Gleason's Theorem and Quantum Logic

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Introduction

In 1932, John von Neumann noticed that the projection operators of Hillbert space could be viewed as quantum mechanical propositions about observables [9]. The principles governing these quantum propositions were then called quantum logic by von Neumann and Birkhoff [2]. Von Neumann also managed to dervie the Born rule in his textbook [9]. However, his assumptions were regarded to not be well-motivated by John Bell [1], [3].

By the late 1940s, George Mackey was wondering whether the Born rule was the only possible rule for calculating probabilities in a theory that represented measurements as orthonormal bases on a Hilbert space. In his investigation of mathematical foundation of quantum mechanics, Mackey had proposed the following problem [8, p.50-51][4, p.129]: Determine all measures on the closed subspaces of a Hilbert space. Gleason provided the answer to this problem in the same year [5], which is later called Gleason's theorem. Gleason succeeded in decribing all σ-additive probability measures on the logical of all closed subspaces of a separable Hilbert space and in showing that, except for the obvious two-dimensional counterexamplaes, all probability measures can be identified with normal states in the sense of von Neumann approach. Gleason's achievement confirmed von Neumann's original insight and put the calculus of Hilbert space quantum mechanics on natural physical grounds [6, p.87-88]. Gleason's theorem is of particular importance for the field of quantum logic and its attempt to find a minimal set of mathematical axioms for quantum theory.

After a considerable effort the Gleason's theorem was established in the early 90's for finitely additive vector measures on the projection lattices of von Neumann algebras. It has turned out that the lattice homomorphisms on nonabelian von Neumann algebras are σ-additve, or that finitely additive measures on projection lattices whose kernels are lattice ideal enjoy many continuity properties [6, p.3-4].

The present paper is intended to serve as an introduction to topic of the relation between Gleason's theorem and the theory of quantum logic.Most of the contents were referred to chapters 1, 2, and 3 of [4] and chapters 1, 2, and 3 of [6].

Chapter 1

Preliminaries

1.1 Elements of Operator Algebras

A C^*-algebra A is a Banach algebra over the field $\mathbb{C}$ together with a map $* : A \to A$, $a \mapsto a^*$ which following properties:

1. $a^{**} = a$ for all $a \in A$,
2. $(a+b)^* = a^* + b^*$ for all $a, b \in A$,
3. $(ab)^* = b^* a^*$ for all $a, b \in A$,
4. $(\lambda a)^* = \bar{\lambda} a^*$ for all $\lambda \in \mathbb{C}$ and $a \in A$,
5. $\|aa^*\| = \|a\| \, \|a^*\|$ for all $a \in A$.

A bounded linear map $\pi : A \to B$ between C^*-algebras A and B is called a $*$-homomorphism if

1. $\pi(ab) = \pi(a)\pi(b)$
2. $\pi(a^*) = \pi(a)^*$

for all $a, b \in A$.

Let A be a C^*-algebra. A *state* in C^*-algebra is a positive functional ρ which is $\|\rho\| = 1$. We shall denote the set of all states on A by $S(A)$ which is called the *state space* of A.

A $*$-representation of a C^*-algebra A on a Hilbert space H is a ring homomorphism $\pi : A \to B(H) = \{f \mid f \text{ is a bounded linear operator on } H\}$ such that

1. $\pi(a^*) = \pi(a)^*$ for all $a \in A$,
2. π is nondegenerate.

Theorem 1 (GNS representation)**.** *Given a state ρ of A, there is a $*$-representation π of A acting on a Hilbert space H with unit cylic vector ψ such that*

$$\rho(a) = \langle \pi(a)\psi, \psi \rangle$$

for every a in A. Moreover, the representation is unique up to unitary equivalence.

Proof. [7, Theorem 4.5.2] $\square$

Let X be an inner product space. A *spilitting subspace* of X is the subspace M of X which is the following property holds,

$$M + M^{\perp} = S.$$

Then any vector $x \in X$ can be uniquely expressed in the form

$$x = x_M + x_{M^{\perp}},$$

where $x_M \in M$ and $x_{M^{\perp}} \in M^{\perp}$. We denote by $E(X)$ the set of all splitting subspaces. The map $P_M : X \to X$ such that $P_M(x) = x_M$ for all $x \in X$ is a bounded linear operator with $\|P_M\| = 1$ whenever $M \neq \{0\}$. Moreover, $P_M^2 = P_M$, P_M is self-adjoint, and $\operatorname{ran} P_M = M$. The operator P_M is called the *orthoprojector* or *projection* from X onto M. It is clear that $I - P_M$ is an orthoprojector onto $M^{\perp}$.

