Generatoare rectilinii

(1) Formați ecuațiile generatoarelor rectilinii ale hiperboloidului cu o pânză

$$\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$$

paralele cu planul

$$\pi: 6x+4y+3z-17=0$$

Soluție $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1 \Leftrightarrow$

$$\Leftrightarrow \frac{x^2}{4}-\frac{z^2}{16}=1-\frac{y^2}{9} \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{x}{2}-\frac{z}{4}\right)\left(\frac{x}{2}+\frac{z}{4}\right)=\left(1-\frac{y}{3}\right)\left(1+\frac{y}{3}\right)$$

$$\text{I}\ \begin{cases} \alpha\left(\frac{x}{2}+\frac{z}{4}\right)=\beta\left(1+\frac{y}{3}\right) & \mid\cdot 12 \\ \beta\left(\frac{x}{2}-\frac{z}{4}\right)=\alpha\left(1-\frac{y}{3}\right) & \mid\cdot 12 \end{cases}$$

$$\text{II}\ \begin{cases} \alpha\left(\frac{x}{2}+\frac{z}{4}\right)=\beta\left(1-\frac{y}{3}\right) & \mid\cdot 12 \\ \beta\left(\frac{x}{2}-\frac{z}{4}\right)=\alpha\left(1+\frac{y}{3}\right) & \mid\cdot 12 \end{cases}$$

Cazul I

$$d_{\alpha,\beta}: \begin{cases} 6\alpha x-4\beta y+3\alpha z-12\beta=0 \\ 6\beta x+4\alpha y-3\beta z-12\alpha=0 \end{cases}$$

$d_{\alpha,\beta} \parallel \pi \Leftrightarrow \vec{d}_{\alpha,\beta} \perp \vec{n} \Leftrightarrow \vec{n} \cdot \vec{d}_{\alpha,\beta} = 0$

$\vec{n}(6,4,3)$ vectorul normal al planului π

$$\begin{array}{ccc} p & q & r \\ 6\alpha & -4\beta & 3\alpha \\ 6\beta & 4\alpha & -3\beta \end{array}$$

$$p = \begin{vmatrix} -4\beta & 3\alpha \\ 4\alpha & -3\beta \end{vmatrix} = 12(\beta^2 - \alpha^2)$$

$$q = -\begin{vmatrix} 6\alpha & 3\alpha \\ 6\beta & -3\beta \end{vmatrix} = 36\alpha\beta$$

$$r = \begin{vmatrix} 6\alpha & -4\beta \\ 6\beta & 4\alpha \end{vmatrix} = 24(\alpha^2 + \beta^2)$$

$\vec{d}_{\alpha,\beta}\left(12(\beta^2-\alpha^2),\ 36\alpha\beta,\ 24(\alpha^2+\beta^2)\right)$ sau

$\vec{d}'_{\alpha,\beta}\left(\beta^2-\alpha^2,\ 3\alpha\beta,\ 2(\alpha^2+\beta^2)\right)$

$\vec{n} \cdot \vec{d}'_{\alpha,\beta} = 0 \Leftrightarrow 6(\beta^2-\alpha^2) + 12\alpha\beta + 6(\alpha^2+\beta^2) = 0$

$\Leftrightarrow 12\beta^2 + 12\alpha\beta = 0 \Leftrightarrow 12\beta(\alpha+\beta) = 0$

$I_1 \rightarrow \beta = 0 \quad (\alpha \neq 0)$

$\Leftrightarrow d_{\alpha,0}: \begin{cases} 6\alpha x + 3\alpha z = 0 \\ 4\alpha y - 12\alpha = 0 \end{cases} \Rightarrow$

$\Leftrightarrow \boxed{d_{\alpha,0}: \begin{cases} 2x + z = 0 \\ y - 3 = 0 \end{cases}}$

II_2. $\alpha + \beta = 0 \Leftrightarrow \beta = -\alpha$

$d_{\alpha,-\alpha}: \begin{cases} 6\alpha x + 4\alpha y + 3\alpha z + 12\alpha = 0 & \mid :\alpha \\ -6\alpha x + 4\alpha y + 3\alpha z - 12\alpha = 0 & \mid :(-\alpha) \end{cases}$

$d_{1,-1}: \begin{cases} 6x + 4y + 3z + 12 = 0 \\ 6x - 4y - 3z + 12 = 0. \end{cases}$

Analog cazul II.

(2) Verificați că $A(-2,0,1)$ aparține paraboloidului hiperbolic $\frac{x^2}{4} - \frac{y^2}{9} = z$ și determinați unghiul ascuțit format de generatoarele rectilinii care trec prin A.

