Magnetic monopoles

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1 Introduction

Our first goal in this presentation is to convince you that magnetic monopoles represent serious physics. There are two kinds of monopoles: Dirac and 't Hooft-Polyakov. The second must exist if any grand unified theory is correct, and the first explains charge quantization but requires nontrivial physical assumptions. A quick tabular summary of their differences is as follows:

Dirac monopole	't Hooft-Polyakov monopole
$U(1)$ electromagnetism	Nonabelian gauge theories
Point singularity	Finite-size soliton ($\approx 10^{-30}$ m)
Infinite field strength	Finite field strength
Relies on a physical argument	Follows mathematically from field equations
Explains charge quantization	Has to exist if any GUT is correct

By a "magnetic monopole" we mean an object generating nonzero magnetic flux through a closed surface containing it. For convenience, we will often refer to magnetic monopoles simply as "monopoles." By the "magnetic charge" of a monopole we mean the magnitude of the magnetic flux it generates.

Maxwell's electromagnetism explicitly postulates the non-existence of magnetic monopoles with its equation $\nabla \cdot B = 0$ – however, we may consider other more general theories. Allowing the magnetic potential A to be defined only locally, but not globally, produces the Dirac monopole. Treating electrodynamics as a $U(1)$ gauge theory and going from $U(1)$ to nonabelian gauge groups produces the 't Hooft-Polyakov monopole.

1.1 The Dirac monopole

The Dirac monopole is a point singularity of the magnetic field. Imagine the point charge of electromagnetism and replace the electric field for the magnetic field – roughly speaking, this is all a Dirac monopole is. The field of a monopole with *magnetic charge* g is given by

$$\mathbf{B}(\mathbf{r}) = \frac{g}{4\pi}\frac{\hat{\mathbf{r}}}{r^2}. \tag{1}$$

1.1.1 Dirac monopole with the string

Let the magnetic potential be

$$\mathbf{A} = \frac{g}{4\pi}\frac{1-\cos\theta}{r\sin\theta}\hat{\phi}, \tag{2}$$

where we work in spherical coordinates r, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$, and $\hat{\phi}$ is the unit vector "circling around" the z axis. (We leave it as an exercise to check that $\nabla^2\mathbf{A} = 0$.) At $\theta \neq \pi$, the magnetic field

$$\mathbf{B} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\partial_\theta\left(\frac{g}{4\pi}\frac{1-\cos\theta}{r\sin\theta}\sin\theta\right) = \frac{g}{4\pi}\frac{\hat{\mathbf{r}}}{r^2} \tag{3}$$

is the desired monopole field, and at $\theta = \pi$, the field is undefined. Notice that by not defining the field at $\theta = \pi$, we were able to achieve nonzero magnetic flux while still having $\mathbf{B} = \nabla \times \mathbf{A}$. The line $\theta = \pi$, where the field is undefined and which in Cartesian coordinates can be described as the negative z axis, is called the *Dirac string.* The Dirac string does not have to be the z axis; modifications of equations 2 and 3 for any shapes of strings can be written down.

Surely by the word "monopole" we don't mean a monopole with a string attached (which would be a "tadpole"!). Therefore, the Dirac string should be undetectable. The magnetic field of the Dirac monopole is spherically symmetric wherever it is defined; the only way the string could be detected is through the Aharonov-Bohm effect, the dependence of the Schrodinger equation on the magnetic potential $\mathbf{A}$. The phase shift between an electron going to the left of the string and an electron going to the right of the string is the exponential of a contour integral,

$$\exp\left(-ie\oint \mathbf{A}\cdot\mathbf{dl}\right) = \exp\left(-ieg\right),$$

and this must be equal to 1. Therefore,

$$eg = 2\pi n \tag{4}$$

for some integer n. This condition determines the *Dirac charge* g_D,

$$g_D \equiv \frac{2\pi}{e}, \tag{5}$$

which is the smallest possible magnetic charge. If one or more magnetic monopoles were to exist, equation 4 dictates that their magnetic charges g must necessarily be multiples of the Dirac charge g_D.

1.1.2 Dirac monopole without the string

Rui will later talk in detail about how monopoles can be realized topologically – but here, we will at least briefly mention a basic construction, different from the Dirac string construction presented above. Instead of making the magnetic field undefined along some string, we can alternatively choose to make $\mathbf{A}$ a multi-valued function. Suppose we define the magnetic potential piecewise in the top and bottom hemispheres as

$$\mathbf{A} = \hat{\phi}\,\frac{g/4\pi}{r\sin\theta}\begin{cases}1-\cos\theta & 0 \le \theta \le \pi/2,\\ -1-\cos\theta & \pi/2 \le \theta \le \pi.\end{cases} \tag{6}$$

Now, because $\mathbf{A} = 0$ at $\theta = 0$ and $\theta = \pi$, the magnetic field is well-defined at the poles. And at the equator $\theta = \pi/2$, where the two hemispheres are glued together,

$$\mathbf{A}^{\text{top}} - \mathbf{A}^{\text{bottom}} = \frac{g/2\pi}{r}\hat{\phi}. \tag{7}$$

If $eg = 2\pi n$ for some integer n, then the right-hand side of equation 7 is just the pure gauge $\frac{1}{ie}(\frac{\hat{\phi}}{r}\partial_\phi\Omega)\Omega^{-1}$, where $\Omega(\phi)$ is the gauge transformation $\exp(in\phi)$. It is important that n is an integer, because otherwise $\Omega(\phi)$ would not be equal to $\Omega(\phi+2\pi)$. The condition

$$eg = 2\pi n, \quad n \in \mathbb{Z} \tag{8}$$

again tells us that magnetic charge is quantized in units of g_D (equation 5).

1.1.3 Dirac's original construction

The fact that equations 4 and 8 present the same quantization conditions for g is no accident. To derive both of these equations, we have used that the electromagnetic field is a $U(1)$ gauge field, and that the "going around the $U(1)$ circle" n times is equivalent to staying at the identity. Dirac was acutely aware of the relationship between monopoles and gauge theory, citing "essentially just Weyl's Principle of Gauge Invariance" as the crux of his 1931 monopole article.

Dirac's construction is quite complicated, and one might also question whether it is even correct. But, as Dirac believes, his construction simply follows from the formalism of quantum mechanics when one considers it without imposing unnecessary restrictions. Dirac identifies electromagnetic field with the phase of the wavefunction and, using that the phase can be discontinuous when the amplitude vanishes, identifies the magnetic monopole with a certain kind of discontinuity of the phase. An approximate reconstruction of Dirac's argument is given below.

1. The wavefunction can be written as a product of a magnitude $A \in \mathbb{R}^{>0}$ and a phase $\gamma \in \mathbb{R}$ as $\psi(x) = A(x)e^{i\gamma(x)}$.

2. The absolute phase is undetectable – that is, changing $\gamma(x) \mapsto \gamma(x) + c$ for any constant $c \in \mathbb{R}$ does not change any physical observables.

3. (Nontrivial physical claim.) Because of (2), we must think of γ as a kind of generalized function whose value $\gamma(x)$ at a point is undefined, and even whose differences $\gamma(x) - \gamma(y)$ between distant points are undefined, but whose *derivatives* $\partial_\mu \gamma(x)$ are defined.

4. One can show that if γ is a generalized function as in (3), then the effect of the electromagnetic field is simply a change in γ. Thus, **electromagnetic field is identified with a change in phase** of the wavefunction.

5. Because ψ is continuous, the phase γ is also continuous when $A \neq 0$. When the amplitude A is zero, γ has no physical meaning. The phase γ is allowed to have a discontinuity in the region where $A = 0$, and this region is called the *nodal line*. (The nodal line is a one-dimensional object because the complex equation $\psi = 0$ removes two degrees of freedom from $\mathbb{R}^3$.)

6. The discontinuity in γ permitted by (5) is identified with a singularity in the electromagnetic field. The nodal line is the **Dirac string**, and at the end of a nodal line (provided that it does terminate, which is not always the case) is the **Dirac monopole**.

Quantization of magnetic charge is deduced from the fact that γ at a point is undetermined up to a multiple of 2π and that therefore, the contour integral of $\nabla\gamma$ around a closed curve must be a multiple of 2π. The principle used here is the same as the principle used when deriving equations 4 and 8, namely that $\exp(i\gamma) = 1$ if and only if γ is a multiple of 2π.

1.1.4 Dirac charge in physical units

One might wonder how big the Dirac charge actually is. Writing $\mathbf{B} = \frac{g\mu_0}{4\pi}\frac{\hat{\mathbf{r}}}{r^2}$ and restoring the factors of $\hbar, \varepsilon_0, \mu_0$ by dimensional analysis, we obtain for g_D the formula

$$g_D = \frac{2\pi\hbar}{e\mu_0} \approx 3.29 \times 10^{-9}\ \mathrm{amp} \cdot \mathrm{m}, \tag{9}$$

so that the monopole magnetic flux is $\mu_0 g_D \approx 4.4 \times 10^{-15}\ m^2 T$, approximately 5% of the flux of the Earth's magnetic field through the cross section of a human hair. As another magnitude estimation, note that the ratio of the fundamental charge (multiplied by c to match the units) to the Dirac charge is

$$\frac{ce}{g_D} = 2\frac{e^2}{4\pi\hbar c\varepsilon_0} = 2\alpha. \tag{10}$$

That is, the Dirac charge is about $137/2 = 68.5$ times greater than the fundamental charge. If a magnetic monopole of one Dirac charge were placed at the center of a hydrogen atom (directly on top of the proton), the ratio of the electric and magnetic forces experienced by the electron would be, since $\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$,

$$F_B/F_E \approx \frac{g_D\mu_0 v}{4\pi r^2} / \frac{e}{4\pi\varepsilon_0 r^2} = \frac{g_D}{ce}(v/c) \approx \frac{1}{2\alpha}\alpha = 1/2, \tag{11}$$

where we use that for an orbiting electron, $v/c \approx \alpha$. Thus, the magnetic force becomes the same order of magnitude as Coulomb attraction! This points to the fact that g_D is very large, because normally, magnetic force can compare with Coulomb force only in relativistic scanarious or when coils or magnets are involved.

