(1) $T(n) = 3T(n-1) + 12n$ Given

$T(0) = 5$

$\rightarrow T(1) = 3T(0) + 12 \times 1$

$T(1) = 3 \times 5 + 12 \times 1$

$T(1) = 15 + 12 = 27$

$\rightarrow T(2) = 3T(1) + 12 \times 2$

$T(2) = 3 \times 27 + 24$

$T(2) = 81 + 24$

$T(2) = 105$

(2) (a) $T(n) = T(n-1) + c$ ...... (1)

Putting $n-1$ in place of $n$ in eqn (1):-

$$T(n-1) = T(n-2) + c \quad \text{— (2)}$$

Putting eqn (2) in eqn (1):-

$$T(n) = T(n-2) + c + c$$

$$T(n) = T(n-2) + 2c \quad \text{— (3)}$$

Putting $n-2$ in place of $n$ in eqn (1):-

$$T(n-2) = T(n-3) + c \quad \text{— (4)}$$

Putting eqn (4) in eqn (3):-

$$T(n) = T(n-3) + c + 2c$$

$$T(n) = T(n-3) + 3c \quad \text{— (5)}$$

So, the general solution is :-

$$T(n) = T(n-i) + ic$$

[putting $i$ in eqn (5)]

$\Rightarrow T(n-i) = T(0)$ [$i$ is a constant]

$\Rightarrow n - i = 0$

$n = i$

Now, the general equation becomes :-

$$T(n) = T(i-i) + nc$$

$$T(n) = T(0) + cn$$

$$T(n) = O(n)$$

Time complexity

(b) $T(n) = 2T(n/2) + n$ — (1)

Putting $n/2$ in place of $n$ in eqn (1):—

$$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + \frac{n}{2} \quad -(2)$$

Putting eqn. (2) in eqn. (1):—

$$T(n) = 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n$$

$$T(n) = 4T\left(\frac{n}{4}\right) + n + n$$

$$T(n) = 2^2 T\left(\frac{n}{4}\right) + 2n \quad \ldots\ldots (3)$$

Putting $\frac{n}{4}$ in place of $n$ in eqn (1):—

$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + \frac{n}{4}$$

$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + \frac{n}{4} \quad — (4)$$

Putting eqn (4) in eqn (3):—

$$T(n) = 2^2\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + 2n$$

$$T(n) = 8T\left(\frac{n}{8}\right) + n + 2n$$

$$T(n) = 2^3 T\left(\frac{n}{8}\right) + 3n \quad -(5)$$

So, the general solution is:—

$$T(n) = 2^i T\left(\frac{n}{2^i}\right) + in$$

$\Rightarrow \dfrac{n}{2^i} = 1$ [$i$ is a constant]

$\Rightarrow n = 2^i$

$\Rightarrow i = \log_2 n$.

So, new general equation becomes,

Assuming [$T(1)=1$]

$T(n) = nT(1) + n\log_2 n$

$T(n) = n + n\log_2 n$

Time Complexity = $O(n\log n)$.

(c) $T(n) = 2T(n/2) + c \quad \cdots\cdots (1)$

Putting $\frac{n}{2}$ in place of $n$ in eqn (1):-

$$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + c \quad — (2)$$

Putting eqn (2) in eqn (1):-

$$T(n) = 2\left[2T\left(\frac{n}{4}\right) + c\right] + c$$

$$T(n) = 2^2 T\left(\frac{n}{4}\right) + 2c + c \quad — (3)$$

Putting $\frac{n}{4}$ in eqn. (1):-

$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + c \quad — (4)$$

Putting eqn (4) in eqn (3):-

$$T(n) = 2^2\left[2T\left(\frac{n}{8}\right) + c\right] + 2c + c$$

$$T(n) = 2^3 T\left(\frac{n}{8}\right) + 2^2 c + 2c + c$$

$$T(n) = 2^3 T\left(\frac{n}{2^3}\right) + 2^2 c + 2c + c$$

$\vdots$

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + kc$$

Assume:-

$\Rightarrow T\left(\frac{n}{2^k}\right) = T(1)$

$\Rightarrow \frac{n}{2^k} = 1$

$\Rightarrow n = 2^k$

$\Rightarrow k = \log_2 n$

Assuming $T(1)=1$, we get,

$$T(n) = n\,T(1) + \log_2 n\, c$$

$$T(n) = n \times 1 + c\log_2 n$$

$$T(n) = n + c\log_2 n$$

$$T(n) = O(n\log_2 n)$$

Time Complexity.

