Gleason's Theorem and Quantum Logic

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November 29, 2024

Table of Contents

Table of Contents

Introduction

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P="the particle has momentum in the interval $[0, \frac{1}{6}]$",
Q="the particle is in the interval $[-1, 1]$",
R="the particle is in the interval $[1, 3]$",then

$$P \wedge (Q \vee R) = 1,$$
$$(P \wedge Q) \vee (P \wedge R) = 0.$$

by the uncertainty principle $\sigma_x \sigma_p \geq \hbar/2 = \frac{1}{2}$.

Table of Contents

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A Hilbert space $\mathcal{H}$ is **separable** if it has countable dense subset.

Theorem

Hilbert space is separable if and only if it has orthonormal basis.

Operator Algebras

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Example

$\mathcal{B}(H)$: The set of all bounded linear operators on a complex Hilbert space H is a C^*-algebra.

Operator Algebras

An element A of a C^*-algebra $\mathcal{A}$ is said to be

- **normal** if $AA^* = A^*A$
- **self-adjoint** if $A = A^*$
- **projection** or **orthoprojector** if self-adjoint and $A^2 = A$
- **positive** if $A = B^*B$ for some $B \in \mathcal{A}$, and we write $A \geq 0$.

A positive linear functional $f : A \to \mathbb{C}$ is a **state** if $\|f\| = 1$. Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras; a mapping linear map $\pi : \mathcal{A} \to \mathcal{B}$ such that

1. $\pi(AB) = \pi(A)\pi(B)$,
2. $\pi(A^*) = \pi(A)^*$

is said to be a $*$-**morphism**. A **representation** of a C^*-algebra $\mathcal{A}$ is defined to be a pair (H, π), where H is a complex Hilbert space and π is a $*$-morphism from $\mathcal{A}$ into $\mathcal{B}(H)$.

Operator Algebras

A **cyclic representation** of a C^*-algebra $\mathcal{A}$ is defined to be a triple (H, π, x) where (H, π) is a representation of $\mathcal{A}$ and $x \in H$ is a **cyclic vector** for $\{\pi(A) : A \in \mathcal{A}\}$, i.e., $\{\pi(A)x \mid A \in \mathcal{A}\}$ is dense in H.

GNS representation

Let ω be a state over the C^*-algebra $\mathcal{A}$. Then there exists a cyclic representation $(H_\omega, \pi_\omega, x_\omega)$ of $\mathcal{A}$ such that

$$\omega(A) = \langle x_\omega, \pi_\omega(A)x_\omega \rangle$$

for all $A \in \mathcal{A}$, where $\|x_\omega\| = \|\omega\| = 1$

Operator Algebras

Let H be a Hilbert space. For any $\mathcal{M} \subset \mathcal{B}(H)$, let

$$\mathcal{M}' = \{A \in \mathcal{M} \mid AM = MA \text{ for all } M \in \mathcal{M}\}$$

A **von Neumann algebra** on a Hilbert space H is a subset $\mathcal{A}$ of $\mathcal{B}(H)$ such that

1. $I \in \mathcal{A}$,
2. $\alpha A + \beta B \in \mathcal{A}$ whenever $A, b \in \mathcal{A}$, $\alpha, \beta \in \mathbb{C}$,
3. if $A \in \mathcal{A}$, then $A^* \in \mathcal{A}$,
4. $\mathcal{A}$ is closed in the weak operator topology.

Table of Contents

Dirac-von Neumann Axioms

The Dirac-von Neumann axioms give a mathematical formulation of quantum mechanics in terms of operators on a Hilbert space. They were introduced by Paul Dirac in 1930 and John von Neumann in 1932.

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- The expectation value of an observable A for a system in a state ψ is given by the inner product $\langle \psi, A\psi \rangle$.

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Example

For C^*-algebra $\mathcal{A}$ on Hilbert space $\mathcal{H}$, $\omega_x(A) = \langle x, Ax \rangle$ for $x \in \mathcal{H}, A \in \mathcal{A}$ is called **vector state**.

Table of Contents

Quantum Logic

A **lattice** $\mathcal{L} = (L, \vee, \wedge)$ is a structure over a poset $(L, \leq)$ such that

$$x \wedge y = \inf\{x, y\},$$
$$x \vee y = \sup\{x, y\}.$$

exist in L for all $x, y \in L$.

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$$\bigwedge_{i \in I} x_i = \inf\{x_i \mid i \in I\},$$
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exist in L for any sequence $(x_i)_{i \in I}$ with a countable or arbitary index set I.A **bounded lattice** $\mathcal{L} = (L, \vee, \wedge, 0, 1)$ is a lattice which satisfy

$$0 \leq x \leq 1 \quad (\forall x \in L)$$

Quantum Logic

We say that a lattice $\mathcal{L}$ is **distributive** if for all $a, b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$
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A lattice $\mathcal{L}$ is **modular** if

$$a \vee (b \wedge c) = (a \vee b) \wedge c \quad (a \leq c, a, b, c \in L)$$

We say that for a unary operation $' : L \to L$ the **de Morgan laws** holds if for any sequence (a_i) of elements from L we have

$$(\bigwedge_i a_i)' = \bigvee_i a_i',$$
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We say $\perp: L \to L, a \mapsto a^{\perp}$ is said to be an **orthocomplementation** on a poset L with 0 and 1 if

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The poset L with the orthocomplementation $\perp$ is said to be **orthocomplemented**.Two elements a and b are said to be **orthogonal**, and we write $a \perp b$, if $a \leq b^\perp$.