1.1.5 Dirac monopole and the quantization of charge

Why the electric charge of (color-neutral) particles quantizes in units of e is a great mystery – a mystery that the magnetic monopole, turns out, is able to solve. This seems to be the original motivation behind Dirac's 1931 paper.

In subsubsection 1.1.2, we were able to show that every magnetic monopole is necessarily a multiple of the Dirac charge. In this section, we will derive a somewhat more general equation that will quantize both magnetic and electric charge.

Let us consider the scattering of an electrically charged particle off a magnetic monopole. If the charge of the monopole is g and the charge of the particle is q, the change in angular momentum of the particle as it scatters off the monopole is $gq/2\pi$ (independent of the mass of the particle or its initial angular momentum). As angular momentum is quantized in units of $\hbar$, one can show that electric charge q must be quantized in units of $2\pi/g$. Knowing already that g is quantized in units of $2\pi/e$, we see that electric charge is quantized in units of e.

That the Dirac monopole leads to charge quantization is significant because the monopole is no longer idle talk, but a consequential theoretical construct. Much like the theory of the Big Bang has gained popularity because it explains cosmological observations, one might have some ground to consider the magnetic monopole as the explanation for charge quantization. This would be an explanation only partly satisfactory, of course, because the monopole only explains why quantization occurs, and not why it occurs in units of e with $e^2/4\pi \approx 1/137$.

1.2 'tHooft-Polyakov monopole

This monopole is nonsingular and follows simply as a mathematical consequence of the (nonlinear) field equations in certain gauge theories. As will be discussed later, the 't Hooft - Polyakov monopole has to exist if any GUT is true. Indeed, many interesting things can be said about the relationship of 't Hooft-Polyakov monopoles to topology, symmetry breaking, and more. But before that, let us provide a simple explicit construction of this monopole. Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \frac{1}{2}D_\mu\Phi^a D^\mu\Phi^a - U, \tag{12}$$

where a runs from 1 to 3, $\Phi^1, \ldots, \Phi^3$ is a Higgs field of three scalars, W is a gauge boson, and

$$\begin{aligned} U &= \frac{\lambda}{4}\left(\Phi^a\Phi^a - v^2\right)^2, \\ D_\mu\Phi^a &= \partial_\mu\Phi^a - e\varepsilon^{abc}W^b_\mu\Phi^c, \\ F^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - e\varepsilon^{abc}W^b_\mu W^c_\nu, \end{aligned} \tag{13}$$

where the "vev" v is a parameter of the Lagrangian, and it will determine the mass scale of the monopole. The field equations are

$$\begin{aligned} D_\nu F^{\mu\nu}_a &= e\varepsilon^{abc}\Phi^b D^\mu\Phi^c, \\ D_\mu D^\mu\Phi^a &= \lambda\left(\Phi^b\Phi^b - v^2\right)\Phi^a. \end{aligned} \tag{14}$$

The Lagrangian 12 has two important features: nonlinear field equations and spontaneous symmetry breaking. The 't Hooft-Polyakov monopole will appear as a so-called topological soliton. Under normal conditions, the value of Φ will be concentrated around some vev, say $(0, 0, v)$, and we would extract the post-SSB physics via the usual procedure. However, this is not the only possibility. It is possible to have a "hedgehog" configuration of Φ, where, at large distances,

$$\Phi^a \sim v\hat{\mathbf{r}}^a, \tag{15}$$

and somewhere in the middle Φ changes directions. (Recall that $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector in the $\mathbf{r}$ direction.) Note that equation 15 mixes color and position indexes. The "hedgehog" configuration cannot be unraveled into the usual $\Phi \approx (0, 0, v)$ configuration within a finite amount of work, so it is stable in time.

To find an explicit soliton solution, let us try the ansatz

$$\Phi^a(r) = h(r)\hat{\mathbf{r}}^a, \quad W_i^a = -\frac{1}{2}(1-k(r))\varepsilon_{ija}\frac{\hat{\mathbf{r}}^j}{r}, \quad W_0^a = 0. \tag{16}$$

That is, we are looking for solutions that are spherically symmetric and stationary in time. Here, h and k are some real-valued functions which are to be determined. Substituting equations 16 into equations 14, we find, very nicely, that the field equations are satisfied provided that

$$\begin{aligned} \frac{d^2h}{dr^2} + \frac{2}{r}\frac{dh}{dr} &= \frac{2}{r^2}k^2h - \lambda v^2(1-h^2)h, \\ \frac{d^2k}{dr^2} &= \frac{1}{r^2}(k^2-1)k + 4v^2h^2k. \end{aligned} \tag{17}$$

Moreover, we impose boundary conditions $h(0) = 0$, $k(0) = 1$ and $h(\infty) = 1$, $k(\infty) = 0$. The first pair of boundary conditions ensures continuity at the origin, and the second ensures that the solution has finite energy. Equations 17 can be solved numerically. What results is a solution with length scale $r \sim 1/v$, plotted in Figure 1.2 for several values of the dimensionless parameter λ.

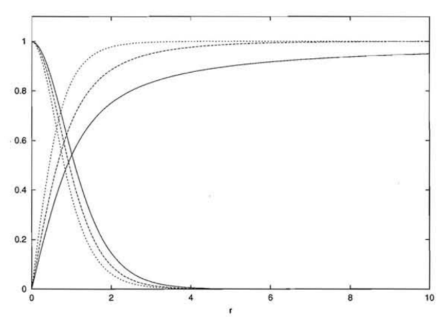

Fig. 8.1. The monopole profile functions $h(r)$ and $k(r)$ for $\lambda = 0$ (solid curves), $\lambda = 0.1$ (dashed curves), and $\lambda = 1.0$ (dotted curves).

This solution represents a concentration of Φ and W fields around the origin, and its characteristic length scale is $1/v$. The magnetic field is identified to be

$$b_i = -\frac{1}{2}\varepsilon_{ijk}f_{jk}, \tag{18}$$

where

$$f_{\mu\nu} = F_{\mu\nu}^a \Phi^a \tag{19}$$

is a kind of "averaged" Faraday tensor. The motivation behind this identification is that if in some region $\Phi^a = \phi\hat{\Phi}^a$, where ϕ is some positive-valued function and $\sum_a(\hat{\Phi}^a)^2 = 1$ and $D\mu\hat{\Phi}^a = 0$ (e.g., in a region of "almost-constant" Higgs field), then $f_{\mu\nu}$ satisfies Maxwell's equations in free space:

$$\begin{aligned} &\partial_\mu f^{\mu\nu} = 0, \\ &\varepsilon^{\sigma\tau\mu\nu}\partial_\tau f_{\mu\nu} = 0. \end{aligned} \tag{20}$$

The identification (equation 18) is not unique; it is motivated by the fact that $f_{\mu\nu}$ behaves like the usual Faraday tensor. But if we are willing to accept this identification, we find that the magnetic field near the origin is

$$b_i = -\frac{\hat{\mathbf{r}}^i}{2er^2}, \tag{21}$$

which is the field of a monopole of charge $-\frac{2\pi}{e}$. Thus, we have exhibited the construction of a 't Hooft-Polyakov monopole in an $SU(2)$ Yang-Mills-Higgs field theory. In many other nonabelian gauge theories with spontaneous symmetry breaking, such monopoles will also occur by similar mechanisms – but more on that in later sections.

1.3 Symmetry between electric and magnetic charges

If we allow monopoles to enter electromagnetism, certain changes to the theory necessarily follow if we wish to preserve momentum conservation, energy conservation, and Lorentz invariance. What emerges is a beautiful unified theory where electric and magnetic charges play complementary roles. In this subsection, we derive electromagnetism with monopoles and study some of its basic properties. A relativistic quantum field theory of magnetic monopoles, complete with a Lagrangian and Feynman rules, was given by Zwanziger [13]. For simplicity, we discuss a classical theory of monopoles instead.

1.3.1 Magnetostatic considerations

Imagine a magnetostatic situation where current I flows around a circular loop of radius R, and a magnetic monopole g is held at the center of the loop. The charge carriers in the wire experience Lorentz force from the field of the monopole equal to $\frac{Ig}{2R}$. By assumption, there is no electric field. Also, because this is a static situation, momentum cannot be carried away by the electromagnetic field. If momentum conservation is to hold, the monopole must experience a force from the loop of wire equal to $\frac{Ig}{2R}$. This turns out to be equal to gB', where B' is the magnetic field that the loop of wire induces at the location of the monopole. Skipping over some details, we see that a stationary magnetic charge g in field $\mathbf{B}$ must experience a force $\mathbf{F} = g\mathbf{B}$.

1.3.2 Electrostatic considerations

The picture where monopoles only create magnetic field, but not electric field, is not satisfactory at least because it violates conservation of energy. Indeed, suppose we take a (superconducting) loop of wire and pass current through it, which takes no power to accomplish. Suppose we took a ring of magnetic monopoles, such that the linear density of monopoles is uniform throughout the ring, and passed it through the loop of wire. In a picture where monopoles only generate magnetic field, the ring would generate uniform magnetic field no matter how fast it spins, so no electromotive force would be excited in the loop of wire to stop the current. Meanwhile, the ring of monopoles would experience torque from the magnetic field of the wire which could be used to generate energy.

A natural thing to do is to require that the theory of monopoles be relativistic. Much like the electromagnetic field of an electric monopole q moving with velocity v is

$$\mathbf{E} = \frac{q}{4\pi r^2}\left(\gamma\hat{\mathbf{r}} - (\gamma-1)(\hat{\mathbf{r}}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}\right), \qquad \mathbf{B} = \frac{q}{4\pi r^2}\left(-\gamma\mathbf{v}\times\hat{\mathbf{r}}\right), \tag{22}$$

we let the electromagnetic field of a magnetic monopole g moving with velocity v be

$$\mathbf{E}' = \frac{g}{4\pi r^2}\left(+\gamma\mathbf{v}\times\hat{\mathbf{r}}\right), \qquad \mathbf{B}' = \frac{g}{4\pi r^2}\left(\gamma\hat{\mathbf{r}} - (\gamma-1)(\hat{\mathbf{r}}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}\right). \tag{23}$$

(These formulas are obtained by Lorentz-transforming the field of a stationary electric and magnetic monopole, respectively, and $\gamma = 1/\sqrt{1-v^2}$.)

