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## PROJECT 1: SNOWFLOW PROBLEM

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## Part I

# The Snowplow Problem

## 1 Problem

Firstly, we shall introduce the problem, as described in [1, pp. 84–85]. Note that for practicality, we have decided to use kilometers instead of miles.

One morning it began to snow very hard and continued snowing steadily throughout the day. A snowplow set out at 9:00 AM to clear a road, clearing 2 km by 11:00 AM. and an additional kilometer by 1:00 PM. At what time did it start snowing?

The source also mentions the following assumptions:

- It is snowing at a constant rate.
- The rate at which the snowplow can clear a road is inversely proportional to the height (or depth) of the snow.

## 2 Solution

We shall now begin the attempt at solving the problem.

Let us consider the following notations:

- $t$  - time since 9:00 AM (hours)
- $x(t)$  - the distance that the snowplow has travelled (km) at time  $t$
- $h(t)$  - the height of the snow (cm) at time  $t$

Additionally:

- $b$  - the number of hours before 9:00 AM when it started snowing
- $r$  - rate of snowfall (cm/h)

Since it is snowing at a constant rate, the height of the snow  $h$  is increasing at a constant rate. Mathematically this means that the derivative of  $h$  is constant, and equal to  $r$ ,  $r \in \mathbb{R}$ .

$$h'(t) = \frac{dh}{dt} = r$$

Thus  $h$  is a linear function with slope  $r$  and with the equation

$$h(t) = rt + C, \quad C \in \mathbb{R}$$

That's a lot of letters and unknowns. Let us attempt to substitute the constant  $C$  in the equation of  $h(t)$  with something more relevant to our problem. We consider  $t = 0$  the moment that the snowplow sets out to clear the road. Similarly,  $t = 0 - b$  is the moment that it starts snowing, and the height of the snow is equal to 0.

$$h(0 - b) = r(0 - b) + C$$

$$0 = -rb + C$$

$$\boxed{C = rb}$$

And now we can substitute  $C$  in the equation of  $h$ :

$$h(t) = rt + rb$$

$$h(t) = r(t + b)$$

We know that the rate at which the snowplow moves (clears a road) is inversely proportional to the height of the snow:

$$x'(t) = \frac{dx}{dt} \propto \frac{1}{h(t)}$$

Then the product of the two is constant:

$$x'(t)h(t) = m, \quad m \in \mathbb{R}$$

$$\iff x'(t) = \frac{m}{h(t)}$$

$$\iff x'(t) = \frac{m}{r(t + b)}$$

Since  $m$  and  $r$  are constants, we can rewrite the equation as follows, separating the constants from the variables and unknowns:

$$x'(t) = \frac{m}{r} \frac{1}{(t + b)}$$

Since we have information about the kilometers the snowplow clears after a known number of hours, let us attempt to find the equation of  $x(t)$ . For this, we will integrate  $x'(t)$ :

$$\begin{aligned} x(t) &= \int x'(t) dt \\ &= \int \frac{m}{r} \frac{1}{(t+b)} dt \\ &= \frac{m}{r} \int \frac{1}{(t+b)} dt \\ &= \frac{m}{r} \ln(t+b) + C_2, \quad C_2 \in \mathbb{R} \end{aligned}$$

Note that the product  $\frac{m}{r}C_2$  is just a product of constants and thus unnecessary to write. Now using the equation for  $x(t)$ , we substitute using known information:

- The snowplow cleared 2km between 9:00 AM and 11:00 AM (the span of two hours):

$$\begin{aligned} x(2) - x(0) &= \frac{m}{r} \ln(t+b) \Big|_0^2 \\ 2 &= \left( \frac{m}{r} \ln(2+b) \right) - \left( \frac{m}{r} \ln(0+b) \right) \\ 2 &= \frac{m}{r} (\ln(2+b) - \ln(0+b)) \\ \boxed{2 &= \frac{m}{r} \ln \left( \frac{2+b}{b} \right)} \end{aligned} \tag{1}$$

