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Considerations Concerning Diffraction Scattering in Quantum Chromodynamics

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A model for diffractive, especially elastic hadron–hadron scattering at high c.m. energies squared s and small momentum transfer squared $|t|$ is developed, based on quantum chromodynamics (QCD). First the scattering of hadron constituents, partons, is considered. In this paper we study mainly the scattering of quarks and antiquarks. The s-dependence of the amplitudes is treated by analytical means using a functional integral approach and an eikonal approximation. The t-dependence of the amplitudes is then shown to be governed by a certain correlation function of gluon string operators. A calculation of this function by non-perturbative methods in the framework of lattice QCD or in the stochastic vacuum field model should be feasible. The transition from the parton to the hadron level is accomplished by constructing an effective S-operator in terms of local tensor operators. In this way we avoid dealing explicitly with hadronic wave functions. In our approach the bound state nature of the hadrons enters through matrix elements of the local operators, where information from deep inelastic lepton–hadron scattering exists. The resulting expression for the elastic hadron–hadron scattering amplitude is discussed and is shown to lead to an understanding of various phenomenological findings. The physical picture emerging is one where single partons of the hadrons interact at a time, i.e., the "Pomeron" couples to single partons. This was suggested previously by phenomenological analyses of experiments and by theoretical investigations of an abelian gluon model. © 1991 Academic Press, Inc.

1. INTRODUCTION

Today it is the general belief that quantum chromodynamics (QCD) is the correct theory of hadrons and their interactions. So far, however, reliable predictions can only be extracted from QCD theory for a restricted class of experimental observables. For short distance phenomena perturbation theory gives reliable results, and for static properties of hadrons one can apply non-perturbative techniques like Monte Carlo calculations in the framework of lattice gauge theories. Neither of these theoretical techniques is, however, as yet applicable to "soft" hadronic reactions at high energies. On the other hand, there exists a tremendous wealth of experimental information on these reactions. Their phenomenological analysis has revealed many empirical regularities, from the additive quark rule for total cross sections [1], known for a long time, to the findings of [2]. Also some startling "irregularities" have been found recently. The yield of soft photons

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measured in high energy hadron–hadron collisions [3] seems to be much higher than expected on the basis of Low's theorem [4]. The real part of the forward elastic proton–antiproton scattering amplitude at high energies seems to be bigger than expected [5]. Without a reliable theoretical framework for calculating the properties of soft hadronic reactions, the above findings are hard to interpret.

In this article we will consider the following class of "soft" hadronic reactions: diffractive and, in particular, elastic scattering of two hadrons at high energies. Phenomenologically these reactions can be described by the Pomeron-exchange model (cf. [6] for reviews). Since the pioneering works of [7] most theorists have agreed that the properties of the "Pomeron" should somehow be derived from multi-gluon exchange [8, 9, and references cited therein], but how to devise a reliable calculational framework based on this idea remained an open problem. A new effort to solve this old problem was made in [10], where in an abelian model (subsequently called the LN-model) we related the Pomeron properties to nonperturbative features of the vacuum like the gluon condensate of SVZ [11]. The LN-model has subsequently been applied with some success to various reactions [12]; however, the basic problem of going from the abelian gluon case to real QCD with its nonabelian gluons remained unsolved.

In the present article we address this issue. In Section 2 we recall, and develop further, the physical picture underlying the LN-model. In Section 3 we treat quark–quark elastic scattering at small momentum transfer starting from the functional integral of QCD. Section 4 is devoted to a discussion of an eikonal approximation to the solution of the Dirac equation in the presence of a nonabelian external gluon field. Section 5 gives our answer for the quark–quark scattering amplitude at high energies and identifies the relevant theoretical object governing the reaction as a certain correlation function of string operators. In Section 6 we propose to go from constituents to hadrons using the effective Lagrangian techniques. Section 7 finally contains a discussion of the results and of open problems, and our conclusions.

Throughout this work we use units $\hbar = c = 1$ and the metric and γ-matrix conventions of [13].

2. The Physical Picture

Let us consider elastic scattering of two hadrons h and h'

$$h(p_1) + h'(p_2) \to h(p_3) + h'(p_4), \tag{2.1}$$

where we indicate the four-momenta in brackets. The four momentum transfer is $q = p_1 - p_3$ and the usual invariant variables are

$$\begin{aligned} s &= (p_1 + p_2)^2, \\ t &= (p_1 - p_3)^2 = q^2, \\ u &= (p_1 - p_4)^2. \end{aligned} \tag{2.2}$$

We are interested in high energies, $s \to \infty$, and small t, $|t| \lesssim 1\ \mathrm{GeV}^2$, say. This is a regime where, in our opinion, neither QCD perturbation theory applies—since $|t|$ is too small—, nor can the usual lattice gauge theory approach give numerical answers directly—since s is too large.

In QCD, hadrons are described as bound states of constituents, partons, which are the quarks and gluons. We will for the time being consider only "light" hadrons h, h', i.e., only those composed of u, d-quarks and their antiquarks and possibly gluons. We will ignore strange quarks and heavier quark flavours in the following. In a microscopic theory of hadron–hadron scattering we should start from parton–parton scattering. The amplitudes for the partonic reactions should then be folded with the hadronic wave functions. But it is well known that the internal wave function of a hadron depends on the reference frame and on the resolution with which we look at the hadron [14, 15]. Thus, our first task is to discuss the "appropriate" wave function for elastic hadron–hadron scattering (2.1).

We take it as an empirical fact that a hadron h at rest or in slow motion, looked at with low resolution, is well described in the constituent quark picture. In order for this picture to make sense, the radius R_q of a constituent quark should be reasonably much smaller than the hadron radius R_h such that

$$R_q^2 \ll R_h^2. \tag{2.3}$$

The radius R_q measures the extension of the "gluon cloud" associated with a slow quark. On the other hand, a correlation length a characterizing the gluonic fluctuations in the "nonperturbative" vacuum of QCD was introduced in [10, 16]. To be specific, consider the—suitably made gauge invariant [16]—correlation function of two gluon field strength tensors at points x and y. The correlation length a characterizes the spatial extension of this function in $|x-y|$. It seems natural to assume

$$a \approx R_q \tag{2.4}$$

and this is also supported by the numerical results obtained from phenomenology [16, 17].

Consider now the hadron h moving in the positive x^3-direction with momentum $P \gg m_h$, where m_h is the hadron's mass. We are interested in the limit $P \to \infty$. Constructing a hadronic wave packet with an uncertainty $\Delta P \sim \varepsilon P$ $(0 < \varepsilon \ll 1,\ \varepsilon$ fixed) for all momentum components, we can localize the position of the hadron's centre of mass with an uncertainty $\Delta P^{-1} \sim 1/(\varepsilon P)$. For $P \to \infty$ the world line of the hadron's c.m. in Minkowski space can thus be localized very precisely.

According to quantum field theory, the partons in the hadron will interact by emitting and reabsorbing other partons. Using equal time quantization, where momentum is conserved but energy is not, the fluctuation times for these processes can be estimated in a well-known way (cf. [14]). Suppose parton ψ in hadron h splits into ψ' and ψ'':

$$\psi(k) \to \psi'(k') + \psi''(k''), \tag{2.5}$$

where

$$k = \begin{pmatrix} E \\ 0 \\ k_{\parallel} \end{pmatrix}, \qquad E = (k_{\parallel}^2 + m^2)^{1/2}$$
$$k' = \begin{pmatrix} E' \\ \mathbf{k}'_T \\ k'_{\parallel} \end{pmatrix}, \qquad E' = (k'^2_{\parallel} + \mathbf{k}'^2_T + m'^2)^{1/2}, \tag{2.6}$$

and similarly for k''. Momentum conservation implies

$$\begin{aligned} &\mathbf{k}'_T + \mathbf{k}''_T = 0, \\ &k'_{\parallel} + k''_{\parallel} = k_{\parallel}. \end{aligned} \tag{2.7}$$

The energy difference ΔE determines the fluctuation time Δt:

$$\Delta t \simeq (\Delta E)^{-1} = (E' + E'' - E)^{-1}. \tag{2.8}$$

Large fluctuation times can only occur if a "hard" parton ψ, i.e., one where $k_{\parallel} = O(P)$ splits into partons ψ', ψ'' of

$$k_T \equiv |\mathbf{k}'_T| = |\mathbf{k}''_T| \ll P.$$

If in case I, ψ splits into two other hard partons ($k'_{\parallel}, k''_{\parallel} = O(P)$, $0 < k'_{\parallel}/k_{\parallel} < 1$, $0 < k''_{\parallel}/k_{\parallel} < 1$) we obtain

$$\Delta t \sim P/k_T^2. \tag{2.9}$$

If in case II, ψ splits into one hard parton, say ψ'' and one "wee" parton ψ', where

$$|k'_{\parallel}| \sim k_T \sim a^{-1}, \tag{2.10}$$

we obtain

$$\Delta t \sim a. \tag{2.11}$$

Here we assume the typical k_T and mass scale of wee partons to be again given by the vacuum correlation length a. Larger wave lengths should be rapidly damped out due to confinement. Two other scales of relevance for us are the transverse and longitudinal distances ψ' and ψ'' can separate during the lifetime Δt of the fluctuation. We obtain

$$\Delta x_T \sim \frac{k_T}{k'_{\parallel}} \Delta t \sim \begin{cases} k_T^{-1} & \text{for case I,} \\ a & \text{for case II,} \end{cases} \tag{2.12}$$

and

$$\Delta x_{\parallel} \sim \left(1 - \frac{k'_{\parallel}}{E'}\right) \Delta t \sim \begin{cases} P^{-1} & \text{for case I,} \\ a & \text{for case II.} \end{cases} \tag{2.13}$$

Consider now an observer watching the fast-moving hadron over a time t_0. He will see all fluctuations with $\Delta t > t_0$ as belonging to the hadron's wave function, i.e., as forming the incoming jet of partons. Such fluctuations will practically never occur over the time t_0. On the other hand, he will describe fluctuations with $\Delta t < t_0$ as interactions of partons with each other during his observation time. Thus the appropriate wave function for observation time t_0 should include all parton modes with $\Delta t \sim (P/k_T^2) > t_0$, i.e., with

$$k_T^2 < P/t_0. \tag{2.14}$$

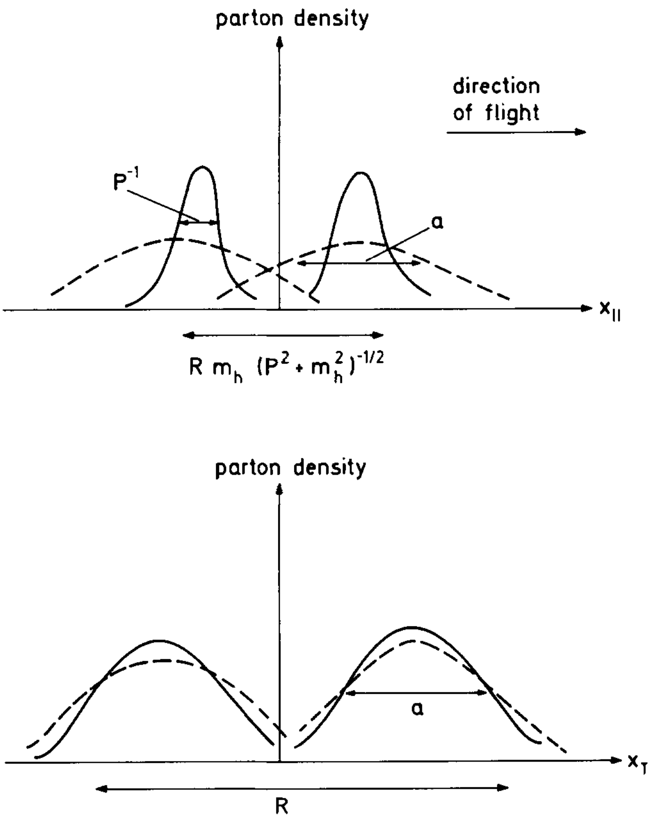

FIG. 1. Sketch of parton densities in a hadron in longitudinal and transverse directions. Full lines: hard partons, with the peaks indicating the nominal positions of the constituent quarks: dashed lines: wee partons.

We imagine now letting the wave function of the hadron evolve, starting from the constituent quark level to the level (2.14). This is in the spirit of models where each valence quark in the hadron has its own cloud of sea quarks and gluons [18]. The corresponding pictures of parton densities obtained from (2.12) and (2.13) are sketched in Fig. 1. Longitudinal distances between hard partons are of order $1/P$ due to Lorentz contraction. Wee partons form a cloud of extension $\sim a$ around them. In transverse space wee and hard partons are similarly distributed. We can argue that the chance to find two partons together with a transverse separation $<a$ is small. For constituent quarks this chance is $\propto (a/R)^2$ and the same applies in our picture to daughter partons of different constituent quarks. Two hard partons from the same constituent quark with $k_T < a^{-1}$ will have $\Delta x_T > a$ according to (2.12). For larger k_T we can have smaller Δx_T, but we estimate the chance for such fluctuations to be small using perturbation theory and invoking a small coupling strength α_s at large k_T. Finally, we note that partons can, of course, split repeatedly producing more and more partons. Assuming this to occur in a random walk fashion, the estimates of Δx_T, $\Delta x_{\|}$ in (2.12), (2.13) are only changed by a factor $\sqrt{n}$. Here n is the parton multiplicity which should, according to standard lore, grow at most as some power of $\ln P$ for $P \to \infty$. Consequently, the powers of P in the estimates (2.12), (2.13) are unaffected.

Let us briefly recall deep inelastic electron hadron scattering (DIS):

$$e + h \to e + X. \tag{2.15}$$

We use standard notation (cf., e.g., [19]), where $Q^2 = -q^2$ is the momentum transfer squared. In the Breit frame the momenta of h and of the initial and final electron are all of order Q except near the kinematic boundaries. In a typical deep inelastic collision the electron thus probes the hadron with a time resolution $1/Q$. If we want to use the impulse approximation, we should choose—as is well known—as appropriate observation time $t_0|_{\mathrm{DIS}} \sim 1/Q$. To choose $t_0 < 1/Q$ would clearly make no sense and for $t_0 \gg 1/Q$ the observer would see many partonic interactions before and after the electron parton scattering, making the description of the reaction in terms of the impulse approximation invalid. The appropriate resolution in k_T of the wavefunction is then according to (2.14), with $P \sim Q$, $t_0|_{\mathrm{DIS}} \sim Q^{-1}$:

$$k_T^2|_{\mathrm{DIS}} \lesssim Q^2. \tag{2.16}$$

The usual scale-dependent parton densities $N(\xi, Q^2)$ correspond indeed to this resolution.

