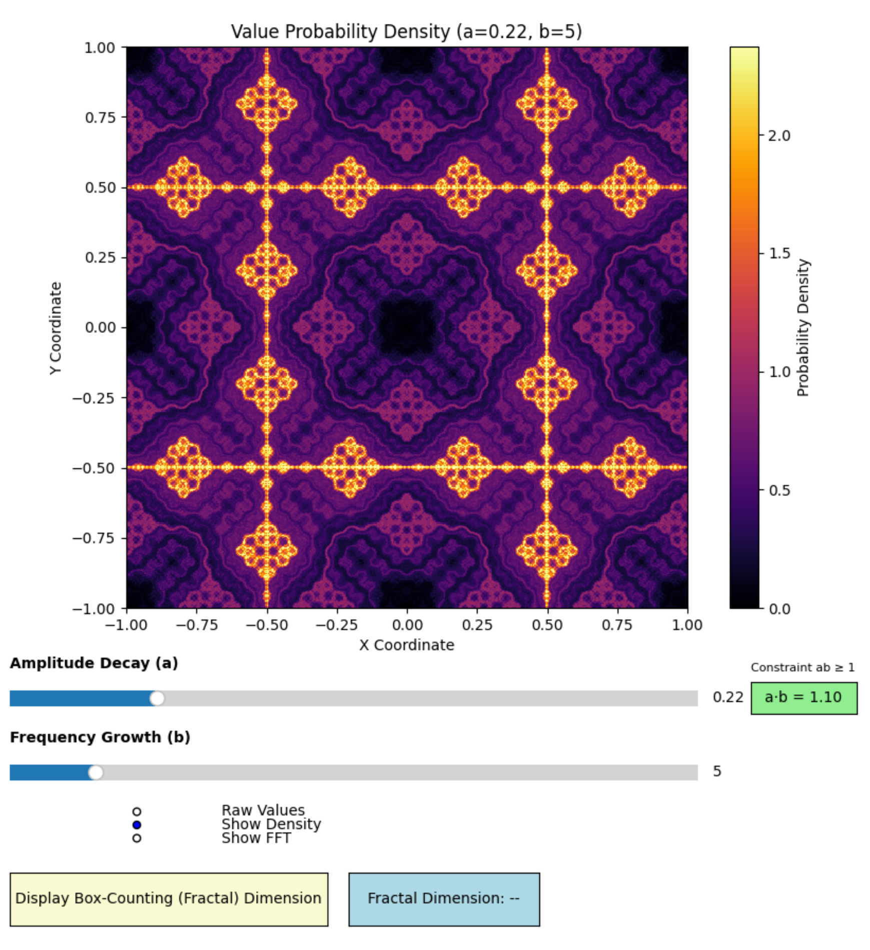
2D-Weierstrass Function Visualization Toolkit



📌 Overview

This interactive Python tool visualizes the **2D Weierstrass function**—a fascinating extension of the classic 1D function known for being **continuous everywhere but differentiable nowhere**. The application provides multiple perspectives on this mathematical curiosity, including:

1. Raw function visualization (2D and 1D)
2. Density distribution mapping (2D)
3. Frequency spectrum analysis (FFT for both 2D and 1D)
4. Fractal dimension calculation (2D)

🚀 Key Features

- **Interactive controls** for parameters *a* (amplitude decay) and *b* (frequency scaling)
- **Multiple visualization modes**:
  - Raw function values (2D and 1D)
  - Value density distribution (2D)
  - Fast Fourier Transform (FFT) spectrum (2D and 1D)
- **Box-counting dimension** calculation for fractal analysis
- **Real-time updates** via Numba-accelerated computation
- **Visual heuristic** indicating when fractal behavior emerges (a·b ≥ 1)

📐 Mathematical Foundation

The 2D Weierstrass function is defined as:

$$W(x,y)=\sum_{n=0}^{N} a^n\cdot\cos(\pi b^n x)\cdot\cos(\pi b^n y)$$

Where:

- $a\in(0,1)$ controls **amplitude decay**
- $b\in\{3,5,7,\dots\}$ (odd integers) controls **frequency growth**
- $N=20$ is the number of terms used for approximation

