Tema 9

Ex 1:

a) $\sum (n+1)^n x^n$

avem $\mathbb{R} \ni$ seria dezv în jurul lui $x_0=0$
termen general $a_n = (n+1)^n$

$$\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+2)^{n+1}}{(n+1)^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)^n \cdot (n+2)$$

$$= \lim \left(\frac{n\left(1+\frac{2}{n}\right)}{n\left(1+\frac{1}{n}\right)}\right)^n (n+2)$$

$$\lambda = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{(n+1)^n} = \infty \Rightarrow$$

$\Rightarrow$ seria e D.

b) $\sum_{n\geq 0} \frac{1}{n} x^n$

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \cdot n+1 = 1 \Rightarrow R = 1$$

Avem $(-1, 1) \subseteq \mathcal{C}$

$x = -1 \Rightarrow \sum (n+1)^n (-1)^n$

Studiem AC

$\sum |(n+1)^n (-1)^n| = \sum (n+1)^n$

Crit rad: $\lim_{n\to\infty} \sqrt[n]{(n+1)^n} = n+1 = \infty > 1 \Rightarrow$

$\Rightarrow \sum D \Rightarrow$ nu e AC

$x = 1 \Rightarrow \sum (n+1)^n$

$\lim_{n\to\infty} \sqrt[n]{(n+1)^n} = n+1 = \infty > 1 \Rightarrow D.$

c) $\sum_{n\geq 0} \frac{(-1)^n}{n} x^n \qquad a_n = \frac{(-1)^n}{n}$

$\lambda = \lim_{n\to\infty} \left|\frac{|a_{n+1}|}{|a_n|}\right| = \lim_{n\to\infty} \frac{1}{n+1} \cdot n = 1 \Rightarrow$

$\Rightarrow R = 1 \Rightarrow (-1, 1) \in \mathcal{C}$

$x = 1 \Rightarrow \sum \frac{(-1)^n}{n} \cancel{(-1)^n}$

Avem $a_n = \frac{1}{n}$, descrescător, $\lim_{n\to\infty} a_n = 0$

$\Rightarrow \sum \frac{(-1)^n}{n}$ C

Leibniz $\Rightarrow$

Studiem AC

$$\sum |(n+1)^n (-1)^n| = \sum (n+1)^n$$

Crit rad: $\lim\limits_{n\to\infty} \sqrt[n]{(n+1)^n} = n+1 = \infty > 1 \Rightarrow$

$\Rightarrow \sum D \Rightarrow$ nu e AC

$x = 1 \Rightarrow \sum (n+1)^n$

$\lim\limits_{n\to\infty} \sqrt[n]{(n+1)^n} = n+1 = \infty > 1 \Rightarrow D.$

c) $\sum\limits_{n\geq 0} \frac{(-1)^n}{n} x^n \qquad a_n = \frac{(-1)^n}{n}$

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$\Rightarrow R = 1 \Rightarrow (-1, 1) \in \mathcal{C}$

$x = 1 \Rightarrow \sum \frac{(-1)^n}{n} \cancel{(-1)^n}$

Avem $a_n = \frac{1}{n}$, descrescător, $\lim\limits_{n\to\infty} a_n = 0$

$\Rightarrow \sum \frac{(-1)^n}{n}$ C

Tlull
$\Rightarrow$

e) $\sum_{n\geq 0} \frac{1}{n!} x^n$

$$\lambda = \lim_{n\to\infty} \frac{\cancel{n!}}{\underbrace{\cancel{(n+1)!}}_{n+1}} = 0$$

Exercițiul 1 - integrale

a) $\int \frac{2x-1}{x^2-3x+2} dx$

$$= \int \frac{2x-3+3-1}{x^2-3x+2} dx$$

$$= \underbrace{\int \frac{2x-3}{x^2-3x+2} dx}_{I_1} + \underbrace{\int \frac{2}{x^2-3x+2} dx}_{I_2}$$

$$t = x^2 - 3x + 2$$

$$dt = (2x-3)\,dx \Rightarrow I_1 = \int \frac{dt}{t} = \ln|t|$$

$$I_1 = \ln|x^2-3x+2|$$

$$I_2 = 2\int \frac{1}{x^2-3x+2} dx =$$

$$= 2\int \frac{1}{x^2-x-2x+2} dx = 2\int \frac{1}{x(x-1)-2(x-1)} dx$$

$$= 2\int \frac{1}{(x-2)(x-1)} dx$$

e) $\sum_{n\geq 0} \frac{1}{n!} x^n$

$\lambda = \lim_{n\to\infty} \frac{\cancel{n!}}{\cancel{(n+1)!}_{n+1}} = 0$

Exercițiul 1 - integrale

a) $\int \frac{2x-1}{x^2-3x+2} dx$

$= \int \frac{2x-3+3-1}{x^2-3x+2} dx$

$= \underbrace{\int \frac{2x-3}{x^2-3x+2} dx}_{I_1} + \underbrace{\int \frac{2}{x^2-3x+2} dx}_{I_2}$