Consider the following static situation. A ring of magnetic monopoles of radius R and uniform magnetic charge linear density λ is forced to rotate with constant angular velocity ω around a fixed axis such that its center does not move in space. An electric charge q is held at the center of the ring. Since the electric field induced by the rotating ring at its center is $E = \lambda\omega/2$, the force experienced by the electric charge is $q\lambda\omega/2$. To make momentum conservation hold, the ring must experience an equal force – and since the electric charge in the middle of the ring does not create a magnetic field, this force has to be due to the electric field of the charge, which is equal to $E = \frac{q}{4\pi R^2}$. The force must be proportional to ω. One way to make this work is by requiring that a magnetic monopole of charge g moving with velocity $\mathbf{v}$ in electric field $\mathbf{E}$ experiences a force $\mathbf{F} == g\mathbf{v}\times\mathbf{E}$, where the minus sign comes from carefully considering the directions of forces. Thus, combining this discussion with the one from Subsubsection 1.3.2, as well as for the Lorentz force law for electric charges, we see that the complete Lorentz force law for a particle of electric charge q and magnetic charge g is given by

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + g(\mathbf{B} - \mathbf{v} \times \mathbf{E}). \tag{24}$$

Already one can see that electric and magnetic charges have very symmetric roles in the theory – but there is more symmetry to come!

1.3.3 Field theory formulation

If we let $\rho_e, \mathbf{J}_e$ and $\rho_g, \mathbf{J}_g$ be, respectively, the electric and magnetic charge densities and currents, so that a point electric charge q moving with velocity $\mathbf{v}$ generates density $\rho_e = q\delta^3(\mathbf{r})$ and current $\mathbf{J}_e = \mathbf{v}\, q$, and likewise for the magnetic charge, a modification of Maxwell's equations that reproduces equations 22 and 23 is

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho_e, \\ \nabla \cdot \mathbf{B} &= \rho_g, \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} &= -\mathbf{J}_g, \\ \nabla \times \mathbf{B} - \partial_t \mathbf{E} &= \mathbf{J}_e. \end{aligned} \tag{25}$$

Because this theory was constructed from equations 22 and 23, which are Lorentz-invariant, it is also Lorentz-invariant, where electric and magnetic fields trasform in the usual way. The energy density and energy current remain, as before,

$$W = \frac{\mathbf{E}^2 + \mathbf{B}^2}{2}, \quad \pi = \mathbf{S}, \tag{26}$$

where $\mathbf{S} = \mathbf{E} \times \mathbf{B}$ is the Poynting vector. Indeed,

$$\begin{aligned} \partial_t \frac{\mathbf{E}^2 + \mathbf{B}^2}{2} &= \mathbf{E} \cdot (\nabla \times \mathbf{B} - \mathbf{J}_e) + \mathbf{B} \cdot (-\nabla \times \mathbf{E} - \mathbf{J}_g) \\ &= (-\mathbf{E} \cdot \mathbf{J}_e - \mathbf{B} \cdot \mathbf{J}_g) - (-\mathbf{E} \cdot \nabla \times \mathbf{B} + \mathbf{B} \cdot \nabla \times \mathbf{E}) \\ &= f - \nabla \cdot \mathbf{S}, \end{aligned}$$

where we identify $f = -\mathbf{E} \cdot \mathbf{J}_e - \mathbf{B} \cdot \mathbf{J}_g$ as the work done by charged particles on the electromagnetic field. Momentum density can be worked out from here by imposing Lorentz invariance.

1.3.4 A new gauge symmetry

Equations 24 and 25 might be familiar to students from Girma Hailu's Physics 153: Electrodynamics, where one of the assigned problems was about monopoles. The problem, among other things, asks the students to prove that electromagnetism with monopoles is invariant under the transformation

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \mapsto A \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}, \qquad \begin{pmatrix} \rho_e \\ \rho_g \end{pmatrix} \mapsto A \begin{pmatrix} \rho_e \\ \rho_g \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{J}_e \\ \mathbf{J}_g \end{pmatrix} = A \begin{pmatrix} \mathbf{J}_e \\ \mathbf{J}_g \end{pmatrix}, \tag{27}$$

where A is a unitary 2×2 real-valued matrix. That is to say, a system of charges $(q_1, g_1), \ldots, (q_n, g_n)$ moving along some trajectories in spacetime would exert on each other the same forces, according to equation 24, as a system of charges $(q_1', g_1'), \ldots, (q_n', g_n')$ moving along the same trajectories, where $(q_i', g_i') = A(q_i, g_i)$, if electric and magnetic fields are likewise "rotated" by A – and the field equations (equation 25) will still be satisfied.

Equation 27 encodes a gauge symmetry, in the sense that the choice of the unitary matrix A does not in any way affect the observable physics – namely, the forces that particles exert on each other. Because of this gauge symmetry, there is a rather broad range of theories that are physically equivalent to "ordinary" monopole-free electromagnetism. Indeed, for an electrically and magnetically charged particle with some charge (q, g), let $\kappa = g/q$ be the ratio of magnetic charge to electric charge. If κ was the same for all charged particles,

$$\kappa = \text{const} \tag{28}$$

then a rotation by $\tan^{-1}(\kappa)$ would bring all particles to $\kappa = 0$, which is the situation with no magnetic charges. As far as we know, magnetic monopoles might actually exist but be completely undetectable if g is always κq for some constant κ shared by all charged particles. Alternatively, the "ordinary" monopole-free electromagnetism can be viewed as the form of monopole-equipped electromagnetism discussed in this section, satisfying equation 28 and presented in a gauge where monopole charge vanishes.

Electromagnetism with monopoles, at the cost of being slightly more complicated, places electric and magnetic fields and charges in very symmetric positions, as opposed to monopole-free electromagnetism where $\mathbf{E}$ and $\mathbf{B}$ are fundamentally different. Thus, perhaps, for some calculations it might be more convenient to work in this extended or "symmetrized" electromagnetism.

2 Monopoles and topology

In this section, we will see how the possible monopole charges predicted by a gauge theory is related to the topology of the gauge group. This is an extremely crucial property of monopoles that further guarantees the stability of the monopole charge. Along the way, we will use some results from algebraic topology, which is a an area of mathematics that provides extremely powerful tools to classify spaces up to continuous deformation.

2.1 Topological classification of Dirac monopoles

Recall from the discussion of Dirac monopoles without strings with gauge group $U(1)$ that we have two gauge potentials that are not defined globally, but instead cover the upper and lower hemispheres in $\mathbb{R}^3$ respectively. We ensure that these two potentials describe the same observable physical quantities in the region that they overlap (the equator) by requiring that they differ by a $U(1)$ gauge transformation. By further requiring that this gauge transformation be single-valued when acting on wavefunctions, we obtain the Dirac quantisation condition $g = ng_D$ where $n \in \mathbb{Z}$ and $g_D = \frac{2}{e}$ is the Dirac magnetic charge. Essentially, the magnetic charge g of the monopole is determined by the winding number n of the gauge transformation in the Lie group $U(1)$. This provides a topological basis for the quantization of magnetic charge in $U(1)$ theories - the magnetic charge must be integer multiples of g_D since it is a winding number. To see why this is useful, we note that as we allow the radius of the sphere upon which this construction lives on to vary, the gauge transformation and consequently the winding number must be a continuous function of r. Since n is an integer but must also be continuously varying, it must be a constant against r, with the result that the amount of magnetic charge g contained within a sphere of radius r remains the same even for arbitrarily small r - the monopole is a point singularity! We expect this to be problematic from general considerations of requiring finite energy. A better alternative that guarantees non-singular solutions will be presented in just a bit.

In order to do the same for arbitrary gauge theories, not just $U(1)$, we can generalize the construction above by considering gauge fields in the Lie algebra of an arbitrary Lie group H, that are defined on a sphere in space enclosing a magnetic monopole. As before, the minimal set of gauge fields that cover the sphere are given by non-singular gauge potentials A^U and A^L on the upper and lower hemispheres respectively. Again, these potentials must be related by a gauge transformation, defined on the equator to produce an element of the Lie algebra, and whose exponentiation produces an element in the Lie group that must be single-valued, providing a constraint on the possible values of the enclosed magnetic charge. The (exponentiated) gauge transformation is thus an element in H, and its action at the equator on the gauge fields themselves (which lie in the adjoint representation) is now specified by mappings from the circle S^1 to H. Provided that H is connected, which it is for any group we will consider, we can always deform any element of the Lie algebra to another element provided that the path of deformation does not cut through another path, where problems of well-definedness arise. Therefore, what we are really interested in are mappings $S^1 \to H$ up to homotopy. This is precisely what the fundamental group $\pi_1(H)$ describes, and happily coincides with our usual understanding of the winding number of all such maps, since the winding number is invariant up to deformations where paths don't cross each other. We will define the topological charge to be the winding numbers that $\pi_1(H)$ gives us, and the magnetic charge will then simply be the topological charge multiplied by the Dirac charge g_D in analogy to the abelian case.

We have seen above that in order to classify all possible topological charges given by a gauge group H, we we simply need to compute $\pi_1(H)$. There is a general strategy to compute $\pi_1(H)$, and in general, any higher homotopy group $\pi_n(H)$ for any Lie group H. The following is not strictly required to understand the rest of the material, but is a quick way to obtain the desired result if one has some familiarity with algebraic topology. Given any topological space H, we can always construct a fiber bundle

$$F \to G \to H \tag{29}$$

where G is another topological space and F is called the fibre over any point of H, such that over any sufficiently small neighbourhood of H, we always have $G \cong H \times F$. Such a fibre sequence (which we also call a fibration) can always be associated a long exact sequence of homotopy groups:

$$\cdots \longrightarrow \pi_{n+1}(H) \longrightarrow \pi_n(F) \longrightarrow \pi_n(G) \longrightarrow \pi_n(H) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots \tag{30}$$

What we mean by an exact sequence is simply that the image of any given map is equal to the kernel of the next map in the sequence. From this, we can obtain many useful relations between each of the homotopy groups.

