Summary of Analysis

Changjae Lee

August 4, 2024

Contents

Part I

Measure and Integration

Chapter 1

Foundations

1.1 Sets

Let A be a nonemptyset and R be a binary relation on A.

Refl Reflexive: $\forall a \in A$, aRa

AntiRefl Antireflexive: $\forall a \in A$, $\neg aRa$

Sym Symmetric: $\forall a, b \in A$, $aRb \Rightarrow bRa$

AntiSym Antisymmetic: $\forall a, b \in A$, aRb and $bRa \Rightarrow a = b$

Asym Asymmetric: $\forall a, b \in A$, $aRb \Rightarrow \neg bRa$

Trans Transitive: $\forall a, b \in A$, aRb and $bRc \Rightarrow aRc$

Total Total: $\forall a, b \in A$, $a \neq b \Rightarrow aRb$ or bRa

Well-founded $\exists m \in A, \forall a \in A$, mRa

Name	Definition
equivalence	**Refl** + **Sym** + **Trans**
preorder	**Refl** + **Trans**
partial order	**Refl** + **AntiSym** + **Trans**
total order	**Refl** + **AntiSym** + **Trans** + **Total**
well-ordering	**Refl** + **Sym** + **Trans** + **Total** + **Well-founded**

Table 1.1: List of transitive relations.

proset The pair $(A, \precsim)$ is called **preordered set (proset)** if $\precsim$ is a preorder on A.

poset The pair $(A, \preccurlyeq)$ is called **partially ordered set (poset)** if $\preceq$ is a partial order on A.

toset The pair $(A, \preceq)$ is called **totally ordered set (toset)** if $\preceq$ is a total order on A.

woset The pair $(A, \leq)$ is called **well-ordered set (woset)** if $\leq$ is a well-ordering on A.

1.2 Systems of Sets

1.3 Nets and Filters

Definition 1.3.1 (directed set)**.** Let $(J, \precsim)$ be a proset.

(upward) directed $\forall a, b \in J, \exists c \in J,\ a \precsim c$ and $b \precsim c$.

downward directed $\forall a, b \in J, \exists c \in J,\ c \precsim a$ and $c \precsim b$.

Definition 1.3.2. A **net** $x_\bullet = (x_\alpha)_{\alpha \in J}$ in X is a function $x_\bullet : J \to X$ where J is a directed set.

1.4 Topological Properties

Chapter 2

Measures

2.1 Abstract Measures

If Σ is a σ-algebra on X, then (X, Σ) is called a **measurable space**.

Definition 2.1.1. A **measure** on (X, Σ) is a function $\mu : \Sigma \to [0, \infty]$ s.t.

1. $\mu(\varnothing) = 0$
2. (countable additivity) If $(E_j)_1^\infty$ disjoint sequence in Σ, then

$$\mu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j)$$

If μ is a measure on (X, Σ), then (X, Σ, μ) is called **measure space**.

Definition 2.1.2. Let (X, Σ, μ) is a measuable space. Then μ is

finite $\mu(X) < \infty$

σ-finite $\exists\, (E_j)_{j=1}^{\infty} \in \mathrm{Seq}(\Sigma)$, $X = \bigcup_{j=1}^{\infty} E_j$ and $\mu(E_j) < \infty$

semifinite $\forall E \in \Sigma, \exists F \in \Sigma$, $F \subset E$ and $0 < \mu(F) < \infty$

Iverson bracket Let P be a statement,

$$[\![P]\!] = \begin{cases} 1, & P \text{ is true} \\ 0, & \text{otherwise} \end{cases}$$

Then the **indicator function** of a subset A of a set X is a function $\mathbb{1}_A : X \to \{0, 1\}$ defined as

$$\mathbb{1}_A(x) = [\![x \in A]\!] = \begin{cases} 1, & x \in A \\ 0, & \text{otherwise} \end{cases}$$

Example 2.1.1. Let X be any nonempty set

- The **counting measure** $\mu : \mathcal{P}(X) \to [0, \infty]$ is defined by

$$\mu(E) = \begin{cases} |E|, & E \text{ is finite} \\ \infty, & \text{otherwise} \end{cases}$$

- $\forall (X, \Sigma), \forall x \in X$, a **Dirac measure** $\delta_x : \Sigma \to [0, \infty]$ defined by

$$\delta_x(E) = \mathbb{1}_E(x) = [\![x \in E]\!]$$

2.2 Borel Measures

2.3 Outer Measures

Definition 2.3.1. An **outer measure** on a nonempty set X is a function $\mu^* : \mathcal{P}X \to [0, \infty]$ s.t.

1. $\mu^*(\varnothing) = 0$
2. $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$
3. $\mu^*(\bigcup_{j=1}^{\infty} A_j) \leq \sum_1^{\infty} \mu^*(A_j)$

2.4 Complete Measures

Chapter 3

Abstract Integration

3.1 Measurable Functions

3.2 Integration of Nonnegative Functions

3.3 Integration of Complex Functions

3.4 Product Measures

Chapter 4

Signed Measure and Differentiation

4.1 Signed Measures

4.2 Complex Measures

4.3 Differentiation on Euclidean Space

4.4 Bounded Variations

Part II

Functional Analysis

Throughout this part $\mathbb{F}$ will denote either $\mathbb{C}$ or $\mathbb{R}$

Chapter 5

Topological Vector Spaces

Chapter 6

Normed Vector Spaces

Chapter 7

Banach Spaces

Chapter 8

Hilbert Spaces

8.1 pre-Hilbert Spaces

Let $\mathcal{X}$ be a $\mathbb{F}$-vector space. Consider some conditions on $u : \mathcal{X} \times \mathcal{X} \to \mathbb{F}$;

(a) $u(\alpha x + \beta y, z) = \alpha u(x, z) + \beta u(y, z)$

(b) $u(x, y) = \overline{u(y, x)}$

(c) $u(x, x) \geq 0$

(d) $u(x, x) = 0 \Rightarrow x = 0$

If u satisfies (a)-(c), then we call u a **semi-inner product** on $\mathcal{X}$. An **inner product** is a semi-inner product that also satisfies (d) and denote it by $\langle \cdot, \cdot \rangle$.

Theorem 1 (The Cauchy-Bunyakowsky-Schwart Inequality)**.** *If u is a semi-inner product on $\mathcal{X}$, then*

$$\forall x, y \in \mathcal{X},\ |u(x, y)|^2 \leq u(x, x)u(y, y)$$

Moreover, equality occurs iff

$$\exists \alpha, \beta \in \mathbb{F},\ u(\alpha x + \beta y, \alpha x + \beta y) = 0$$

Index