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For OMP L, if we have

$$\bigvee_{i=1}^{\infty} a_i \in L$$

whenever $a_i \perp a_j$ for $i \neq j$, $\{a_i\} \subset L$, then $\mathcal{L} = (L, \vee, \wedge, 0, 1, \perp)$ is said to be a **quantum logic**.

Quantum Logic

- A **(finitely additive) measure** μ on an OMP L is a map $\mu : L \to \mathbb{C}$ such that $\mu(x \vee y) = \mu(x) + \mu(y)$ whenever $x \perp y$.

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  for any index set I and for any sequence (a_i) for $a_i \perp a_j$ for $i \neq j$.

Quantum Logic

Example

Let H be a real or complex Hilbert space. Denote by $L(H)$ the system of all closed subspaces of H. Then $L(H)$ is a quantum logic (called the quantum logic of a Hilbert space H), where the partial ordering is determined by the set-theoretic inclusion, the meet and join are defined as follows

$$\bigvee_t M_t = \bigcap_t M_t, \quad \bigvee_t M_t = cl(sp(\bigcup_t M_t)),$$

and the orthocomplementation

$$\perp : M \mapsto M^\perp = \{x \in H \mid \langle x, y \rangle = 0 \text{ for all } y \in M\}.$$

Example

Suppose that $\mathcal{A}$ is a von Neumann algebra of operators action on a real or complex Hilbert space H. Denote by $L_{\mathcal{A}(H)}$ the set of all closed subspaces of H whose orthoprojectors belongs to $\mathcal{A}$. Then $L_{\mathcal{A}(H)}$ is a sublogic of $L(H)$.

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Table of Contents

Gleason's Theorem

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If H be a separable, real or complex Hilbert space, $\dim H \neq 2$, then for any state μ on $L(H)$ there exists a unique positive Hermitian trace operator T on H with $\operatorname{tr} T = 1$, such that

$$\mu(M) = \operatorname{tr}(TP_M)$$

for all $M \in L(H)$.

proof. [Dvu93, p.131-149]
The operator T is called a **density matrix**.

Gleason's Theorem

Remark

Gleason's theorem can be reformulated into an equivalent form: For any state μ on $L(H)$ of a separable Hilbert space H, $\dim H \neq 2$, there exists an orthornormal system of vectors (x_i) and a system of positive numbers $\{\lambda_i\}$ such that $\sum_i \lambda_i = 1$, and

$$\mu(M) = \sum_i \lambda_i \mu_{x_i}$$

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Table of Contents

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proof. [Ham03, Theorem 3.4.1] The nonexistence of 0-1 state on an OML is connected with a long dispute on hidden variables in quantum theory.The existence of sufficiently many hidden variables would mean that the probabilistic character of quantum mechnics disappears by adding certain new parameters.The theorem says theat the projection lattice of a Hilbert space cannot be embedded into a Boolean algebra because any such embedding would induce a 0-1 state.

Hidden Variable Problem

Theorem

Let H be a Hilbert space with $\dim H \geq 3$. There is no 0-1 finitely additive state, i.e., no probability measure attains only values 0 and 1 on the projection lattice $\mathcal{P}(H)$.

proof. [Ham03, Theorem 3.4.1] The nonexistence of 0-1 state on an OML is connected with a long dispute on hidden variables in quantum theory.The existence of sufficiently many hidden variables would mean that the probabilistic character of quantum mechnics disappears by adding certain new parameters.The theorem says theat the projection lattice of a Hilbert space cannot be embedded into a Boolean algebra because any such embedding would induce a 0-1 state.It is therefore an important consequence of Gleason Theorem that the Hilbert space model cannot be completed by considering axiliary hidden variables to a classical model without violating its inner structure.

Hidden Variable Problem

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Indeterminacy Priciple

Let p and q be atomic nonorthogonal projections in a Hilbert space H. For every completely additive state μ on $\mathcal{P}(H)$ the following equivalence holds

$$\mu(p),\, \mu(q) \in \{0, 1\} \Leftrightarrow \mu(p) = \mu(q) = 0$$

In other words, no quantum state assigns sharp probability zero or one to two atomic nonorthogonal projection unless they are both false.

Table of Contents

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A state ϕ on a von Neumann algebra $\mathcal{M}$ is said to be **disperision free** if $\phi(A^2) - \phi(A)^2 = 0$, for all self-adjoint $A \in \mathcal{M}$. A nonabelian factor admits no dispersion free state. That is one of characteristic of noncommutative probability.

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Thank You!