Consider now hadron–hadron scattering (2.1) in the overall c.m. system and assume $h(h')$ to travel in positive (negative) x^3-direction with momentum $P \gg m_h, m_{h'}$. Let the worldlines of the centres of mass of h and h' be localized to $O(1/P)$ as discussed above. We assume their projections onto the $x^0 - x^3$-plane to cross at $x^0 = x^3 = 0$ (Fig. 2). An observer in the "femtouniverse" will see the incoming partons of h and h' interacting for a time $-a \lesssim t \lesssim a$ and afterwards separating and forming the final hadron states. What will be the appropriate obser-

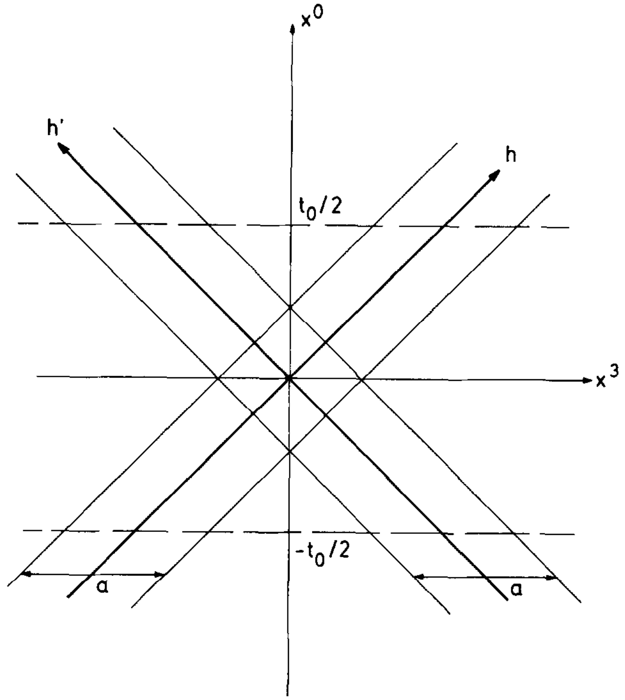

FIG. 2. An elastic collision of two hadrons h and h'. The thick lines are the projections of the hadron's c.m. worldlines onto the $x^0 - x^3$ plane. The thin lines indicate the longitudinal extensions of the hadron's wave functions.

vation time t_0 here, allowing us (hopefully) a simple description of the reaction? We argue as follows. Scattering processes with $k_T \gg a^{-1}$, corresponding to a very short time resolution, will occur, but are rare, being governed by a small coupling strength α_s. Such events lead to high k_T jets and can be treated perturbatively. We will neglect them in a first approximation in our discussion of soft reactions where typical momentum transfers should be of order a^{-1}, i.e., nonperturbative interactions. Clearly, we should choose then $t_0 > a$. In fact, we will require $t_0 \gg a$, but not too large, such that the description of the parton reactions is "simple." We would like to have a situation where (i) the parton state does not change qualitatively and (ii) partons travel essentially on straight lines and are not splitting over the time interval $(-t_0/2, t_0/2)$. The observer in the femtouniverse will then see quasi-free partons incoming, having interactions with a time and x_T scale a.

To meet condition (i), we choose $t_0/2$ smaller than the time t_c when in an inelastic collision, the first "protohadrons" appear. We are thinking here of the string picture in the Lund model [20] where the "first" hadrons appear with $k_\| \approx 0$. The "protohadrons" are presumably the clusters seen in inelastic events [21], with typical cluster mass

$$m_c \approx 1.3 \text{ GeV}. \tag{2.17}$$

We estimate then in a standard way

$$t_c \sigma \approx m_c \tag{2.18}$$

where $\sigma \simeq 0.16\ \text{GeV}^2$ is the string tension. This leads to

$$\begin{aligned} t_c &\approx 1.6\ \text{fm}, \\ t_0 &< 2t_c = 3.3\ \text{fm}. \end{aligned} \tag{2.19}$$

To meet condition (ii) we consider first the splitting of a hard parton into two hard ones (cf. (2.5)). For $k_T > \sqrt{P/t_0}$ the fluctuation time Δt (2.9) is smaller than t_0 and the transverse splitting Δx_T is given by (2.12):

$$\Delta x_T \sim 1/k_T < \sqrt{t_0/P}. \tag{2.20}$$

For $k_T < \sqrt{P/t_0}$ we obtain $\Delta t > t_0$. For the transverse splitting over the time interval $(-t_0/2, t_0/2)$ we now obtain

$$\Delta x_T \sim (k_T/P)\, t_0 < \sqrt{t_0/P}. \tag{2.21}$$

In order to meet (ii), we must therefore require both from (2.20) and (2.21):

$$\sqrt{t_0/P} \ll a. \tag{2.22}$$

This is certainly satisfied for fixed t_0 and $P \to \infty$. Splittings of hard partons into wee and hard or splittings of wee partons have fluctuation times $\Delta t \sim a$ and transverse separations $\Delta x_T \sim a$ and should be part of the partonic description of the process over the time interval $(-t_0/2, t_0/2)$.

We can estimate from (2.22) the minimal value of P, from where on our description should be valid by setting

$$P_{\min} = t_0 \cdot a^{-2}. \tag{2.23}$$

Determinations of the correlation length a from low energy hadron spectroscopy [16] and from high-energy scattering using the abelian model [12] or the methods proposed in the present paper [22] lead to

$$a \approx 0.4\ \text{fm}. \tag{2.24}$$

Choosing $t_0 \simeq 2$ fm we satisfy (2.19) and $t_0 \gg a$ and obtain

$$\begin{aligned} P_{\min} &\approx 2.5\ \text{GeV}, \\ \sqrt{s_{\min}} &\approx 2P_{\min} \approx 5\ \text{GeV}. \end{aligned} \tag{2.25}$$

Quite consistently, this c.m. energy marks the upper end of the resonance region where, clearly, conditions (i) and (ii) above no longer hold.

To summarize, we argued that for high energy elastic hadron–hadron scattering

(2.1) an observation time $t_0 \approx 2$ fm (cf. Fig. 2) should be appropriate. From (2.14) this corresponds to a resolution $k_T^2 < P/t_0$ in the wave function and to an "equivalent" resolution (2.16) in deep inelastic scattering of

$$Q_0^2 = P/t_0. \tag{2.26}$$

Over the time interval $(-t_0/2, t_0/2)$ we can use the partonic description with the partons undergoing "soft" collisions of momentum transfers $k_T \lesssim a^{-1}$ only. In the next section we will, therefore, study parton–parton scattering, in particular, quark–quark scattering at large s and small t. The physical picture for this reaction as developed in [10] and above sees each quark travelling through the "nonperturbative" vacuum of QCD, carrying a cloud of gluons with it (Fig. 3). The gluon cloud is assumed to have a transverse extension characterized by the length a. It is then clear that the two quarks will only scatter if they approach each other in the transverse directions to a distance of less than or about the length a. In the present paper we will show how something like this picture emerges from the functional integral approach with the length a characterizing the behaviour of the correlation function of certain string operators.

Suppose now that the problem of parton–parton scattering is solved. How can we go from the parton to the hadron level? In principle we face at this point all the complications of the hadronic wave functions. Our basic hypothesis in [10] was that the length a is much smaller than the radius R of a light hadron. In this way we could easily understand the additive quark rule [1] and many other experimentally observed features of the Pomeron, especially the recent results [23] which give strong support to the picture of the Pomeron coupling to single quarks in hadrons. We will show in the present paper that with our hypothesis $a^2 \ll R^2$ the complications of the hadronic wave functions can be circumvented in a certain sense. All we

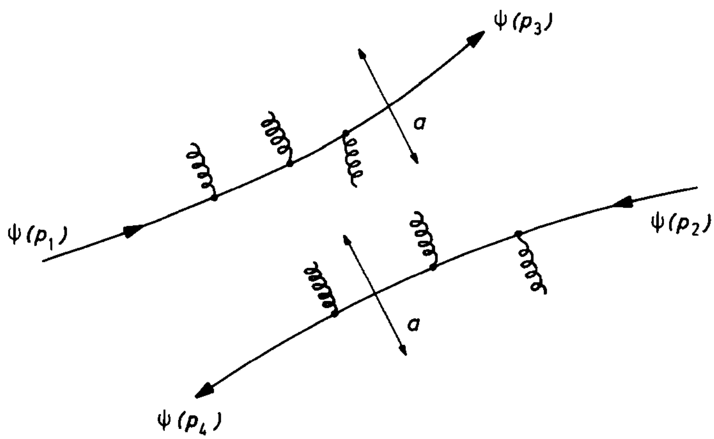

FIG. 3. A quark–quark collision taking place in the nontrivial gluon background provided by the "nonperturbative" vacuum fluctuations of QCD.

have to know about the hadrons are some of their matrix elements of local tensor operators. And these operators turn out to be just the well-known ones from deep inelastic lepton–hadron scattering.

3. Quark Quark Scattering in the Functional Integral Approach

Our starting point is the Lagrangian density of QCD

$$\mathscr{L}(x) = -\tfrac{1}{2}\,\mathrm{Tr}(G_{\lambda\rho}(x)\,G^{\lambda\rho}(x)) + \bar{\psi}(x)(i\gamma^{\lambda}D_{\lambda} - m)\,\psi(x), \tag{3.1}$$

where $\psi(x)$ is the quark field with mass m. For simplicity here we consider only one quark flavour, the generalization to more flavours being obvious. The matrices of the gluon potential and field strength tensor are denoted by $G_{\lambda}(x)$ and $G_{\lambda\rho}(x)$, the covariant derivative by D_{λ}. In standard notation (see, e.g., [19])

$$\begin{aligned} G_{\lambda}(x) &= G^{a}_{\lambda}(x)(\lambda_a/2), \\ G_{\lambda\rho}(x) &= \partial_{\lambda}G_{\rho}(x) - \partial_{\rho}G_{\lambda}(x) + ig[G_{\lambda}(x), G_{\rho}(x)], \end{aligned} \tag{3.2}$$

$$D_{\lambda} = \partial_{\lambda} + igG_{\lambda}. \tag{3.3}$$

All quantities in (3.1) are the unrenormalized ones.

Consider now quark–quark scattering

$$\psi(p_1) + \psi(p_2) \to \psi(p_3) + \psi(p_4), \tag{3.4}$$

where we suppress spin and colour indices for the moment. As explained in Section 2, it should make sense for the femtouniverse observer with observation time $t_0 \gg a$ to speak about "free" incoming quarks. In this spirit, we will consider here the limit $t_0 \to \infty$ and work with free asymptotic quark states, imagining adiabatic turnoff of the interaction at times $x^0 \to \pm\infty$. We also assume that the gauge is specified in an identical way for $x^0 \to \pm\infty$, let it be, for instance, the Feynman gauge.

Before studying the quark–quark scattering amplitude we will consider the quark 2-point function. The corresponding unrenormalized function is given by the functional integral

$$\langle 0|\, T(\psi(x_1)\,\bar{\psi}(x_2))\,|0\rangle = \mathscr{Z}^{-1}\int \mathscr{D}(G, \psi, \bar{\psi})\,\psi(x_1)\,\bar{\psi}(x_2) \times \exp\left[i\int dx\,\mathscr{L}(x)\right], \tag{3.5}$$

where

$$\begin{aligned} \mathscr{Z} &= \langle 0\ \mathrm{out} \mid 0\ \mathrm{in}\rangle \\ &= \int \mathscr{D}(G, \psi, \bar{\psi})\exp\left[i\int dx\,\mathscr{L}(x)\right]. \end{aligned} \tag{3.6}$$

The integration over the fermion degrees of freedom in (3.5) and (3.6) can be carried out explicitly in the standard way, since $\mathscr{L}$ in (3.1) is bilinear in ψ and $\bar{\psi}$, and we obtain

$$\mathscr{Z} = \int \mathscr{D}(G) \exp\left[-i \int dx \frac{1}{2} \mathrm{Tr}(G_{\lambda\rho} G^{\lambda\rho})\right] \times \det[-i(i\gamma^\lambda D_\lambda - m + i\varepsilon)], \tag{3.7}$$

$$\langle 0 | T(\psi(x_1)\bar{\psi}(x_2)) | 0 \rangle = \left\langle \frac{1}{i} S_F(x_1, x_2; G) \right\rangle. \tag{3.8}$$

Here S_F is the Green's function for a quark in an external gluon potential G_λ and the brackets $\langle\ \rangle$ mean integration over all gluon fields with the measure dictated by the functional integral. To be precise, let $F(G)$ be an arbitrary functional of G; then we define

$$\langle F(G) \rangle = \mathscr{Z}^{-1} \int \mathscr{D}(G) F(G) \exp\left[-i \int dx \frac{1}{2} \mathrm{Tr}(G_{\lambda\rho} G^{\lambda\rho})\right] \times \det[-i(i\gamma^\lambda D_\lambda - m + i\varepsilon)]. \tag{3.9}$$

Note that expectation of 1 is 1,

$$\langle 1 \rangle = 1, \tag{3.10}$$

but that, in general,

$$\langle F(G) \rangle^* \neq \langle F^*(G) \rangle. \tag{3.11}$$

Gauge fixing terms and the, in general, ensuing Faddeev–Popov ghost terms are not written explicitly in (3.7), (3.9), but are assumed to be lumped into the measure $\mathscr{D}(G)$.

Let us define the renormalized quark mass m', taken to be the pole mass, and the quark wave function renormalization constant Z_ψ. We write

$$\langle 0 | T(\psi(x_1)\bar{\psi}(x_2)) | 0 \rangle = i \int \frac{dk}{(2\pi)^4} e^{-ik(x_1 - x_2)} \tilde{S}_F(\not{k}) \tag{3.12}$$

and have as defining equations for m' and Z_ψ:

$$\begin{aligned} &\tilde{S}_F^{-1}(\not{k} = m') = 0, \\ &\tilde{S}_F(\not{k})|_{\text{poleterm}} = \frac{Z_\psi}{\not{k} - m' + i\varepsilon}. \end{aligned} \tag{3.13}$$

The renormalized quark field operator ψ' and the mass shift δm are then given by

$$\begin{aligned} &\psi'(x) = Z_\psi^{-1/2} \psi(x), \\ &\delta m = m' - m. \end{aligned} \tag{3.14}$$

Next we must have a closer look at Green's functions for a quark in an external gluon field, i.e., at the functions satisfying the equation

$$(i\gamma^\lambda D_\lambda - m)\, S_I(x, y; G) = -\delta(x-y). \tag{3.15}$$

Here we write a general index I for inhomogenous. We will be concerned with the Feynman ($I=F$) and retarded ($I=r$) boundary conditions. The corresponding free Green's functions, but for the renormalized mass m', are

$$\begin{aligned} S_F^0(x, y) &= -\int \frac{dk}{(2\pi)^4}\, e^{-ik(x-y)}\, \frac{\not{k}+m'}{k^2-m'^2+i\varepsilon}, \\ S_r^0(x, y) &= -\int \frac{dk}{(2\pi)^4}\, e^{-ik(x-y)}\, \frac{\not{k}+m'}{k^2-m'^2+i\varepsilon k^0}. \end{aligned} \tag{3.16}$$