The 1D slice, typically taken at $x=0$, simplifies to:

$$W(0,y)=\sum_{n=0}^{N} a^n\cdot\cos(0)\cdot\cos(\pi b^n y)=\sum_{n=0}^{N} a^n\cdot\cos(\pi b^n y)$$

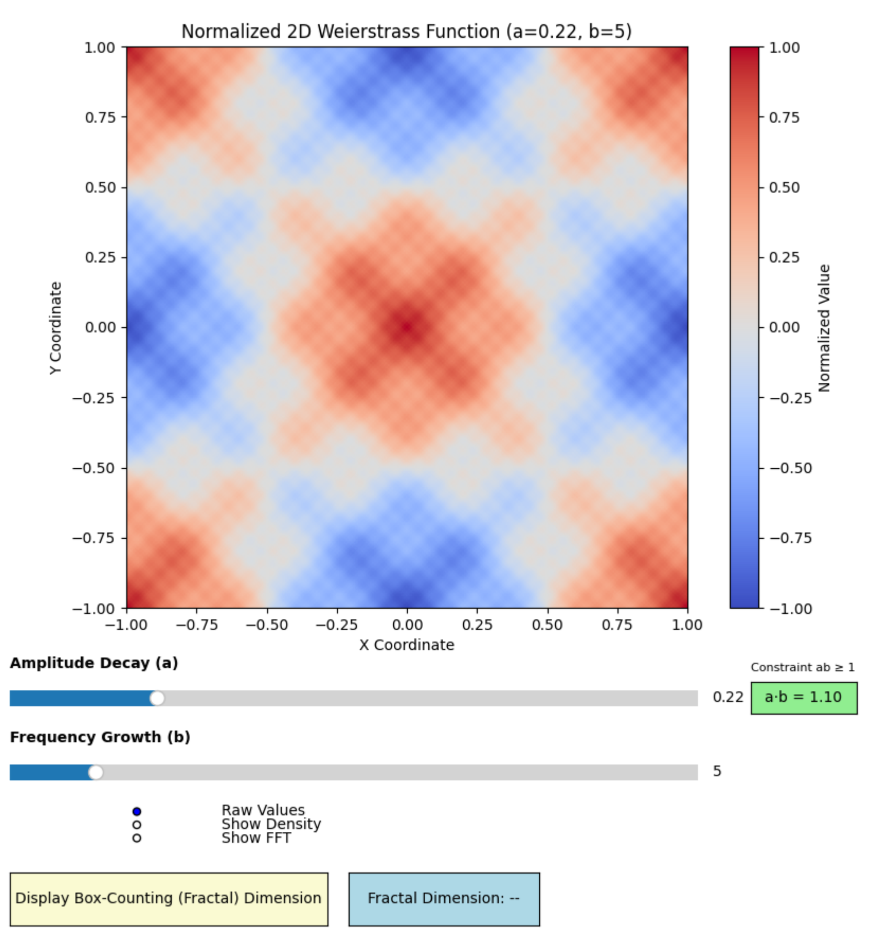
This highlights that the 1D function is a sum of cosine waves with geometrically increasing frequencies and exponentially decreasing amplitudes.