Proposition 1. *Let P is an idempotent linear operator on X which is self-adjoint. If $M = \{x \in X \mid Px = x\}$, then $M \in E(S)$ and $P_M = P$.*

Proof. We have

$$\langle x, z - Pz \rangle = \langle x, z \rangle - \langle x, Pz \rangle = \langle x - Px, z \rangle = 0$$

for all $x \in M$ and for all $z \in X$. Therefore, $z - Pz \in M^{\perp}$, and

$$z = Pz + (I - P)z$$

$\square$

Corollary 1. *If P is the orthoprojector onto a splitting subspaces M of X, then I_P is the orth projector onto $M^{\perp}$ and $M^{\perp} = \{x \in X \mid Px = 0\}$.*

Proposition 2. *Let $\{e_1, \ldots, e_n\}$ is a finite orthonormal system in X. Then $M = \{\mathrm{sp}(e_1, \ldots, e_n) \in E(S)\}$ and*

$$P_M(x) = \sum_{i=1}^{n} (x, e_i) e_i$$

for all $x \in X$.

Lemma 1. *A splitting subspace M of X is invariant under an operator T on X if and only if $AP_M = P_M T P_M$.*

Lemma 2. *Let M and N be two splitting subspaces. Then the followings are equivalent.*

1. $M \perp N$.

2. $P_M P_N = O$

Proof. If $M \perp N$, then $N \subset M^\perp$. We have $P_N x \in N$ for all $x \in X$, so that $P_M P_N = 0$ for all $x \in X$. If $P_M P_N = O$, conversely, then $P_M x = P_M P_N x = 0$ for all $x \in N$. Therefore $N \subset M^\perp$, so that $M \perp N$. □

We say that two orthoporjectors P and Q on X are *orthogonal* if $PQ = O$, and we write $P \perp Q$.

Lemma 3. *Let M and N be two splitting subspaces of X. The following statements are equivalent:*

1. $P_M \leq P_N$.

2. $\|P_M\| \leq \|P_N x\|$ *for all* $x \in X$.

3. $M \subset N$.

4. $P_M P_N = P_M$.

5. $P_N P_M = P_M$.

1.2 Quantum Logic

Let L be a poset. For a sequence $(a_i)_{i\in I}$ in L, the *join* is defined by

$$\bigvee_{i\in I} a_i = \sup\{a_i \mid i \in I\},$$

and the *meet* is defined by

$$\bigwedge_{i\in I} a_i = \inf\{a_i \mid i \in I\}.$$

Definition 1. Let L be a poset. Then

- L is a *lattice* if $a \vee b$ and $a \wedge b$ exist in L for any $a, b \in L$.

- L is a *σ-lattice* if $\bigvee_{i\in I} a_i$ and $\bigwedge_{i\in I} a_i$ exist in L for any sequence $(a_i)_{i\in I}$ with a countable index set I.

- L is a *complete lattice* if $\bigvee_{i\in I} a_i$ and $\bigwedge_{i\in I} a_i$ exist in L for any sequence $(a_i)_{i\in I}$ with an arbitrary index set I.

Definition 2. A poset L is said to be *bounded* if there exist elements 0 and 1 in L,

$$0 \leq x \leq 1$$

for all $x \in L$.

A unary operation $\perp\colon L \to L, a \mapsto a^{\perp}$ is said to be an *orthocomplementation* on a bounded poset L if

1. $(a^{\perp})^{\perp} = a$,
2. if $a \leq b$, then $b^{\perp} \leq a^{\perp}$,
3. $a \vee a^{\perp} = 1$

for any $a, b \in L$. A bounded poset L with the orthocomplementation $\perp$ is called *orthocomplemented*.

Definition 3. An orthocomplemented poset L is said to be an *orthomodular poset* (in short OMP) if

1. $a \vee b \in L$ if $a \perp b$,
2. (orthomodularity) $b = a \vee (b \wedge a^{\perp})$ if $a \leq b$,

hold for all $a, b \in L$.

