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Project 1: Snowplow Problem

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October 15, 2025
Academic Year 2025-2026

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The Snowplow Problem

1 Introduction to the problem

Firstly, we shall introduce the problem, as described in [1]. Note that for practicality, we have decided to use kilometers instead of miles.

> One morning it began to snow very hard and continued snowing steadily throughout the day. A snowplow set out at 9:00 AM to clear a road, clearing 2 km by 11:00 AM. and an additional kilometer by 1:00 PM. At what time did it start snowing?

The source also mentions the following assumptions:

- It is snowing at a constant rate.
- The rate at which the snowplow can clear a road is inversely proportional to the height (or depth) of the snow.

2 Solution

We shall now begin the attempt at solving the problem.
Let us consider the following notations:

- t - time since 9:00 AM (hours)
- $x(t)$ - the distance that the snowplow has travelled (km) at time t
- $h(t)$ - the height of the snow (cm) at time t

Additionally:

- b - the number of hours before 9:00 AM when it started snowing
- r - rate of snowfall (cm/h)

Since it is snowing at a constant rate, the height of the snow h is increasing at a constant rate. Mathematically this means that the derivative of h is constant, and equal to r, $r \in \mathbb{R}$.

$$h'(t) = \frac{\mathrm{d}h}{\mathrm{d}t} = r$$

Thus h is a linear function with slope r and with the equation

$$h(t) = rt + C, \quad C \in \mathbb{R}$$

That's a lot of letters and unknowns. Let us attempt to substitute the constant C in the equation of $h(t)$ with something more relevant to our problem. We consider $t = 0$ the moment that the snowplow sets out to clear the road. Similarly, $t = 0 - b$ is the moment that it starts snowing, and the height of the snow is equal to 0.

$$\begin{aligned}
&h(0-b) = r(0-b) + C \\
&0 = -rb + C \\
&\boxed{C = rb}
\end{aligned}$$

And now we can substitute C in the equation of h:

$$\begin{aligned}
h(t) &= rt + rb \\
h(t) &= r(t+b)
\end{aligned}$$

We know that the rate at which the snowplow moves (clears a road) is inversely proportional to the height of the snow:

$$x'(t) = \frac{\mathrm{d}x}{\mathrm{d}t} \propto \frac{1}{h(t)}$$

Then the product of the two is constant:

$$\begin{aligned}
&x'(t)h(t) = m, \quad m \in \mathbb{R} \\
\iff &x'(t) = \frac{m}{h(t)} \\
\iff &x'(t) = \frac{m}{r(t+b)}
\end{aligned}$$

Since m and r are constants, we can rewrite the equation as follows, separating the constants from the variables and unknowns:

$$x'(t) = \frac{m}{r}\frac{1}{(t+b)}$$

Since we have information about the kilometers the snowplow clears after a known number of hours, let us attempt to find the equation of $x(t)$. For this, we will integrate $x'(t)$:

$$\begin{aligned} x(t) &= \int x'(t)\,\mathrm{d}t \\ &= \int \frac{m}{r}\frac{1}{(t+b)}\,\mathrm{d}t \\ &= \frac{m}{r}\int \frac{1}{(t+b)}\,\mathrm{d}t \\ &= \frac{m}{r}\ln(t+b) + C_2, \quad C_2 \in \mathbb{R} \end{aligned}$$

Note that the product $\frac{m}{r}C_2$ is just a product of constants and thus unnecessary to write. Now using the equation for $x(t)$, we substitute using known information:

- The snowplow cleared 2km between 9:00 AM and 11:00 AM (the span of two hours):

$$\begin{aligned} x(2) - x(0) &= \frac{m}{r}\ln(t+b)\bigg|_0^2 \\ 2 &= \left(\frac{m}{r}\ln(2+b)\right) - \left(\frac{m}{r}\ln(0+b)\right) \\ 2 &= \frac{m}{r}\left(\ln(2+b) - \ln(0+b)\right) \\ & \boxed{2 = \frac{m}{r}\ln\left(\frac{2+b}{b}\right)} \end{aligned} \tag{1}$$