- The snowplow cleared 1km between 11:00 AM (2 hours after it set out) and 1:00 PM (4 hours after it set out):

$$\begin{aligned} x(4) - x(2) &= \frac{m}{r} \ln(t+b) \Big|_2^4 \\ 1 &= \left( \frac{m}{r} \ln(4+b) \right) - \left( \frac{m}{r} \ln(2+b) \right) \\ 1 &= \frac{m}{r} (\ln(4+b) - \ln(2+b)) \\ \boxed{1 &= \frac{m}{r} \ln \left( \frac{4+b}{2+b} \right)} \end{aligned} \tag{2}$$

We now have two equations with  $b$  as our unknown. We shall now try to combine the two and solve for  $b$ :

If we multiply (2) by 2:

$$\begin{aligned}
 1 &= \frac{m}{r} \ln \left( \frac{4+b}{2+b} \right) \Big| \times 2 \\
 2 &= \frac{m}{r} 2 \ln \left( \frac{4+b}{2+b} \right) \\
 2 &= \frac{m}{r} \ln \left( \frac{4+b}{2+b} \right)^2 \\
 \boxed{\frac{2r}{m} &= \ln \left( \frac{4+b}{2+b} \right)^2} \tag{3}
 \end{aligned}$$

Then if we solve for the ln in (1):

$$\begin{aligned}
 \frac{m}{r} \ln \left( \frac{2+b}{b} \right) &= 2 \\
 \boxed{\frac{2r}{m} &= \ln \left( \frac{2+b}{b} \right)} \tag{4}
 \end{aligned}$$

Then, equating the right hand side members of (3) and (4):

$$\begin{aligned}
 \ln \left( \frac{2+b}{b} \right) &= \ln \left( \frac{4+b}{2+b} \right)^2 \\
 \frac{2+b}{b} &= \left( \frac{4+b}{2+b} \right)^2 \\
 \frac{2+b}{b} &= \frac{(4+b)^2}{(2+b)^2} \\
 (2+b)^3 &= b(4+b)^2 \\
 b^3 + 6b^2 + 12b + 8 &= b(b^2 + 8b + 16) \\
 b^3 + 6b^2 + 12b + 8 &= b^3 + 8b^2 + 16b \\
 2b^2 + 4b - 8 &= 0 \quad \Big| \div 2 \\
 b^2 + 2b - 4 &= 0 \\
 \stackrel{\text{quadratic formula}}{\Longleftrightarrow} b_1 &= \sqrt{5} - 1 \approx 1.236 \quad b_2 = -\sqrt{5} - 1 \approx -3.236
 \end{aligned}$$

Since  $b$  is the number of hours before 9:00 AM, the moment at which the snowplow set out to clear the snow,  $b$  must be a positive number so  $b > 0$  and the second solution  $b_2$  is rejected.

In conclusion, it started snowing 1.236 hours before 9:00 AM. If we want to be specific:

$$0.236 \text{ hours is } 0.236 \cdot 60 = 14.16 \text{ minutes}$$

$$0.16 \text{ minutes is } 0.16 \cdot 60 = 9.6 \text{ seconds}$$

so 1.236 hours equate to 1 hour, 14 minutes and 9.6 seconds

$$9:00:00 - 1:14:9.6 = 7:45:50.4 \text{ AM}$$

It started snowing at 7:45:50.4 AM.

## Part II

# Two Snowplows

Now we shall move on to the second part of the project, respectively problem D from [1, pp. 84–85]

## 3 Problems

One day it began to snow exactly at noon at a heavy and steady rate. A snowplow left its garage at 1:00 PM., and another one followed in its tracks at 2:00 PM.

### 3.1 First subpoint

At what time did the second snowplow crash into the first?