Going back to our case of calculating $\pi_1(H)$, note that for any topological group H we can try to find a universal cover $\tilde{H}$, which means that $\pi_1(H) = 0$. The group structure is special, and an analogy of Lagrange's theorem guarantees that we can always find a subgroup K such that $H = \tilde{H}/K$, which we can express as a fiber bundle

$$K \to \tilde{H} \to H \tag{31}$$

where K is the fibre over any point of H. Applying this to 30 with $n = 1$, we find that

$$\pi_1(H) \cong \pi_0(K) \tag{32}$$

where $\pi_0(K)$ is the set of path-connected components of K. (Here, it must be a group because it is isomorphic to $\pi_1(H)$.) Going back down to earth, we can understand this result by noting that, by choosing the identity element as the basepoint in H, we can compute $\pi_1(H)$ by considering paths in H that begin and end at the identity element by definition. Since $H = \tilde{H}/K$, all such paths correspond to paths in H that begin at the identity and end at any element of K, not just the identity alone, since all elements in K eventually get identified with the identity. The continuity of the paths connecting the identity and the elements in K, however, must be preserved under the quotient map. In other words, paths that end at different elements of K which are not path-connected also cannot be continuously deformed into each other in H under the quotient map. This way, we see that the homotopy classes of loops in H correspond exactly to the path-connected components of K, as we see in Eq. 32.

As an example, consider $H = U(1)$. To apply Eq. 32, recall that a universal covering of $U(1) \cong S^1$ is given by $\mathbb{R}$, with fiber given by $\mathbb{Z}$. Then,

$$\pi_1(U(1)) \cong \pi_0(\mathbb{Z}) \cong \mathbb{Z} \tag{33}$$

which again tells us that the topological charge of a magnetic monopole in $U(1)$ gauge theories can be any integer. This approach wasn't very useful since it is already very well-known that $\pi_1(U(1)) \cong \pi_1(S^1) \cong \mathbb{Z}$. For a more useful example of a non-abelian gauge group with interesting monopole charges, we consider the group $SO(3)$, the group of orthogonal transformations in 3 dimensions with determinant 1. This group is clearly non-abelian, since this can be thought of as a group of matrices with matrix multiplication as the group composition law, which is non-commutative. To apply Eq. 32, we note that $SU(2)$ is a double covering of $SO(3)$, such that $SO(3) = SU(2)/\mathbb{Z}_2$. Therefore, we have

$$\pi_1(SO(3)) \cong \pi_0(\mathbb{Z}_2) \cong \mathbb{Z}_2 \tag{34}$$

which tells us that the topological charge, and subsequently the magnetic charge, can only be 0 or g_D! Specifically, we do not have antimonopoles, or equivalently, antimonopoles would be indistinguishable from monopoles.

As a last remark, we note from the classification of simple Lie algebras that all simple Lie algebras H take the form $H = \tilde{H}/\mathbb{Z}_N$ where $\tilde{H}$ is simply-connected and $N \in \mathbb{N}$. In other words, for gauge theories with arbitrary simple gauge groups (which comprise a majority of what we care about in physics) the magnetic charge is still quantized nicely, with the only catch being the disappearance of antimonopoles, as mentioned above.

Why is this definition of the non-abelian magnetic charge sensible? Essentially, all the properties that we care about are unchanged by the fact that H is now not necessarily abelian.

1. For one, magnetic charge is conserved under continuous time evolution, for a discrete quantity that varies continuously can only be constant.

2. Similarly, we can repeat the analysis in exactly the same way as in the abelian case to see that the topological charge is independent of the radius of the sphere r. Physically, the magnetic charge cannot be carried by the long-range field associated with a monopole. Following our logic, we conclude that if we shrink r all the way down and approach $r = 0$, the magnetic charge must again be concentrated at a point singularity!

3. Furthermore, the topological charge is gauge invariant, since if we were to make a gauge transformation on the gauge fields that lie on the sphere, what we really care about is what goes on inside the Lie group - this gauge transformation is just an automorphism that shifts each group element by the same element corresponding to the specified gauge transformation α. Consequently, if we label our path in H as $f(\lambda)$ where λ parametrizes the path, then under the gauge transformation, we have the transformed path $\alpha f(\lambda)\alpha^{-1}$. Provided that H is path-connected, we can always deform α continuously to the identity, and our transformed path is still homotopic to the original one.

We begin by ending this section by stressing the emergence of quantized magnetic charge as a consequence of the topology associated with the gauge theory. It is the quantization of magnetic charge together with the fact that it must be a continuous variable of space and time that guarantees its stability. Other methods of constructing monopoles fail to guarantee that the monopoles they give rise to must be stable.

2.2 Topological classification of 't Hooft-Polyakov monopoles

We notice that our constructions above eventually conclude that all of the magnetic charge in a gauge theory must be contained within a point singularity. At this point, we can be satisfied with this definition, or if we choose to believe that singularities are unphysical, which is actually really compelling since this implies infinite energy density at the origin, we are forced to conclude that something must be slightly off about our reasoning above.

't Hooft and Polyakov's brilliant insight was that we can avoid having a singularity by taking advantage of the topology associated with the gauge groups in spontaneously broken gauge theories. Specifically, given a spontaneously broken gauge theory described by the breaking pattern $G \to H$, where G and H are Lie groups, the 't Hooft-Polyakov construction is a soliton solution that manages to avoid singular monopole by having gauge field configurations associated with G that are confined within a monopole core of some finite radius. These gauge fields are massive as a result of the spontaneous breaking of the gauge symmetry associated to some GUT. Meanwhile, outside the monopole core, the long range gauge field is massless such that it can propagate out to infinity, where it agrees with the gauge field of the Dirac monopole discussed previously. In this paper, we will not give an explicit construction of t' Hooft and Polyakov's soliton solutions, but we will instead leverage on topological principles to show why they must exist if we impose certain conditions. Doing so will also let us see why our claims about the properties of t' Hooft-Polyakov monopoles are true.

2.2.1 Topological solitons

The basic idea behind topological solitons is to construct field configurations with finite energy, so that we can avoid scenarios with divergent energy like the Dirac monopole. Furthermore, these field configurations should somehow involve the spontaneous breaking of a gauge theory. We begin as usual by considering a gauge theory with high energy gauge group G and a scalar multiplet Φ transforming under G. When Φ acquires a vacuum expectation value Φ_0, the gauge group is broken down to H, the subgroup of G that leaves Φ_0 invariant. As usual, we will assume that G and H are path-connected.

In order to construct finite energy solutions to the classical field equations of this gauge theory, we need to specify field configurations of excitations around Φ_0 with energy density that dies off at spatial infinity. We can avoid the problem of explicitly constructing such configurations that are also solutions to the classical Yang-Mills equations by simply positing that we require maps with some specific boundary configuration. Mathematically speaking, we need to specify maps from the sphere at spatial infinity, $S^2 \subset \mathbb{R}^3$ to Φ_0. In general, the vacuum manifold is now of the form

$$G/H = \{\Phi' \mid \Phi' = \Omega\Phi_0\,\Omega \in G\} \tag{35}$$

if Φ transforms in the fundamental representation of G. One can see that the vacuum manifold is still G/H, just defined in a slightly different way, if the order parameter Φ now transforms in the adjoint representation, rather than the fundamental. The vacuum manifold above is the manifold of all equivalent vacua that are related by G. Since we can always apply a gauge transformation in G to relate any point Φ' in G/H back to $\Phi_0 = \Omega^{-1}\Phi'$, we see that if we fix a map $f : S^2 \to \Phi_0$, we can obtain another map $\Omega \circ f : S^2 \to \Phi'$ that is homotopic to f, since Ω can always be continuously deformed to the identity in G which is path-connected. Therefore, to specify finite-energy field configurations at infinity, it is enough to to consider maps $S^2 \to G/H$, which is precisely the definition for $\pi_2(G/H)$, the second homotopy group of G/H.

Now, in order to determine $\pi_2(G/H)$, we can use the LES in 30 for $n = 2$ along with the (quite remarkable) result from Lie theory that $\pi_2(G) \cong 0$ for any Lie group G. Now, in general, G need not be simply-connected, in which case the exactness of the LES gives us

$$\pi_2(G/H) \cong \pi_1(H)/\pi_1(G) \tag{36}$$

However, if G is simply connected, as it is in most GUTs, then $\pi_1(G) = 0$ and we have

$$\pi_2(G/H) \cong \pi_1(H) \tag{37}$$

In this case, we see that this matches the topological charge classification we have for magnetic charges in the case of the Dirac monopole, and we have good reason to suspect that the topological charge of the t' Hooft-Polyakov monopole coincides with its magnetic charge.

All we have shown, however is that the topological charge associated with the finite-energy condition on Φ at spatial infinity is equal to the topological charge of loops in H. To go further and show that this is in fact equivalent to the magnetic charge, we recall that the finite-energy condition on a field configuration Φ implies that we have

$$D_i\Phi = (\partial_i - igA_i^aT^a)\Phi = 0 \tag{38}$$

on the sphere at spatial infinity, where T^a corresponds to the generators of G in the representation that Φ transforms in. By picking an appropriate gauge, we can enforce $\Phi = \Phi_0$ which is constant such that $\partial_i\Phi = 0$ on the sphere. Thus, we require that

$$A_i^aT^a\Phi_0 = 0 \tag{39}$$

which is true only if A_i^a is a gauge field corresponding to the unbroken gauge group H. To see why, observe that since H acts trivially on Φ_0 by definition, we have that the infinitesimal action on Φ_0 satisfies

$$(1 + i\varepsilon A_i^aT^a)\Phi_0 = \Phi_0 \tag{40}$$

$$\iff A_i^aT^a\Phi_0 = 0 \tag{41}$$

which is true as long as A_i^a correspond to the gauge fields of H. This tells us that the only gauge fields that can be excited out to spatial infinity are the gauge fields of H, which is not surprising since they are the only ones that remain massless after spontaneous symmetry breaking. Now, we are reduced back to the case of the Dirac monopole with gauge group H, only now we are at spatial infinity and can safely ignore all the stuff going on inside the monopole core. Everything works the same regardless of the radius of the sphere as we argued earlier, and we can repeat the work we did in Section 4.1 to conclude that this is exactly the same magnetic charge as in the Dirac monopole. Hence, we see that the topological charge of finite-energy solitons corresponds exactly to the magnetic charge that Dirac initially envisioned.

