Gleason's Theorem and Quantum Logic

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Introduction

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Q="the particle is in the interval $[-1, 1]$",
R="the particle is in the interval $[1, 3]$",then

$$P \wedge (Q \vee R) = 1,$$
$$(P \wedge Q) \vee (P \wedge R) = 0.$$

by the uncertainty principle $\sigma_x \sigma_p \geq \hbar/2 = \frac{1}{2}$.

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Operator Algebras

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A Hilbert space $\mathcal{H}$ is **separable** if it has countable dense subset.

Theorem

Hilbert space is separable if and only if it has orthonormal basis.

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Example

$\mathcal{B}(H)$: The set of all bounded linear operators on a complex Hilbert space H is a C^*-algebra.

Operator Algebras

An element A of a C^*-algebra $\mathcal{A}$ is said to be

- **normal** if $AA^* = A^*A$
- **self-adjoint** if $A = A^*$
- **projection** or **orthoprojector** if self-adjoint and $A^2 = A$
- **positive** if $A = B^*B$ for some $B \in \mathcal{A}$, and we write $A \geq 0$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras; a mapping linear map $\pi : \mathcal{A} \to \mathcal{B}$ such that

1. $\pi(AB) = \pi(A)\pi(B)$,
2. $\pi(A^*) = \pi(A)^*$

is said to be a $*$-**morphism**. A **representation** of a C^*-algebra $\mathcal{A}$ is defined to be a pair (H, π), where H is a complex Hilbert space and π is a $*$-morphism from $\mathcal{A}$ into $\mathcal{B}(H)$.

A **cyclic representation** of a C^*-algebra $\mathcal{A}$ is defined to be a triple (H, π, x) where (H, π) is a representation of $\mathcal{A}$ and $x \in H$ is a cyclic vector for $\{\pi(A) : A \in \mathcal{A}\}$.

GNS representation

Let ω be a state over the C^*-algebra $\mathcal{A}$. then there exists a cyclic rperesentation $(H_\omega, \pi_\omega, x_\omega)$ of $\mathcal{A}$ such that

$$\omega(A) = \langle x, \pi(A) x_\omega \rangle$$

for all $A \in \mathcal{A}$, where $\|x_\omega\| = \|\omega\| = 1$

Operator Algebras

Let H be a Hilbert space. For any $\mathcal{M} \subset \mathcal{B}(H)$, let

$$\mathcal{M}' = \{A \in \mathcal{M} \mid AM = MA \text{ for all } M \in \mathcal{M}\}$$

A **von Neumann algebra** on a Hilbert space H is a subset $\mathcal{A}$ of $\mathcal{B}(H)$ such that

1. $I \in \mathcal{A}$,
2. $\alpha A + \beta B \in \mathcal{A}$ whenever $A, b \in \mathcal{A}$, $\alpha, \beta \in \mathbb{C}$,
3. if $A \in \mathcal{A}$, then $A^* \in \mathcal{A}$,
4. $\mathcal{A}$ is closed in the weak operator topology.

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The Dirac-von Neumann axioms give a mathematical formulation of quantum mechanics in terms of operators on a Hilbert space. They were introduced by Paul Dirac in 1930 and John von Neumann in 1932.

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- The expectation value of an observable A for a system in a state ψ is given by the inner product $\langle \psi, A\psi \rangle$.

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Example

For C^*-algebra $\mathcal{A}$ on Hilbert space $\mathcal{H}$, $\omega_x(A) = \langle x, Ax \rangle$ for $x \in \mathcal{H}, A \in \mathcal{A}$ is called **vector state**.

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Quantum Logic

A **lattice** $\mathcal{L} = (L, \vee, \wedge)$ is a structure over a poset $(L, \leq)$ such that

$$x \wedge y = \inf\{x, y\},$$
$$x \vee y = \sup\{x, y\}.$$

exist in L for all $x, y \in L$.

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$$\bigwedge_{i \in I} x_i = \inf\{x_i \mid i \in I\},$$
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exist in L for any sequence $(x_i)_{i \in I}$ with a countable or arbitary index set I.

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exist in L for any sequence $(x_i)_{i \in I}$ with a countable or arbitary index set I.A **bounded lattice** $\mathcal{L} = (L, \vee, \wedge, 0, 1)$ is a lattice which satisfy

$$0 \leq x \leq 1 \quad (\forall x \in L)$$

Quantum Logic

We say that a lattice $\mathcal{L}$ is **distributive** if for all $a, b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$
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A lattice $\mathcal{L}$ is modular if

$$a \vee (b \wedge c) = (a \vee b) \wedge c \quad (a \leq c, a, b, c \in L)$$

We say that for a unary operation $' : L \to L$ the **de Morgan laws** holds if for any sequence (a_i) of elements from L we have

$$(\bigwedge_i a_i)' = \bigvee_i a_i',$$
$$(\bigvee_i a_i)' = \bigwedge_i a_i'$$

We say $\perp: L \to L, a \mapsto a^\perp$ is said to be an **orthocomplementation** on a poset L with 0 and 1 if

1. $(a^\perp)^\perp = a$ for any $a \in L$,

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The poset L with the orthocomplementation $\perp$ is said to be **orthocomplemented**.Two elements a and b are said to be **orthogonal**, and we write $a \perp b$, if $a \leq b^\perp$.