We keep the same assumption as part I:

- The rate at which a snowplow can clear a road is inversely proportional to the height (or depth) of the snow, and thus to the time elapsed since the road was clear of snow.

### 3.2 Second subpoint

Could the crash have been avoided by dispatching the second snowplow at 3:00 PM instead?

## 4 Solutions

### 4.1 Time of crash

We consider the following notations:

- $t$  - time since noon (12:00 PM) (hours)
- $r$  - rate of snowfall (cm/h)
- $x(t)$  - the distance that the first snowplow has travelled (km) at time  $t$
- $y(t)$  - the distance that the second snowplow has travelled (km) at time  $t$
- $h(t)$  - the height of snow (cm) at time  $t$

The time of the crash is the moment in time  $t$  such that

$$x(t) = y(t)$$

Firstly, we must determine the equations  $x(t)$  and  $y(t)$ .

Since the rate at which a snowplow can clear a road is inversely proportional to the height (or depth) of the snow, the product of the speed (*the derivative*  $x'(t)$  or  $y'(t)$  of the position  $x(t)$  or  $y(t)$ ) of any of the two snowplows with the height  $h(t)$  of the snow is constant, and we shall denote it with  $k$ :

$$\begin{aligned}x'(t) \cdot h(t) &= k, & k \in \mathbb{R} \\y'(t) \cdot h(t) &= k\end{aligned}$$

The rate at which it is snowing is steady i.e. constant, thus  $h(t)$  is a linear function with slope  $r$  and equation

$$h(t) = rt + C, \quad C \in \mathbb{R}$$

Since it began snowing at noon (12:00 PM), which we consider our starting time  $t = 0$



$$\begin{aligned}
h(0) &= 0 \\
r \cdot 0 + C &= 0 \\
\boxed{C = 0}
\end{aligned}$$

So,  $h(t)$  becomes

$$h(t) = rt$$

and the proportionality equation for the first snowplow becomes:

$$\begin{aligned}
x'(t) \cdot rt &= k \\
x'(t) &= \frac{k}{rt}
\end{aligned}$$

Let us denote the fraction of the two constants  $\frac{k}{r}$  with  $A$ :

$$x'(t) = \frac{A}{t}$$

We're getting closer to finding the equations  $x(t)$  and  $y(t)$ .

Let us integrate  $x(t)$ :

$$\begin{aligned}
\int x'(t) dt &= \int \frac{A}{t} dt \\
x(t) &= A \ln |t| + C \\
t \geq 0 &\implies x(t) = A \ln t + C
\end{aligned}$$

Let us use our given information:

- The first snowplow leaves the garage at 1:00 PM, 1 hour after noon (12:00 PM), so

$$\begin{aligned}
x(1) &= 0 \\
A \ln 1 + C &= 0 \\
C &= 0
\end{aligned}$$

and  $x(t)$  becomes

$$x(t) = A \ln t \tag{5}$$

- The second snowplow left at 2:00 PM.

If the second snowplow has cleared  $y(t)$  kilometers of snow at time  $t$ , then the first snowplow cleared the same amount of snow at time  $\tau$ <sup>1</sup>,  $\tau \in \mathbb{R}^+$ ,  $\tau < t$ .

$$\begin{aligned}y(t) &= x(\tau) \\ y(t) &= A \ln \tau\end{aligned}$$

Solving for  $\tau$ , so we can substitute it to have less unknowns in our expressions:

$$\begin{aligned}\ln \tau &= \frac{y(t)}{A} \\ \tau &= e^{\frac{y(t)}{A}}\end{aligned}$$

The height of snow that the second snowplow must clear at time  $t$  is the snow that has fallen since the first snowplow has passed, time equal to  $t - \tau$ . And since the rate of snowfall is constant ( $r$ ), the height of the snow that has fallen during  $t - \tau$  hours is equal to  $r \cdot (t - \tau)$ .