🔍 Visualization Modes

2D Visualizations

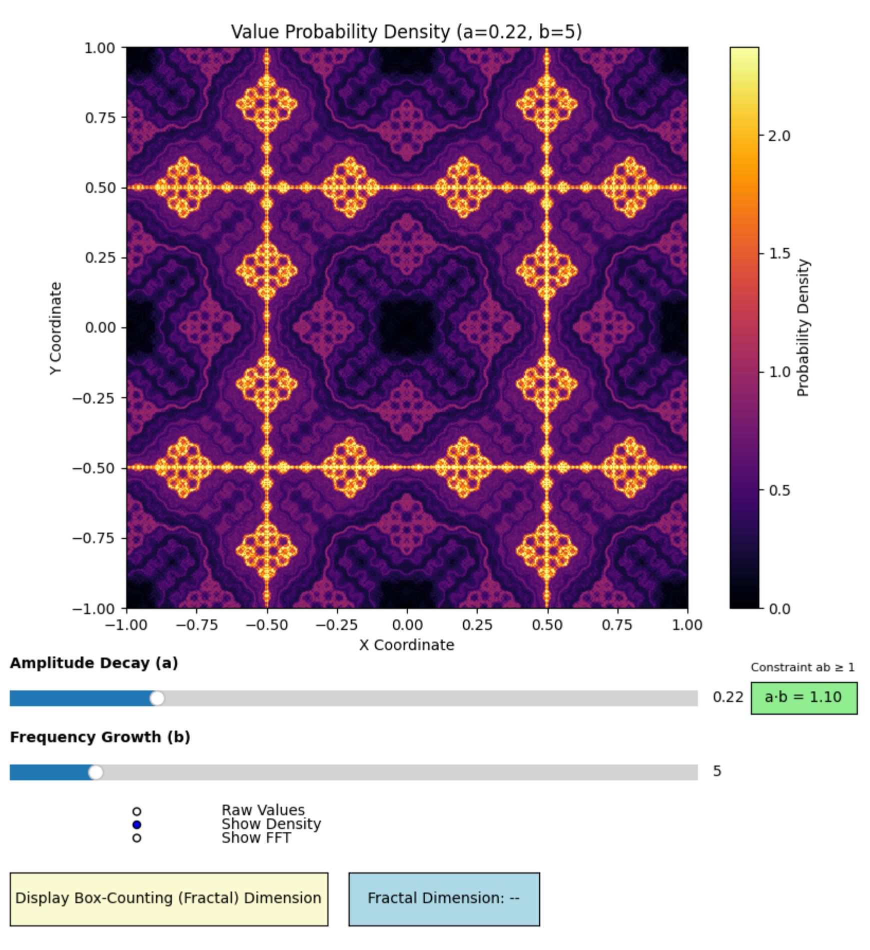
These modes apply to the full 2D function surface.

1. Raw Function Values



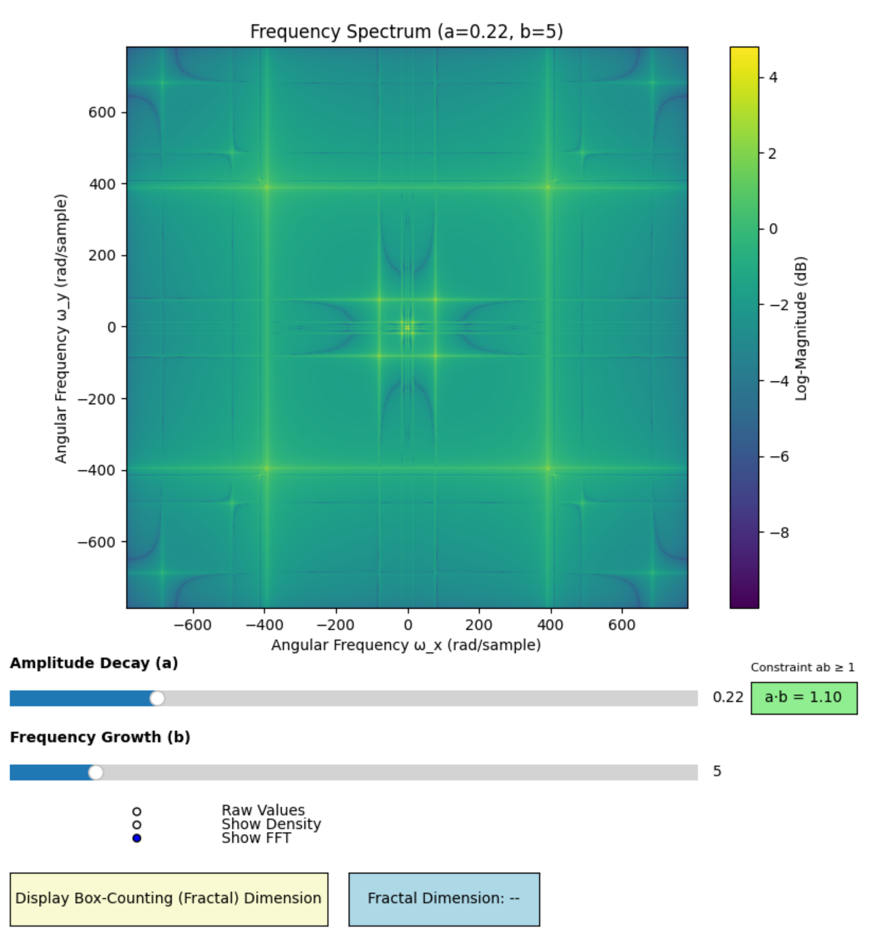
- **X/Y Axes**: Spatial coordinates in [-1, 1] range
- **Color**: Normalized function value (blue = negative, red = positive)
- **Title**: "Normalized 2D Weierstrass Function"
- Shows actual output of the mathematical function

