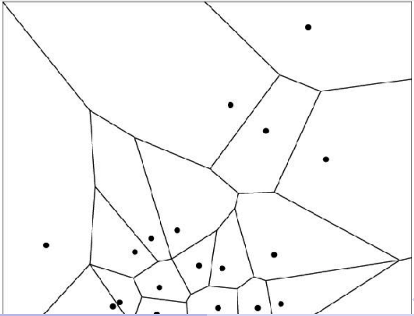
$$f(x,y) = c_1 x + c_2 y + c_3$$

- We can just solve $\mathbf{Ac} = \mathbf{f}$, where $\mathbf{f} = (f_1, f_2, f_3)^{\mathrm{T}}$ and $\mathbf{A} = \{(x_i, y_i, 1)\}_{1 \le i \le 3}$.

Natural neighbour

Voronoi tesselation

- Voronoi tesselation divides the data into cells defined by the positions of the measurements.
- We use the interpolation point to define a new Voronoi cell.
- The value of this cell can be evaluated as the weighted sum of the contributions from its overlapping cells.



Splining

Bayes Theorem

$$p(D|M,\Theta) = \frac{L(M,\Theta|D)\pi(\Theta)}{E}$$

$$posterior = \frac{likelihood \times prior}{Evidence}$$

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Maximum likelihood

We maximise the likelihood of the **DATA** w.r.t the **MODEL**. When we perform a χ^2 test we are in fact calculating a Gaussian ML.

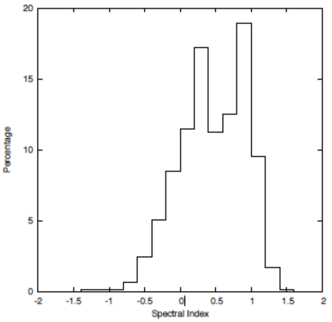
$$N(D_{\rm i}, \sigma) \propto \exp \frac{-(D_{\rm i} - M_{\rm i})^2}{2\sigma^2}$$

Maximum A Posteriori

If we know something about our parameters then we can utilise that prior information.

Example

Say we are fitting a spectral index... $S = A\nu^{-\alpha}$



$$\ln L = -0.5\frac{(d_i - m_i)^2}{\sigma_i^2}$$

$$+\ln \pi$$

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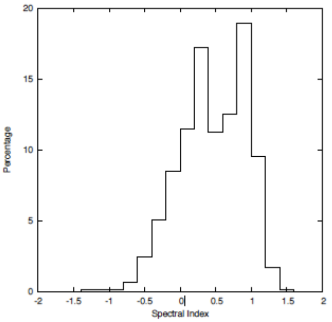

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Model comparison

What if we have **more than one** model...?

The Evidence:

$$E = \int L(\Theta)\pi(\Theta)d^D\Theta$$

The model selection ratio:

$$R = \frac{\mathrm{Pr}(H_1|D)}{\mathrm{Pr}(H_2|D)} = \frac{\mathrm{Pr}(D|H_1)\mathrm{Pr}(H_1)}{\mathrm{Pr}(D|H_1)\mathrm{Pr}(H_1)} = \frac{Z_1}{Z_2}\frac{\mathrm{Pr}(H_1)}{\mathrm{Pr}(H_2)}$$

Worked Example

Let's look at Kriging again...

Kriging variance

$$\gamma(\mathbf{h}) = \frac{1}{2}\left(f(\mathbf{x}) - f(\mathbf{x}+\mathbf{h})\right)^2$$

$$\mathrm{TEC}(\mathbf{x_0}) = \sum_{i=1}^{n} \lambda_i \mathrm{TEC}(\mathbf{x_i})$$

$$\Delta = 2\sum_{i=1}^{n} \lambda_i \gamma(\mathbf{x_i}, \mathbf{x_0}) - \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j \gamma(\mathbf{x_i}, \mathbf{x_0})$$

$$\Gamma_{ij}\lambda = \Gamma_{i0}$$

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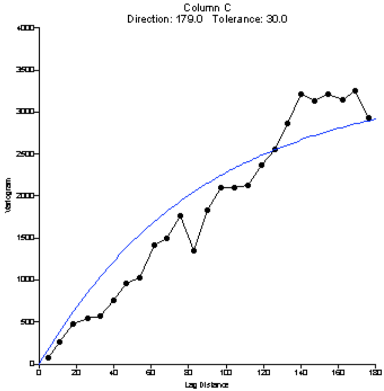
We need a model for γ

- We can calculate γ directly from the data:

$$\gamma(\mathbf{x_i} - \mathbf{x_j}) = \frac{1}{2}(\mathrm{TEC}(\mathbf{x_i}) - \mathrm{TEC}(\mathbf{x_j}))^2$$

- But we need an **analytic** form for γ
- So we have to pick a model...
- Typical models would be:

$$\begin{aligned} \gamma_1 &= a + b * \mathbf{h} \\ \gamma_2 &= a + b * \mathbf{h}^{\alpha} \end{aligned}$$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Variogram
Lag Distance
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1500
2000
2500
3000
3500
4000
0
20
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60
80
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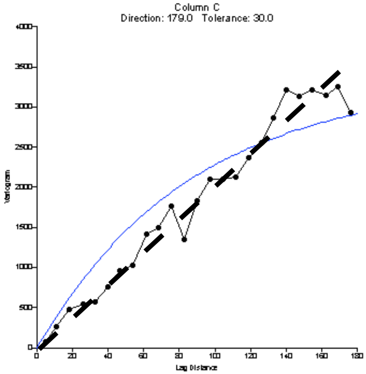
120
140
160
180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For Further Reading I

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