$$\begin{aligned}y'(t) \cdot h(t) &= k \\ y'(t) \cdot r(t - \tau) &= k\end{aligned}$$

replacing  $\tau$  with  $e^{\frac{y(t)}{A}}$  we obtain

$$y'(t) = \frac{k}{r(t - e^{\frac{y(t)}{A}})}$$

and denoting  $\frac{k}{r}$  with  $A$ :

$$y'(t) = \frac{A}{t - e^{\frac{y(t)}{A}}}$$

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<sup>1</sup>We use  $\tau$  purely for notation purposes, its true value is out of the scope for this exercise.

which is a **first-order linear differential equation** with  $y$  as the dependent variable, and  $t$  as the independent variable. We must solve it to obtain the equation of  $y(x)$ .

$$\frac{dy}{dt} = \frac{A}{t - e^{\frac{y}{A}}}$$

[1] hints to setting  $t$  as the dependent variable and  $y$  as the independent variable. This way we will obtain  $t(y)$  (*the moment in time  $t$  that the second snowplow has cleared  $y$  kilometers of snow*).

$$\frac{dt}{dy} = \frac{1}{dy/dt} = \frac{t - e^{\frac{y}{A}}}{A}$$

Since this is a first-order linear differential equation, we shall solve it using the integrating factor. We bring the differential equation into standard form  $\frac{dt}{dy} + P(y)t = Q(y)$

$$\begin{aligned}\frac{dt}{dy} &= \frac{t - e^{\frac{y}{A}}}{A} \\ \frac{dt}{dy} &= \frac{t}{A} - \frac{e^{\frac{y}{A}}}{A} \\ \frac{dt}{dy} - \frac{1}{A}t &= -\frac{e^{\frac{y}{A}}}{A}\end{aligned}$$

The integrating factor  $\mu(y)$  is equal to:

$$\begin{aligned}\mu(y) &= e^{\int P(y)dy} \\ \mu(y) &= e^{\int -\frac{1}{A}dy} \\ \mu(y) &= e^{-\frac{y}{A}}\end{aligned}$$

Now, we multiply both sides of the differential equation with the integrating factor:

$$\begin{aligned}\left. \frac{dt}{dy} - \frac{1}{A}t = -\frac{e^{\frac{y}{A}}}{A} \right| \times e^{-\frac{y}{A}} \\ e^{-\frac{y}{A}} \frac{dt}{dy} - e^{-\frac{y}{A}} \frac{1}{A}t = \frac{-e^{\frac{y}{A}} \cdot e^{-\frac{y}{A}}}{A} \\ e^{-\frac{y}{A}} \frac{dt}{dy} - e^{-\frac{y}{A}} \frac{1}{A}t = -\frac{1}{A}\end{aligned}$$

If we write the equation replacing  $\frac{dt}{dy}$  with  $t'$ , we notice that the left hand side resembles the product derivation rule:

$$\underbrace{e^{-\frac{y}{A}}t' - e^{-\frac{y}{A}}\frac{1}{A}t}_{e^{-\frac{y}{A}}t' - e^{-\frac{y}{A}}\frac{1}{A}t} = -\frac{1}{A}$$

$$e^{-\frac{y}{A}}t' - e^{-\frac{y}{A}}\frac{1}{A}t \equiv (e^{-\frac{y}{A}})t' + (e^{-\frac{y}{A}})'t \equiv (t \cdot e^{-\frac{y}{A}})' \equiv \frac{d(t \cdot e^{-\frac{y}{A}})}{dy}$$

So, we have

$$\frac{d(t \cdot e^{-\frac{y}{A}})}{dy} = -\frac{1}{A}$$

and we integrate both sides with respect to  $y$ :

$$\int \frac{d(t \cdot e^{-\frac{y}{A}})}{d}y = \int -\frac{1}{A}dy$$

$$(t \cdot e^{-\frac{y}{A}}) = -\frac{y}{A} + C, \quad C \in \mathbb{R}$$

Solving for  $t$  we get the formula for  $t(y)$

$$t(y) = \frac{1}{e^{-\frac{y}{A}}} \left( C - \frac{y}{A} \right) = e^{\frac{y}{A}} \left( C - \frac{y}{A} \right) \quad (6)$$

