Introduction to Numerical Analysis - Chapter 1: Root-Finding for a Function of One Variable

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Objective:

To study numerical methods that approximate roots of equations of the form $f(x) = 0$, particularly when analytical solutions are infeasible. The chapter introduces iterative techniques and analyzes their convergence behavior.

Key Concepts:

- Root-Finding Problem: Solving $f(x) = 0$ for a given function f.
- Iterative Methods: Generate a sequence (x_n) to approach the root x^*.
- Convergence:
  * Defined by $\lim_{n} x_n = x^*$
  * Order of Convergence: Linear, sublinear, quadratic, etc.

Methods Covered:

1. Bisection Method:

- Based on the Intermediate Value Theorem
- Splits the interval [a, b] and iteratively narrows down where the sign of $f(x)$ changes
- Guaranteed convergence but linear speed
- Includes a computable error bound

2. Fixed-Point Iteration:

- Transforms $f(x) = 0$ into $x = g(x)$
- Iterates $x_{n+1} = g(x_n)$
- Requires g to be a contraction for global convergence

- Convergence speed depends on $|g'(x)|$

3. Newton-Raphson Method:

- Uses tangent line approximation:

  $x_{n+1} = x_n - f(x_n) / f'(x_n)$

- Quadratic convergence if the derivative is non-zero and the guess is close

- Requires calculation of f', not globally convergent

Case Studies:

1. Gas Volume (Van der Waals Equation)

   - Models non-ideal gas behavior; solved with bisection.

2. Investment Fund

   - Finds the minimum interest rate for retirement savings using bisection/Newton.

3. Population Growth

   - Estimates the growth rate of France's population using Newton's method.

Supporting Theory:

- Intermediate Value Theorem ensures existence of a root

- Error bounds and stopping criteria are crucial for practical implementations

- Graphical and numerical diagnostics (like error vs. iteration) help assess convergence