Soluție $\frac{x^2}{4} - \frac{y^2}{9} = z \Leftrightarrow \left(\frac{x}{2} + \frac{y}{3}\right)\left(\frac{x}{2} - \frac{y}{3}\right) = z$

$(-2,0,1)$ $\frac{4}{4} - \frac{0}{9} = 1 \Leftrightarrow 1 = 1.$

$$\text{I} \quad \begin{cases} \alpha\left(\frac{x}{2}+\frac{y}{3}\right) = \beta z \\ \beta\left(\frac{x}{2}-\frac{y}{3}\right) = \alpha \end{cases}$$

$$\text{II} \quad \begin{cases} \alpha\left(\frac{x}{2}-\frac{y}{3}\right) \pm \beta z \\ \beta\left(\frac{x}{2}+\frac{y}{3}\right) = \alpha \end{cases}$$

$$\text{I} \quad \begin{cases} \alpha\left(\frac{-2}{2}+\frac{0}{3}\right) = \beta \cdot 1 \\ \beta\left(\frac{-2}{2}-\frac{0}{3}\right) = \alpha \end{cases}$$

$$\begin{cases} -\alpha = \beta \\ -\beta = \alpha \end{cases}$$

$$\begin{cases} \alpha\left(\frac{x}{2}+\frac{y}{3}\right) = -\alpha z & |:\alpha \\ -\alpha\left(\frac{x}{2}-\frac{y}{3}\right) = \alpha & |:(-\alpha) \end{cases}$$

$$\begin{cases} \frac{x}{2}+\frac{y}{3}+z = 1 & |\cdot 6 \\ \frac{x}{2}-\frac{y}{3}+1 = 0 & |\cdot 6 \end{cases}$$

$$\boxed{d_1 : \begin{cases} 3x+2y+6z-6=0 \\ 3x-2y \qquad +6=0 \end{cases}}$$

p_1	q_1	r_1
3	2	6
3	-2	0

$$p_1 = \begin{vmatrix} 2 & 6 \\ -2 & 0 \end{vmatrix} = 12$$

$$q_1 = -\begin{vmatrix} 3 & 6 \\ 3 & 0 \end{vmatrix} = 18$$

$$r_1 = \begin{vmatrix} 3 & 2 \\ 3 & -2 \end{vmatrix} = -12$$

$\vec{d_1}\,(12, 18, -12)$ sau $\vec{d_1'}\,(2, 3, -2)$.

II. $d_2: \begin{cases} \alpha\left(\frac{-2}{2} - \frac{0}{3}\right) = \beta \cdot 1 \\ \beta\left(-\frac{2}{2} + \frac{0}{3}\right) = \alpha \end{cases} \Leftrightarrow \begin{cases} -\alpha = \beta \\ -\beta = \alpha \end{cases}$

$$\begin{cases} \alpha\left(\frac{x}{2} - \frac{y}{3}\right) = -\alpha z & \mid : \alpha \\ -\alpha\left(\frac{x}{2} + \frac{y}{3}\right) = \alpha & \mid : (-\alpha) \end{cases}$$

$$\begin{cases} \frac{x}{2} - \frac{y}{3} = -z & \mid \cdot 6 \\ \frac{x}{2} + \frac{y}{3} = -1 & \mid \cdot 6 \end{cases} \Leftrightarrow \boxed{d_2: \begin{cases} 3x - 2y + 6z = 0 \\ 3x + 2y \quad + 6 = 0 \end{cases}}$$

$$d_2: \left.\begin{array}{ccc} p_2 & q_2 & r_2 \\ 3 & -2 & 6 \\ 3 & 2 & 0 \end{array}\right\} \Rightarrow \begin{array}{l} p_2 = \begin{vmatrix} -2 & 6 \\ 2 & 0 \end{vmatrix} = -12 \\ q_2 = -\begin{vmatrix} 3 & 6 \\ 3 & 0 \end{vmatrix} = 18 \\ r_2 = \begin{vmatrix} 3 & -2 \\ 3 & 2 \end{vmatrix} = 12 \end{array}$$

$\vec{d}_2(-12, 18, 12)$ sau $\vec{d}_2'(-2, 3, 2)$

$$\cos\left(\widehat{\vec{d}_1', \vec{d}_2'}\right) = \frac{\vec{d}_1' \cdot \vec{d}_2'}{\|\vec{d}_1'\| \cdot \|\vec{d}_2'\|} = \frac{2\cdot(-2) + 3\cdot 3 + (-2)\cdot 2}{\sqrt{4+9+4} \cdot \sqrt{4+9+4}} =$$

$$= \frac{-4+9-4}{\sqrt{17}\cdot\sqrt{17}} = \frac{1}{17}$$

$$\cancel{\cos}\left(\widehat{\vec{d}_1', \vec{d}_2'}\right) = (\widehat{d_1, d_2}) = \arccos\frac{1}{17}.$$