2.2.2 't Hooft-Polyakov vs Dirac

The results above give a good physical picture of what the soliton solution actually is, and how they relate back to Dirac monopoles. Given some arbitrary gauge group G which undergoes spontaneous symmetry breakdown to an unbroken subgroup H, we should consider a monopole as being contained within some core of finite radius $R \sim (ev)^{-1}$ that is to be determined by the mass scale v associated with the symmetry breakdown. Outside R, SSB has occurred, and we are only left with the long-range, massless gauge fields associated with the unbroken symmetry group H that propagate out to infinity and give us a prediction of the monopole charges that agree with that of the Dirac monopole. At distance scales this large, the monopole core appears essentially like a point singularity as in the case of the Dirac monopole. In fact, if we didn't have the means to probe deeper into the monopole core, one would never be able to measure the effect of G and would think that this is exactly a Dirac monopole.

However, as we zoom into the singularity and eventually penetrate into the monopole core, we have to consider the full gauge group G. To see why this must be the case, assume that there were only H gauge fields. From the argument that led to us considering the fundamental group of the gauge group as the indicator of the topological charge, we find that the element of $\pi_1(H)$ must be a function of the radius of a sphere r. As we shrink $r \to 0$, the associated element in $\pi_1(H)$ must be 0, since it is now the constant path. We then have two possibilities:

1. The path along the loop is continuous, from which it follows that it must have been the trivial homotopy class in the first place. This means that the topological charge is always zero.
2. The path along the loop is discontinuous, which means that we have a singularity. Since the topological charge is gauge invariant by construction, this singularity is a genuine one that cannot simply be gauged away.

Thus, if we want to retain the possibility of non-trivial topological charges while still avoiding singularities, there cannot only be H gauge fields inside the monopole core. The only other sensible option is to include G gauge fields inside the monopole core. This must be the case, since within the monopole core, energy scales are high enough such that gauge fields belong to the full unbroken symmetry group can be excited. In fact, this is what resolves the problem of a singular monopole for us. We can have loops that are topologically non-trivial in the subgroup H , but ultimately trivial in the total group G. Hence, the topological charge defined by H at asymptotic distances can still be non-trivial, but as we shrink radius of a sphere and penetrate within the monopole core, loops can become trivial within the larger group G, avoiding a singularity altogether. A schematic representation of the physical picture so far is given in Fig. 2.2.2

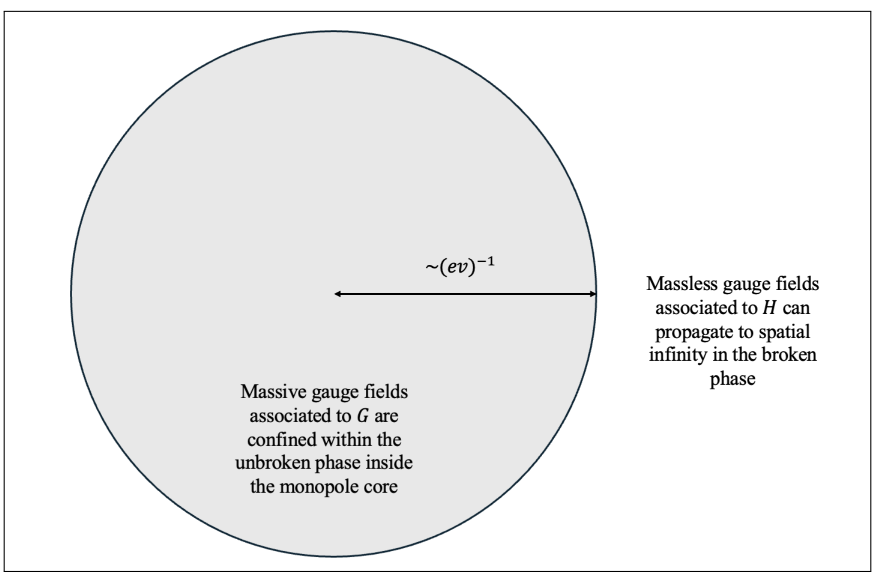

Figure 1: Monopole core for a spontaneously broken gauge theory $G \to H$, where v is the mass scale at which SSB occurs.

To summarize, we initially started with a nice topological description of Dirac monopoles, but that suffered from singularities at the origin, where physics breaks down. Now we have something better - 't Hooft-Polyakov monopoles manage to avoid the singularity by introducing a larger gauge group G that is relevant within a monopole cloud of some finite radius determined by the scale of the symmetry breaking from G to H. These field configurations, called solitons, acquire a topological charge from the finite-energy condition that we require from them, which corresponds exactly to the magnetic charge of Dirac monopoles, as we would want. Since this holds for any $H \subset G$, we would expect the preceding discussion to hold for any GUT and thus make predictions about the possibility of t' Hooft-Polyakov monopoles. We would be able to do this by calculating $\pi_2(G/H)$. In the case where G is simply connected, we can make the following definition:

Definition 1 (Non-singular monopole)**.** For a GUT with a simply-connected gauge G, the allowed topological charges of non-singular monopoles after the symmetry breaking $G \to H$ are given by the elements of $\pi_2(G/H) \cong \pi_1(H)$.

Note that this definition only holds in the presence of SSB, when there is a larger unified gauge group G to talk about. What about gauge theories which do not arise from any sort of SSB? In cases such as these, we can calculate $\pi_1(H)$ where H is the associated gauge group of the theory, but we will need to be careful about how to interpret these calculations, for in the absence of GUTs, we do not have the t' Hooft-Polyakov construction and we only have Dirac monopoles which are not guaranteed to be non-singular and physical.

2.3 Symmetry breaking properties

Having seen how GUTs should predict the existence of monopoles, we now turn to examples in order to better illustrate the machinery we have developed above. Along the way, we will see several more properties of topological solitons that are made clearer by considering examples of spontaneously broken gauge theories.

2.3.1 Importance of global topology

We begin this section by illustrating an example of a theory where symmetry breaking occurs at several stages, which is often the case in realistic GUTs that describe the Standard Model. Consider a model with gauge group $G = SU(3)$ and a scalar field Φ that transforms in the adjoint representation of $SU(3)$. Suppose that the $SU(3)$ symmetry is spontaneously broken by Φ acquiring an expectation value $\Phi_0 = v\text{diag}(1/2, 1/2, -1)$ where v is the mass scale of the symmetry breaking.

As we learnt in class, we can identify the unbroken subgroup H as the subgroup of G whose adjoint action leaves Φ_0 invariant. Precisely, we want to find a group such that the generators of its Lie algebra commute with Φ_0. As usual, we can identify this as $SU(2) \times U(1)$. Note, however, that identifying H as being isomorphic to $SU(2) \times U(1)$ as Lie groups is too quick - observe from the procedure above that there is some ambiguity on the level of Lie groups, where several different Lie groups can have the same Lie algebra. Instead, when we have Lie groups that are not necessarily isomorphic but have isomorphic Lie algebras, we say that they are **locally isomorphic**, an appropriate name since Lie algebras are defined locally as the tangent plane at a point. Here, we see one such example: to determine the true unbroken gauge group H, we should define it as the minimal group that retains the symmetry we require, or in simpler terms, we should get rid of any redundant gauge groups hiding in $SU(2) \times U(1)$ which act trivially on all fields in our theory. A good candidate would be the center of $SU(2) \times U(1)$, which is given by $\mathbb{Z}_2$, since the adjoint action given by the subgroup isomorphic to $\mathbb{Z}_2$ in $SU(2) \times U(1)$ would be trivial on all adjoint fields. We will need to go one step further to check that we can pick a specific generator for this subgroup such that its action on fundamental fields is also trivial, but we will not do this in this paper, but simply accept that it is sensible to write that

$$H \cong (SU(2) \times U(1))/\mathbb{Z}_2 \tag{42}$$

is the true unbroken gauge group.

After going through so much trouble to determine this "true gauge group", we can ask ourselves why this matters so much. Indeed, since the Lie algebra of $SU(2) \times U(1)$ is isomorphic to that of $(SU(2) \times U(1)/\mathbb{Z}_2$, the physics that we describe in terms of gauge fields and correlation functions should not care about what the "true" Lie group is, since the fields are always defined starting from the Lie algebra. To this extent, we recall that the topological classification of monopole solutions is given by $\pi_2(G/H) \cong \pi_1 nh(H)$, which clearly gives us very different results depending on what H is! In this example, if we had forgotten about the global structure of H and taken $H = SU(2) \times U(1)$, then $\pi_1(H) \cong \mathbb{Z}$, and we would think that we have our usual integer quantization of magnetic charge. However, we now know that we really should take $H = (SU(2) \times U(1))/\mathbb{Z}_2$. Analyzing the long exact sequence of homotopy groups again, we get

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \pi_1(H) \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

From this, we only conclude for now that $\mathbb{Z}_2 \cong \pi_1(H)/\iota(\mathbb{Z})$. It would be nice to know what $\pi_1(H)$ is, but the general problem of calculating what $\pi_1(H)$ is, is in general very difficult if we do not know what the map ι is exactly. This is known as the "extension problem" in mathematics. What we can say, however, is that there appears to be more than just the normal magnetic monopole charge given by $\mathbb{Z}$. Now, $\pi_1(H)$ is larger than $\mathbb{Z}$, and what we can say definitely is that the cardinality of $\pi_1(H)$ is twice the cardinality of $\mathbb{Z}$, or in other words, the number of allowed monopole charges here should be twice that of the number of magnetic monopole charges arising from $U(1)$ electromagnetism. We should therefore interpret this as some additional, $\mathbb{Z}_2$ non-abelian magnetic charge that monopoles can carry in addition to $U(1)_{EM}$ magnetic charge, similar to the case of colour magnetic charge discussed at the beginning. We see here that accounting for the global structure of the true gauge group has given rise to some very interesting results!