2. Density Approximation



- **X/Y Axes**: Spatial coordinates in [-1, 1] range
- **Color**: Probability density of values
- **Title**: "Value Probability Density"
- Reveals value distribution independent of location

3. FFT Spectrum (2D)

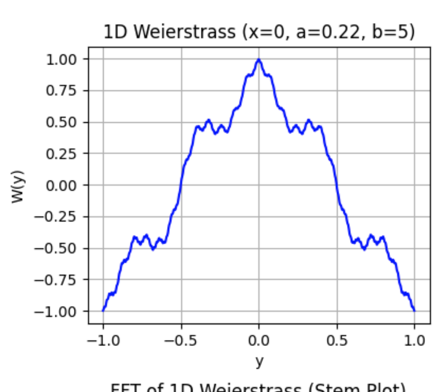


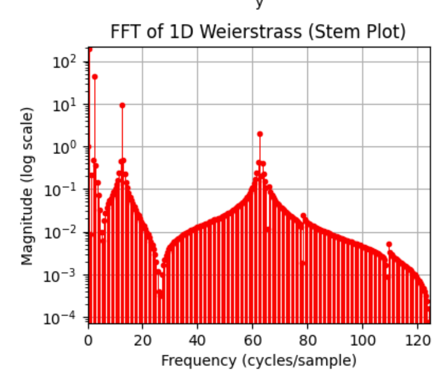
- **X/Y Axes**: Angular frequency (rad/sample)
- **Color**: Log-magnitude (dB scale)
- **Title**: "Frequency Spectrum"
- Shows dominant spatial frequencies and orientations present in the 2D surface. For the Weierstrass function, you'll observe energy concentrated along axes and diagonals, reflecting its separable cosine components.

1D Visualization

This section focuses on a 1D slice of the Weierstrass function (specifically, $x=0$) and its frequency content.

1. 1D Weierstrass Function (x=0)





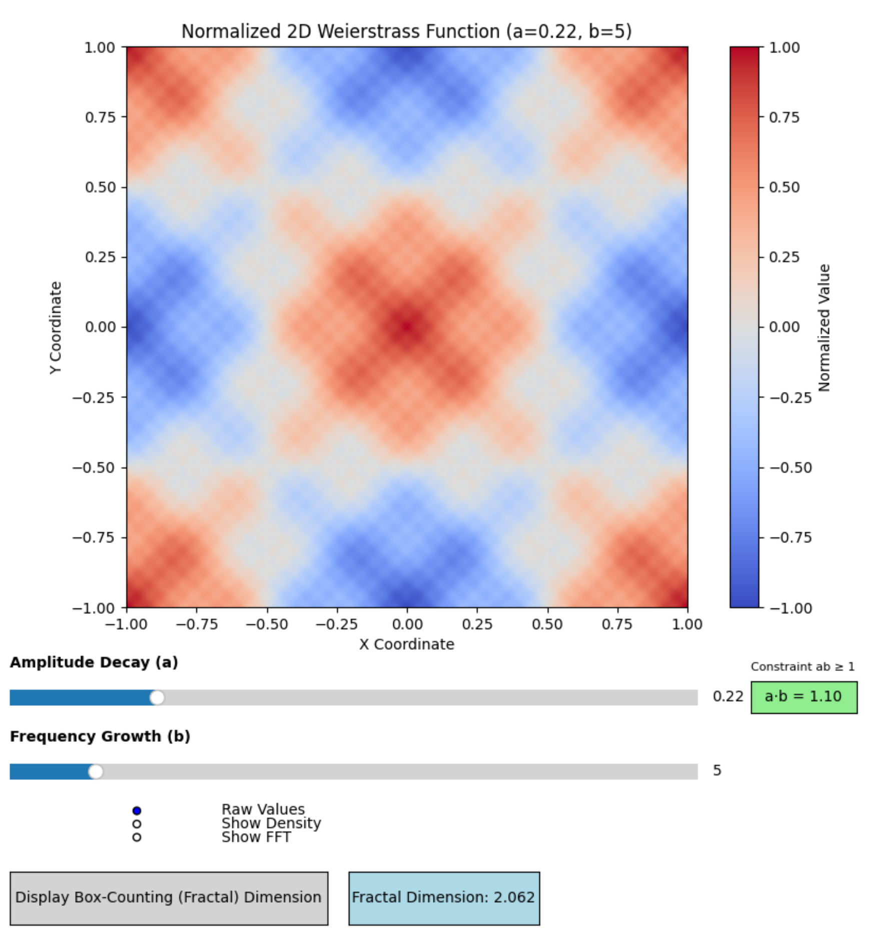
- **X-axis**: Spatial coordinate y
- **Y-axis**: Normalized function value `W(0, y)`
- **Title**: "1D Weierstrass Function (x=0)"
- Displays a cross-section of the 2D surface, revealing the intricate, non-differentiable oscillations characteristic of the function. This plot directly shows the summation of cosine waves at various frequencies.