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For OMP L, if we have

$$\bigvee_{i=1}^{\infty} a_i \in L$$

whenever $a_i \perp a_j$ for $i \neq j$, $\{a_i\} \subset L$, then $\mathcal{L} = (L, \vee, \wedge, 0, 1, \perp)$ is said to be a **quantum logic**.

Quantum Logic

- A **(finitely additive) measure** μ on an OMP L is a map $\mu : L \to \mathbb{C}$ such that $\mu(x \vee y) = \mu(x) + \mu(y)$ whenever $x \perp y$.

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  for any index set I and for any sequence a_i for $a_i \perp a_j$ for $i \neq j$.

Quantum Logic

Example

Let H be a real or complex Hilbert space. Denote by $L(H)$ the system of all closed subspaces of H. Then $L(H)$ is a quantum logic (called the quantum logic of a Hilbert space H), where the partial ordering is determined by the set-theoretic inclusion, the meet and join are defined as follows

$$\bigvee_t M_t = \bigcap_t M_t, \quad \bigvee_t M_t = cl(sp(\bigcup_t M_t)),$$

and the orthocomplementation

$$\perp : M \mapsto M^{\perp} = \{x \in H \mid \langle x, y \rangle = 0 \text{ for all } y \in M\}.$$

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Example

Suppose that $\mathcal{A}$ is a von Neumann algebra of operators action on a real or complex Hilbert space H. Denote by $L_{\mathcal{A}(H)}$ the set of all closed subspaces of H whose orthoprojectors belongs to $\mathcal{A}$. Then $L_{\mathcal{A}(H)}$ is a sublogic of $L(H)$.

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Example

Let $\mathcal{P}(\mathcal{A})$ denote the set of all projections in a C^*-algebra $\mathcal{A}$. Projections P_1 and P_2 are called orthogonal if $P_1 P_2 = 0$. Give an ordering on $\mathcal{P}(\mathcal{A})$ by $P_1 \leq P_2$ if and only if $P_1 P_2 = P_2 P_1 = P_1$. If $\mathcal{A}$ is unital, then the structure $\mathcal{P}(\mathcal{A})$ is an OMP with the complement $P^p erp = 1 - P$. The structure $(\mathcal{P}(\mathcal{A}), \leq, 0, 1, \perp)$ is a Boolean algebra if and only if, A is abelian.

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Gleason's Theorem

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If H be a separable, real or complex Hilbert space, dim $H \neq 2$, then for any state μ on $L(H)$ there exists a unique positive Hermitian trace operator T on H with $\operatorname{tr} T = 1$, such that

$$\mu(M) = \operatorname{tr}(TP_M)$$

for all $M \in L(H)$.

proof. [Dvu93, p.131-149]

Remark

Gleason's theorem can be reformulated into an equivalent form: For any state μ on $L(H)$ of a separable Hilbert space H, $\dim H \neq 2$, there exists an orthornormal system of vectors (x_i) and a system of positive numbers $\{\lambda_i\}$ such that $\sum_i \lambda_i = 1$, and

$$\mu(M) = \sum_i \lambda_i \mu_{x_i}$$

for all $M \in L(H)$.

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Hidden Variable Problem

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Indeterminancy Priciple

Let p and q be atomic nonorthogonal projections in a Hilbert space H. For every completely additive state μ on $\mathcal{P}(H)$ the following equivalence holds

$$\mu(p),\ \mu(q) \in \{0,1\} \Leftrightarrow \mu(p) = \mu(q) = 0$$

In other words, no quantum state assigns sharp probability zero or one to two atomic nonorthogonal projection unless they are both false.

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Quantum Probability

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[RS06, p.8] In classical probability theory, the probability space is a triple (X, Σ, μ) where X is a nonempty set, Σ is a σ-algebra on X, and probability measure $\Sigma \to [0, 1]$.
In **noncommutative probability theory** or **qunatum probability theory**, a probability space is a triple $(\mathcal{M}, P(\mathcal{M}), \phi)$ consisting of a von Neumann algebra, its lattice of orthogonal projections and a normal state on the algebra.If $\mathcal{M}$ is abelian, then $(\mathcal{M}, P(\mathcal{M}), \phi)$ is just classical probability space.
A state ϕ on a von Neumann algebra $\mathcal{M}$ is said to be **disperision free** if $\phi(A^2) - \phi(A)^2 = 0$, for all self-adjoint $A \in \mathcal{M}$. A nonabelian factor admits no dispersion free state. That is one of characteristic of noncommutative probability.

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Thank You!