In matrix notation, where matrix multiplication includes an integration over space-time we have for the Green's functions the well-known relations

$$S_I = S_I^0 - S_I^0(g\not{G} - \delta m)\, S_I \tag{3.17a}$$

$$= S_I^0 - S_I(g\not{G} - \delta m)\, S_I^0 \tag{3.17b}$$

$$= \sum_{n=0}^{\infty} [-S_I^0(g\not{G} - \delta m)]^n\, S_I^0. \tag{3.17c}$$

Now we return to reaction (3.4). We use the reduction formula to relate the scattering amplitude to the 4-point function of the quark field ψ. This function in turn can be expressed by a functional integral. Carrying through these steps in the standard way we obtain, after integrating out the fermion degrees of freedom,

$$\begin{aligned} &\langle \psi(p_3)\, \psi(p_4)\, |S|\, \psi(p_1)\, \psi(p_2)\rangle \equiv S_{fi} \\ &\quad = -Z_\psi^{-2} \langle (p_3|\, (i\overrightarrow{\not{\partial}} - m')\, S_F(i\overleftarrow{\not{\partial}} + m')\, |p_1)(p_4|\, (i\overrightarrow{\not{\partial}} - m') \\ &\quad\quad \times S_F(i\overleftarrow{\not{\partial}} + m')\, |p_2) - (p_3 \leftrightarrow p_4)\rangle. \end{aligned} \tag{3.18}$$

Here $|p_j)$ is a shorthand notation for the Dirac wave functions:

$$|p_j) = u(p_j)\, e^{-ip_j x}. \tag{3.19}$$

Our Dirac spinors are normalized to

$$\bar{u}_r(p_j)\, \gamma^\mu u_s(p_j) = 2p_j^\mu\, \delta_{rs}, \tag{3.20}$$

where r, s are the spin indices. In (3.18) we use a convenient inner product notation. For two Dirac wave functions $\phi_1(x)$, $\phi_2(x)$ we define

$$(\phi_1\,|\,\phi_2) = \int dx\, \bar{\phi}_1(x)\, \phi_2(x). \tag{3.21}$$

Using (3.17a) we can rewrite (3.18) for $(p_1, p_2) \neq (p_3, p_4)$, i.e., in nonforward directions as

$$S_{fi} = -Z_\psi^{-2}\langle (p_3|\,(g\not{G} - \delta m)|\,\psi^F_{p_1})(p_4|\,(g\not{G} - \delta m)|\,\psi^F_{p_2}) - (p_3 \leftrightarrow p_4)\rangle. \quad (3.22)$$

Here we define

$$|\psi^F_{p_j}) = S_F(i\overleftarrow{\not{\partial}} + m')\,|p_j) \qquad (j = 1, 2). \quad (3.23)$$

Let us discuss (3.22) in detail. The first term written out explicitly corresponds to all t-channel exchange diagrams (including quark loops), the term $(p_3 \leftrightarrow p_4)$ corresponds to the u-channel exchanges (Fig. 4). For $s \to \infty$ and t small, a large momentum has to flow through the gluon lines in the u-channel exchange diagrams. They correspond to large angle scattering and their contributions relative to the t-channel exchange ones are suppressed at least by a power of s. Therefore we will neglect the u-channel exchange diagrams in the following. For the scattering of different quark flavours they are not present anyhow.

The wave functions $|\psi^F_{p_j})$ of (3.23) satisfy the Dirac equation in the presence of the external gluon potential G_λ:

$$(i\gamma^\lambda D_\lambda - m)\,|\psi^F_{p_j}) = 0 \qquad (j = 1, 2). \quad (3.24)$$

The amplitudes

$$\mathcal{M}^F_{kj}(G) = (p_k|\,(g\not{G} - \delta m)\,|\psi^F_{p_j}) \quad (3.25)$$

with $j = 1$, $k = 3$, and $j = 2$, $k = 4$ can be considered as scattering amplitudes for a quark in a given external gluon potential. Indeed, $\mathcal{M}^F_{kj}(G)$ is an integral over a complete incoming wave function multiplied by the potential and the free outgoing wave function. Thus we obtain from (3.22) the following prescription for calculating the quark–quark scattering amplitude. Evaluate first the scattering amplitudes

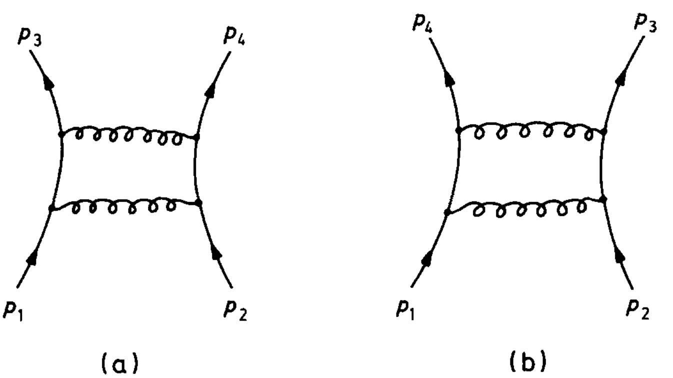

FIG. 4. Examples of a t-channel exchange diagram (a) and a u-channel exchange diagram (b).

$\mathcal{M}^F_{31}(G)$ and $\mathcal{M}^F_{42}(G)$ for each initial quark scattering independently on a given gluon potential G_λ. Take then the average over all gluon potentials in the sense of the functional integral (3.9), indicated by the brackets $\langle\ \rangle$. In this way the physical picture of Fig. 3 has a precise meaning.

Our further strategy will be to search for high energy approximations for the amplitudes $\mathcal{M}^F_{kj}(G)$ (3.25) and insert those in (3.22). Here some sort of eikonal approximation suggests itself. Before we can come to this, however, we have to deal with one more problem. The wave functions $|\psi^F_{p_j})$ are solutions of the Dirac equation with an external gluon potential, but they do not satisfy a simple boundary condition for times x^0 going to plus or minus infinity. Similarly to (3.23) we can, however, define retarded functions

$$\begin{aligned} |\psi^r_{p_j}) &= S_r(i\overleftarrow{\not\partial} + m')\,|p_j) \\ &= |p_j) - S_r(g\not G - \delta m)\,|p_j) \qquad (j = 1, 2), \end{aligned} \tag{3.26}$$

where we used (3.17b) to get the second line. These functions satisfy a simple boundary condition

$$|\psi^r_{p_j}) \to |p_j) \qquad \text{for} \quad x^0 \to -\infty, \tag{3.27}$$

and for them will we be able to find simple approximate forms in the next section. Can we replace $\mathcal{M}^F_{kj}(G)$ by $\mathcal{M}^r_{kj}(G)$ defined as in (3.25) but with ψ^r instead of ψ^F?

The answer to this question is affirmative under certain conditions. With the help of (3.17c) let us start by writing a series expansion for $|\psi^I_{p_j})$ $(I = F, r)$,

$$|\psi^I_{p_j}) = \sum_{n=0}^{\infty} |p_j, I, n), \tag{3.28}$$

where

$$\begin{aligned} &|p_j, I, 0) = |p_j), \\ &|p_j, I, n) = -S^0_I(g\not G - \delta m)\,|p_j, I, n-1) \qquad (n \geqslant 1). \end{aligned} \tag{3.29}$$

To zeroth order we clearly have

$$|p_j, F, 0) - |p_j, r, 0) = 0. \tag{3.30}$$

To first order we easily find

$$\begin{aligned} |p_j, F, 1) - |p_j, r, 1) = -i \int \frac{dk}{(2\pi)^3}\, e^{-ikx}\theta(-k^0)\,\delta(k^2 - m'^2) \\ \times \int dy\, e^{i(k-p_j)y}(\not k + m')(g\not G(y) - \delta m)\,u(p_j). \end{aligned} \tag{3.31}$$

If $G(y)$ is a slowly varying function of y, the integral over y in (3.31) restricts k to lie near p_j, which for $s \to \infty$ has a very large positive zero-component, $p_j^0 \to \infty$. On the other hand, (3.31) says that k must lie on the negative energy part of the mass hyperboloid. For $s \to \infty$ and gluon potentials G with a fixed frequency cutoff, we therefore obtain

$$|p_j, F, 1) - |p_j, r, 1) \approx 0. \tag{3.32}$$

By mathematical induction the argument can be extended to all n leading to

$$|\psi^F_{p_j}) \approx |\psi^r_{p_j}) \tag{3.33}$$

under the above conditions.

We have argued at length in Section 2 that for the parton scattering processes over the time interval $(-t_0/2, t_0/2)$, indeed, the relevant scale for the frequency of the exchanged quanta is given by a^{-1} independent of s. In the following we can, therefore, assume that the functional integral (3.22) is dominated by the contribution from gluon potentials with fixed frequency-cutoff[1] (in the c.m. system of the reaction) such that for $s \to \infty$ we can use (3.33). Doing this and neglecting u-channel exchanges we obtain for the quark–quark scattering amplitude:

$$S_{fi} \approx -Z_\psi^{-2} \langle \mathcal{M}^r_{31}(G)\, \mathcal{M}^r_{42}(G) \rangle. \tag{3.34}$$

4. The Dirac Equation with External Gluon Potential

We study in this section the Dirac equation with external gluon potential

$$(i\gamma^\lambda D_\lambda - m' + \delta m)\, \psi^r_{p_j}(x) = (i\gamma^\lambda \partial_\lambda - g\not{G}(x) - m' + \delta m)\, \psi^r_{p_j}(x) = 0$$
$$(j = 1, 2), \tag{4.1}$$

imposing the boundary condition

$$\psi^r_{p_j}(x) \to u(p_j)\, e^{-ip_j x} \tag{4.2}$$

for $x^0 \to -\infty$. Spin and colour indices are not written explicitly. The gluon potential $G(x)$ in (4.1) is assumed to vary slowly on the scale set by the wavelength of the incoming waves. The mass shift δm is assumed to be turned off adiabatically for $x^0 \to \pm\infty$. In the following we will make use of the gauge invariance of (4.1). For arbitrary $SU(3)$ matrix function $U(x)$, where

$$\begin{aligned} U(x)\, U^\dagger(x) &= 1, \\ \det U(x) &= 1, \end{aligned} \tag{4.3}$$

[1] We emphasize that we do *not* make this assumption for the scattering amplitude of real quarks—whatever that means—where all the splitting processes discussed in Section 2 are included. In our approach these splittings are put into the wave functions which we assume to be given.

we define

$$\psi'^{r}_{p_j}(x) = U(x)\,\psi^{r}_{p_j}(x),$$
$$G'_\lambda(x) = U(x)\,G_\lambda(x)\,U^\dagger(x) - \frac{i}{g}\,U(x)\,\partial_\lambda U^\dagger(x). \tag{4.4}$$

Then the $\psi'^{r}_{p_j}$ satisfy (4.1) with G replaced by G'.

We will work in the c.m. system of reaction (3.4), choosing the coordinate system as in [10], such that the third coordinate axis lies symmetrically with respect to incoming and outgoing quarks (Fig. 5). Then

$$p_{1,2} = \begin{pmatrix} E_+ + \dfrac{\mu^2}{4E_+} \\ \pm\frac{1}{2}\mathbf{q}_T \\ \pm E_+ \mp \dfrac{\mu^2}{4E_+} \end{pmatrix}, \tag{4.5}$$

where

$$E_+ = \tfrac{1}{2}(p_1^0 + p_1^3) = \tfrac{1}{4}(\sqrt{s} + \sqrt{-u}),$$
$$\mu^2 = m'^2 + \tfrac{1}{4}\mathbf{q}^2. \tag{4.6}$$

In this system the momentum transfer q is purely transverse:

$$q = p_1 - p_3 = p_4 - p_2 = \begin{pmatrix} 0 \\ \mathbf{q} \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} \mathbf{q}_T \\ 0 \end{pmatrix}. \tag{4.7}$$

We will also use light cone coordinates where

$$x_\pm = x^0 \pm x^3, \tag{4.8}$$

and similarly for other four-vectors.

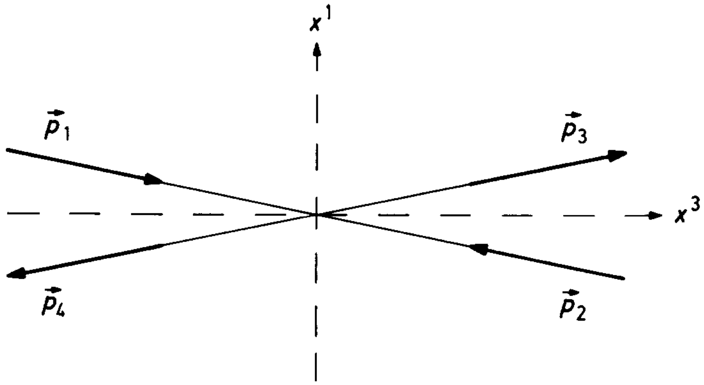

FIG. 5. The reaction (3.4) in the c.m. system and the coordinate system used in Section 4.

We find it convenient to transform the first-order differential equation (4.1) into a second-order one for a "potential" of ψ. In this way the problem to find a high energy approximation for the Dirac wave function is in essence played back to the analogous problem for a scalar field which is easier to handle. To give the details, let us work with ψ', G' of (4.4) keeping $U(x)$ arbitrary for the moment, and let us make the ansatz,

$$\begin{aligned}\psi'^{r}_{p_j}(x) &= (i\gamma^\lambda D'_\lambda + m' - \delta m)\,\phi'_j(x)\\ &= (i\gamma^\lambda \partial_\lambda - g\not{G}'(x) + m' - \delta m)\,\phi'_j(x),\end{aligned} \tag{4.9}$$

where ϕ'_j is to be determined. From (4.1) and (4.4) we find the differential equation for ϕ'_j to be

$$(i\gamma^\mu D'_\mu - m' + \delta m)(i\gamma^\lambda D'_\lambda + m' - \delta m)\,\phi'_j(x) = 0 \tag{4.10}$$

and the boundary condition (4.2) can be met by requiring

$$\phi'_j(x) \to \frac{1}{p_j^0 + m'}\,\frac{1+\gamma^0}{2}\,u(p_j)\,e^{-ip_j x} \tag{4.11}$$

for $x^0 \to -\infty$. (This choice is not unique but convenient.) Splitting off the fast variation with x in $\phi'_j(x)$ we write

$$\phi'_j(x) = e^{-ip_j x}\tilde{\phi}_j(x) \tag{4.12}$$

and obtain

$$\begin{aligned}\{\Box - 2i(p_j^\lambda - gG'^\lambda)\,\partial_\lambda + 2gp_j^\lambda G'_\lambda + ig\gamma^\lambda(\partial_\lambda \not{G}') - g^2\not{G}'\not{G}'\\ - 2m'\,\delta m + (\delta m)^2\}\,\tilde{\phi}_j(x) = 0.\end{aligned} \tag{4.13}$$

Consider now $j = 1$, i.e., the quark coming from the left in Fig. 5. Introducing the light cone variables (4.8) and choosing here as gauge condition

$$G'_+(x) \equiv G'^0(x) + G'^3(x) = 0, \tag{4.14}$$

we find that (4.13) simplifies drastically in the limit $E_+ \to \infty$. We obtain

$$\left\{\frac{\partial}{\partial x_+} + \frac{i}{2}\,gG'_-(x) + O(E_+^{-1})\right\}\tilde{\phi}_1(x) = 0. \tag{4.15}$$