(d) $T(n) = T\left(\frac{n}{2}\right) + c$ — (1)

Putting $n/2$ in place of $n$ in eqn (1):-

$$T(n/2) = T\left(\frac{n}{4}\right) + c \quad \text{— (2)}$$

Putting eqn (2) in eqn (1):-

$$T(n) = T\left(\frac{n}{4}\right) + c + c$$

$$T(n) = T\left(\frac{n}{4}\right) + 2c \quad \text{— (3)}$$

$$T(n) = T\left(\frac{n}{2^2}\right) + 2c$$

Putting $n/4$ in place of $n$ in eqn (1):-

$$T\left(\frac{n}{4}\right) = T\left(\frac{n}{8}\right) + c \quad \text{— (4)}$$

Putting eqn (4) in eqn (3):-

$$T(n) = T\left(\frac{n}{8}\right) + c + 2c$$

$$T(n) = T\left(\frac{n}{2^3}\right) + 3c \quad \text{— (5)}$$

Now, the general solution is :-

$$T(n) = T\left(\frac{n}{2^i}\right) + ic \quad \text{— (6)}$$

$$T\left(\frac{n}{2^i}\right) = T(1) \qquad [i \text{ is a constant}]$$

$$\frac{n}{2^i} = 1$$

$$n = 2^i$$

$$i = \log_2 n \quad \text{— (7)}$$

