Gleason's Theorem and Quantum Logic

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Introduction

Quantum logic began in 1936 when John von Neumann and Garret Birkhoff described the quantum propositions, through a lattice of projection operators in Hilbert space. This was a new logic that, unlike the Boolean logic, did not satisfy the distribution law.

Examples

Choose a metric system $\hbar = 1$, and let
P="the particle has momentum in the interval $[0, \frac{1}{6}]$",
Q="the particle is in the interval $[-1, 1]$",
R="the particle is in the interval $[1, 3]$", then

$$P \wedge (Q \vee R) = 1,$$
$$(P \wedge Q) \vee (P \wedge R) = 0.$$

by the uncertainty principle $\sigma_x \sigma_p \geq \hbar/2$.

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Dirac-Von Neumann Axioms

The Dirac-von Neumann axioms give a mathematical formulation of quantum mechanics in terms of operators on a Hilbert space. They were introduced by Paul Dirac in 1930 and John von Neumann in 1932.

Hilbert Space Formulation

Let $\mathcal{H}$ be a fixed complex Hilbert space of countably infinite dimension.

- The observables of a qunatum system are defined to be the self-adjoint operators A on $\mathcal{H}$.
- A state ψ of the quantum system is a unit vector of $\mathcal{H}$, up to scalar multiples.
- The expectation value of an observable A for a system in a state ψ is given by the inner product $\langle \psi, A\psi \rangle$.

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Operator Algebras

C^*-algebras were first considered primarily for their use in quantum mechanics to model algebras of physical observables. John von Neumann attepted to establish a general framework for these algebras, which culminated in a series of papers on rings of operators. These papers considered a special class of C^*-algebras that are now known as von Neumann algebras.

Definition

C^*-Algebra is a complex Banach algebra A together with an involution $* : A \to A, x \mapsto x^*$ which satisfies

1. $x^{**} = x$,
2. $(x + y)^* = x^* + y^*$
3. $(xy)^* = y^* x^*$,
4. $(\lambda x)^* = \bar{\lambda} x^*$ for all $\lambda \in \mathbb{C}$,
5. $\|xx^*\| = \|x\| \, \|x^*\|$

Operator Algebras

Operator Algebra Formalism

The Dirac-von Neumann axioms can be formulated in terms of a C^*-algebra as follows.

- The bounded observables of the quantum mechanical system are defined to be the self-adjoint elements of the C^*-algebra.
- The states of the quantum mechanical system are defined to be positive functional ρ such that $\|\rho\| = 1$.
- The value $\rho(x)$ of a state ρ on an element x is the expectation value of the observable x if the quantum system is in the state ρ.

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A **lattice** $\mathcal{L} = (L, \vee, \wedge)$ is a structure over a poset $(L, \leq)$ such that

$$x \wedge y = \inf\{x, y\},$$
$$x \vee y = \sup\{x, y\}.$$

exist in L for all $x, y \in L$. More generally, *σ-lattice* or a **complete lattice** whenever

$$\bigwedge_{i \in I} x_i = \inf\{x_i \mid i \in I\},$$
$$\bigvee_{i \in I} x_i = \sup\{x_i \mid i \in I\}.$$

exist in L for any sequence $(x_i)_{i \in I}$ with a countable or arbitary index set I. A **bounded lattice** $\mathcal{L} = (L, \vee, \wedge, 0, 1)$ is a lattice which satisfy

$$0 \leq x \leq 1 \quad (\forall x \in L)$$

Quantum Logic

We say that a lattice $\mathcal{L}$ is **distributive** if for all $a, b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

A lattice $\mathcal{L}$ is modular if

$$a \vee (b \wedge c) = (a \vee b) \wedge c \quad (a \leq c, a, b, c \in L)$$

We say that for a unary operation $' : L \to L$ the **de Morgan laws** holds if for any sequence (a_i) of elements from L we have

$$(\bigwedge_i a_i)' = \bigvee_i a_i',$$
$$(\bigvee_i a_i)' = \bigwedge_i a_i'$$

Quantum Logic

We say $\perp\colon L \to L, a \mapsto a^\perp$ is said to be an **orthocomplementation** on a poset L with 0 and 1 if

1. $(a^\perp)^\perp = a$ for any $a \in L$,
2. if $a \leq b$, then $b^\perp \leq a^\perp$,
3. $a \vee a^\perp = 1$ for any $a \in L$.

The poset L with the orthocomplementation $\perp$ is said to be **orthocomplemented**. Two elements a and b are said to be **orthogonal**, and we write $a \perp b$, if $a \leq b^\perp$.

Quantum Logic

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For OMP L, if we have

$$\bigvee_{i=1}^{\infty} a_i \in L$$

whenever $a_i \perp a_j$ for $i \neq j$, $\{a_i\} \subset L$, then $\mathcal{L} = (L, \vee, \wedge, 0, 1, \perp)$ is said to be a **quantum logic**.

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Gleason's Theorem

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$. Then any bounded completely additive measure μ on the projection lattice $P(\mathcal{H})$ extends uniquely to a normal functional on the algebra $B(\mathcal{H})$ of all bounded operators acting on H.

Since any normal functional on a von Neumann algebra can be represented by a trace class operator, Gleason's theorem gives a characterization of any bounded completely additive measure μ on $P(\mathcal{H})$ in the form

$$\mu(p) = \mathrm{tr}(Tp) \tag{1}$$

for all $p \in P(\mathcal{H})$ where the T is a trace class operator on $\mathcal{H}$. The operator T appearing in the formula is called the **density meatrix** of given physical state.

Remark

The equation $\mu(p) = \mathrm{tr}(Tp)$ can be rewritten in the form

$$\mu(p) = \sum_{n=1}^{\infty} \alpha_n \omega_{x_n}(p) \tag{2}$$

where $\omega_{x_n} = \langle x_n, px_n \rangle$ are vector states, corresponding to the orthonormal sequence (x_n) of eigenvectors of T. Therefore, any completely additive probability measure on the Hilbert space structure is a σ-convex combination of pure vector states.

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Hidden Variable Problem

Theorem

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Hidden Variable Problem

Gleason's theorem says that the probability structure is determined by quantum logic, i.e. by the ordered structure of projections in a Hilbert space. The fact that only linear restriction can qualify for being quantum state has, besides hidden variable theory, other interesting physical consequences. One of them is indeterminacy principle.

Indeterminancy Priciple

Let p and q be atomic nonorthogonal projections in a Hilbert space $\mathcal{H}$. For every completely additive state μ on $P(\mathcal{H})$ the following equivalence holds

$$\mu(p),\ \mu(q) \in \{0, 1\} \Leftrightarrow \mu(p) = \mu(q) = 0$$

In other words, no quantum state assigns sharp probability zero or one to two atomic nonorthogonal projection unless they are both false.

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