2.3.2 Heirarchical symmetry breaking

In models that most closely describe reality, it is often the case that symmetry breaking of a GUT occurs in several stages, as illustrated below.

$$G \xrightarrow{v_1} H_1 \xrightarrow{v_2} H_2 \tag{43}$$

Here, v_1, v_2 are the mass scales at which each stage of symmetry breaking occurs. From now on, we will like to refer to different stages of the symmetry breaking as different phases belonging to G, H_1 or H_2. This is not such a bold move to make, since the acquisition of a vacuum expectation value during spontaneous symmetry breaking is effectively described as a phase transition, where the order parameter acquires the vacuum expectation value.

We get from the above three fiber bundles, two associated to each phase H_1 and H_2 in relation to the total gauge group G, and one associated to the second stage of symmetry breaking. What we get exactly will depend on how the gauge groups are related to each other. The most common relation, and the one we shall analyse here, is given by $H_2 \subset H_1 \subset G$. In this case, we have

$$H_1 \to G \to G/H_1 \tag{44}$$
$$H_2 \to G \to G/H_2 \tag{45}$$
$$H_2 \to H_1 \to H_1/H_2 \tag{46}$$

We can analyse the homotopy groups associated to each fiber bundle above again using the LES 30 associated to each fibration. Using the magical theorem that $\pi_2(G) = 0$ for any Lie group, the fibrations 44 and 45 merely tell us at this point that

$$\pi_2(G/H_1) \cong \pi_1(H_1) \tag{47}$$
$$\pi_2(G/H_2) \cong \pi_1(H_2) \tag{48}$$

We have seen previously that Eq. 47 can be interpreted as the allowable topological charge associated to the finite energy long range field configurations associated to the phase H_1. The topological charge is associated to the magnetic charge of a non-singular monopole by virtue of a finite radius core within which massive gauge fields belonging to G reside. This gives us a natural way to interpret Eq. 48 as well - it is just the allowable topological charge associated to the finite energy long range field configurations associated to the phase H_2. Of course, when we say "long range", we mean distances at which energies are low enough such that the full gauge group has broken down all the way to H_2. Naturally, we expect there to be a finite monopole core within which massive G and H fields reside. Repeating the analysis in Section 4.2.1 to find out which gauge fields are allowed to propogate in each phase, we end up with the following physical picture in Fig . 2.3.2

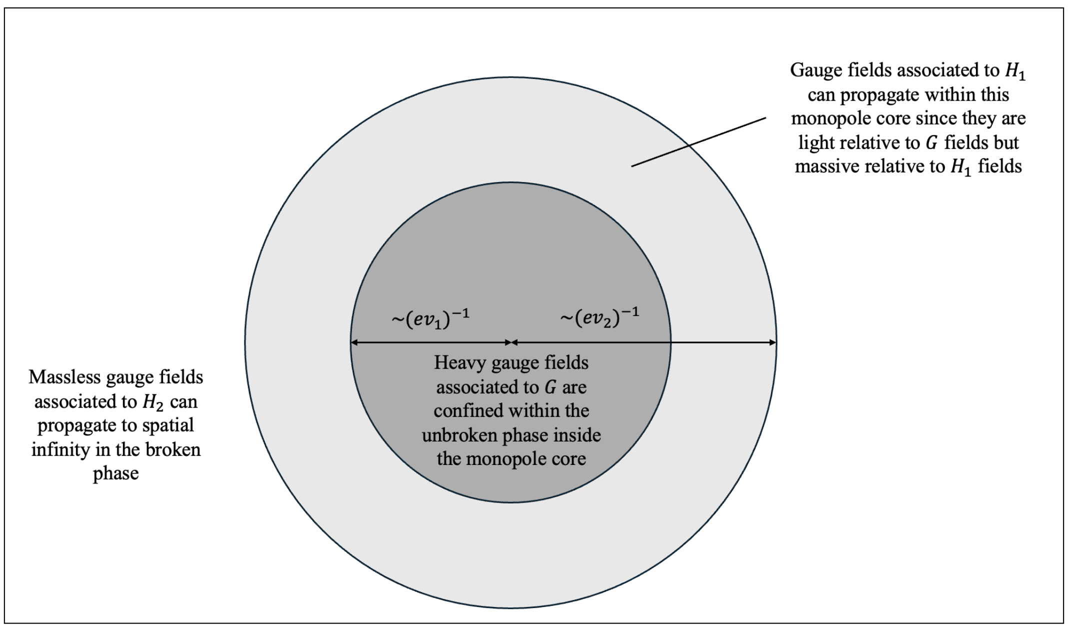

Figure 2: Monopole core for a spontaneously broken gauge theory $G \to H$, where v is the mass scale at which SSB occurs.

Now, what about the fiber sequence associated to 46? In general, $H_1 \neq H_2$, which gives us yet more interesting physics. We can focus on one part of the sequence:

$$\cdots \longrightarrow \pi_2(H_1) \longrightarrow \pi_2(H_1/H_2) \longrightarrow \pi_1(H_2) \longrightarrow \pi_1(H_1) \longrightarrow \cdots$$

Since $\pi_2(H_1) = 0$, the exactness of the sequence says that the map $\pi_2(H_1/H_2) \to \pi_(H_2)$ is injective. Hence, the image of the map in $\pi_(H_2)$ is isomorphic to $\pi_2(H_1/H_2)$. Then, the exactness at $\pi_1(H_2)$ tells us that

$$\pi_2(H_1/H_2) \cong \ker(\pi_1(H_2) \to \pi_1(H_1)) \tag{49}$$

In physical terms, we can interpret this as $\pi_2(H_1/H_2)$ characterizing topological charge that is non-trivial in the H_2 phase but trivial in the H_1 phase. Thus we find that if H_1/H_2 is not 2-connected

($\pi_2(H_1/H_2) \neq 0$), then we can have new allowable values of magnetic charge appear after the phase transition from H_1 to H_2! These new monopoles in the H_2 phase will be lighter compared to the monopoles in the H_1 phase, where the former has core radius of order $(ev_2)^{-1}$ and mass v_2/e compared to the latter with core radius of order $(ev_1)^{-1}$ and mass v_1/e. This is due to the fact that they were not around in the H_1 phase but suddenly sprang to life in the H_2 phase. Of course, elements that do not belong to the kernel in Eq. 49 are non-trivial topological charges that still carry the old topological charges associated to H_1, and may or may not carry new topological charges associated to H_2.

Definition 2 (New light monopoles after symmetry breaking)**.** In a spontaneously broken gauge theory with a phase transition from H_1 to H_2, the second homotopy group $\pi_2(H_1/H_2)$ describes new, lighter magnetic monopoles in H_2 that were not present in H_1.

As an example, consider the symmetry breaking pattern

$$G = SU(3) \xrightarrow{v_1} H_1 = SO(3) \xrightarrow{v_2} H_2 = U(1) \tag{50}$$

from which we can compute

$$\pi_2(G/H_1) \cong \pi_1(SO(3)) \cong \mathbb{Z}_2 \tag{51}$$

Thus, in the H_1 phase, we have Z_2 monopoles carrying $SO(3)$ magnetic charge with allowable charges of 0 or 1, in units of g_D. Now, as we approach the second transition v_2 smoothly, possibly by turning down the temperature slowly or zooming out to greater distances, we have

$$\pi_2(G/H_2) \cong \pi_1(U(1)) \cong \mathbb{Z} \tag{52}$$

Here, we see that as we continuously approach the transition scale until we hit the transition itself, something needs to happen to the Z_2 monopoles such that we suddenly have integer charge monopoles after the transition. Clearly, some of these new charges need to appear spontaneously, and thus should be lighter than the previous $\mathbb{Z}_2$ monopoles. However, will any of the $\mathbb{Z}_2$ monopoles survive the phase transition at all? If we insist on sticking to our topological philosophy, we can consider the map $\pi_2(G/H_1) \cong \mathbb{Z}_2 \to \mathbb{Z} \cong \pi_2(G/H_2)$. Recalling that there is no non-trivial group isomorphism $\mathbb{Z}_2 \to \mathbb{Z}$, we conclude that this must be the zero map, where all the $\mathbb{Z}_2$ monopoles get mapped to zero during the phase transition. One such way that this can happen is that the $\mathbb{Z}_2$ monopoles, which are their own antiparticles, start annihilating each other rapidly during the phase transition until none remain.

Of course, we can also see whether we have monopoles that did not exist in H_1 but now exist in H_2 by calculating the following.

$$\pi_2(SO(3)/U(1)) \cong \pi_2(\mathbb{R}P^2/S^1) \tag{53}$$

The exact homotopy group is quite difficult to calculate here, but we can get a sense of it by looking at the long exact sequence of fibrations yet again - here, we see that $\mathbb{Z} \cong \pi_2(\mathbb{R}P^2) \subset \pi_2(SO(3)/U(1))$. In other words, we will have lighter monopoles in H_2 that were not there in the first place in H_1, and they should at least be all of the integer valued monopoles, as we anticipated above!

2.3.3 Flux tubes

We can now consider a similar question, albeit in the opposite sense - is it possible for magnetic charges to be forced to assume integer multiples of itself after a phase transition? Certainly, there is nothing preventing this from happening. A reasonable guess might be that pairs of monopoles might become connected through flux tubes, forming composite objects with a total magnetic charge similar to how quark confinement can be modelled using flux tubes. We can discuss flux tubes more precisely by defining them to be time-independent solutions to our field equations with finite energy per unit length. Equivalently, we can regard them as topological solitons in two spatial dimensions with a finite energy condition. This is very similar to our definition for magnetic monopoles in terms of topological solitons in three spatial dimensions, and we can follow an analogous procedure to define them as follows.

We start by defining flux tubes to be specified by a finite energy condition in a two dimensional cylinder. We can characterize this by a maps from the circle S^1 at two-dimensional spatial infinity into the vacuum manifold G/H as in the case of magnetic monopoles. In other words, flux tubes can be assigned a topological charge with elements in $\pi_1(G/H)$. In the case that G is connected and simply connected (which is usually the case as always in GUTs), we have

$$\pi_1(G/H) \cong \pi_0(H) \tag{54}$$

Thus, we see that we have the following definition for the topological charge of a flux tube.

Definition 3 (Flux tube)**.** For a GUT with a simply-connected gauge group G, the allowed topological charges of flux tubes after symmetry breaking are given by the elements of $\pi_1(G/H) \cong \pi_0(H)$.

