We can now multiply the solution of least squares to get a system of equations as follows:

$$\begin{bmatrix} N & N_1\mu_1^T + N_2\mu_2^T \\ N_1\mu_1 + N_2\mu_2 & (N-2)\Sigma - N\mu_1\mu_1^T - N_2\mu_2\mu_2^T \end{bmatrix} \begin{pmatrix} B_o \\ B \end{pmatrix} = \begin{bmatrix} 0 \\ -N\mu_1 + N\mu_2 \end{bmatrix} \tag{15}$$

$$NB_o + (N_1\mu_1^T + N_2\mu_2^T)B = 0 \tag{16}$$

$$(N_1\mu_1 + N_2\mu_2)B_o + [(N-2)\Sigma + N_1\mu_1\mu_1^T + N_2\mu_2\mu_2^T]B = -N\mu_1 + N\mu_2 \tag{17}$$

We solve for B_o in the first equation and substitute it into the second equation and factor out B. We eventually want to get this second equation into the form stated in the question prompt.

$$B_o = -\frac{1}{N}(N_1\mu_1^T + N_2\mu_2^T)B$$

$\frac{1}{N}(N_1\mu_1 + N_2\mu_2) + (N-2)\Sigma + N_1\mu_1\mu_1^T + N_2\mu_2\mu_2^T$
$B = -N\mu_1 + N\mu_2$
(N-2) $\Sigma - \frac{1}{N}(N_1\mu_1 + N_2\mu_2) + N_1\mu_1\mu_1^T + N_2\mu_2\mu_2^T$
$B = N(\mu_2 - \mu_1)$(18)

We massage the equation to look more like the one given in the promt and we see that 2 out 3 of the terms match already.

$$[(N-2)\Sigma + (N_1N_2)/N\Sigma_B]B = N(\mu_2 - \mu_1) \tag{19}$$

Written 3. SVD of rank deficient matrix

In this exercise we calculate the singular value decomposition of a matrix as well as an approximation of the original matrix using the largest eigenvalue. We see that this approximation is very close to the original matrix and computed an energy of .908.

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 4 & 3 \\ 0 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix}$$

$$M^\intercal = \begin{bmatrix} 1 & 3 & 5 & 0 & 1 \\ 2 & 4 & 4 & 2 & 3 \\ 3 & 5 & 3 & 4 & 5 \end{bmatrix}$$

$$MM^\intercal = \begin{bmatrix} 14 & 26 & 22 & 16 & 22 \\ 26 & 50 & 46 & 28 & 40 \\ 22 & 46 & 50 & 20 & 32 \\ 16 & 28 & 20 & 20 & 26 \\ 22 & 40 & 32 & 26 & 35 \end{bmatrix}$$

$$M^\intercal M = \begin{bmatrix} 36 & 37 & 38 \\ 37 & 49 & 61 \\ 38 & 61 & 84 \end{bmatrix}$$

Eigenvalues for $MM^\intercal$ and $M^\intercal M$ are the following:

$$\lambda_1 \approx 153.567$$
$$\lambda_2 \approx 15.433$$
$$\lambda_3 \approx 0$$
$$\lambda_4 \approx 0$$
$$\lambda_5 \approx 0$$

Normalized eigenvectors for $MM^\intercal$ and $M^\intercal M$ are the following:

$MM^\intercal$:

$$v_1 \approx (0.297696, 0.570509, 0.520742, 0.322578, 0.458985)$$
$$v_2 \approx (0.159064, -0.0332002, -0.735857, 0.510392, 0.414259)$$
$$v_3 \approx (-0.912871, 0.182574, 0, 0, 0.365148)$$
$$v_4 \approx (-0.904534, 0.301511, 0, 0.301511, 0)$$
$$v_5 \approx (0.784465, -0.588348, 0.196116, 0., 0.)$$

$M^\intercal M$:

$$v_1 \approx (0.409283, 0.56346, 0.717636)$$
$$v_2 \approx (-0.815978, -0.125885, 0.56421)$$
$$v_3 \approx (0.408248, -0.816497, 0.408248)$$

$$U = \begin{bmatrix} 0.297696 & 0.784465 & 0.159064 & -0.912871 & -0.904534 \\ 0.570509 & -0.588348 & -0.0332002 & 0.182574 & 0.301511 \\ 0.520742 & 0.196116 & -0.735857 & 0 & 0 \\ 0.322578 & 0. & 0.510392 & 0 & 0.301511 \\ 0.458985 & 0 & 0.414259 & 0.365148 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 0.409283 & -0.815978 & 0.408248 \\ 0.56346 & -0.125885 & -0.816497 \\ 0.717636 & 0.56421 & 0.408248 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 12.3922 & 0 & 0 \\ 0 & 3.92849 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(square root of eigenvalues)}$$

$$Aprox = \begin{bmatrix} 1.509889 & 2.0786628 & 2.64743661 \\ 2.89357443 & 3.98358126 & 5.0735881 \\ 2.64116728 & 3.63609257 & 4.63101787 \\ 1.63609257 & 2.25240715 & 2.86872172 \\ 2.32793529 & 3.20486638 & 4.08179747 \end{bmatrix} \text{(Approximation with first eigenvalue)}$$