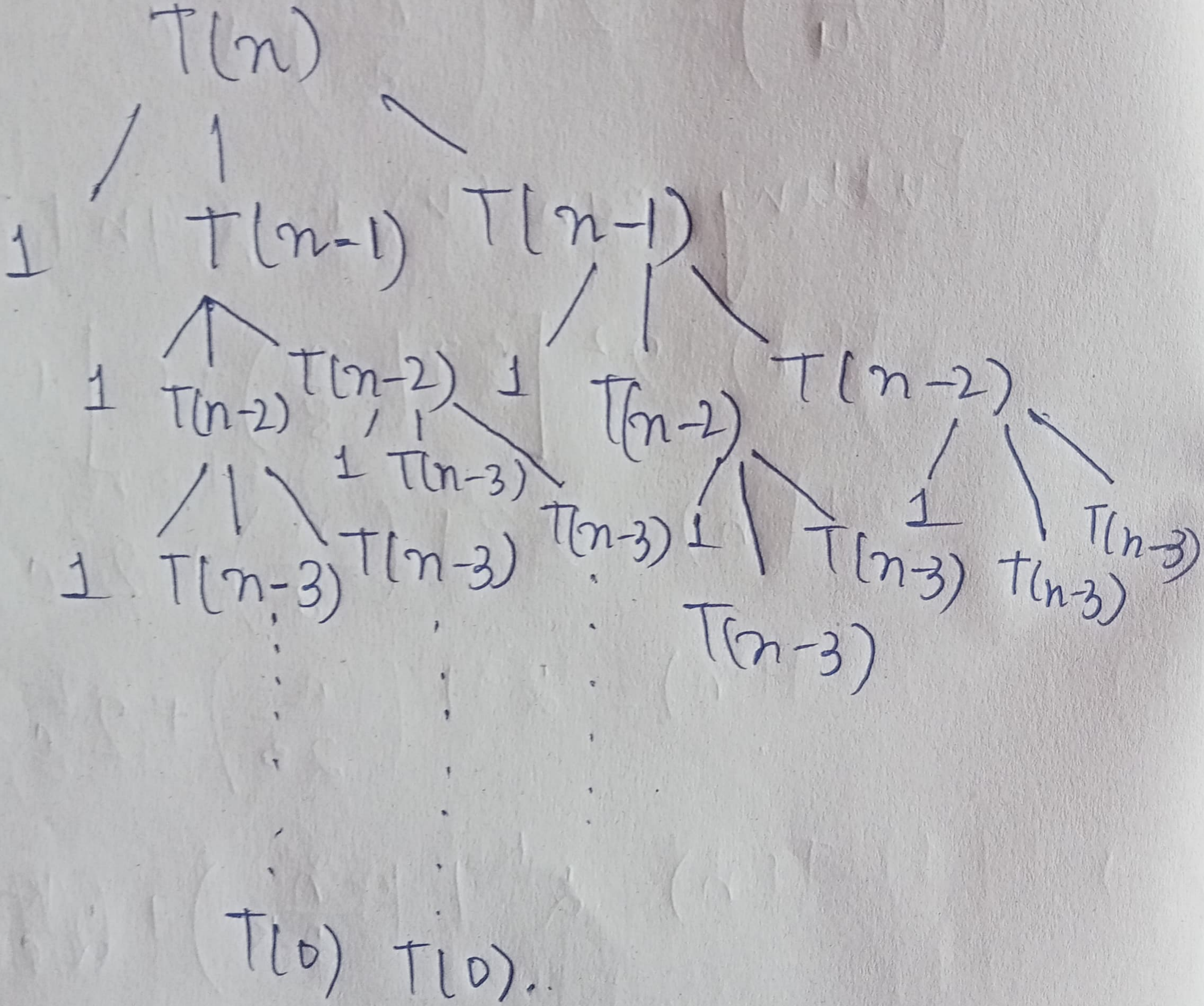
So, new general equation becomes:–

Substituting eqn ($\overset{7}{\cancel{6}}$) in eqn (6):–

$$T(n) = T(1) + \log_2 n \, c$$

$$T(n) = K + c\log_2 n$$ [K is a constant]

Time complexity $T(n)$ is $O(\log_2 n)$.


T(n)
1
T(n-1)
T(n-1)
1
T(n-2)
T(n-2)
1
T(n-2)
T(n-2)
1
T(n-3)
1
T(n-3)
T(n-3)
T(n-3)
1
T(n-3)
1
T(n-3)
T(n-3)
T(n-3)
T(n-3)
T(0)
T(0)

3) a) $T(n) = 2T(n-1) + 1$ — (1) $T(n) = \begin{cases} 1 & n = 0 \\ 2T(n-1)+1 & n > 0 \end{cases}$

$T(n) = 2[2T(n-2) + 1] + 1$

$T(n) = 2^2 T(n-2) + 3$

$T(n) = 2^2 T(n-2) + 2 + 1$ — (2)

$T(n) = 2^2 [2T(n-3) + 1] + 2 + 1$

$T(n) = 2^3 T(n-3) + 2^2 + 2 + 1$ ..... (3)

⋮

$T(n) = 2^k T(n-k) + 2^{k-1} + 2^{k-2} + \dots + 2^2 + 2 + 1$ — (4)

$\rightarrow n - k = 0$

$n = k$. [k is a constant]

Putting $n = k$ in eqn (4)