The solutions for (4.15) and $U(x)$ in (4.4) leading to (4.14) are easily written down. We find

$$\begin{aligned}\tilde{\phi}_1(x) = P\left\{\exp\left[-\frac{i}{2}\,g\int_{-\infty}^{x_+} dx'_+\,G'_-(x'_+, x_-, \mathbf{x}_T)\right]\right\}\\ \times\{1 + O(E_+^{-1})\}(p_1^0 + m')^{-1}\,\frac{1}{2}(1+\gamma^0)\,u(p_1),\end{aligned} \tag{4.16}$$

$$U^\dagger(x) = P\left\{\exp\left[-\frac{i}{2}\,g\int_{-\infty}^{x_-} dx'_-\,G_+(x_+, x'_-, \mathbf{x}_T)\right]\right\}. \tag{4.17}$$

Here P means path ordering. Putting everything together and transforming from ψ' back to ψ with the help of (4.4), we obtain

$$\begin{aligned}\psi'^{r}_{p_1}(x) &= U^{\dagger}(x)\{i\gamma^{\lambda}D'_{\lambda} + m' - \delta m\} \cdot e^{-ip_1x}\tilde{\phi}_1(x) \\ &= \{i\gamma^{\lambda}D_{\lambda} + m' - \delta m\}\ U^{\dagger}(x)\, e^{-ip_1x}\tilde{\phi}_1(x).\end{aligned} \tag{4.18}$$

Now it is easy to see that the path-ordered integral in (4.16) gauge-transformed with $U(x)$ of (4.17) gives the path-ordered integral of the gauge-transformed field. Using this and defining

$$V_{-}(x_{+}, x_{-}, \mathbf{x}_T) = P\left\{\exp\left[-\frac{i}{2}g\int_{-\infty}^{x_{+}} dx'_{+}\, G_{-}(x'_{+}, x_{-}, \mathbf{x}_T)\right]\right\}, \tag{4.19}$$

we thus find from (4.18) the final result

$$\psi'^{r}_{p_1}(x) = V_{-}(x_{+}, x_{-}, \mathbf{x}_T)\{1 + O(E_{+}^{-1})\}\, e^{-ip_1x}u(p_1). \tag{4.20}$$

The solution (4.20) has a simple physical interpretation. The quarks of the incoming wave with four-momentum p_1 travel for $E_{+} \to \infty$ essentially along rays $x_{-} = \text{const}$ (Fig. 6). The wave accordingly picks up the nonabelian phase factor V_{-}

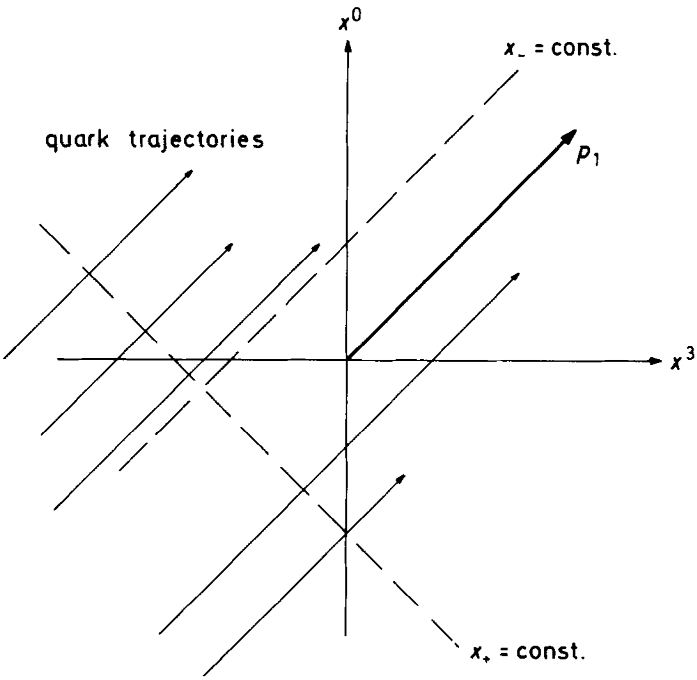

FIG. 6. Quark trajectories in Minkowski space corresponding to the quark wave with four momentum p_1 (cf. (4.5)) in the limit $E_{+} \to \infty$.

(4.19) obtained by integration over x'_+. Note that no gauge condition is any longer imposed in (4.20). This will be important in the following. We also note that one has to be careful when using (4.20) in inner products like

$$\bar{u}(p_3)\, \Gamma \psi'^{r}_{p_1}(x), \tag{4.21}$$

where Γ is a matrix from the Dirac algebra. For $\Gamma = \gamma^\mu$ the leading term in (4.20) indeed gives the leading term in (4.21) for $E_+ \to \infty$, as one can easily check. For $\Gamma = 1$, on the other hand, this is not the case and the leading term as well as the term of $O(E_+^{-1})$ in (4.20) give contributions of the same order in (4.21).

The solution for $\psi'^{r}_{p_2}(x)$ is obtained from (4.19), (4.20) by interchanging the role of the plus and minus components. We obtain

$$\psi'^{r}_{p_2}(x) = V_+(x_+, x_-, \mathbf{x}_T)\{1 + O(E_+^{-1})\}\, e^{-ip_2 x} u(p_2), \tag{4.22}$$

where we define

$$V_+(x_+, x_-, \mathbf{x}_T) = P\left\{\left[-\frac{i}{2} g \int_{-\infty}^{x_-} dx'_-\, G_+(x_+, x'_-, \mathbf{x}_T)\right]\right\}. \tag{4.23}$$

We note that $V_\pm(x_+, x_-, \mathbf{x}_T)$ are $SU(3)$ matrices acting in colour space for fixed x. They satisfy the differential equations and boundary conditions as follows:

$$\frac{\partial}{\partial x_\mp} V_\pm(x_+, x_-, \mathbf{x}_T) = -\frac{i}{2} g G_\pm(x_+, x_-, \mathbf{x}_T) \cdot V_\pm(x_+, x_-, \mathbf{x}_T),$$
$$V_\pm(x_+, x_-, \mathbf{x}_T) \to 1 \qquad \text{for} \quad x_\mp \to -\infty. \tag{4.24}$$

In the next section we will use the solutions (4.20) and (4.22) to evaluate the S-matrix element (3.34) for quark–quark scattering. Let us close this section with a few remarks on other works on the eikonal approximation for particles in an external time-varying potential. The nonrelativistic case was treated in [24], the case of relativistic scalar particles in [25], of relativistic spin $\frac{1}{2}$ particles in interaction with an abelian vector field in [26]. In essence the results obtained there agree with ours when suitably specialised.

5. The Quark–Quark and Quark–Antiquark Scattering Matrices at High Energies

5.1. *Quark–Quark Scattering*

We consider now the amplitude

$$\mathcal{M}'^{r}_{31}(G) = (p_3|\, (g\not{G} - \delta m)\, |\psi'^{r}_{p_1}) \tag{5.1}$$

occurring in (3.34). Inserting the approximate solution (4.20) for $\psi^r_{p_1}$, we obtain for $E_+ \to \infty$:

$$\mathcal{M}^r_{31}(G) \to \int dx\, e^{i(p_3-p_1)x} \bar{u}(p_3)(g\not{G}(x) - \delta m)\, V_-(x_+, x_-, \mathbf{x}_T)\, u(p_1). \tag{5.2}$$

With the definitions (cf. (4.6) and [10])

$$\begin{aligned} p &= \tfrac{1}{2}(p_1 + p_3), \\ p' &= \tfrac{1}{2}(p_2 + p_4), \\ v &= p \cdot p' = \frac{1}{2} s - \mu^2 = 2E_+^2 + \frac{1}{2}\left(\frac{\mu^2}{2E_+}\right)^2, \end{aligned} \tag{5.3}$$

and the relations (A.35), (A.37) of [10], we obtain now for $v \to \infty$:

$$\begin{aligned} \mathcal{M}^r_{31}(G) &\to \int dx\, e^{i(p_3-p_1)x}\, g p^\rho G_\rho(x_+, x_-, \mathbf{x}_T)\, V_-(x_+, x_-, \mathbf{x}_T) \\ &\quad \times v^{-1} \bar{u}(p_3)\, \not{p}' u(p_1) \\ &\to \int dx\, e^{i(p_3-p_1)x} g G_-(x_+, x_-, \mathbf{x}_T)\, V_-(x_+, x_-, \mathbf{x}_T) \\ &\quad \times \frac{1}{\sqrt{2v}} \bar{u}(p_3)\, \not{p}' u(p_1). \end{aligned} \tag{5.4}$$

Similarly, we find for $\mathcal{M}^r_{42}(G)$:

$$\begin{aligned} \mathcal{M}^r_{42}(G) &\to \int dy\, e^{i(p_4-p_2)y} g G_+(y_+, y_-, \mathbf{y}_T)\, V_+(y_+, y_-, \mathbf{y}_T) \\ &\quad \times \frac{1}{\sqrt{2v}} \bar{u}(p_4)\, \not{p} u(p_2). \end{aligned} \tag{5.5}$$

Now we insert (5.4) and (5.5) in (3.34). Writing out the color indices $A_1 \cdots A_4$ of the quarks and using (4.24) we obtain

$$\begin{aligned} &\langle \psi(p_3, A_3)\, \psi(p_4, A_4)\, |S|\, \psi(p_1, A_1)\, \psi(p_2, A_2) \rangle \equiv S_{fi} \\ &\to Z_\psi^{-2} \int dx\, dy\, e^{i(p_3-p_1)x} e^{i(p_4-p_2)y} \frac{2}{v} \bar{u}(p_3)\, \not{p}' u(p_1)\, \bar{u}(p_4)\, \not{p} u(p_2) \\ &\quad \times \left\langle \left(\frac{\partial}{\partial x_+} V_-(x_+, x_-, \mathbf{x}_T) \right)_{A_3 A_1} \left(\frac{\partial}{\partial y_-} V_+(y_+, y_-, \mathbf{y}_T) \right)_{A_4 A_2} \right\rangle. \end{aligned} \tag{5.6}$$

Translational invariance of the functional integral allows us to split off the

δ-function of energy-momentum conservation as it should be and to write the T-matrix element in a compact form. To be explicit, with the change of variables

$$\begin{aligned} x &= X + \tfrac{1}{2}z, \\ y &= X - \tfrac{1}{2}z, \end{aligned} \tag{5.7}$$

we obtain

$$S_{fi} \to \int dX\, e^{i(p_3+p_4-p_1-p_2)X} \int dz\, e^{-iqz} \frac{2}{v} \bar{u}(p_3)\, \not{p}' u(p_1)\, \bar{u}(p_4)\, \not{p} u(p_2)$$

$$\times (-Z_\psi^{-2}) \left\langle \left(\frac{\partial}{\partial z_+} V_-(z_+, 0, \mathbf{z}_T) \right)_{A_3 A_1} \left(\frac{\partial}{\partial z_-} V_+(0, -z_-, 0) \right)_{A_4 A_2} \right\rangle, \tag{5.8}$$

$$S_{fi} = i(2\pi)^4\, \delta(p_3 + p_4 - p_1 - p_2)\, \mathscr{T}_{fi}, \tag{5.9}$$

$$\mathscr{T}_{fi} \to i \int d^2 z_T\, e^{i\mathbf{q}\mathbf{z}_T} \int dz_+ \int dz_- \frac{1}{v} \bar{u}(p_3)\, \not{p}' u(p_1)\, \bar{u}(p_4)\, \not{p} u(p_2)$$

$$\times Z_\psi^{-2} \left\langle \left(\frac{\partial}{\partial z_+} V_-(z_+, 0, \mathbf{z}_T) \right)_{A_3 A_1} \left(\frac{\partial}{\partial z_-} V_+(0, -z_-, 0) \right)_{A_4 A_2} \right\rangle. \tag{5.10}$$

Here we evaluate $\mathscr{T}_{fi}$ in the coordinate system defined in Section 4 (cf. (4.5), (4.7), and Fig. 5) and use

$$dz = \tfrac{1}{2} d^2 z_T\, dz_+\, dz_-. \tag{5.11}$$

We also remember that we work in non-forward directions, i.e., for $q \neq 0$. The integrations over $z_\pm$ in (5.10) can easily be performed. We also have

$$\frac{1}{v} \bar{u}(p_3)\, \not{p}' u(p_1)\, \bar{u}(p_4)\, \not{p} u(p_2) \to \bar{u}(p_3)\, \gamma^\mu u(p_1)\, \bar{u}(p_4)\, \gamma_\mu u(p_2) \tag{5.12}$$

for $v \to \infty$. Inserting this we obtain

$$\mathscr{T}_{fi} \to i\bar{u}(p_3)\, \gamma^\mu u(p_1)\, \bar{u}(p_4)\, \gamma_\mu u(p_2) (-Z_\psi^{-2}) \int d^2 z_T\, e^{i\mathbf{q}\mathbf{z}_T}$$

$$\times \langle [V_-(\infty, 0, \mathbf{z}_T) - 1]_{A_3 A_1} [V_+(0, \infty, 0) - 1]_{A_4 A_2} \rangle. \tag{5.13}$$

Here the neglected terms are at most of relative order $1/\sqrt{v}$.

Now we remember that we are, up to this point, working in a fixed gauge, for instance, Feynman gauge (cf. Section 3). Rotational and global $SU(3)$-colour invariance allow us then to write the following decomposition for the part of (5.13) involving the colour indices:

$$-Z_\psi^{-2} \int d^2 z_T\, e^{i\mathbf{q}\mathbf{z}_T} \langle [V_-(\infty, 0, \mathbf{z}_T) - 1]_{A_3 A_1} [V_+(0, \infty, 0) - 1]_{A_4 A_2} \rangle$$

$$= J^{(qq)}(-\mathbf{q}^2)\, \delta_{A_3 A_1}\, \delta_{A_4 A_2} + K^{(qq)}(-\mathbf{q}^2) (\tfrac{1}{2}\lambda_a)_{A_3 A_1} (\tfrac{1}{2}\lambda_a)_{A_4 A_2}. \tag{5.14}$$

The term with $J^{(qq)}$ corresponds to no colour exchange between the quarks; the term with $K^{(qq)}$ corresponds to colour exchange. In diffractive scattering of hadrons only the J-term can contribute at the parton level. The K-term presumably leads to multiparticle production as envisaged in [27]. In the following we will concentrate on the diffractive term and leave the study of the colour exchange term to a further work. We obtain then from (5.13), (5.14),

$$\begin{aligned}&\langle \psi(p_3, A_3), \psi(p_4, A_4) \,|\mathcal{T}|\, (\psi(p_1, A_1), \psi(p_2, A_2))\rangle|_{\text{diffr.}} \equiv \mathcal{T}_{fi},\\ &\mathcal{T}_{fi} \to i\bar{u}(p_3)\,\gamma^\mu u(p_1)\,\bar{u}(p_4)\,\gamma_\mu u(p_2)\,J^{(qq)}(q^2)\,\delta_{A_3A_1}\,\delta_{A_4A_2},\end{aligned} \tag{5.15}$$

where

$$\begin{aligned}J^{(qq)}(q^2) \equiv J^{(qq)}(-\mathbf{q}^2) = -Z_\psi^{-2} \int d^2z_T\, e^{i\mathbf{q}\mathbf{z}_T} \frac{1}{9} \langle \mathrm{Tr}[V_-(\infty, 0, \mathbf{z}_T) - 1]\\ \cdot \mathrm{Tr}[V_+(0, \infty, 0) - 1]\rangle.\end{aligned} \tag{5.16}$$

Here and in the following $\mathcal{T}_{fi}$ is understood as the non-colour exchange part of the T-matrix element.

