# Characterizing Minkowski Functionals

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In this article, we fix a field $\mathbb{K}$ by $\mathbb{R}$ or $\mathbb{C}$ and $X$ be a $\mathbb{K}$-vector space.

**Definition 1.** For $x, y \in X$, $A \subseteq \mathbb{K}$, $\alpha \in \mathbb{K}$, and $S, T \subseteq X$,

1. $[x, y] = \{tx + (1-t)y \mid 0 \leq t \leq 1\}$,
2. $AS = \{\alpha s \mid \alpha \in \mathbb{K} \text{ and } s \in S\}$,
3. $\alpha S = \{\alpha\}S$,
4. $S + T = \{s + t \mid s \in S \text{ and } t \in T\}$

**Definition 2.** Let $S \subseteq X$. We say that the set $S$ is

$$\begin{array}{rrl} \textit{star-shaped} & (\forall s \in S, \exists s_0 \in S) & [s_0, s] \subseteq S \\ \textit{convex} & (\forall x, y \in S) & [x, y] \subseteq S \\ \textit{absorbing} & (\forall x \in X, \exists r_x > 0) & |c| \geq r_x \Rightarrow x \in cS \\ \textit{balanced} & (\forall \alpha \in \mathbb{K}) & |\alpha| \leq 1 \Rightarrow \alpha S \subseteq S \end{array}$$

**Proposition 1.** *If $B \subseteq X$ is balanced, then*

$$(\forall \alpha \in \mathbb{K}) \quad |\alpha| = 1 \Rightarrow \alpha B = B$$

*Proof.*

$$B = \alpha(\overline{\alpha}B) \subseteq \alpha B \subseteq B$$

□

**Proposition 2.** *Suppose $S \subseteq X$ is convex or balanced. If $(0, \infty)S = X$, then $S$ is absorbing.*

*Proof.* Suppose $(0, \infty)S = X$ and fix nonezero $x \in X$. Let

$$T_x = \{t > 0 \mid x \in tS\}$$

Since $(0, \infty)S = X$, $T_y$ is nonempty for each $y \in X$. Choose $r_y$ by

$$r_y = \inf T_y$$

.

$$r = \inf\{r_{\alpha x} \mid \alpha \in \mathbb{K} \text{ and } |\alpha| = 1\}$$

. Note that $r > 0$. If not $\alpha x = 0$ for some $|\alpha| = 1$. It contradicts to the fact that $x$ is nonzero.

1. If $S$ is convex, then $0 \in S$ since $0 \in r_0 S$ for some $r_0 > 0$. If $|\alpha| \geq r$,

$$\frac{|\alpha|}{\alpha} x \in rS \Rightarrow \frac{r}{|\alpha|}\frac{|\alpha|}{\alpha} x + \frac{|\alpha| - r}{|\alpha|} 0 = \frac{r}{\alpha} x \in rS \Leftrightarrow x \in \alpha S$$

   by the convexity of $S$.

2. If $S$ is balanced. If $|\alpha| \geq r$,

$$\frac{|\alpha|}{\alpha} x \in rS \Leftrightarrow x \in \alpha \left( \frac{r}{|\alpha|} S \right) \subseteq \alpha S$$

   by the balancedness of $S$.

□

**Definition 3.** Let $S \subseteq X$ and let $f : X \to [0, +\infty]$. We say that the function $g$ is

| | | |
|---|---|---|
| *positively homogeneous* | $(\forall r > 0, \forall x \in X)$ | $f(rx) = rf(x)$ |
| *absolutely homogenous* | $(\forall \alpha \in \mathbb{K}, \forall x \in X)$ | $f(\alpha x) = \lvert\alpha\rvert f(x)$ |
| *real-valued* | $(\forall x \in X)$ | $f(x) \in \mathbb{R}$ |
| *subadditive* | $(\forall x, y \in X)$ | $f(x + y) \leq f(x) + f(y)$ |

**Definition 4.** Let $S \subseteq X$. A *Minkowski functional* of $S$ is a function

$$\mu_S : X \to [0, +\infty]$$

defined by

$$\mu_S(x) = \inf\{t \in (0, \infty) \mid t^{-1} x \in S\}$$

Here we define

$$\inf \varnothing = +\infty \quad \text{and} \quad \sup \varnothing = -\infty$$

**Proposition 3.** *Let $S \subseteq X$.*

*(a) $\mu_S$ is positively homogeneous.*

*(b) $\mu_S$ is subadditive iff $(0,1)S$ is convex.*

*(c) $\mu_S$ is absolutely homogeneous iff $S$ is balanced.*

*Proof.* (a) $\forall r, t \in (0, \infty)$

$$r^{-1}x \in S \Leftrightarrow (tr)^{-1}(tx) \in S$$

So, $\mu_S(tx) = t\mu_S(x)$.