We denote an orthomodular lattice by OML. If an OMP L have the following property

$$\bigvee_{i=1}^{\infty} a_i \in L$$

for any pairwise orthgonal sequence $(a_i)_{i=1}^{\infty}$, L is called a *quantum logic*.

Definition 4. Let L_1 amd L_2 be two OMPs. A mapping $h : L_1 \to L_2$ is called a *homomorphism* if

1. $h(1) = 1$,
2. $h(a) \perp h(b)$ if $a \perp b$ in L_1,
3. $h(a \vee b) = h(a) \vee h(b)$ if $a \perp b$ in L_1.

In addition, we call h a σ-homomorphism if

$$h(\bigvee_{i=1}^{\infty} a_i) = \bigvee_{i=1}^{\infty} h(a_i)$$

holds for any sequence of pairwise orthogonal elements $(a_i)_{i=1}^{\infty}$ of L_1.

Definition 5. Let L be an OMP. Consider a map $\mu : L \to [-\infty, \infty]$ such that

$$\mu(0) = 0, \tag{1.1}$$

$$\mu(\bigvee_{i\in I} a_i) = \sum_{i\in I} \mu(a_i), \text{ for any pairwise orthogonal sequence } (a_i)_{i\in I}. \tag{1.2}$$

Then,

1. μ is called a *finitely additive measure* or a *charge* if (1.2) holds for any finite set I,
2. μ is called a *signed measure* if (1.2) holds for any countably infinite set I,
3. μ is called a *completely additive signed measure* if (1.2) holds for any arbitrary set I.

If μ is nonnegative and $\mu(1) = 1$, then we call μ a *state*.

Example 1. Let H be a Hilbert space over a real or complex. The class

$$L(H) = \{M \mid M \text{ is a closed subspace of } H\}$$

is a quantum logic, where the partial ordering is the inclusion, the meet and join are defined as follows

$$\bigwedge_{i\in I} M_i, \quad \bigvee_{i\in I} M_i = \mathrm{cl}(\mathrm{span}(\bigcup_{\mathrm{i\in I}} \mathrm{M_i})),$$

and the orthocomplementation $\perp$: $M \mapsto M^\perp$.

Example 2. Let H be a Hilbert space over a real or complex. The class

$$\mathcal{P}(H) = \{P \mid P \text{ is a orthoprojector on } H\}$$

is a quantum logic where $P \leq Q$ if and only if

$$\langle Px, x\rangle \leq \langle Qx, x\rangle$$

for all $x \in H$ and $P^\perp = I - P$.

Chapter 2

Gleason's Theorem

2.1 Frame Functions

Definition 6. Let H be a Hilbert space, and let $\mathcal{S}(H) = x \in H \mid \|x\| = 1$. A map $f : \mathcal{S}(H) \to \mathbb{R}$ is a frame function on H if there is a constant w, called *weight* of f, such thet for any orthonormal basis $\{x_i \mid i \in I\}$ of H we have

$$\sum_{i \in I} f(x_i) = w.$$

Remark. Let f be a frame function on H. Then,

1. $f(cx) = f(x)$ for any sacalar $|c| = 1$,
2. $f|_{\mathcal{S}(M)}$ is a frame function on M for any closed subspace M of H.

A frame function f on H is *bounded* if

$$\sup_{x \in \mathcal{S}(H)} |f(x)| < \infty$$

, f is *semibounded* if

$$\inf_{x \in \mathcal{S}(H)} |f(x)| > -\infty$$

and f is regular if there is a Hermitian operator T on H such that

$$f(x) = \langle Tx, x \rangle$$

for all $x \in \mathcal{S}(H)$.

We will prove the regularity of the semibounded frame function for general Hilbert spaces of dimension 3 or higher, starting from the case where $H = \mathbb{R}^3$.

- $\theta(p, q)$ denotes the angle between vectors p and q on $\mathcal{S}(\mathbb{R}^3)$.