2. FFT of 1D Weierstrass Function (Stem Plot)

- **X-axis**: **Frequency (cycles/sample)**
  - This axis represents the spatial frequencies present in the 1D function. "Cycles/sample" indicates how many complete cycles of a wave occur within the span of one discrete data point (sample) in the y dimension.
  - For a digitally sampled signal, the frequency is often normalized to the sampling rate. A value of 0.5 cycles/sample is the Nyquist frequency, representing the highest frequency that can be uniquely resolved.
- **Y-axis**: **Magnitude (log scale)**
  - This shows the **amplitude** or **strength** of each frequency component. A logarithmic scale is used to better visualize the wide range of magnitudes.
- **Plot Type**: **Stem Plot**
  - A stem plot is ideal here because the FFT of the Weierstrass function yields **discrete, distinct frequency components**. Each "stem" corresponds to one of the cosine terms in the sum $a^n\cdot\cos(\pi b^n y)$.
- **Interpretation**:
  - You will observe **distinct stems (peaks)** at frequencies corresponding to $\pi, b\pi, b^2\pi, \dots, b^{N-1}\pi$ (transformed into cycles/sample units by the FFT). These are the fundamental frequencies and their harmonics that build the Weierstrass function.
  - The **heights of these stems will progressively decrease** as frequency increases. This directly reflects the a^n term in the Weierstrass definition, where higher frequencies have smaller amplitudes, contributing to the function's fine, self-similar details.
  - The presence of these clearly defined, geometrically spaced peaks, with amplitudes that diminish exponentially, is a hallmark of the Weierstrass function's construction and its multi-frequency nature.

Fractal Dimension Calculation

4. Box-Counting Dimension (2D)



- Calculates fractal dimension using box-counting method on the 2D surface.
- Requires `a·b ≥ 1` (fractal condition for the 1D Weierstrass function; this condition is extended heuristically to 2D in this visualization).
- Displayed in a dedicated box when calculated, turning green if the condition is met and calculation is possible.

⚙️ Technical Implementation

Optimized Computation

```
@njit
def compute_weierstrass_2d_precomputed(X, Y, a_powers, b_freqs):
    W = np.zeros_like(X)
    for n in range(len(a_powers)):
        W += a_powers[n] * np.cos(b_freqs[n] * X) * np.cos(b_freqs[n] * Y)
    return W

@njit
def compute_weierstrass_1d(y, a_powers, b_freqs):
    total = np.zeros_like(y)
    for n in range(len(a_powers)):
        total += a_powers[n] * np.cos(b_freqs[n] * y)
    return total
```

- Uses Numba JIT compilation for 100x speedup.
- `compute_weierstrass_1d` is used for the 1D plot and its FFT.
- Precomputes power series for efficiency.