In our approach the diffractive amplitude at the quark level is thus governed by the correlation function of two string operators $V_+(0, \infty, 0)$ and $V_-(\infty, 0, \mathbf{z}_T)$ (Fig. 7). Several comments are now in order.

If we work in the abelian gluon model [10], all the same steps can be carried

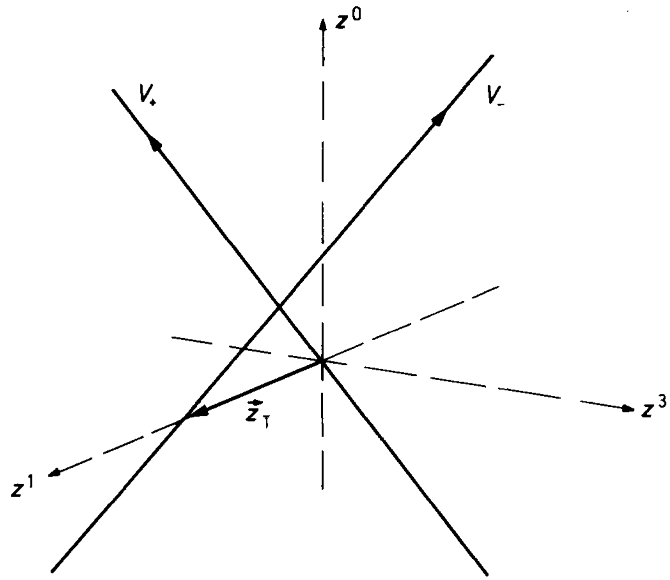

FIG. 7. Illustration of the string operators, the correlation function of which governs diffractive scattering at the quark level (cf. (5.15), (5.16)).

through. The details are given in Appendix A. Expanding there up to fourth order in the coupling constant and making for the abelian gluon propagator the same Ansatz as in (3.8), (3.13) of [10], we reproduce easily the result (4.10) of [10] for the T-matrix element. We have in this way a check of our present calculational scheme by an explicit Feynman diagram evaluation in a special case. Our expression (5.15) constitutes an extension of the results of [10] to the non-abelian case and to all orders in the coupling constant.

The correlation function

$$Z_{\psi}^{-2}\langle \mathrm{Tr}[V_-(\infty, 0, \mathbf{z}_T) - 1] \cdot \mathrm{Tr}[V_+(0, \infty, 0) - 1] \rangle \tag{5.17}$$

should rapidly fall off to zero for $|\mathbf{z}_T| \to \infty$ if the physical picture developed in [10] is correct. A convenient parameter measuring this falloff should be identified with the correlation length a of [10] and Section 1. It should be possible to evaluate (5.16) by available nonperturbative methods, since the external scale in (5.16) is given by $|\mathbf{q}| \lesssim 1$ GeV, whereas the large scale s which goes to infinity no longer enters. In Monte Carlo simulations in the framework of lattice gauge theories for instance a spatial resolution of the order of $1/|\mathbf{q}|$ should then be sufficient to get an estimate of $J^{(qq)}$. In principle we should thus be able to check in the future whether the hypothesis $a^2 \ll R^2$, where R is the radius of a light hadron is correct or not. In the present paper as in [10] we adopt $a^2 \ll R^2$ as a working hypothesis.

Let us consider next gauge transformations (4.4) where we require

$$\lim_{x^0 \to \pm\infty} U(x) = U_0 \tag{5.18}$$

with some fixed $SU(3)$-matrix U_0. We find easily that $V_\pm$ transforms then as

$$\begin{aligned} V_+(0, \infty, 0) &\to U_0 V_+(0, \infty, 0)\, U_0^\dagger, \\ V_-(\infty, 0, \mathbf{z}_T) &\to U_0 V_-(\infty, 0, \mathbf{z}_T)\, U_0^\dagger, \end{aligned} \tag{5.19}$$

and that $Z_{\psi}^2 \cdot J^{(qq)}(q^2)$ (5.16) is invariant. We can achieve $Z_{\psi}^2 \cdot J^{(qq)}(q^2)$ to be invariant under *arbitrary* gauge transformations if we impose periodic boundary conditions in Minkowski space by identifying the following points (Fig. 8)

$$\begin{aligned} (x_+, x_-, \mathbf{x}_T) &\equiv (x_+ + T, x_-, \mathbf{x}_T) \\ &\equiv (x_+, x_- + T, \mathbf{x}_T). \end{aligned} \tag{5.20}$$

Here T is some large length ($T \gg R \gg a$). In essence (5.16) represents then the correlation function of two closed strings (loops). These loops are similar to the ones considered in [28] but our loops live in Minkowski instead of Euclidean space and run along light-like rays. We can give a plausible physical argument for imposing (5.20), i.e., closing our strings. In a theory of hadron–hadron elastic scattering we should start in principle with the reduction formula for hadrons, not for quarks, as we did in Section 3. Suitable interpolating operators for hadrons are

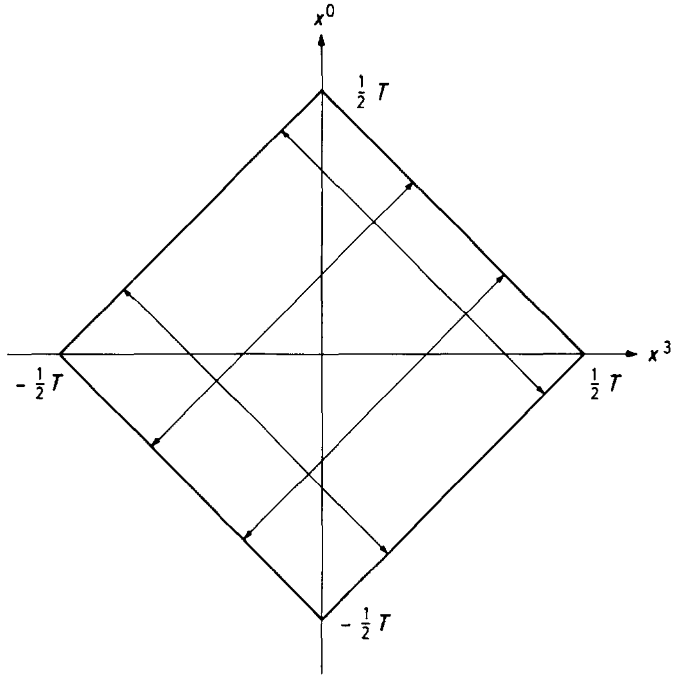

FIG. 8. Periodicity domain and identification of points in Minkowski space when imposing (5.20).

string operators. In the S-matrix element for hadron–hadron elastic scattering we will presumably encounter string operators as in (5.13) for all the hadron's constituents with the strings closed at $x^0 \to \pm\infty$ by the strings representing the hadron's wave function. If now $a^2 \ll R^2$ all what matters should be the interaction of one constituent of a hadron with one constituent of the other hadron at a time. Only the region near $z = 0$ in Fig. 7 should thus contribute significantly to the correlation function (5.16) and the precise way how the strings are closed—i.e., by attaching the hadron's strings or by imposing (5.20)—should not matter. As an empirical support of this point of view we can cite the approximate validity of the additive quark rule [1] for baryon and meson scattering.

Finally we should stress again that our result (5.15), (5.16) for quark–quark scattering at high energies depends crucially on the assumption that only gluon modes up to a fixed frequency contribute in the functional integral. We have argued at length in Section 2 that this should be a valid approximation for the scattering of our partons over the time interval $(-t_0/2, t_0/2)$. In the following we will only use (5.15) folded with the hadronic wave functions. The hypothetical scattering amplitude for "real" quarks should *not* be identified with (5.15). In such an amplitude clearly all the splitting processes with long time scales will play an important role as discussed in Section 2.

5.2. The Wave Function Renormalization Constant Z_ψ

We will now calculate Z_ψ occurring in (5.16) in a similar way as the scattering amplitude. Let us start with

$$J_\mu(x) = \bar{\psi}(x)\,\gamma_\mu \psi(x) \tag{5.21}$$

which is three times the baryon number current. This current is colour neutral and conserved, the renormalized current J'_μ can thus be taken equal to the unrenormalized one:

$$J'_\mu(x) = J_\mu(x). \tag{5.22}$$

This is a consequence of the usual Ward identity telling us that the vertex function of J_μ gets renormalized by Z_ψ. We find it convenient to calculate Z_ψ approximately starting from this vertex function.

To give the details, let us consider the matrix element of $J'_\mu(0)$ for a quark state and apply the reduction formula. We find

$$\begin{aligned}
&\langle \psi(p_1, A)\,|J'_\mu(0)|\,\psi(p_1, B)\rangle \\
&\quad = \bar{u}(p_1)\,\gamma_\mu u(p_1)\,\delta_{AB} \\
&\quad = Z_\psi^{-1} \int dx \int dy\, \bar{u}(p_1)\, e^{ip_1 x}(i\overrightarrow{\not\partial}_x - m') \\
&\qquad \times \langle 0|\,T(\psi(x)\,\bar{\psi}(0)\,\gamma_\mu\psi(0)\,\bar{\psi}(y))\,|0\rangle(i\overleftarrow{\not\partial}_y + m')\,u(p_1)\,e^{-ip_1 y},
\end{aligned} \tag{5.23}$$

$$\begin{aligned}
&Z_\psi \bar{u}(p_1)\,\gamma_\mu u(p_1)\,\delta_{AB} \\
&\quad = \int dx \int dy \langle \bar{u}(p_1)\, e^{ip_1 x}(-i\overrightarrow{\not\partial}_x + m') \\
&\qquad \times [S_F(x, 0; G)\,\gamma_\mu S_F(0, y; G))]_{AB}\,(i\overleftarrow{\not\partial}_y + m')\,u(p_1)\,e^{-ip_1 y}\rangle.
\end{aligned} \tag{5.24}$$

Here A, B are the colour indices. In analogy to (3.23) we can define a wave function

$$(\bar{\Psi}^F_{p_1}| = (p_1|\,(-i\overrightarrow{\not\partial} + m')\,S_F, \tag{5.25}$$

satisfying

$$(\bar{\Psi}^F_{p_1}|\,(i\overleftarrow{\not\partial} + g\not{G} + m) = 0. \tag{5.26}$$

We then have

$$Z_\psi \bar{u}(p_1)\,\gamma_\mu u(p_1)\,\delta_{AB} = \langle \bar{\Psi}^F_{p_1,A}(0)\,\gamma_\mu \psi^F_{p_1,B}(0)\rangle. \tag{5.27}$$

Using arguments similar to those in Section 3, (3.28)–(3.33), we can show that for slowly varying gluon potentials $(\bar{\psi}^F_{p_1}|$ can be replaced by the corresponding advanced function, defined by

$$(\bar{\psi}^{\text{adv}}_{p_1}| = (p_1|\,(-i\overleftarrow{\not\partial} + m')\,S_r, \tag{5.28}$$

which satisfies

$$(\bar{\psi}^{\text{adv}}_{p_1}|\,(i\overleftarrow{\not\partial} + g\not{G} + m) = 0, \tag{5.29}$$

and as boundary condition

$$(\bar{\psi}^{\text{adv}}_{p_1}| \to (p_1| \tag{5.30}$$

for $x^0 \to +\infty$. Note that in (5.28) we must use the retarded Green's function S_r to get the advanced wave function. The differential equation (5.29) with the boundary condition (5.30) can be solved approximately using the methods of Section 4 and we find

$$\bar{\psi}^{\text{adv}}_{p_1}(x) = \tilde{V}_-(x_+, x_-, \mathbf{x}_T)\{1 + O(E_+^{-1})\}\, e^{-ip_1x} u(p_1). \tag{5.31}$$

Here

$$\tilde{V}_-(x_+, x_-, \mathbf{x}_T) = \bar{P}\left\{\exp\left[\frac{i}{2} g \int_{x_+}^{\infty} dx'_+\, G_-(x'_+, x_-, \mathbf{x}_T)\right]\right\}, \tag{5.32}$$

and $\bar{P}$ means anti-path-ordering. It is easy to show that

$$\frac{\partial}{\partial x_+}\,[\tilde{V}^\dagger_-(x_+, x_-, \mathbf{x}_T)\, V_-(x_+, x_-, \mathbf{x}_T)] = 0, \tag{5.33}$$

where V_- is defined in (4.19). From this we obtain

$$\tilde{V}^\dagger_-(x_+, x_-, \mathbf{x}_T)\, V_-(x_+, x_-, \mathbf{x}_T) = V_-(\infty, x_-, \mathbf{x}_T), \tag{5.34}$$

since $\tilde{V}_-$ goes to 1 for $x_+ \to \infty$.

Now inserting everything in (5.27) we find

$$\begin{aligned} Z_\psi \bar{u}(p_1)\,\gamma_\mu u(p_1)\,\delta_{AB} &\approx \langle \bar{\psi}^{\text{adv}}_{p_1,A}(0)\,\gamma_\mu \psi^r_{p_1,B}(0)\rangle \\ &\to \bar{u}(p_1)\,\gamma_\mu u(p_1) \langle [\tilde{V}^\dagger_-(0,0,0)\, V_-(0,0,0)]_{AB}\rangle \end{aligned} \tag{5.35}$$

for $E_+ \to \infty$. With (5.34) we are led to

$$Z_\psi \approx \tfrac{1}{3}\langle \mathrm{Tr}\; V_-(\infty, 0, 0)\rangle. \tag{5.36}$$

This is our result for the wave function renormalization constant. Also here we will impose the periodicity conditions (5.20). We find then (5.36) to be invariant under arbitrary gauge transformations.