- The snowplow cleared 1km between 11:00 AM (2 hours after it set out) and 1:00 PM (4 hours after it set out):

$$\begin{aligned} x(4) - x(2) &= \frac{m}{r}\ln(t+b)\bigg|_2^4 \\ 1 &= \left(\frac{m}{r}\ln(4+b)\right) - \left(\frac{m}{r}\ln(2+b)\right) \\ 1 &= \frac{m}{r}\left(\ln(4+b) - \ln(2+b)\right) \\ & \boxed{1 = \frac{m}{r}\ln\left(\frac{4+b}{2+b}\right)} \end{aligned} \tag{2}$$

We now have two equations with b as our unknown. We shall now try to combine the two and solve for b:

If we multiply (2) by 2:

$$\begin{aligned}
1 &= \frac{m}{r}\ln\left(\frac{4+b}{2+b}\right)\Bigg|\times 2 \\
2 &= \frac{m}{r}2\ln\left(\frac{4+b}{2+b}\right) \\
2 &= \frac{m}{r}\ln\left(\frac{4+b}{2+b}\right)^2
\end{aligned}$$

$$\boxed{\frac{2r}{m} = \ln\left(\frac{4+b}{2+b}\right)^2} \tag{3}$$

Then if we solve for the ln in (1):

$$\frac{m}{r}\ln\left(\frac{2+b}{b}\right) = 2$$

$$\boxed{\frac{2r}{m} = \ln\left(\frac{2+b}{b}\right)} \tag{4}$$

Then, equating the right hand side members of (3) and (4):

$$\begin{aligned}
\ln\left(\frac{2+b}{b}\right) &= \ln\left(\frac{4+b}{2+b}\right)^2 \\
\frac{2+b}{b} &= \left(\frac{4+b}{2+b}\right)^2 \\
\frac{2+b}{b} &= \frac{(4+b)^2}{(2+b)^2} \\
(2+b)^3 &= b(4+b)^2 \\
b^3+6b^2+12b+8 &= b(b^2+8b+16) \\
b^3+6b^2+12b+8 &= b^3+8b^2+16b \\
2b^2+4b-8 &= 0 \Bigg| \div 2 \\
b^2+2b-4 &= 0
\end{aligned}$$

$$\overset{\text{quadratic formula}}{\iff} b_1 = \sqrt{5}-1 \approx 1.236 \qquad b_2 = -\sqrt{5}-1 \approx -3.236$$

Since b is the number of hours before 9:00 AM, the moment at which the snowplow set out to clear the snow, b must be a positive number so $b > 0$ and the second solution b_2 is rejected.

In conclusion, it started snowing 1.236 hours before 9:00 AM. If we want to be specific:

$$0.236 \text{ hours is } 0.236 \cdot 60 = 14.16 \text{ minutes}$$
$$0.16 \text{ minutes is } 0.16 \cdot 60 = 9.6 \text{ seconds}$$

so 1.236 hours equate to 1 hour, 14 minutes and 9.6 seconds

$$9{:}00{:}00 - 1{:}14{:}9.6 = 7{:}45{:}50.4 \text{ AM}$$

It started snowing at 7:45:50.4 AM.

Part II
Two Snowplows

3 The Problem

One morning it began to snow very hard and continued snowing steadily throughout the day. A snowplow set out at 9:00 AM to clear a road, clearing 2 km by 11:00 AM. and an additional kilometer by 1:00 PM. At what time did it start snowing?

4 The Solution

References

[1] R. Nagle, E. Saff, and A. Snider, *Fundamentals of Differential Equations*. Pearson Education, 2012, ISBN: 9780321849236. [Online]. Available: `https://books.google.com.cy/books?id=IdYsAAAAQBAJ`.