Notice how this is in direct analog with our definition for non-singular monopoles, just one dimension lower as we expect. Naturally, when we have several heirarchies of symmetry breaking, we again can interpret $\pi_0(H_1)$ and $\pi_0(H_2)$ as the possible topological charges associated to flux tubes in the symmetry broken phases.

Furthermore, in the case that $\pi_1(H)$ is simply connected (which might sometimes be the case), we can again look at the homotopy fiber sequence to obtain the following

$$\pi_1(H_1/H_2) = \ker(\pi_0(H_2) \to \pi_0(H_1)) \tag{55}$$

Again, the kernel above corresponds to topological flux that was trivial in H_1 but now non-trivial in H_2. These new flux tubes will have thickness and energy per unit length determined by the appropriate symmetry breaking scale.

Definition 4 (New light flux tubes after symmetry breaking)**.** In a spontaneously broken gauge theory with a phase transition from H_1 to H_2 with H_1 simply connected, the first homotopy group $\pi_1(H_1/H_2)$ describes new, lighter(thinner) flux tubes in H_2 that were not present in H_1.

Of course, what we are really interested in is how magnetic charge eventually ends up in flux tubes after symmetry breaking, or the other way round. We can assign a precise mathematical criterion to this by from the long exact sequence:

$$\mathrm{im}(\pi_1(H_2) \to \pi_1(H_1)) \cong \ker(\pi_1(H_1) \to \pi_1(H_1/H_2) \tag{56}$$

Translating this to physics, the above says that monopoles in the H_1 phase that are in the image of $\pi_1(H_2) \to \pi_1(H_1)$ do not develop a flux tube in H_2, as they are mapped trivially to $\pi_1(H_1/H_2)$. Naturally, these monopoles either survive the phase transition or are annihilated through various mechanisms. Conversely, the elements in $\pi_1(H_1)$ that do not fall in the image of the map above will develop flux tubes. Of course, this entails that we know how the homotopy groups are mapped exactly, which might not be very clear at times. The best way to study the formation and dissipation of flux tubes would be to just compute $\pi_1(G/H)$ in each phase and compare them.

2.4 The Standard Model

After all ideas we talked about above, the obvious question in the air is whether we can apply such ideas to figure out whether the Standard Model (SM) allows for magnetic monopoles or not. As we will see, this will depend critically on whether GUTs are allowed or not, and even more critically on the breaking mechanisms behind SSB.

2.4.1 No GUTs

We begin by just considering the SM as it is currently established and verified experimentally, in the absence of GUTs. The SM gauge group in this case is given by $G = SU(3)_C \times SU(2)_W \times U(1)_Y$. This spontaneously breaks to $H = SU(3)_C \times U(1)_{EM}$ as we have seen in class via the acquisition of a vev by the Higgs doublet H with the potential $V(H) = -m^2|H|^2 + \lambda|H|^4$. The obvious thing to do here is to first calculate the topological charge of G:

$$\pi_1(SU(3) \times SU(2) \times U(1)) \cong \pi_1(U(1)) \cong \mathbb{Z} \tag{57}$$

This seems to suggest that the high energy theory of the SM allows for integer charged monopoles. However, we must be careful with how we interpret this, since these might be singular monopoles. In fact, the topological charge has to be zero if we want to avoid singularities. We can do better by looking at the topological charge of the theory after SSB instead by calculating $\pi_2(G/H)$. Here, note that since $\pi_1(G) \neq 0$, we cannot write $\pi_2(G/H) \cong \pi_1(H)$. Instead, we have to work with the second homotopy group in full generality. To proceed with the calculation, note that we have the following

$$G/H \cong \frac{SU(3) \times SU(2) \times U(1)}{U(1)} \ncong SU(3) \times SU(2) \tag{58}$$

Instead, we have to work out the full quotient $\pi_2(G/H)$ using the LES 30. But this is not so bad, since $\pi_2(G) = 0$ by the magical theorem for Lie groups, and $\pi_1(H) = 0$, so we end up with $\pi_2(G/H) = 0$. In other words, at the low energy phase, we do not have magnetic monopoles. This is consistent with our experience in low energy physics and also at the LHC.

2.4.2 With GUTs

Things, however, get extremely interesting once we allow for the possibility of GUTs, where the SM gauge group we see phenomenologically is actually a broken phase of a grand unified theory at higher energies. For this paper, we will consider the minimal $SU(5)$ model, where the breaking pattern is given by

$$G = SU(5) \xrightarrow{v_1} H_1 = [SU(3) \times SU(2) \times U(1)]/\mathbb{Z}_6 \xrightarrow{v_2} H_2 = [SU(3) \times U(1)]/\mathbb{Z}_3 \tag{59}$$

Here, we note that in contrast to the phenomenological SM gauge group seen in the preceeding section, the appearance of quotients by cyclic groups appears as a consequence of the specific breaking patterns in the minimal $SU(5)$ model. In particular, the unbroken group is determined by the subgroup of $SU(5)$ whose generators commute with the vev, given by $\Phi_1 = v_1 \mathrm{diag}(1/3, 1/3, 1/3, -1/2, -1/2)$. While it is easy to assemble together the generators of $SU(5)$ so that the unbroken group is $SU(3)\times SU(2)\times U(1)$, we must note that this is **not** a normal subgroup of $SU(5)$. From the theory of groups, we can turn this into a normal subgroup by quotienting out by the center, which is $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$. Another way to see this is that the vev is invariant under the adjoint action of the subgroup isomorphic to $\mathbb{Z}_6$ in $SU(5)$. A similar line of reasoning can be applied to the symmetry breaking at v_2, where the unbroken $U(1)$ subgroup of $SU(5)$ is the one generated by $\mathrm{diag}(1/3, 1/3, 1/3, 0, -1)$, from which one can work out that the vev is invariant under the $\mathbb{Z}_3$ isomorphic subgroup of $SU(5)$.

We can now go on to evaluate the relevant topological charges. Since $G = SU(5)$ is simply connected, we have

$$\pi_2(G/H_1) = \pi_1(H_1) = \mathbb{Z} \tag{60}$$

Here, the fact that it is $\mathbb{Z}$ is not very obvious, but what the equation above tells us is that the magnetic monopole charges correspond to the homotopy classes of loops in H_1. We know at least that $\pi_1(H_1) \neq 0$ since $SU(3) \times SU(2) \times U(1)$ is not simply-connected, and the quotient by $\mathbb{Z}_6$ will not change that fact. To further see that the homotopy classes of loops is classified by $\mathbb{Z}$, we note that a path that can go around H_1 is constructed explicitly in [7], where they pick generators of the individual $U(1)$ subgroups of the three components $SU(3), SU(2), U(1)$ in H_1 and build a path that goes through all 3 subgroups. In other words, we can always pick generators and exponentiate to construct loops that do not necessarily complete a full loop in any single $U(1)$ subgroup, but manage to complete a full loop through the other components of H_1.

We can repeat a similar analysis for the second broken phase H_2, which gives rise to the low energy standard model gauge group $SU(3)_C \times U(1)_{EM}$. Again, $G = SU(5)$ is simply connected, and we have

$$\pi_2(G/H_2) = \pi_1(H_2) = \mathbb{Z} \tag{61}$$

Here, we reach the same conclusion using the same observations as above - H_2 cannot be simply connected since $SU(3) \times U(1)$ is not simply connected, and taking a quotient of it by a cyclic group cannot make it simply connected. Furthermore, we can again pick generators of the $U(1)$ subgroups in H_2 to build paths that always wind around an integer number of times. From the above, we see that the low energy SM gauge group is predicted to contain monopoles, in contrast to the case with no GUTs. Essentially, the presence of non-trivial topological solitons is caused by the cyclic group quotients that are only possible when we consider the SM gauge group as one that is induced from the spontaneous breaking of a GUT gauge group.

We end off this section on topology by commenting on the power of topology in classifying so much information about the existence and behaviour of monopoles across phase transitions without having to resort to explicit soliton constructions. All it took was the imposing of a boundary condition on gauge fields, which is in itself quite remarkable. Even though the latter half of the chapter was devoted to just developing the theory without explicit examples, we note that one can carry out a deeper analysis of the minimal $SU(5)$ model presented above, and even variants of it, in order to analyse certain aspects of the monopoles they can accomodate, including but not limited to the formation of flux tubes and cosmic strings, how monopoles in H_1 do not change during the phase transition to H_2, and the implications it has for the existence of light monopoles and their electroweak interactions with fermions.

We further note that given the clear contrast in the possibility of monopoles between just the phenomenological SM gauge group and the one derived from the $SU(5)$ GUT, the existence of monopoles will lend extremely strong support to the validity of GUTs as a step in the right direction to describe our universe. This is good motivation for the next section where an exposition of how monopoles can form and be detected is given.