- $N_p = \{s \in \mathcal{S}(H) \mid \theta(p,s) \leq \pi/2\}$ is called the *northern hemisphere* with respect to p.
- $E_p = \{s \in \mathcal{S}(H) \mid \theta(p,s) = \pi/2\}$ is called the *equator* with respect to p.
- A function l_p on N_p given by

$$l_p(s) = \cos^2\theta(p,s) = \langle p, s\rangle^2$$

  which is called the *latitude function*.
- Let $s \in N_p \setminus \{p\}$ with $l_p(s) > 0$. There exists exactly one great circle $C(s)$, having s as its northern most point.
- The great half-circle $D(s) = C(S) \cap N_p$ is called *decscent* through s.
- A *frame* is a triple (p,q,r) such that $\{p,q,r\}$ is orthonormal basis in $\mathbb{R}^3$.

Theorem 2. *Let f be a semibounded frame function on $\mathbb{R}^3$ with the weight w. Let*

$$M = \sup_{x\in\mathcal{S}(\mathbb{R}^3)} f(x),$$
$$m = \inf_{x\in\mathcal{S}(\mathbb{R}^3)} f(x),$$
$$\alpha = w - M - m.$$

Then, there is a frame p,q,r such that

$$f(s) = Mx^2 + \alpha y^2 + mz^2$$

where (x,y,z) is the coordinate of s with respect to the frame (p,q,r).

Proof. [4, Theorem 3.2.13] □

Let H be a complex Hilbert space, a closed real-linear subspace M of H is *completely real* if the inner product $\langle\cdot,\cdot\rangle$ takes only real values on $M \times M$.

Lemma 4. *Let f be a semibounded frame function on an n-dimensional Hilbert space H_n which is regular on each completely real subsapce. Then f is regular.*

Proof. Suppose f is a semibounded frame function on H_n which is regular on each completely real subspace. Let $M = \sup_{x\in\mathcal{S}(H)} f(x)$. Choose a sequence (x_k) in $\mathcal{S}(H_n)$ such that $\lim_k f(x_k) = M$. Since $\mathcal{S}(H_n)$ is compact in the strong topology, we can assume $\lim_k x_k = x, x \in \mathcal{S}(H_n)$. Let

$$\lambda_k = \frac{\langle x, x_k\rangle}{|\langle x, x_k\rangle|}$$

$\lambda_k \to 1$ as $k \to \infty$ since $x_k \to x$. So, $\langle \lambda_k x_k, x \rangle = |\langle x, x_k \rangle|$ is real and the linear subspace M generated by $\{\lambda_k x_k, x\}$ over $\mathbb{R}$ is a completely real subspace. Therefore, there is a Hermitian operator T on M such that $f(x) = \langle Tx, x \rangle$ by the assumpion. Since $f(x_k) = f(\lambda_k x_k)$, we have

$$\begin{aligned} |f(x) - M| &\leq |f(x) - f(\lambda_k x_k)| + |f(\lambda_k x_k) - M| \\ &= |\langle Tx, x \rangle - \langle Tx_k, x_k \rangle| + |f(x_k) - M| \\ &\leq 2\,\|T\|\,\|x - x_k\| + |f(x_k) - M| \to 0 \end{aligned}$$

Now, extend the function f to the whole H via

$$F(z) = \begin{cases} 0 & \text{if } x = 0, \\ \|z\|^2 f\left(\dfrac{z}{\|z\|}\right) & \text{if } z \neq 0, \end{cases}$$

for all $z \in H$. Then we have

$$F(cz) = |c|\,F(z)$$

for all scalar c and for all $z \in H$. If y is any vector of H orthogonal to x, then

$$F(cx + y) = |c|^2\,M + F(y)$$

for all scalar c. Let L be the two-dimensional real linear subspace generated by x and y. Then L is completely real. Since f is regular on L, there is a Hermitian bilinear form t_L. But x is the point at which the maximum of the bilinear form, and y is orthogonal to x in L. Hence the matrix of the bilinear form t_L is diagonal with respect to the orthonormal basis $\{x, y\}$ of L. So,

$$F(rx + sy) = |r|^2\,M + |s|^2\,F(y)$$

for all real numbers r, s. Assume z is any vector of H orthogonal to x, and c is a non-zero scalar. Then,

$$\begin{aligned} F(cx + z) &= F(c(x + c^{-1}z)) \\ &= |c|^2\,F(x + c^{-1}z) \\ &= |c|^2\,F(x + |c|^{-1}\,|c|\,c^{-1}z) \end{aligned}$$

and $y = |c|^{-1}\,cz$ is orthogonal to x. Then

$$F(x + c^{-1}z) = M + |c|^{-2}\,F(|c|\,c^{-1}z) = M + |c|^{-1}\,F(z)$$

which gives

$$F(cx + z) = |c|^2\,M + F(z).$$

We have proved that if the restriction of a semibounded frame function to any completely real subspace is regular, so is f on H_2.