Box-Counting Algorithm

```
@njit
def box_counting_dimension(Z, epsilons):
    # Normalize Z to [0,1]
    # Create 3D grid (x, y, value)
    # Count occupied boxes at different scales
    # Calculate dimension via log-log regression
```

- Operates in normalized value space.
- Uses linear regression on log-scale data.

FFT Analysis

```
def compute_fft(Z): # For 2D FFT
    fft_Z = np.fft.fft2(Z)
    fft_shifted = np.fft.fftshift(fft_Z)
    return np.log10(np.abs(fft_shifted) + 1e-10)
```

- `np.fft.fft2` computes the 2D Fourier transform.
- `np.fft.fft` computes the 1D Fourier transform (used for the 1D FFT plot).
- Shifts zero-frequency to center (for 2D) or provides appropriate frequency bins (for 1D).
- Applies logarithmic scaling for better visualization of magnitude.

📊 Visualization Legend

Element	Raw View (2D)	Density View (2D)	FFT View (2D)	1D Plot (x=0)	1D FFT (Stem)
X-axis	X Coordinate	X Coordinate	ω_x (rad/sample)	Y Coordinate	Freq (cycles/sample)
Y-axis	Y Coordinate	Y Coordinate	ω_y (rad/sample)	W(0,y) Value	Magnitude (log)
Color/Lines	Function value	Probability	Log-magnitude (dB)	Blue line	Red stems
Range (X/Y)	[-1, 1]	[-1, 1]	[-π, π] rad/sample	[-1, 1]	[0, max Freq]
Aspect Ratio	1:1	1:1	1:1	N/A	N/A

🔑 Key Clarifications

1. **Two distinct "frequency" concepts:**
   - **Parameter b**: Controls the scaling of frequencies in the *mathematical definition* of the Weierstrass function itself (e.g., $\pi, b\pi, b^2\pi$).
   - **FFT analysis**: Measures the spatial frequencies present in the *visual output* of the rendered function.
   - These are intrinsically related but represent different stages of understanding: one is a *design parameter*, the other is a *measured property* of the result.
2. **Density vs FFT:**
   - **Density** (2D) shows the **frequency of occurrence of function values** (e.g., how often does the function output a value of 0.5 vs. 0.1?).
   - **FFT** (2D and 1D) shows the **frequency of spatial patterns or oscillations** (e.g., how often does a specific pattern of peaks and valleys repeat across the surface or along the 1D line?).
3. **Practical interpretation:**
   - Higher b → More fine details in the function → More energy at higher frequencies in the FFT.
   - Higher *a* → Sharper contrasts and more pronounced oscillations → Wider value distribution and potentially more high-frequency energy.
   - `a·b ≥ 1` → Fractal behavior is typically expected (for the 1D function, this is the condition for non-differentiability and fractal dimension > 1) → Valid fractal dimension calculation.

🧩 Parameter Effects

Parameter Change	Raw View	Density View	FFT View (2D)	1D Plot	1D FFT (Stem)	Dimension
a ↑	Sharper contrasts	Wider distribution	More high-frequency energy	Larger amplitudes	Higher stems overall	↑ (0.1-0.3)
b ↑	Finer details	More complex peaks	Energy shifts outward	More oscillations	Stems shift to higher frequencies	↑ (0.1-0.4)
a·b ≥ 1	Fractal patterns	Heavy tails	Power-law spectrum	Highly jagged	Clear, distinct peaks	Valid result

▶️ Getting Started

- Install requirements:

```
pip install numpy matplotlib numba
```

- Run the script:

```
python weierstrass_fractal_FFT_Box_Count.py
```

- Interact with controls:
  - Adjust *a* and b sliders.
  - Toggle visualization modes to explore different aspects (Raw, Density, FFT).
  - Click "Display Box-Counting (Fractal) Dimension" to compute the fractal dimension when `a·b ≥ 1`.

📚 References

1. Weierstrass, K. (1872). On continuous functions of a real argument that do not have a well-defined differential quotient.
2. Falconer, K. (2013). Fractal Geometry: Mathematical Foundations and Applications.
3. Mandelbrot, B. B. (1982). The Fractal Geometry of Nature.