5.3. *Quark–Antiquark and Antiquark–Antiquark Scattering*

Let us consider first quark–antiquark scattering:

$$\psi(p_1, A_1) + \bar{\psi}(p_2, A_2) \to \psi(p_3, A_3) + \bar{\psi}(p_4, A_4). \tag{5.37}$$

The analysis of this reaction runs along similar lines as for quark–quark scattering in Section 3. The result is that in analogy to (3.34) the S-matrix element for (5.37) can be expressed for $s \to \infty$ and small t as

$$S_{fi} \approx -Z_{\psi}^{-2} \langle \mathcal{M}^r_{31}(G)\, \mathcal{M}'^r_{42}(G) \rangle. \tag{5.38}$$

Here s-channel exchange diagrams have been neglected with respect to t-channel exchange ones and $\mathcal{M}'^r_{42}(G)$ is the matrix element for the antiquark $\bar{\psi}$ in (5.37) scattering on the given gluon potential G. But the scattering of an antiquark on G can be related to the scattering of a quark on the charge conjugated (C-transformed) gluon potential G' defined by

$$G'_\lambda(x) = -G^{\mathrm{T}}_\lambda(x) = -G^*_\lambda(x). \tag{5.39}$$

We thus obtain

$$\mathcal{M}'^r_{42}(G)|_{\text{antiquarks}} = \mathcal{M}^r_{42}(G')|_{\text{quarks}}. \tag{5.40}$$

It is also easy to see that the non-abelian phase factors $V'_\pm$ defined as in (4.19), (4.23) but with G replaced by G' are just the complex conjugates of $V_\pm$:

$$V'_\pm(x_+, x_-, \mathbf{x}_T) = V^*_\pm(x_+, x_-, \mathbf{x}_T). \tag{5.41}$$

Putting everything together we now find from (5.13), (5.14) for $\nu \to \infty$:

$$\begin{aligned} &\langle \psi(p_3, A_3)\, \bar{\psi}(p_4, A_4)\, |\mathcal{T}|\, \psi(p_1, A_1)\, \bar{\psi}(p_2, A_2) \rangle \\ &\quad \to i\bar{u}(p_3)\, \gamma^\mu u(p_1)\, \bar{v}(p_2)\, \gamma_\mu v(p_4) \\ &\qquad \times [J^{(q\bar{q})}(q^2)\, \delta_{A_3A_1}\, \delta_{A_2A_4} + K^{(q\bar{q})}(q^2)(\tfrac{1}{2}\lambda_a)_{A_3A_1}\, (\tfrac{1}{2}\lambda_a)_{A_2A_4}], \end{aligned} \tag{5.42}$$

where $J^{(q\bar{q})}$ and $K^{(q\bar{q})}$ are defined by

$$\begin{aligned} &-Z_{\psi}^{-2} \int d^2z_T\, e^{i\mathbf{q}\mathbf{z}_T} \langle [V_-(\infty, 0, \mathbf{z}_T) - 1]_{A_3A_1}\, [V^*_+(0, \infty, 0) - 1]_{A_4A_2} \rangle \\ &\quad = J^{(q\bar{q})}(-\mathbf{q}^2)\, \delta_{A_3A_1}\, \delta_{A_2A_4} + K^{(q\bar{q})}(-\mathbf{q}^2)(\tfrac{1}{2}\lambda_a)_{A_3A_1}\, (\tfrac{1}{2}\lambda_a)_{A_2A_4}. \end{aligned} \tag{5.43}$$

For antiquark–antiquark scattering we can directly use charge conjugation (C) invariance to relate it to quark–quark scattering. It is also easy to check that our approximation scheme respects C-invariance.

At the end of this section we summarize our findings for the quark and antiquark scattering amplitudes at $\nu \to \infty$ and small t ($0 < |t| \lesssim 1\ \mathrm{GeV}^2$). The diffractive, i.e., no colour exchange parts of these amplitudes are governed by correlation functions

$$J^{(qq)}(-\mathbf{q}^2) = -Z_\psi^{-2} \int d^2 z_T \, e^{i\mathbf{q}\mathbf{z}_T}$$

$$\times \frac{1}{9} \langle \mathrm{Tr}[V_-(\infty, 0, \mathbf{z}_T) - 1] \, \mathrm{Tr}[V_+(0, \infty, 0) - 1] \rangle, \quad (5.44)$$

$$J^{(q\bar{q})}(-\mathbf{q}^2) = -Z_\psi^{-2} \int d^2 z_T \, e^{i\mathbf{q}\mathbf{z}_T}$$

$$\times \frac{1}{9} \langle \mathrm{Tr}[V_-(\infty, 0, \mathbf{z}_T) - 1] \, \mathrm{Tr}[V_+^*(0, \infty, 0) - 1] \rangle. \quad (5.45)$$

Here Z_ψ can be expressed as follows using (5.36), translational, rotational, and C-invariance:

$$\begin{aligned} Z_\psi &= \tfrac{1}{3} \langle \mathrm{Tr}\, V_-(\infty, 0, 0) \rangle \\ &= \tfrac{1}{3} \langle \mathrm{Tr}\, V_-(\infty, 0, \mathbf{z}_T) \rangle \\ &= \tfrac{1}{3} \langle \mathrm{Tr}\, V_+(0, \infty, 0) \rangle \\ &= \tfrac{1}{3} \langle \mathrm{Tr}\, V_+^*(0, \infty, 0) \rangle. \end{aligned} \quad (5.46)$$

Let us define

$$J_\pm(q^2) = \tfrac{1}{2}(J^{(qq)}(q^2) \pm J^{(q\bar{q})}(q^2)). \quad (5.47)$$

We then have

$$\langle \psi(p_3, A_3)\, \psi(p_4, A_4) \,|\mathcal{T}|\, \psi(p_1, A_1)\, \psi(p_2, A_2) \rangle$$

$$\to i\bar{u}(p_3)\, \gamma^\mu u(p_1)\, \bar{u}(p_4)\, \gamma_\mu u(p_2)[J_+(q^2) + J_-(q^2)]\, \delta_{A_3 A_1}\, \delta_{A_4 A_2}, \quad (5.48)$$

$$\langle \psi(p_3, A_3)\, \bar{\psi}(p_4, A_4) \,|\mathcal{T}|\, \psi(p_1, A_1)\, \bar{\psi}(p_2, A_2) \rangle$$

$$\to i\bar{u}(p_3)\, \gamma^\mu u(p_1)\, \bar{v}(p_2)\, \gamma_\mu v(p_4)[J_+(q^2) - J_-(q^2)]\, \delta_{A_3 A_1}\, \delta_{A_2 A_4}, \quad (5.49)$$

$$\langle \bar{\psi}(p_3, A_3)\, \bar{\psi}(p_4, A_4) \,|\mathcal{T}|\, \bar{\psi}(p_1, A_1)\, \bar{\psi}(p_2, A_2) \rangle$$

$$\to i\bar{v}(p_1)\, \gamma^\mu v(p_3)\, \bar{v}(p_2)\, \gamma_\mu v(p_4)[J_+(q^2) + J_-(q^2)]\, \delta_{A_1 A_3}\, \delta_{A_2 A_4}. \quad (5.50)$$

From Sections 5.1 and 5.2 we deduce that $J_\pm(q^2)$ are gauge-invariant functions if we impose the periodicity conditions (5.20) as we will always do in the following. Due to (3.11) $J_\pm(q^2)$ are not necessarily real, the scattering amplitudes (5.48)–(5.50) may have real parts.

6. From quarks to Hadrons

Our aim in this section is first to write down an effective S-operator or effective Lagrangian which reproduces the amplitudes (5.48)–(5.50). This effective S-operator will then be sandwiched between hadron states to arrive at the hadronic amplitudes. In this way we will avoid dealing explicitly with hadronic wave functions.

Looking at the amplitudes (5.48)–(5.50), one might naively conclude from the $\gamma^\mu \otimes \gamma_\mu$ structure that they all correspond to the exchange of an effective vector particle. However, this is *not* the case. The exchange of a vector particle in the field theoretic sense leads to a sign change in the amplitudes when going from quark–quark to quark–antiquark scattering. (This is as for one photon exchange which leads to a repulsive interaction for electron–electron but an attractive one for electron–positron scattering.) This sign change is present in (5.48)–(5.50) in the J_--terms which correspond thus to a vector exchange ($C=-1$), but no sign change occurs for the J_+-terms which represent $C=+1$ exchanges. Our trick will be to write the J_+-terms as an infinite series of field-theoretic couplings corresponding to the exchange of tensors of rank 2, 4, 6, ... having $C=+1$.

In this and the following sections we will take into account u and d quarks, the correct number of light flavours. We continue to neglect the s-quark and heavier quark flavours.

6.1. *Twist-Two Operators*

Let us start by recalling the usual twist 2 tensor operators (cf., e.g., [29, 30]), familiar from deep inelastic lepton–nucleon scattering. We define the unrenormalized tensor operators of rank n as

$$O_{n,q}^{\mu_1 \cdots \mu_n}(x) = \frac{i^{n-1}}{n!\, 2^{n-1}} \sum_{\psi=u,d} \bar{\psi}(x)[\gamma^{\mu_1} \overleftrightarrow{D}^{\mu_2} \cdots \overleftrightarrow{D}^{\mu_n} + \text{permutations} - \text{traces}]\, \psi(x) \qquad (n=1,2,3,...); \tag{6.1}$$

$$O_{n,G}^{\mu_1 \cdots \mu_n}(x) = \frac{i^{n-2}}{n!\, 2^{n-2}} [G^{\alpha\mu_1\lambda}(x)\, \overleftrightarrow{D}^{\mu_3} \cdots \overleftrightarrow{D}^{\mu_n} G^{\alpha\mu_2}_{\lambda}(x) + \text{permutations} - \text{traces}] \qquad (n=2,4,...). \tag{6.2}$$

Here D^μ denotes the covariant derivative of quark and gluon fields, respectively, and $\overleftrightarrow{D}^\mu = \vec{D}^\mu - \overleftarrow{D}^\mu$. By construction the operators, (6.1), (6.2) are totally symmetric and traceless. The notation is as in [29] except for a factor of 2 in $O_{n,G}$.

The quark-operators $O_{n,q}$ are even (odd) under the C-transformation for n even (odd). The gluon operators $O_{n,G}$ are even under C.

We will consider now renormalization prescriptions for our operators. Let us, on the one hand, introduce renormalized operators O' defined by one of the standard renormalization schemes with scale Q_0^2, appropriate for the description of DIS. We have then with certain renormalization constants Z':

$$O_{n,q}(x) = Z'_n(Q_0^2)\, O'_{n,q}(x; Q_0^2) \qquad (n=1,3,...) \tag{6.3}$$

$$\begin{pmatrix} O_{n,q}(x) \\ O_{n,G}(x) \end{pmatrix} = \mathscr{Z}'_n(Q_0^2) \begin{pmatrix} O'_{n,q}(x; Q_0^2) \\ O'_{n,G}(x; Q_0^2) \end{pmatrix} \qquad (n=2,4,...), \tag{6.4}$$

where

$$\mathscr{Z}'_n(Q_0^2)=\begin{pmatrix} Z'_{n,qq}(Q_0^2) & Z'_{n,qG}(Q_0^2) \\ Z'_{n,Gq}(Q_0^2) & Z'_{n,GG}(Q_0^2) \end{pmatrix} \tag{6.5}$$

is a matrix taking into account the mixing of quark and gluon operators.

The operator $O^{\mu}_{1,q}$ of (6.1) is, of course, three times the baryon number current $B^{\mu}(x)$

$$\begin{aligned} B^{\mu}(x) &= \tfrac{1}{3} O^{\mu}_{1,q}(x) \\ &= \tfrac{1}{3} \sum_{\psi=u,d} \bar{\psi}(x)\gamma^{\mu}\psi(x). \end{aligned} \tag{6.6}$$

This current needs no renormalization (cf. (5.22)) and we can thus set $Z'_1(Q_0^2)\equiv 1$ in (6.3).

Let now h be a hadron. We can define the parton densities of h at the scale Q_0^2 through the following moment relations where as usual Σ' means the spin average:

$$\begin{aligned} &\sum_{h-\text{spin}}' \langle h(p_1)\,|O'^{\mu_1\cdots\mu_n}_{n,q}(0;Q_0^2)|\,h(p_1)\rangle \\ &\quad = [2p_1^{\mu_1}\cdots p_1^{\mu_n} - \text{traces}] \\ &\quad \times \int_0^1 d\xi\, \xi^{n-1}[N_q^h(\xi;Q_0^2)+(-1)^n N_{\bar{q}}^h(\xi;Q_0^2)] \qquad (n=1,2,3,\ldots); \end{aligned} \tag{6.7}$$

$$\begin{aligned} &\sum_{h-\text{spin}}' \langle h(p_1)\,|O'^{\mu_1\cdots\mu_n}_{n,G}(0;Q_0^2)|\,h(p_1)\rangle \\ &\quad = [2p_1^{\mu_1}\cdots p_1^{\mu_n} - \text{traces}] \\ &\quad \times \int_0^1 d\xi\, \xi^{n-1} N_G^h(\xi;Q_0^2) \qquad (n=2,4,6,\ldots). \end{aligned} \tag{6.8}$$

Here $N_q^h(N_{\bar{q}}^h)$ is the sum of the u- and d-quark ($\bar{u}$- and $\bar{d}$-antiquark) distribution functions of hadron h. In the field-theoretic parton model of [15] the functions $N^h_{q,\bar{q},G}(\xi, Q_0^2)$ are the number densities of the effective partons seen at the scale Q_0^2. This implies that the forward matrix elements of the operators

$$O'_{n,q}(0,Q_0^2), \qquad O'_{n,G}(0,;Q_0^2)$$

between the effective parton states seen at scale Q_0^2 are given by the corresponding free field values. Here the effective parton states are understood as in Section 3.

As a generalization of this renormalization scheme let us introduce a scheme where we define operators $O'(x;Q_0^2,t)$ by requiring their matrix elements for these quark and gluon states to be given by the free-field values at squared momentum transfer t:

$$\langle\psi(p_3, A, r)\,|O'^{\mu_1\cdots\mu_n}_{n,q}(0; Q_0^2, t)|\,\psi(p_1, B, s)\rangle$$
$$= [2p^{\mu_1}\cdots p^{\mu_n} - \text{traces}]\,\delta_{AB}\,\delta_{rs} + \text{terms with } q^\mu, \tag{6.9}$$
$$\langle G(p_3, a, j)\,|O'^{\mu_1\cdots\mu_n}_{n,q}(0; Q_0^2, t)|\,G(p_1, b, k)\rangle$$
$$= 0 \qquad \text{up to terms with } q^\mu \qquad (n = 1, 2, 3, \ldots); \tag{6.10}$$
$$\langle\psi(p_3, A, r)\,|O'^{\mu_1\cdots\mu_n}_{n,G}(0; Q_0^2, t)|\,\psi(p_1, B, s)\rangle$$
$$= 0 \qquad \text{up to terms with } q^\mu, \tag{6.11}$$
$$\langle G(p_3, a, j)\,|O'^{\mu_1\cdots\mu_n}_{n,G}(0; Q_0^2, t)|\,G(p_1, b, k)\rangle$$
$$= [2p^{\mu_1}\cdots p^{\mu_n} - \text{traces}]\,\delta_{ab}\,\delta_{jk} + \text{terms with } q^\mu \qquad (n = 2, 4, 6, \ldots). \tag{6.12}$$

Here $\psi = u, d$ and A, B, r, s (a, b, j, k) are the colour and spin indices of the quarks (gluons). As in Section 5, we set $p = (p_1 + p_3)/2$, $q = p_1 - p_3$, and $t = q^2$. The transition from $O'(x; Q_0^2, t)$ to $O'(x; Q_0^2)$ corresponds to a change of renormalization scheme and we have

$$O'_{n,q}(x; Q_0^2, t) = z_n(t, Q_0^2)\, O'_{n,q}(x; Q_0^2) \qquad (n = 1, 3, \ldots); \tag{6.13}$$
$$\begin{pmatrix} O'_{n,q}(x; Q_0^2, t) \\ O'_{n,G}(x; Q_0^2, t) \end{pmatrix} = \zeta_n(t; Q_0^2) \begin{pmatrix} O'_{n,q}(x; Q_0^2) \\ O'_{n,G}(x; Q_0^2) \end{pmatrix} \qquad (n = 2, 4, \ldots), \tag{6.14}$$

where the z_n are finite numbers, the ζ_n finite 2×2 matrizes. By construction we have

$$\begin{aligned} O'_{n,q}(x; Q_0^2, t))|_{t=0} &\equiv O'_{n,q}(x; Q_0^2), \\ O'_{n,G}(x; Q_0^2, t))|_{t=0} &\equiv O'_{n,G}(x; Q_0^2), \end{aligned} \tag{6.15}$$
$$\begin{aligned} z_n(t, Q_0^2)|_{t=0} &= 1, \\ \zeta_n(t, Q_0^2)|_{t=0} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{6.16}$$

The calculation of the z_n and ζ_n should be possible with the methods of Section 5.2.