3 Production of Monopoles

3.1 Kibble-Zurek Mechanism

While our analysis in section 4 demonstrates that the existence of magnetic monopoles is consistent with fields described by GUT symmetry groups (and that they are quite natural objects to consider), that alone does not necessitate their existence. For that, we require a mechanism that could exploit the non-trivial topology of the manifold resultant from symmetry breaking. As the value of the field in the unbroken GUT symmetry is taken to be continuous, it is energetically favorably for a domain of unbroken symmetry to inherit the vacuum configuration of the broken domains neighboring it. Therefore, if the universe remained at the transition temperature for long enough, the correlations between distant regions of space would require that the fields predominately form topologically trivial configurations. However, if the transition were to happen on a small enough timescale, inhomogeneities in the spatial temperature distribution would lead to the formation of domains

with uncorrelated field configurations. Depending on the values of the various domains coming into contact after nucleation and on the group structure left over after symmetry breaking, various types of topological defects can be formed at the boundaries [6].
Taking the original symmetry group of the theory to be $\mathbf{G}$ and the remaining symmetry group after breaking to be $\mathbf{H}$, the manifold of degenerate vacuum state is $\mathbf{M} = \mathbf{G}/\mathbf{H}$. If $\mathbf{M}$ is not connected it becomes possible for two domains to take disconnected field values, resulting in a domain wall of the field in its unbroken energy state as it interpolates between the planar boundary of the broken regions. Interesting behavior can still arise when the domains take values of the field from mutually connected region, but more domains are typically required. If three domains with different field values meet at an edge, the planar boundaries between them will relax into a smooth transition between vacua, but it is possible for a loop around the edge to take a non-contractable path through the manifold, resulting in a 'string' of the unbroken symmetry along the edge. One instance of the most relevant topological defect can be formed when four domains with different, yet connected, vacua states meet at a point. The connectedness of the region of the manifold they lie in allows their boundaries to become smoothly connected, but it remains possible for a volume around the central point to be incapable of contraction, resulting in a excited point of unbroken field i.e. a monopole [9]. In the language from earlier in the paper, strings require a non-trivial $\pi_1(\mathbf{M})$ homotopy group while monopoles require a non-trivial $\pi_2(\mathbf{M})$. For simplicity, we will restrict our analysis to behavior of this type under SU(5), though an equivalent argument under any GUT that breaks into a manifold with a non-trivial second homotopy, π_2, holds, which differs in the vacuum expectation value after symmetry breaking and, as described in section 4, the charges of the monopoles. After the first symmetry breaking, separating the strong and electroweak forces, the second homotopy group of the manifold resultant from SU(5) is

$$\pi_2(\mathbf{M_5}) \equiv \pi_2(\frac{SU(5)}{(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6}) = \mathbb{Z} \tag{62}$$

, allowing us to expect the creation of magnetic monopoles and anti monopoles during SU(5) symmetry breaking with charges of integer multiple of the Dirac charge. The presence of $\mathbb{Z}_6$ as a redundant guage group is derived from the same principles used in Eq. 42.

Now that we trust that monopoles should form, what remains in analyzing the topological picture is the probability that a set of domain field configurations wrap around $\mathbf{M}$ in a non-trivial way. In the simplified case in which $\mathbf{M}$ is a 2-sphere, the probability for a set of 4 domains to behave this way is 1/8 since that is the probability that 4 unit vectors define a tetrahedron which includes the origin. Lacking a similar geometric intuition for the shape of $\mathbf{M}$, the explicit calculation of the probability for our 12-dimensional manifold will not be undertaken, but we do not expect it to differ much from order one.

After the Planck epoch of the early universe ended ($\sim 10^{-43}$ s), the universe, if any GUT is correct, entered into a phase where the strong and electroweak forces had not yet broken. The generally accepted value for this phase transition under SU(5) is at an energy scale of around 10^{15} GeV or after 10^{-36} s. As we are focused on topological phenomena, it makes sense to focus on the coherence length ξ as the dominant property of the transition governing monopole formation. However, the expected behavior of the coherence length around 2nd order phase transitions is diverging, so since causality constraints provide a finite length scale, we take that to be the relevant upper bound on the coherence length. One approximation of this takes the timescale over which the transition occurs to be the time after the Planck time at which it occurred

$$\xi \lesssim \tau_{GUT} \approx C\frac{M_{pl}}{T_c^2} = 10^{-28} \text{ m} \tag{63}$$

, where C is a constant dependent on the massless spin degrees of freedom at the temperature T_c and is about 1/20 in SU(5) [9]. If one monopole were produced per causation-independent volume of the field after symmetry breaking, we would expect $\gtrsim 10^{84}$ monopoles per cubic meter to have been produced at the time of SU(5) symmetry breaking. Expressing this as a unit-less ratio to the cube of temperature to provide an adiabatically conserved constant throughout the universe's expansion yields

$$\frac{n}{T^3} \gtrsim \frac{\xi^{-3}}{T_c^3} \approx 10^{-11} \tag{64}$$

Which is one order of magnitude less than the same parameter for baryons. Since baryons have mass on the order of 1 GeV, our non-detection of magnetic monopoles under this naive calculation is entirely unreasonable. Given that we might expect that GUTs are real, we therefore turn to inflation to find a way to decrease the density of monopoles. Various estimates of inflation derived from the homogeneity of the CMB and the curvature of the universe place the number of e-folds, the exponent in the power of e the universe expanded by during inflation, to be 30 to 60 [5]. The density of monopoles in the universe would therefore be reduced by a factor of e^{-90} to e^{-180} or 10^{-39} to 10^{-78}, though these factors are highly model dependent. This yields an upper bound on the density of monopoles coming from the first breaking of the SU(5) symmetry of,

$$\frac{n}{T^3} \lesssim 10^{-50} \tag{65}$$

3.2 Other Sources and Loss Mechanisms

After inflation, it is hypothesized that the energy stored in the inflaton field would be transferred into the matter fields, causing a period of reheating, the temperature of which may reach temperatures prior to inflation. Is it possible that during this phase, parts of the universe reached the requisite temperatures to revert to vacuua of the GUT and undergo symmetry breaking again, allowing for the creation of magnetic monopoles. Without the process of inflation after this to re-dilute the newly produced monopoles, the possibility arises that the resultant density of monopoles could over-close the early universe, leading to collapse. Bounds on the production of magnetic monopoles during this time are not well defined since the reheating temperature is not well constrained by theory or cosmology.

An alternative mechanism of monopole production lies on the conceptual basis of the duality between the electric and magnetic field. The Schwinger Effect allows for the production of electron positron pairs in the presence of a sufficiently strong electric field [8]. Arguments stemming from the symmetry between the description of electric and magnetic charge motivate the possibility that a sufficiently strong magnetic field could do the same. The possibility that the magnetic fields currently present in the universe were present in the early universe and therefore much stronger presents another mechanism for large scale production of monopoles. However, the connection between monopoles produced through this mechanism and those produced through symmetry breaking of GUTs is not clear. The production of 't Hooft-Polyakov monopoles would require the spontaneous creation of topological defects at energies much lower than the GUT scale.

Since opposite magnetic monopole charges attract each other, there exist two other ways that their observed density could have been reduced from the calculated value. First, they could annihilate each other, and, second, they could form neutral bound pairs. While this is an effective way of reducing the density of free monopoles when the monopoles being considered have less mass, for masses near the GUT scale the capture process is too slow to keep up with the adiabatic expansion of the universe [9].

3.3 Experimental and Observational Bounds

Fortunately, for the sake of the experimenter, magnetic monopoles are not expected to behave too quantum mechanically. Unlike many other particles, the magnetic monopole as formed through symmetry breaking has an effective radius larger than its Compton wavelength. While the relative length scale of the monopole is given by the inverse of the symmetry-breaking energy, M_{GUT}, its mass (being related to the magnetostatic energy present in the configuration) is given by $\frac{1}{\alpha}M_{GUT}$, where α is the running coupling renormalized at the unification scale, causing its Compton wavelength to be smaller by about 2 orders of magnitude [12]. This allows for phenomenologically simple predictions for their behavior in detectors. Unfortunately, for particle physicists, the mass of the monopole, 10^{15} GeV, is 12 orders of magnitude above the LHC's 13 TeV center of mass energy, leaving the possibility of creating magnetic monopoles through the SU(5) and SU(10) Kibble-Zurek mechanism out of the question.

One mechanism used to detect magnetic monopoles utilizes a Superconducting Quantum Interference Device (SQUID). These devices consist of a small loop of superconducting wire with Josephson junctions in parallel connected to a coil with a certain number of loops. The Josephson junctions are generally defects or interruptions in the superconducting circuit that allow for the measurement of a difference in current between the two sides of the superconducting loop. When a magnetic field is applied to the superconductor, a screening current is generated to cancel the field inside the material. As the applied field approaches the magnitude of the flux quanta allowed in the superconductor, the current changes either to increase the field (if the field is more than a half-integer above a flux quantum) or to decrease the field (if it is less than a half integer above a flux quantum). In superconductors, the charge carriers are Copper pairs with a net charge of $2e$. As such, the value of the magnetic flux quanta is π/e, or half the flux from a Dirac charge, 2π/e, making SQUIDs excellently predisposed in sensitivity to magnetic monopoles. As opposed to magnetic dipoles, the monopole would produce a current in the same direction throughout its travel through the coil given by

$$I(t > 0) = \frac{\phi_0}{L}[1 - \frac{\gamma vt}{[(\gamma vt)^2 + b^2]^{1/2}}] \tag{66}$$

, where L is the self impedance of one loop, b is its radius, and ϕ_0 is the magnetic flux from a Girac charge, $4\pi g_D$. Despite the low densities predicted by inflation, there is still one non-disproven detection of a single magnetic monopole by Blas Cabrera on Valentine's Day, 1982 [3]. Using the SQUID and a 4-coil detector with loops of area 20 cm^2, he measured a flux equal to $4\phi_0$, within the range of experimental error. Finding only one over a total detection time of 151 days placed an upper bound on monopole flux through the surface of Earth at

$$\Phi_{mon} \leq 6.1 \cdot 10^{-10} cm^{-2} s^{-1} sr^{-1} \tag{67}$$

The Monopole Astrophysics and Cosmic Ray Observatory (MACRO) is a liquid scintillator experiment with a total acceptance region of $\sim$ 10000 m^2·sr [1]. It observed no monopoles over its lifetime and was able to set an upper bound on the flux at

$$\Phi_{mon} \leq 1.4 \cdot 10^{-16} cm^{-2} s^{-1} sr^{-1} \tag{68}$$

, thereby calling into question the validity of the 'Valentine's Day' event.

The phrase "magnetic force does no work" due to it (classically) acting on electrically charged particles only perpendicular to their direction of motion would allow us to to say that the amount of energy stored in galactic and extragalactic magnetic fields should remain approximately constant over time. Without magnetic monopoles, dissipation would only happen through expansion of the universe and dipole moment interactions. Therefore, a bound can be placed on the abundance of magnetic monopoles based on the requirement that magnetic field are able to remain at the strength they are today. This is called the Parker Limit, as calculated in [11], and sets the following upper bound on monopole flux

$$\Phi_{mon} \lesssim \cdot 10^{-16} cm^{-2} s^{-1} sr^{-1} \tag{69}$$

, which corresponds to a monopole density of

$$\frac{n}{T^3} \lesssim 10^{-19} \tag{70}$$

, which is still 30 orders of magnitude above what we would expect from magnetic monopoles produced in GUT symmetry less than one second after the big bang.

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