Quantum Logic

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Part I

Lattice Theory

Chapter 1

Lattices

1.1 Posets and Lattices

Definition 1. A poset L is a *lattice* if

$$\forall a, b \in L,\ \sup\{a, b\} \in L \text{ and } \inf\{a, b\} \in L$$

Definition 2. Let A be a nonempty set. An *n-ary operation* f on the A is a map $A^n \to A$. We define $A^0 = \emptyset$.

number	name
$n = 0$	nullary operation
$n = 1$	unary opeartion
$n = 2$	binary operation
$\vdots$	$\vdots$

Definition 3. A *universal algebra*, or simply *algebra*, consists of a nonempty set A and a set F of operations; each $f \in F$ is an n-ary operation for some n (depending on f). We denote this algebra by $\mathfrak{A}$ or $(A; F)$.

A *type* τ of algebras is a sequence $(n_0, n_1, \dots, n_\gamma, \dots)$ of nonnegative integers, $\gamma < o(\tau)$, where $o(\tau)$ is an ordinal called the *order* of τ. An algebra $\mathfrak{A}$ of type τ is an ordered pari $(A; F)$, where A is a nonempty set and F is a sequence $(f_0, \dots, f_\gamma, \dots)$, where f_γ is an n_γ-ary operation on A for $\gamma < o(\gamma)$.

Definition 4. Let $A; \circ$ be an alebra of type (2). If

(Idem)	Idempotent:	$a \circ a = a$
(Comm)	Commutativity:	$a \circ b = b \circ a$
(Assoc)	Associativity:	$(a \circ b) \circ c = a \circ (b \circ c)$

holds, then we call $(L, \circ)$ a *semilattice*.

Definition 5. Let $(L, \wedge, \vee)$ be an algebra of type (2,2) is called a *lattice* if L is a nonempty set, $(L; \vee)$ and $(L; \wedge)$ are semilattices, and

(Asorp) Absorption: $\forall a, b \in L,\ a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

Theorem 1. *1. Let the poset* $\mathfrak{L} = (L; \leq)$ *be a lattice. Set*

$$\begin{aligned} a \vee b &= \sup\{a, b\}, \\ a \wedge b &= \inf\{a, b\}. \end{aligned}$$

Then the algebra $\mathfrak{L}^{\mathrm{alg}} = (L; \vee, \wedge)$ *is a lattice.*

2. Let the algebra $\mathfrak{L} = (L; \vee, \wedge)$ *be a lattice. Set*

$$a \vee b = b \Rightarrow a \leq b$$

Then $\mathfrak{L}^{\mathrm{ord}}$ *is a poset, and the* $\mathfrak{L}^{\mathrm{ord}}$ *is a lattice.*

3. Let the poset $\mathfrak{L}^{\mathrm{ord}} = (L; \leq)$ *be a lattice. Then* $(\mathfrak{L}^{\mathrm{alg}})^{\mathrm{ord}} = \mathfrak{L}$.

4. Let the algebra $\mathfrak{L} = (L; \vee, \wedge)$ *be a lattice. Then* $(\mathfrak{L}^{\mathrm{ord}})^{\mathrm{alg}} = \mathfrak{L}$.