6.2. *The Effective S-Operator*

Let us consider (5.48) and use the following identity, valid for $\nu > 0$:

$$\begin{aligned} J_+(q^2) &= J_+(q^2)\frac{\nu}{\sqrt{\nu^2}} \\ &= J_+(q^2)(pp')\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\alpha\, e^{-\alpha^2\nu^2} \\ &= J_+(q^2)\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\alpha \sum_{n=0}^{\infty} \frac{(-\alpha^2)^n}{n!}\, p^{\mu_1}\cdots p^{\mu_{2n+1}} p'_{\mu_1}\cdots p'_{\mu_{2n+1}}. \end{aligned} \tag{6.17}$$

Inserting this in (5.48) leads to

$$\langle \psi(p_3, A_3), \psi(p_4, A_4) \,|S|\, \psi(p_1, A_1), \psi(p_2, A_2) \rangle$$

$$= -(2\pi)^4\, \delta(p_3 + p_4 - p_1 - p_2) \left\{ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\alpha \sum_{n=0}^{\infty} \frac{(-\alpha^2)^n}{n!} J_+(q^2) \right.$$

$$\times \bar{u}(p_3)\, \gamma^{\mu_1} p^{\mu_2} \cdots p^{\mu_{2n+2}} u(p_1)\, \delta_{A_3 A_1} \bar{u}(p_4)\, \gamma_{\mu_1} p'_{\mu_2} \cdots p'_{\mu_{2n+2}} u(p_2)\, \delta_{A_4 A_2}$$

$$\left. + J_-(q^2)\, \bar{u}(p_3)\, \gamma^{\mu} u(p_1)\, \delta_{A_3 A_1} \bar{u}(p_4)\, \gamma_{\mu} u(p_2)\, \delta_{A_4 A_2} \right\}$$

$$= -(2\pi)^4\, \delta(p_3 + p_4 - p_1 - p_2) \left\{ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\alpha \sum_{n=0}^{\infty} \frac{(-\alpha^2)^n}{n!} J_+(q^2) \right.$$

$$\times \langle \psi(p_3, A_3) \,|O'^{\mu_1 \cdots \mu_{2n+2}}_{2n+2,q}(0; Q_0^2, q^2)|\, \psi(p_1, A_1) \rangle$$

$$\times \langle \psi(p_4, A_4) \,|O'_{2n+2,q,\mu_1 \cdots \mu_{2n+2}}(0; Q_0^2, q^2)|\, \psi(p_2, A_2) \rangle$$

$$+ J_-(q^2) \langle \psi(p_3, A_3) \,|O'^{\mu}_{1,q}(0; Q_0^2, q^2)|\, \psi(p_1, A_1) \rangle$$

$$\left. \times \langle \psi(p_4, A_4) \,|O'_{1,q,\mu}(0; Q_0^2, q^2)|\, \psi(p_2, A_2) \rangle \right\}. \qquad (6.18)$$

Here $\psi = u, d$, and we used (6.9), (6, 11), and (A.35), (A.37) of [10]. We also recall that we are working in non-forward directions and for $\nu \to \infty$. In (6.18) and the following, the scale parameter Q_0^2 is given by (2.26).

We will now write (6.18) as the matrix element of an effective S-operator between quark states. For this purpose let us define the following scalar and matrix functions:

$$P_-(x; Q_0^2) = \int \frac{dk}{(2\pi)^4} e^{-ikx} J_-(k^2) [z_1(k^2, Q_0^2)]^2 \qquad (6.19)$$

$$P_+^{2n+2}(x; Q_0^2) = \int \frac{dk}{(2\pi)^4} e^{-ikx} J_+(k^2)\, \zeta^{\mathrm{T}}_{2n+2}(k^2, Q_0^2) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \zeta_{2n+2}(k^2, Q_0^2) \qquad (6.20)$$

We assume here that the integrands can be suitably continued to arbitrary k^2. With the help of (6.13), (6.14), (6.6), it is now easy to write down the desired S-operator as

$$S_{\text{eff}} = 1 - \frac{1}{2} \int dx \int dy \left\{ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\alpha \sum_{n=0}^{\infty} \frac{(-\alpha^2)^n}{n!} \right.$$

$$\times (O'^{\chi}_{2n+2,q}(x; Q_0^2), O'^{\chi}_{2n+2,G}(x; Q_0^2))\, P_+^{2n+2}(x - y; Q_0^2) \begin{pmatrix} O'_{2n+2,q,\chi}(y; Q_0^2) \\ O'_{2n+2,G,\chi}(y; Q_0^2) \end{pmatrix}$$

$$\left. + 9 B^{\mu}(x)\, P_-(x - y; Q_0^2)\, B_{\mu}(y) \right\}. \qquad (6.21)$$

The spin indices $\mu_1, ..., \mu_{2n+2}$ of the operators O'_{2n+2} are collectively denoted by χ. It is easy to check that the effective S-operator (6.21) when sandwiched between quark–quark, quark–antiquark, and antiquark–antiquark states reproduces *all* the T-matrix elements (5.48)–(5.50) of parton–parton scattering, including the case where the two partons have different flavours. Here and in the following we make the convention that the matrix elements of the products of operators occurring in (6.21) should be factorized into the product of the matrix elements for right- and left-moving particles:

$$\begin{aligned}&\langle\psi(p_3)\,\psi(p_4)\,|O(x)\,O(y)|\,\psi(p_1)\,\psi(p_2)\rangle\\&\quad\to\langle\psi(p_3)\,|O(x)|\,\psi(p_1)\rangle\langle\psi(p_4)\,|O(y)|\,\psi(p_2)\rangle+(x\leftrightarrow y).\end{aligned} \tag{6.22}$$

We also recall that we are working in the high energy approximation, neglecting terms of relative order $\nu^{-1/2}$.

The effective S-operator (6.21) embodies all the information on high energy quark–quark scattering contained in (5.48)–(5.50), but it has important advantages. In (6.21) no reference to free quark states is made any more, and only physical operators O'_{2n+2}, B_μ occur, whose matrix elements between hadron states are observable quantities. It is, however, clear that S_{eff} (6.21) is not yet the complete effective S-operator. Terms describing gluon–quark and gluon–gluon scattering are still missing. We expect them to have a similar structure: a sum of twist-2 times twist-2 operators. This will be dealt with in a separate paper. What about inelastic quark–quark scattering? As discussed in Section 2, following Eq. (2.22) only the emission of wee partons has to be considered over the time interval $(-t_0/2, t_0/2)$. Such amplitudes are describable by an effective S-operator as in (6.21), but with higher twist instead of twist-2 operators. For dimensional reasons the contributions of higher twist terms to S_{eff} have to come with extra powers of the vacuum correlation length a or the inverse condensate mass M_c^{-1}, which are both small compared to the hadron radius R. In the following we will take matrix elements of S_{eff} between hadron states. The matrix elements of higher twist operators are proportional to higher point parton densities in the hadrons [31]. But the chance to find two or more partons at the same point should be small, being governed by R, and we expect, therefore, higher twist terms in S_{eff} to lead to contributions to the hadronic amplitudes being suppressed by powers of a^2/R^2.

6.3. *Hadron–Hadron Elastic Scattering*

Consider now elastic scattering of two hadrons h, h' of spin 0 or $\frac{1}{2}$:

$$h(p_1)+h'(p_2)\to h(p_3)+h'(p_4). \tag{6.23}$$

We will sandwich S_{eff} (6.21) between these hadron states, keeping in mind (6.22), and obtain expressions for the high energy small angle scattering amplitude. The relevant matrix elements of the operators of (6.21) can be written as follows using Poincaré and parity invariance:

$$\begin{aligned}
&\langle h(p_3, s_3)\,|B^\mu(x)|\,h(p_1, s_1)\rangle \\
&\quad = 2p^\mu e^{i(p_3-p_1)x}\,\delta_{s_3s_1} F_B^h(q^2) + \text{terms with } q^\mu; && (6.24)\\
&\langle h(p_3, s_3)\,|O'^{\chi}_{n,q}(x; Q_0^2)|\,h(p_1, s_1)\rangle \\
&\quad = 2[p^{\mu_1}\cdots p^{\mu_n} - \text{traces}]\,e^{i(p_3-p_1)x}\,\delta_{s_3s_1} F^h_{n,q}(q^2; Q_0^2)\\
&\quad\quad \times \int_0^1 d\xi\,\xi^{n-1}[N_q^h(\xi; Q_0^2) + (-1)^n N_{\bar q}^h(\xi; Q_0^2)]\\
&\quad\quad + \text{terms with } q^\mu \qquad (n = 2, 3, 4, \ldots); && (6.25)\\
&\langle h(p_3, s_3)\,|O'^{\chi}_{n,G}(x, Q_0^2)|\,h(p_1, s_1)\rangle \\
&\quad = 2[p^{\mu_1}\cdots p^{\mu_n} - \text{traces}]\,e^{i(p_3-p_1)x}\,\delta_{s_3s_1} F^h_{n,G}(q^2; Q_0^2)\\
&\quad\quad \times \int_0^1 d\xi\,\xi^{n-1} N_G^h(\xi; Q_0^2) + \text{terms with } q^\mu \qquad (n = 2, 4, 6, \ldots). && (6.26)
\end{aligned}$$

Here s_1, s_3 are the spin indices in the case of a spin $\frac{1}{2}$ hadron h. For a spin 0 hadron these indices have to be dropped and $\delta_{s_3s_1}$ is replaced by 1. In (6.24), $F_B^h(q^2)$ is the Dirac form factor of h for the baryon number current, where

$$F_B^h(0) = B^h, \tag{6.27}$$

with B^h as the baryon number of hadron h. The parton densities in (6.25), (6.26) are as defined in (6.7), (6.8) and $F^h_{n,q}$, $F^h_{n,G}$ are form factors normalized to 1 at $q^2 = 0$:

$$F^h_{n,q}(0; Q_0^2) = F^h_{n,G}(0; Q_0^2) = 1. \tag{6.28}$$

Expressions analogous to (6.24)–(6.28) hold for h' with p replaced by p', ξ by ξ', etc.

Now we take the matrix element of (6.21)

$$S_{fi} \equiv \langle h(p_3, s_3), h'(p_4, s_4)\,|S_{\text{eff}}|\,h(p_1, s_1), h'(p_2, s_2)\rangle, \tag{6.29}$$

insert (6.24)–(6.28) and the analogue for h', make the substitution

$$\alpha \to \frac{\alpha}{\xi\xi' v}, \tag{6.30}$$

and find, after some algebra:

$$S_{fi} = \delta_{fi} + i(2\pi)^4\, \delta(p_3 + p_4 - p_1 - p_2)\, \mathscr{T}_{fi}, \tag{6.31}$$

$$\mathscr{T}_{fi} = 4iv\, \delta_{s_3 s_1}\, \delta_{s_4 s_2} \Big\{ J_+(q^2) \int_0^1 d\xi \int_0^1 d\xi' (N^h_{q+\bar q}(\xi; Q_0^2), N^h_G(\xi; Q_0^2))\, \mathscr{E}^{h,h'}(q^2, Q_0^2)$$
$$\times \begin{pmatrix} N^{h'}_{q+\bar q}(\xi'; Q_0^2) \\ N^{h'}_G(\xi'; Q_0^2) \end{pmatrix} + J_-(q^2)[z_1(q^2)]^2\, 9F^h_B(q^2)\, F^{h'}_B(q^2) \Big\}. \tag{6.32}$$

Here $\mathscr{E}$ is the matrix function:

$$\mathscr{E}^{h,h'}(q^2, Q_0^2) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\alpha \sum_{n=0}^{\infty} \frac{(-\alpha^2)^n}{n!} \begin{pmatrix} F^h_{2n+2,q}(q^2; Q_0^2) & 0 \\ 0 & F^h_{2n+2,G}(q^2; Q_0^2) \end{pmatrix}$$
$$\times \zeta^{\mathrm{T}}_{2n+2}(q^2, Q_0^2) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \zeta_{2n+2}(q^2, Q_0^2)$$
$$\times \begin{pmatrix} F^{h'}_{2n+2,q}(q^2, Q_0^2) & 0 \\ 0 & F^{h'}_{2n+2,G}(q^2; Q_0^2) \end{pmatrix}. \tag{6.33}$$

For hadrons h and/or h' having spin 0, the corresponding Kronecker δ's for the spin indices in (6.32) have to be replaced by 1.

The result for the elastic hadron–hadron scattering amplitude (6.32) looks more complicated than it really is. Let us discuss its physical content.

According to (6.32) the scattering occurs on individual partons, each process involving one parton from hadron h and one from hadron h'. The elementary parton scattering amplitudes are to be calculated by nonperturbative methods as explained in Sections 5.1 and 5.2. The bound state aspects of the hadrons enter in (6.32) and (6.33) through the quark and gluon distribution functions, well known from deep inelastic lepton–nucleon scattering, and through the form factors $F(q^2; Q_0^2)$. The latter are in principle measurable quantities; however, only a few of them, like the electromagnetic form factors of the hadrons, are well known. Here we will presumably have to rely (for $q^2 \neq 0$) on model calculations and/or make plausible Ansätze in order to compare our theory with experiment. We emphasize again that S_{eff} and $\mathscr{T}_{fi}$ (6.32) are incomplete, since the gluonic terms are still missing. But, as argued above, this should not change the general structure of the result. Of course, this has to be seen in detail. In the following discussion we should keep such a caveat in mind.

The amplitude (6.32) has a structure reminiscent of the models of [32, 33]. However, there is an important difference. Continuing the amplitude (6.32) in $t \equiv q^2$ to positive values, $t > 0$, we should find *simple* poles at the positions of the squared masses t_M of the mesons M which can be exchanged in the t-channel. (For unstable mesons M these poles will, of course, be away from the real axis in the second sheet.) Now $J_\pm(t)$ *and* the form factors $F^{h,h'}(t)$ will have simple poles at t_M. The factors $z_1(t)$, $\zeta(t)$, on the other hand, representing essentially the inverse of the quark form factors (cf. (6.9)–(6.14)) are expected to have simple *zeros* at t_M. Thus $\mathscr{T}_{fi}$ in (6.32) will indeed have simple poles at $t=t_M$, as it should. In the original model of Wu and Yang [32] one assumes

$$\mathscr{T}_{fi} \propto F^h(t)\, F^{h'}(t) \tag{6.34}$$

and $\mathscr{T}_{fi}$ would have a double pole[2] at $t=t_M$.