$t = x^2-3x+2$

$dt = (2x-3)dx \Rightarrow I_1 = \int \frac{dt}{t} = \ln|t|$

$I_1 = \ln|x^2-3x+2|$

$I_2 = 2\int \frac{1}{x^2-3x+2} dx =$

$= 2\int \frac{1}{x^2-x-2x+2} dx = 2\int \frac{1}{x(x-1)-2(x-1)} dx$

$= 2\int \frac{1}{(x-2)(x-1)} dx$

$$J_1$$

$$= \int \left(\frac{1}{x-1} - \frac{1}{x+1} + \frac{-2}{(x+1)^2} \right) dx$$

$$J_1 = \int \frac{-2}{(x+1)^2} dx = -2 \int \frac{1}{(x+1)^2} = -2 \int \frac{1}{u^2}$$

$u = x+1$

$du = dx$

$$= -2 \int u^{-2}$$

$$= -2 \ \frac{-1}{u}$$

$$= \frac{+2}{x+1}$$

$$J = \ln|x-1| - \ln|x+1| + \frac{2}{x+1} + \mathcal{C}$$

c) $\int \frac{1}{x^3 - x^4} dx$

$$\int \frac{1}{x^3(1-x)} dx = \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{1-x} \right) dx$$

$$= \int \frac{A(1-x)x^2 + B \cdot x(1-x) + C(1-x) + Dx^3}{x^3(1-x)} dx$$

$$= \int \frac{Ax^2 - Ax^3 + Bx - Bx^2 + C - Cx + Dx^3}{x^3(1-x)} dx$$

$$= \int \frac{x^2(A-B) + x^3(D-A) + x(B-C) + C}{x^3(1-x)} dx$$

$$\begin{cases} A - B = 0 \Rightarrow A = 1 \\ D - A = 0 \rightarrow D = 1 \\ B - C = 0 \rightarrow B = 1 \\ C = 1 \end{cases}$$

$$\Rightarrow \int \left(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{1-x} \right) dx$$

$$= \int \frac{1}{x} dx + \int \frac{1}{x^2} dx + \int \frac{1}{x^3} dx + (-1) \int \frac{1}{x-1} dx$$

$$= \ln|x| + \frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} - \ln|x-1|$$

$$= \ln|x| - \frac{1}{x} - \frac{1}{2x^2} - \ln|x-1| + \mathcal{C}$$

$$x > 1 \quad \Rightarrow$$

$$\Rightarrow \ln x - \frac{1}{x} - \frac{1}{2x^2} - \ln(x-1) + \mathcal{C}$$

d) $\int \frac{2x+5}{x^2+5x+10} dx \quad x \in \mathbb{R}$

$u = x^2 + 5x + 10$

$du = (2x+5)\, dx$

$\Rightarrow \int \frac{du}{u} = \ln|u| \Rightarrow$

$\Rightarrow I = \ln|x^2+5x+10| + \mathcal{C}$

e) $\int \frac{1}{x^2+x+1}\,dx,\ x\in\mathbb{R}$

$= \int \frac{1}{x^2+2\cdot\frac{1}{2}\cdot x+\frac{1}{4}-\frac{1}{4}+1}\,dx$

$= \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$

$\frac{1}{a}\operatorname{arctg}\frac{x}{a}$

fie $u = x+\frac{1}{2}$

$du = dx$ $\Rightarrow \int \frac{du}{u^2+\frac{3}{4}}$ (=)

$= \int \frac{du}{\frac{4u^2+3}{4}} = \int \frac{4\,du}{4u^2+3}$

$= 4\int \frac{du}{4u^2+3} = 4\int \frac{\frac{\sqrt{3}}{2}\,dt}{3\left(\frac{4}{3}u^2+1\right)}$

fie $u = \frac{\sqrt{3}}{2}t$

$du = \frac{\sqrt{3}}{2}\,dt$

$= 4\int \frac{\frac{\sqrt{3}}{2}\,dt}{3\left(\frac{4}{3}\cdot\frac{3}{4}\cdot t^2+\ldots\right)}$