(b) Suppose that $(0,1)S$ is convex. If one of $\mu_s(x)$ and $\mu_S(y)$ is $+\infty$, then inequalrity holds. Now suppose both of them are finite. If $s > \mu_S(x)$ and $t > \mu_S(y)$ for some real numbers $s$ and $t$, then

$$\frac{x+y}{s+t} = \left(\frac{s}{s+t}\right)\frac{x}{s} + \left(\frac{t}{s+t}\right)\frac{y}{t} \in S$$

So, $\mu_S(x+y) \leq \mu_S(x) + \mu_S(y)$

Conversely, suppose that $\mu_S(x+y) > \mu_S(x)+\mu_S(y)$ for some $x, y \in X$. Then, both $s^{-1}x$ and $t^{-1}y$ in $X$ but $(s+t)^{-1}(x+y) \notin X$ for some $s, t \in (0, \infty)$ So,

$$\frac{x+y}{s+t} = \left(\frac{s}{s+t}\right)\frac{x}{s} + \left(\frac{t}{s+t}\right)\frac{y}{t} \notin S$$

This shows $(0,1)S$ is not convex.

(c) Suppose that $S$ is balanced.

For any $\alpha \in \mathbb{K}$, therefore,

$$\mu_S(\alpha x) = \mu_S\left(|\alpha|\frac{\alpha}{|\alpha|}x\right) = |\alpha|\mu_S\left(\frac{\alpha}{|\alpha|}x\right) = |\alpha|\mu_S(x)$$

the last equality holds by the proposition 1

Conversely, if $S$ is not balanced, then $\alpha S \nsubseteq S$ for some $0 < |\alpha| \leq 1$. So, there exists $x \in S$ and $r \in (\mu_S(x), \infty)$

$$r^{-1}\alpha x \notin S$$

Hence,

$$\mu_S(x) < \mu_S(\alpha x)$$

□

**Proposition 4.** *Let $f : X \to [0, +\infty]$ be any function and $S \subseteq X$ be any subset. The following statements are equivalent:*

1. *$f$ is positive homogeneous, $f(0) = 0$, and*

$$A = \{x \in X \mid f(x) < 1\} \subseteq S \subseteq B = \{x \in X \mid f(x) \leq 1\}$$

2. *$f = \mu_S$, $S$ contains the origin, and $S$ is star-shpaed at the origin.*

**Theorem 1.** *Let $S \subseteq X$. Then $\mu_S$ is a seminorm on $X$ iff all of the following conditions hold.*

1. *$(0, \infty)S = X$ (or equivalently, $\mu_S$ is real-valued).*

2. *$(0, 1)S$ is convex (or euivalently, $\mu_S$ is subadditive).*

3. *$(0, 1)\alpha S \subseteq (0, 1)S$ for all $\alpha \in \mathbb{K}$ s.t. $|\alpha| = 1$.*

*Conversley, if $p$ is a seminorm on $X$ then the set*

$$V = \{x \in X : f(x) < 1\}$$

*satisfies all three of the above conditions and also, $p = \mu_V$; moreover, $V$ is necessarily convex, balanced, absorbing and satisfies*

$$(0, 1)V = V = [0, 1]V$$

**Corollary 1.** *If $A \subseteq X$ is convex, balanced, and absorbing, then $\mu_A$ is a seminorm on $X$.*

**Definition 5.** A positive *sublinear function* is a positive homogeneous subadditive function $f : X \to [0, \infty)$.

**Theorem 2.** *Suppose $X$ is a topological vector space. Then the nonempty open convex subsets of $X$ are exactly those sets that are of the form*

$$z + \{x \in X \mid p(x) < 1\} = \{x \in X \mid p(x - z) < 1\}$$

*for some $z \in X$ and some positive continuous sublinear function $p$ on $X$.*