$T(n) = 2^n T(0) + 2^{n-1} + 2^{n-2} + \dots + 2^2 + 2 + 1$

$= 2^n T(0) + 1 + 2 + 2^2 + \dots + 2^{n-1}$

$= 2^n \times 1 + 2^n - 1$

$= 2^n + 2^n - 1$

$= 2^n(1+1) - 1$ $\left[ 1 + 2 + 2^2 + \dots + 2^{n-1} = 2^n - 1 \right]$

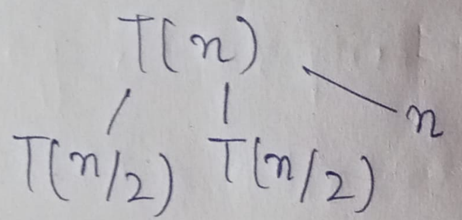
$= 2^n \cdot 2^1 - 1$

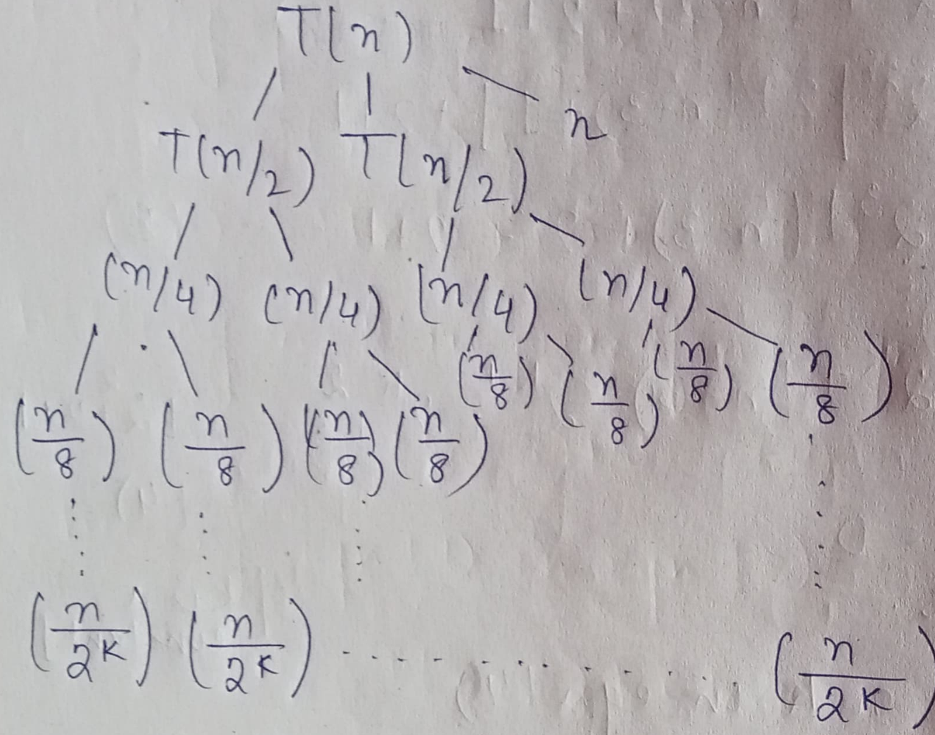
$\Rightarrow 2^{n+1} - 1$.

Time Complexity $T(n) = O(2^n)$

(b) $T(n) = 2T(n/2) + n$

$$T(n) = \begin{cases} 1 & n = 1 \\ 2T\left(\frac{n}{2}\right) + n, & n > 1 \end{cases}$$




Here k steps and each steps n time :-

$n \longrightarrow$ n times

$\left(\frac{n}{2}\right) + \left(\frac{n}{2}\right) \longrightarrow$ n times

$\left(\frac{n}{4}\right) + \left(\frac{n}{4}\right) + \left(\frac{n}{4}\right) + \left(\frac{n}{4}\right) \rightarrow$ n times.

Solve:

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$= 2\left[2T\left(\frac{n}{4}\right) + \frac{n}{2}\right] + n$$

$$T(n) = 2^2 T(n/2^2) + 2n$$

$$T(n) = 2^3 T\left(\frac{n}{2^3}\right) + 3n$$

$$\vdots$$

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + kn$$

$T\left(\frac{n}{2^k}\right) = T(1)$ [K is a constant]

$\frac{n}{2^k} = 1$

$n = 2^k$

$k = \log_2 n$

$T(n) = 2^k \, T(1) + kn$

$= n \times T(1) + n \log_2 n$

$\Rightarrow n + n \log_2 n$

Time complexity $T(n)$ $\Rightarrow O(n \log n)$