The restriction of f to $K_{n-1} = \text{span}(x)^\perp$ is a semibounded frame function on K_{n-1}. To use induction, assume that the lemma is true for any $k < n$. Then $F|_{K_{n-1}}$ is regular by the induction hypothesis and there is an orthonormal basis $\{e_1, \dots, e_{n-1}\}$ for K_{n-1} and real numbers $M_1, \dots, M_{n-1}$ such that

$$F(c_1 e_1 + \cdots + c_{n-1} e_{n-1}) = |c_1|^2 M_1 + \cdots + |c_{n-1}|^2 M_{n-1}$$

for any scalars $c_1, \cdots, c_{n-1}$. We see that

$$F(cx + c_1 e_1 + \cdots + c_{n-1} e_{n-1}) = |c|^2 M + |c_1|^2 M_1 + \cdots + |c_{n-1}|^2 M_{n-1}$$

□

Therefore, $f(x) = \langle Ux, x\rangle$ where U is a Hermitian operator on H_n such that $Ux = Mx$ and $Ue_j = M_j e_j$ for $j = 1, \dots, n-1$.

Theorem 3. *For $n \geq 3$, any semibounded frame function on H_n is regular.*

Proof. Since $n \geq 3$, there is a completly real three-dimensional subspace M of H_n, and $f|_{\mathcal{S}(M)}$ is regular. On the other hand, every completely real two-dimensional subspace N of H_n can be embedded in M. So there is a bilinear form t_M on $M \times M$ such that

$$f(x) = t_M(x, x)$$

for all $x \in \mathcal{S}(M)$, and $t_M|_{N\times N}$ gives a bilinear form. Threfore $f|_{\mathcal{S}(H)}$ is regular. By the lemma 4, f is regular. □

Definition 7. Let H be a Hilbert space.

- $P(H)$ is the set of all finite-dimensional subspaces of H
- $P_1(H)$ is the set of all one-dimensional subspaces of H
- A charge μ is called *bounded* if $\sup_{M\in L(H)} |\mu(M)| < \infty$
- A charge μ is called *semibounded* if $\inf_{M\in L(H)} \mu(M) > -\infty$
- A charge μ is called $P(H)$*-bounded* if $\sup_{M\in P(H)} |\mu(M)| < \infty$
- A charge μ is called $P(H)$*-semibounded* if $\inf_{M\in H(H)} \mu(M) > -\infty$
- A charge μ is called $P_1(H)$*-bounded* if $\sup_{M\in P_1(H)} |\mu(M)| < \infty$
- A charge μ is called $P_1(H)$*-semibounded* if $\inf_{M\in P_1(H)} \mu(M) > -\infty$

Theorem 4 (Gleason's theorem for $L(H_n)$)**.** *For every $P_1(H)$-semibounded charge μ on $L(H_n)$ for $n \geq 3$, there is a unique Hermitian operator T on H_n such that*

$$\mu(M) = \text{tr}\,(TP_M)$$

for all $M \in L(H_n)$.

Proof. Let $f_\mu : \mathcal{S}(H) \to \mathbb{R}$ be a frame function on H_n with the weight $\mu(H)$ given by

$$f_\mu = \mu(\mathrm{span}(x))$$

By theorem 3, there is a unique Hermitian operator T on H_n such that

$$f_\mu(x) = \langle Tx, x \rangle$$

Let $M \in L(H_n), M \neq \{0\}$, and let $\{e_1, \ldots, e_k\}$ be an orthonormal basis of M. Then,

$$\begin{aligned}
\mu(M) &= \sum_{i=1}^{k} \mu(\mathrm{span(e_i)}) \\
&= \sum_{i=1}^{k} \langle Te_i, e_i \rangle \\
&= \sum_{i=1}^{k} \mathrm{tr}(Te_i \otimes \bar{e}_i) \\
&= \mathrm{tr}\left(T \left(\sum_{i=1}^{k} e_i \otimes \bar{e}_i \right) \right) \\
&= \mathrm{tr}(TP_M)
\end{aligned}$$

□

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