Consider now forward scattering. Setting $q^2=0$ in (6.32) gives

$$\begin{aligned}\mathscr{T}_{fi}|_{t=0} = 4iv\, \delta_{s_3 s_1}\, \delta_{s_4 s_2} \Big\{ & J_+(0) \int_0^1 d\xi \int_0^1 d\xi'\, N^h_{q+\bar{q}}(\xi; Q_0^2)\, N^{h'}_{q+\bar{q}}(\xi'; Q_0^2) \\ & + J_-(0)\, 9B^h B^{h'} \Big\}\end{aligned} \tag{6.35}$$

and by the optical theorem for the spin averaged total cross section

$$\begin{aligned}\bar{\sigma}_{\text{tot}}(h, h') \to 2 \Big\{ & \text{Re}[J_+(0)] \cdot \int_0^1 d\xi \int_0^1 d\xi'\, N^h_{q+\bar{q}}(\xi; Q_0^2)\, N^{h'}_{q+\bar{q}}(\xi'; Q_0^2) \\ & + \text{Re}[J_-(0)] \cdot 9B^h B^{h'} \Big\}.\end{aligned} \tag{6.36}$$

Here we used (6.16), (6.27), and (6.28). The T-matrix element (6.35) is not necessarily pure imaginary (cf. the comments at the end of Section 5). Furthermore, the J_--term corresponds to $C=-1$ exchange and thus the amplitudes for hh' and $h\bar{h}'$ scattering will in general not be equal. Our approach thus leads inevitably to an "odderon," discussed at the phenomenological level in [34].

The amplitude (6.35) and the cross section (6.36) depend on the c.m. energy $\sqrt{s}$ through Q_0^2 (2.26). Setting $P \simeq \sqrt{s}/2$, we obtain

$$Q_0^2 = \frac{\sqrt{s}}{2t_0} = \sqrt{s} \cdot 0.05\ \text{GeV} \tag{6.37}$$

for $t_0 = 2\,\text{fm}$. The equivalent DIS-resolution grows linearly with $\sqrt{s}$, not with s, as one might naively think. We get $Q_0^2 = 0.25$, 1.5, 30 GeV^2 for $\sqrt{s} = 5$, 30, 600 GeV. We expect that for $Q_0^2 \lesssim 1.5\ \text{GeV}^2$ the constituent quark model for the hadrons should apply. For a discussion of the "freezing out" of the evolution and the transition to the effective constituent quark theory we refer to [35]. Gluons and sea quarks should thus be negligible for this Q_0^2 range and our formulae (6.35), (6.36) should be directly applicable. We can then set for $Q_0^2 \lesssim 1.5\ \text{GeV}^2$,

$$\int_0^1 d\xi\, N^h_{q+\bar{q}}(\xi; Q_0^2) = \mathscr{N}^h_{\text{c.q.}}$$

where $\mathscr{N}^h_{\text{c.q.}}$ is the total number of constituent quarks in hadron h.

[2] The author is grateful to P. V. Landshoff for pointing out this problem to him.

Assuming the J_--term to be small compared to the J_+-term in (6.36),

$$|\mathrm{Re}\, J_-(0)| \ll |\mathrm{Re}\, J_+(0)|, \tag{6.38}$$

we obtain

$$\bar{\sigma}_{\mathrm{tot}}(h, h') = \sigma_{\mathrm{tot}}(qq)\, \mathcal{N}^{h}_{\mathrm{c.q.}} \mathcal{N}^{h'}_{\mathrm{c.q.}}, \tag{6.39}$$

where

$$\sigma_{\mathrm{tot}}(qq) = 2\, \mathrm{Re}\, J_+(0). \tag{6.40}$$

In (6.39) we recognize the naive additive quark model result [1]. But now we also have an estimate of its range of validity:

$$5\ \mathrm{GeV} \leqslant \sqrt{s} \leqslant 30\ \mathrm{GeV}. \tag{6.41}$$

Here the lower end comes from (2.25) and the upper end from DIS which shows that for $Q^2 \gtrsim 1.5$–$2.0\ \mathrm{GeV}^2$ gluons become important, carrying roughly 50% of a hadron's momentum. Looking at cross section plots [36] we find indeed that over the range (6.41) the cross sections for pp, $\bar{p}p$, πp scattering are reasonably constant and satisfy the additive quark model relation. Below $\sqrt{s} \simeq 5$ GeV we enter the resonance region, above $\sqrt{s} \simeq 30$ GeV the cross sections rise again.

Consider, on the other hand, very high energies: $\sqrt{s} \gtrsim 60$ GeV say, corresponding to $Q_0^2 > 3\ \mathrm{GeV}^2$. In this case folklore tells us that

$$N_{q+\bar{q}}(\xi, Q_0^2) \propto 1/\xi \tag{6.42}$$

for $\xi \to 0$ and naively we get a logarithmic divergence in (6.36). However, for very small values of the parton momenta, i.e., for very small ξ, ξ', the distribution functions no longer scale, we enter the "wee-region" [14, 37]. Also our formulae must now be supplemented by the gluon scattering terms. But we can estimate from (6.36) the quark contribution to the total cross section. The integral

$$\mathcal{N}^{h}_{q+\bar{q}}(Q_0^2) = \int_0^1 d\xi\, N^{h}_{q+\bar{q}}(\xi, Q_0^2) \tag{6.43}$$

is the mean number of q and $\bar{q}$ partons seen in hadron h at resolution Q_0^2. Standard lore [14, 37] suggests that this number should grow logarithmically with Q_0^2. Inserting again $Q_0^2 = \sqrt{s}/(2t_0)$ we obtain

$$\begin{aligned} \mathcal{N}^{h}_{q+\bar{q}}(Q_0^2) &\propto \ln s, \\ \bar{\sigma}_{\mathrm{tot}}(h, h')|_{\text{quark contr.}} &\propto (\ln s)^2 \end{aligned} \tag{6.44}$$

for $s \to \infty$. Thus we expect the quark contribution to the cross section to grow with s at the maximum rate allowed by general principles [38]. Another way to arrive at (6.44) is to assume that the integral (6.43) is related to the mean number of hadrons seen in a typical deep inelastic scattering event with h as target at squared momentum transfer Q_0^2. More precisely, we expect $\mathcal{N}^h_{q+\bar{q}}(Q_0^2)$ to be proportional to the multiplicity $\langle n \rangle$ of hadrons in the target fragmentation region in DIS. Now it is known from experiments [39] that

$$\langle n \rangle \propto \ln Q_0^2 \tag{6.45}$$

for fixed x_{B_j} or fixed W^2 and Q_0^2 up to 50 GeV^2. With $Q_0^2 = \sqrt{s}/(2t_0)$ we obtain again (6.44).

7. Conclusions and Summary

In this article we tried to develop a model for a class of soft hadronic reactions, diffractive processes, based on QCD. Thereby we extended the work of [10] to the case of non-abelian gluons. We avoided the use of perturbation theory, since we see no small coupling constant which could be relevant for soft hadronic processes. Instead, we assumed that a^2/R^2 is a small parameter, where a is the vacuum correlation length and R the radius of a light hadron. This assumption is supported by phenomenology [6, 10, 23]. Using for the description of the QCD vacuum the stochastic field model of [16], one finds both from hadron spectroscopy and from total cross sections at high energies with the methods of the present paper [22] a value of

$$a \simeq 0.4 \text{ fm}. \tag{7.1}$$

Consider now the equivalent uniform density radius R of a hadron which is related to the mean squared charge radius $\langle r^2 \rangle$ by $R^2 = 5\langle r^2 \rangle/3$. The experimental values [40] for the proton and the pion radii are

$$\begin{aligned} R_p^2 &= 1.12 \text{ fm}^2, \\ R_\pi^2 &= 0.73 \text{ fm}^2. \end{aligned} \tag{7.2}$$

With (7.1) this gives

$$\begin{aligned} a^2/R_p^2 &\simeq 0.14, \\ a^2/R_\pi^2 &\simeq 0.22, \end{aligned} \tag{7.3}$$

indicating that corrections to our results should be at most in the 15–20% range.

The assumption $a^2/R^2 \ll 1$ leads to a picture of single partons in hadrons interacting at a time in diffractive scattering, as explained in detail in [10]. Experimental evidence supports this [23].

In Section 2 we analysed the question of the appropriate scale for the wave functions of hadrons in diffractive scattering. We argued that the equivalent resolution in DIS is $Q_0^2 = \sqrt{s}/(2t_0)$ with $t_0 \approx 2$ fm.

In Section 3 we derived an expression for the quark–quark scattering amplitude as an average—in the sense of the functional integral—over scattering amplitudes for fixed gluon potentials. In Section 4 we treated the scattering of quarks in a given gluon background potential in the eikonal approximation. The results of these sections may be of interest by themselves. We should emphasize here that our final expression for the quark–quark scattering amplitude in Section 5 is *not* of the eikonal form (cf. [22]).

In Section 6 we derived an effective S-operator (6.21) which served as basis for our discussion of high energy small angle elastic hadron–hadron scattering in Section 6.3. For a description of other diffractive reactions, $S_{\rm eff}$ (6.21) should be sandwiched between the appropriate hadron states. In (6.21) terms describing gluon–gluon and quark–gluon scattering are still missing. We argued, however, that for the description of diffractive reactions at c.m. energies of 5 to 30 GeV gluons and seaquarks in the hadrons can be neglected. Thus we could directly use (6.21) and obtain the additive quark model result for the total cross sections of hadrons.

Our approach leads to real parts in the forward T-matrix elements and to "odderon" ($C = -1$) exchange. Our "odderon" couples to the baryon number current. It is thus absent in meson–meson and meson–baryon elastic scattering at $t = 0$.

Our model also provides a starting point for the description of nondiffractive soft hadron–hadron interactions through the colour exchange term in (5.14). It seems plausible that this will lead to a string picture of multiparticle production in the spirit of [27].

Of course, there remain many open questions and problems. Let us mention just a few of them.

We have in this work considered at the parton level only the scattering of quarks and antiquarks. What about gluon–gluon and quark–gluon scattering? The S-operator (6.21) leads to the elastic amplitude (6.32) which involves an integration over *all* momenta of the partons in h and h'. But our expressions for the parton–parton amplitudes are only valid at high c.m. energies of the parton–parton collision. It seems physically plausible that some sort of lower cutoff on the parton–parton c.m. energy should be introduced. Is this indeed necessary and how can this be done in a consistent way? In other words, we have to consider the contribution of the scattering of "wee" partons in detail. As argued above, this should, however, only become relevant for $\sqrt{s} \gtrsim 30$ GeV. We will treat these questions along similar lines as in the present work in a subsequent publication.

How can we evaluate the string–string correlation functions $J_\pm(q^2)$ (5.47) with nonperturbative methods, for instance, with Monte Carlo simulations in the framework of lattice gauge theories? The calculations of [22] using the stochastic vacuum model may show a way.

Nontrivial phases and a connection between phases and energy variation are predicted for the amplitudes by Regge theory [41]. Are such phases already hidden in (6.21) or not yet? More generally we must ask if our amplitudes satisfy—at least approximately—all the standard analyticity requirements and unitarity constraints [42]. The physics of a shrinking diffraction peak is incorporated in our model: With increasing s the effective resolution of the hadronic wave functions grows proportional to $\sqrt{s}$, more partons contribute and the effective hadronic radius should also increase. It remains to be seen how this works out quantitatively.

Finally, we come to the most important question of all. Does our Ansatz lead to agreement of the theoretical predictions with experiment? We have made some considerations in this connection in Section 6.3, but much more theoretical work is needed before we will know a definite answer here.

Appendix

In this appendix we discuss the abelian gluon model and show how we arrive at the results of [10] using our present techniques.

Let the Lagrangian be as in (3.1)–(3.3) of [10] and in this appendix let G_μ denote the abelian gluon field. The analogues of (5.15), (5.16) for the abelian theory read then

$$\langle \psi(p_3)\, \psi(p_4)\, |\mathcal{T}|\, \psi(p_1)\, \psi(p_2)\rangle \to i\bar{u}(p_3)\, \gamma^\mu u(p_1)\, \bar{u}(p_4)\, \gamma_\mu u(p_2) \cdot J(q^2), \tag{A.1}$$

where

$$\begin{aligned} J(q^2) = -Z_\psi^{-2} \int d^2 z_T\, e^{i\mathbf{q}\mathbf{z}_T} \Big\langle \Big\{ \exp\Big[-\frac{i}{2} g \int_{-\infty}^{\infty} dz_+\, G_-(z_+, 0, \mathbf{z}_T)\Big] - 1 \Big\} \\ \times \Big\{ \exp\Big[-\frac{i}{2} g \int_{-\infty}^{\infty} dz_-\, G_+(0, z_-, 0)\Big] - 1 \Big\} \Big\rangle. \end{aligned} \tag{A.2}$$

Since renormalization was not considered explicitly in [10] we set $Z_\psi = 1$ in the following. Now we expand in powers of g, considering only the *explicit* g-dependence in (A.2):

$$J(q^2) = g^2 J^{(2)}(q^2) + g^4 J^{(4)}(q^2) + \cdots. \tag{A.3}$$

Note that in the functional integral derived from the Lagrangian (3.1)–(3.3) of [10] and indicated by the brackets $\langle\ \rangle$ in (A.2) there is an implicit g-dependence through the fermion determinant (cf. (3.9)). No expansion in this g-dependence is assumed to be made in (A.3). The term kept in [10] corresponds to the C = +1 part of $g^4 J^{(4)}(q^2)$. We find from (A.2)

$$g^4 J^{(4)}(q^2) = -\left(\frac{g}{2}\right)^4 \int d^2 z_T\, e^{i\mathbf{q}\mathbf{z}_T} \Big\langle \Big\{ \frac{1}{3!} \varphi_- \varphi_+^3 + \frac{1}{4} \varphi_-^2 \varphi_+^2 + \frac{1}{3!} \varphi_-^3 \varphi_+ \Big\} \Big\rangle, \tag{A.4}$$

where

$$\varphi_- = \int_{-\infty}^{\infty} dz_+ \, G_-(z_+, 0, \mathbf{z}_T),$$
$$\varphi_+ = \int_{-\infty}^{\infty} dz_- \, G_+(0, z_-, 0). \tag{A.5}$$

With the Ansätze (3.7), (3.8) of [10] the terms corresponding to C = −1 exchange in (A.4) vanish, $J^{(4)}$ becomes pure C = +1 and is given by

$$g^4 J^{(4)}(q^2) = -\frac{1}{2}\left(\frac{g}{2}\right)^4 \int d^2 z_T \, e^{i\mathbf{q}\mathbf{z}_T} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dz_+ \, dz_- \, dz'_+ \, dz'_-$$
$$\times \langle G_-(z_+, z_-, \mathbf{z}_T) \, G_+(0) \rangle \langle G_-(z'_+, z'_-, \mathbf{z}_T) \, G_+(0) \rangle, \tag{A.6}$$

where

$$g^2 \langle G_-(z) \, G_+(0) \rangle = -2ig^2 \int \frac{dk}{(2\pi)^4} e^{-ikz} D_{np}(k^2). \tag{A.7}$$

Inserting here (3.13) of [10] we finally obtain

$$g^4 J^{(4)}(q^2) = G^4 a^2 I(a^2 q^2). \tag{A.8}$$

Replacing $J(q^2)$ in (A.1) by $g^4 J^{(4)}(q^2)$ from (A.8) gives exactly the T-matrix element (4.10) of [10]. In this way we reproduce the result of the Feynman diagram calculation in [10] with our present technique.

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