At 2:00 PM,  $t = 2$  hours after 12:00, the second snowplow has cleared  $y = 0$  kilometers of snow, thus

$$t(0) = 2$$

$$e^{\frac{0}{A}} \left( C - \frac{0}{A} \right) = 2$$

$$e^0 (C - 0) = 2$$

$$C = 2$$

So, finally, we have the formula for  $t(y)$

$$t(y) = e^{\frac{y}{A}} \left( 2 - \frac{y}{A} \right) \quad (7)$$

Now that we have the equations (5), (7) for  $x(t)$  and  $t(y)$  respectively, we shall combine them.

As we mentioned, the snowplows collide when

$$y = x = A \ln t$$

By substituting  $y$  in equation (7) and solving for  $t$ , we obtain

$$\begin{aligned} t &= e^{\frac{y}{A}} \left( 2 - \frac{y}{A} \right) \\ t &= e^{\frac{A \ln t}{A}} \left( 2 - \frac{A \ln t}{A} \right) \\ t &= e^{\ln t} (2 - \ln t) \\ t &= t(2 - \ln t) \Big| \div t, \quad t > 0 \\ 2 - \ln t &= 1 \\ \ln t &= 1 \\ e^{\ln t} &= e^1 \\ \boxed{t = e} \end{aligned}$$

In conclusion, the two snowplows crashed into one another  $e \approx 2.718$  hours after 12:00 PM (noon). If we want to be specific:

$$0.718 \text{ hours is } 0.718 \cdot 60 = 43.08 \text{ minutes}$$

$$0.08 \text{ minutes is } 0.08 \cdot 60 = 4.8 \text{ seconds}$$

So  $e$  hours equate to 2 hours, 43 minutes and 4.8 seconds

$$12:00:00 + 2:43:4.8 = 14:43:4.8 = 2:43:4.8 \text{ PM}$$

The second snowplow crashed into the first at 2:43:4.8 PM<sup>2</sup>.

## 4.2 Preventing the crash

Could the crash be prevented by dispatching the second snowplow at 3:00 PM?

Formulas remain the same, except for the constant  $C$  in the initial equation

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<sup>2</sup>Note that because of the irrationality of Euler's number  $e$ , the number of seconds might be slightly inaccurate.

of  $t(y)$  (equation (6)). The second snowplow clears 0 kilometers of snow at  $t$  hours after noon at which it gets dispatched, so

$$\begin{aligned}t(0) &= 3 \\e^{\frac{0}{A}} \left(C - \frac{0}{A}\right) &= 3 \\C &= 3\end{aligned}$$

Using the same method as before to find the value of  $t$  for when the two snowplows crash, we get

$$\begin{aligned}t &= e^{\frac{y}{A}} \left(3 - \frac{y}{A}\right) \\t &= e^{\frac{A \ln t}{A}} \left(3 - \frac{A \ln t}{A}\right) \\t &= t(3 - \ln t) \\1 &= 3 - \ln t \\t &= e^2 \approx 5.436\end{aligned}$$

In conclusion, if the first snowplow does not finish its course (or at least deviate from it) by  $e^2$  hours after noon, then a crash would still happen.

As a matter of fact, for an arbitrary number  $T$ ,  $T > 1$  of hours after noon at which the second snowplow would begin its course, the crash will happen

$$t = e^{T-1}$$

hours after noon. Meaning, as long as there is no change in speed, rate of snowfall or the trail that the snowplows follow, *there will always be a crash.*

## References

- [1] R. Kent Nagle, E. B. Saff, and A. D. Snider, *Fundamentals of Differential Equations*, 9th ed. London, England: Pearson Education, 2018.