$= 4\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{3}\int \frac{dt}{t^2+1} = \frac{2\sqrt{3}}{3}\operatorname{arctg} t$

$$\Leftrightarrow \int \frac{du}{u^2+\frac{3}{4}} = \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{u}{\frac{\sqrt{3}}{2}}$$

$$= \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} + \mathcal{C}$$

Exercițiul 2

a) $J = \int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx \quad x \in [0,\infty)$

$|x+1| = x+1$
$|x| = x$

$$J = \int \frac{\sqrt{x+1}-\sqrt{x}}{x+1-x}\,dx$$

$$= \int \left(\sqrt{x+1}-\sqrt{x}\right)dx$$

$$= \int \sqrt{x+1}\,dx - \int \sqrt{x}\,dx$$

$$= \frac{(x+1)^{\frac{1}{2}+1}}{\frac{3}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \mathcal{C}$$

$$= \frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}} + \mathcal{C}$$

b) $J = \int \frac{1}{x+\sqrt{x-1}}\,dx \quad x \in [1,\infty)$

$$\cancel{J = \int \frac{x-\sqrt{x-1}}{x^2-x+1}\,dx}$$

$$\cancel{\begin{cases} u = x-1 \\ du = dx \end{cases} \Rightarrow J = \int \frac{1}{u+1+\sqrt{u}}}$$

$u = \sqrt{x-1} \quad |()^2 \Rightarrow u^2 = x-1$

$du = \frac{1}{2\sqrt{x-1}} dx \qquad x = u^2+1$

$dx = du \cdot 2 \cdot u$

$$I = \int \frac{2u \, du}{u^2+u+1}$$

$$= 2\int \frac{u \, du}{u^2+u+1} = \cancel{\int dt}$$

$$= 2\int \frac{u \, du}{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$\left.\begin{aligned} t &= u+\frac{1}{2} \\ dt &= \frac{1}{2} du \end{aligned}\right\} \Rightarrow$$

$$\int \frac{t}{t^2+\frac{3}{4}} dt = \frac{1}{2}\int \frac{dv}{v}$$

$v = t^2+\frac{3}{4} \qquad \frac{1}{2}\ln|v|$

$dv = 2t\,dt$

$$\Rightarrow I = 2\int \frac{\left(t-\frac{1}{2}\right) 2\,dt}{t^2+\frac{3}{4}}$$

$$I = 2 \cdot 2 \int \frac{t-\frac{1}{2}}{t^2+\frac{3}{4}} dt$$

$$I = 4\left(\int \frac{t}{t^2+\frac{3}{4}} dt - \frac{1}{2}\int \frac{1}{t^2+\frac{3}{4}} dt\right)$$

$$I = 4 \cdot \frac{1}{2} \ln\left|t^2+\frac{3}{4}\right| - 4 \cdot \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \cdot \operatorname{arctg} \frac{t^2+\frac{3}{4}}{\frac{\sqrt{3}}{2}}$$

Ex 3:

a) $I = \int \frac{1}{1+\sqrt{x^2+2x-2}}\,dx \quad x \in \{\sqrt{3}-1, \infty\}$

Cu met. lui Euler:

$R(x, \sqrt{x^2+2x-2})$, $a>0 \Rightarrow$

$\Rightarrow$ substituție de tipul 1

$$\sqrt{ax^2+bx+c} = x\sqrt{a}+t$$

$$\sqrt{x^2+2x-2} = x+t \;/ (\;)^2$$

$$\cancel{x^2}+2x-2 = \cancel{x^2}+2xt+t^2$$

$$2x-2xt = t^2+2$$

$$2x(1-t) = t^2+2$$

$$x = \frac{t^2+2}{2(1-t)}$$

$$dx = \left(\frac{t^2+2}{2(1-t)}\right)' dt$$

$$dx = \frac{2t\cdot 2(1-t)-(t^2+2)(-2)}{4(1-t)^2}$$

$$dx = \frac{4t(1-t)+2(t^2+2)}{4(1-t)^2}\,dt$$

$$dx = \frac{4t-4t^2+2t^2+4}{4(1-t)^2}\,dt$$

$$dx = \frac{-2t^2+4t+4}{4(1-t)^2}\,dt$$

$$\sqrt{x^2+2x-2} = x + t$$

$$\sqrt{x^2+2x-2} = \frac{t^2+2}{2(1-t)} + t$$

$$I = \int \frac{\frac{-2t^2+4t+4}{4(1-t)^2}\,dt}{1 + \frac{t^2+2}{2(1-t)} + t}$$

