Gleason's Theorem and Quantum Logic

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Introduction

In 1932, John von Neumann noticed that the projection operators of Hillbert space could be viewed as quantum mechanical propositions about observables [9]. The principles governing these quantum propositions were then called quantum logic by von Neumann and Birkhoff [2]. Von Neumann also managed to dervie the Born rule in his textbook [9]. However, his assumptions were regarded to not be well-motivated by John Bell [1], [3].

By the late 1940s, George Mackey was wondering whether the Born rule was the only possible rule for calculating probabilities in a theory that represented measurements as orthonormal bases on a Hilbert space. In his investigation of mathematical foundation of quantum mechanics, Mackey had proposed the following problem [8, p.50-51][4, p.129]: Determine all measures on the closed subspaces of a Hilbert space. Gleason provided the answer to this problem in the same year [5], which is later called Gleason's theorem. Gleason succeeded in decribing all σ-additive probability measures on the logical of all closed subspaces of a separable Hilbert space and in showing that, except for the obvious two-dimensional counterexamplaes, all probability measures can be identified with normal states in the sense of von Neumann approach. Gleason's achievement confirmed von Neumann's original insight and put the calculus of Hilbert space quantum mechanics on natural physical grounds [6, p.87-88]. Gleason's theorem is of particular importance for the field of quantum logic and its attempt to find a minimal set of mathematical axioms for quantum theory.

After a considerable effort the Gleason's theorem was established in the early 90's for finitely additive vector measures on the projection lattices of von Neumann algebras. It has turned out that the lattice homomorphisms on nonabelian von Neumann algebras are σ-additve, or that finitely additive measures on projection lattices whose kernels are lattice ideal enjoy many continuity properties [6, p.3-4].

The present paper is intended to serve as an introduction to topic of the relation between Gleason's theorem and the theory of quantum logic. Most of the contents were referred to chapters 1, 2, 3 and 5 of [4] and chapters 1, 2, 3, and 9 of [6].

Chapter 1

Preliminaries

1.1 Elements of Operator Algebras

A C^*-algebra A is a Banach algebra over the field $\mathbb{C}$ together with a map $* : A \to A$, $a \mapsto a^*$ which following properties:

1. $a^{**} = a$ for all $a \in A$,
2. $(a+b)^* = a^* + b^*$ for all $a, b \in A$,
3. $(ab)^* = b^* a^*$ for all $a, b \in A$,
4. $(\lambda a)^* = \bar{\lambda} a^*$ for all $\lambda \in \mathbb{C}$ and $a \in A$,
5. $\|aa^*\| = \|a\| \, \|a^*\|$ for all $a \in A$.

A bounded linear map $\pi : A \to B$ between C^*-algebras A and B is called a $*$-homomorphism if

1. $\pi(ab) = \pi(a)\pi(b)$
2. $\pi(a^*) = \pi(a)^*$

for all $a, b \in A$.

Let A be a C^*-algebra. A *state* in C^*-algebra is a positive functional ρ which is $\|\rho\| = 1$. We shall denote the set of all states on A by $S(A)$ which is called the *state space* of A.

A $*$-representation of a C^*-algebra A on a Hilbert space H is a ring homomorphism $\pi : A \to B(H) = \{f \mid f \text{ is a bounded linear operator on } H\}$ such that

1. $\pi(a^*) = \pi(a)^*$ for all $a \in A$,
2. π is nondegenerate.

Theorem 1 (GNS representation)**.** *Given a state ρ of A, there is a $*$-representation π of A acting on a Hilbert space H with unit cylic vector ψ such that*

$$\rho(a) = \langle \pi(a)\psi, \psi \rangle$$

for every a in A. Moreover, the representation is unique up to unitary equivalence.

Proof. [7, Theorem 4.5.2] $\square$

Let X be an inner product space. A *spilitting subspace* of X is the subspace M of X which is the following property holds,

$$M + M^{\perp} = S.$$

Then any vector $x \in X$ can be uniquely expressed in the form

$$x = x_M + x_{M^{\perp}},$$

where $x_M \in M$ and $x_{M^{\perp}} \in M^{\perp}$. We denote by $E(X)$ the set of all splitting subspaces. The map $P_M : X \to X$ such that $P_M(x) = x_M$ for all $x \in X$ is a bounded linear operator with $\|P_M\| = 1$ whenever $M \neq \{0\}$. Moreover, $P_M^2 = P_M$, P_M is self-adjoint, and $\operatorname{ran} P_M = M$. The operator P_M is called the *orthoprojector* or *projection* from X onto M. It is clear that $I - P_M$ is an orthoprojector onto $M^{\perp}$.

1.2 Quantum Logic

Chapter 2

Gleason's Theorem

2.1 Gleason's Theorem

Example 1. Let H be a two-dimensional, real or complex Hilbert space. If

$$\mu(M) = \begin{cases} 1 & \text{if } M = H \\ 0 & \text{if } M = 0 \\ 0 \text{ or } 1 & \text{if } \dim M = 1 \end{cases}$$

where in the lattercase wse suppose $\mu(M) + \mu(M^{\perp}) = 1$, then μ is a two-valued state on $L(H)$. [4, Example 3.1.1]

Theorem 2 (Gleason's Theorem)**.** *Let H be a separable, real or complex Hilbert space, $\dim H \neq 2$, then for any state μ on $L(H)$ there exists a unique positive Hermitian trace operator T on H with $\operatorname{tr} T = 1$ such that*

$$\mu(M) = \operatorname{tr}(TP_M)$$

for all $M \in L(H)$.

Proposition 1. *Let $\{e_1, \ldots, e_n\}$ is a finite orthonormal system in X. Then $M = \{\operatorname{sp}(e_1, \ldots, e_n) \in E(S)\}$ and*

$$P_M(x) = \sum_{i=1}^{\infty} (x, e_i) e_i$$

for all $x \in X$.

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