$$= \int \frac{\frac{-2t^2+4t+4}{4(1-t)^2}\,dt}{\frac{2-2t+t^2+2+2t-2t^2}{2(1-t)}}$$

$$= \int \frac{-2(t^2-2t-2)}{4(1-t)^2}\,dt \cdot \frac{2(1-t)}{-t^2+4}$$

$$= \int \frac{-t^2+2t+2}{(1-t)(-t^2+4)}\,dt$$

$(1-t)(4-t)(4+t)$

$$= \int \left(\frac{A}{1-t} + \frac{B}{-t^2+4}\right)dt$$

$$= \int \frac{A(-t^2+4)}{(1-t)(-t^2+4)} + \frac{B(1-t)}{(-t^2+4)(1-t)}\,dt$$

$$= \int \frac{-At^2+4A+B-Bt}{(1-t)(-t^2+4)}\,dt$$

$$= \int$$

$t^2 : -1 = -A \Rightarrow A = 1$

$t : 2 = -B \Rightarrow B = -2$

term liber : $2 = 4A + B$

$2 = 4 \cdot 1 + (-2)$

$2 = 2$ (A)

$\int \left(\frac{1}{1-t} + \frac{-2}{-t^2+4} \right) dt$

$= -\ln|1-t| +$

$= \int \frac{-t^2+2t+2}{(1-t)(2-t)(2+t)}$

$= \cancel{\int \frac{-t^2+2t+2}{\quad}}$

$= \int \left(\frac{A}{1-t} + \frac{B}{2-t} + \frac{C}{2+t} \right) dt$

$= \int \frac{A(2-t)(2+t) + B(2+t)(1-t) + C(2-t)(1-t)}{(1-t)(2-t)(2+t)}$

$= \int A(4 + \cancel{2t} - \cancel{2t} - t^2) + B(2 - 2t + t - t^2) +$

$+ C(2 - 2t - t + t^2)$

$= \int 4A - At^2 + 2B - Bt - t^2B + 2C - 3tC + t^2C$

$t^2(-A - B + C) + t(-B - 3C) + 4A + 2B + 2C$

$t^2: \; -1 = C - A - B \Rightarrow A = C - B + 1$

$t: \; 2 = -B - 3C \Rightarrow B = -3C - 2$

$2 = 4A + 2B + 2C$

$A = C + 3C + 2 + 1$

$A = 4C + 3$

$2 = 4(4C+3) + 2(-3C-2) + 2C$

$2 = 16C + 12 - 6C - 4 + 2C$

$2 = 12C + 8$

$12C = -6$

$C = -\frac{1}{2}$ $\qquad A = 4 \cdot \left(-\frac{1}{2}\right) + 3$

$A = -2 + 3$

$= +1$

$B = -3\left(-\frac{1}{2}\right) - 2$

$B = \frac{3}{2} - 2$

$\int \frac{1}{1-t} + \frac{-\frac{1}{2}}{2-t} + \frac{-\frac{1}{2}}{2+t} \, dt \qquad = -\frac{1}{2}$

$-\ln|1-t| + \frac{1}{2}\ln|2-t| - \ln|2+t| + C$

b) $J = \int \frac{1}{(x+1)\sqrt{-4x^2-x+1}}\,dx$

$\Delta =$

$c>0 \Rightarrow$ folosim a II-a substituție

fie $\sqrt{-4x^2-x+1} = xt + \sqrt{1} \quad |(\,)^2$

$$-4x^2 - x + 1 = (xt+1)^2$$

$$-4x^2 - x + \cancel{1} = x^2t^2 + 2xt + \cancel{1}$$

$$-4x^2 - x^2t^2 - x - 2xt = 0$$

$$x^2(-4-t^2) - x(1+2t) = 0$$

~~Exercițiul 4~~

~~a)~~ $\cancel{\int_1^2 \frac{1}{x^3+x^2+x+1}}$

Exercițiul 4

a) $\int_1^2 \frac{1}{x^3+x^2+x+1}\,dx$

$=\int_1^2 \frac{1}{x^2(x+1)+x+1}\,dx$

$=\int_1^2 \frac{dx}{(x+1)(x^2+1)}$

$=\int_1^2 \left(\frac{A}{x+1}+\frac{Bx+C}{x^2+1}\right)dx$

$\cancel{=\int_1^2 \left(\frac{Ax^2+A+Bx+B}{(x+1)(x^2+1)}\right)dx}$

$\cancel{=\int_1^2 \frac{x^2(A)+xB+A+B}{(x+1)(x^2+1)}\,dx}$

$\cancel{x^2 : A=0}$

$\cancel{x : B=0}$

$=\int_1^2 \frac{Ax^2+A+(x+1)(Bx+C)}{(x+1)(x^2+1)}\,dx$

$=\int_1^2 \frac{Ax^2+A+Bx^2+Cx+Bx+C}{(x+1)(x^2+1)}$

$=\int_1^2 \frac{x^2(A+B)+x(C+B)+A+C}{(x+1)(x^2+1)}$

$B=-\frac{1}{2}$

$A+B=0$ $\quad$ $C+B=0$ $\quad$ $A+C=1$

$A=-B$ $\quad$ $C+C-1=0 \Rightarrow C=\frac{1}{2}$ $\quad$ $A=1-C=\frac{1}{2}$

$$= \int_1^2 \left(\frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1} \right) dx$$

$$= \int_1^2 \frac{1}{2(x+1)}\, dx + \int_1^2 \left(-\frac{1}{2}\right) \frac{x+1}{x^2+1}\, dx$$

$$= \frac{1}{2} \int_1^2 \frac{1}{x+1}\, dx - \frac{1}{2} \int_1^2 \frac{x+1}{x^2+1}\, dx$$

$$= \frac{1}{2} \int_1^2 \frac{1}{x+1}\, dx - \frac{1}{2} \left(\int_1^2 \frac{x}{x^2+1}\, dx + \int_1^2 \frac{1}{x^2+1}\, dx \right)$$

$$= \frac{1}{2} \ln(x+1)\Big|_1^2 - \frac{1}{2} \left(\frac{1}{2} \ln|x^2+1|\Big|_1^2 + \operatorname{arctg} x \Big|_1^2 \right)$$

$$\int \frac{x}{x^2+1}\, dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln|u|$$

$u = x^2+1$

$du = 2x\, dx$

$$= \frac{1}{2}(\ln 3 - \ln 2) - \frac{1}{2}\left(\frac{1}{2}(\ln 5 - \ln 2) + \operatorname{arctg} 2 - \operatorname{arctg} 1 \right)$$

$$= \frac{\ln 3}{2} - \frac{\ln 2}{2} - \frac{\ln 5}{4} + \frac{\ln 2}{4} - \frac{\operatorname{arctg} 2}{2} + \frac{\overset{=\frac{\pi}{4}}{\operatorname{arctg} 1}}{2}$$

$$= \frac{1}{4}\left(2\ln 3 - 2\ln 2 - \ln 5 + \ln 2 - 2\operatorname{arctg} 2 + \frac{\pi}{2} \right)$$

b) $\int_1^3 \frac{1}{x(x^2+9)}\,dx$

$= \int_1^3 \left(\frac{A}{x} + \frac{Bx+C}{x^2+9}\right)dx$

$= \int_1^3 \frac{Ax^2+9A+Bx^2+Cx}{x(x^2+9)}$

$= \int_1^3 \frac{x^2(A+B)+Cx+9A}{x(x^2+9)}$

$x^2: \ A+B=0 \Rightarrow A=-B \qquad B=-\frac{1}{9}$

$x: \ C=0$

termen liber: $9A=1 \Rightarrow A=\frac{1}{9}$

$= \int_1^3 \left(\frac{1}{9x} + \frac{-\frac{1}{9}x+0}{x^2+9}\right)dx$

$= \int_1^3 \frac{1}{9x} + \frac{x}{-9(x^2+9)}$

$= \frac{1}{9}\int_1^3 \frac{1}{x}\,dx - \frac{1}{9}\int_1^3 \frac{x}{x^2+9}$

$= \frac{1}{9}\ln|x|\Big|_1^3 - \frac{1}{9}\cdot\frac{1}{2}\cdot\ln|x^2+9|\Big|_1^3$

$= \frac{1}{9}(\ln 3) - \frac{1}{18}(\ln 18)$

$= {}^{3)}\frac{\ln 3}{9} - \frac{\ln 18}{18} = \frac{3\ln 3 - \ln 18}{18}$

$= \frac{\ln\frac{27}{18}}{18} = \frac{1}{18}\ln\frac{3}{2}$

c) $\int_{-1}^{1} \frac{x^2+1}{x^4+1}\,dx$

$f(-x) = \frac{(-x)^2+1}{(-x)^4+1} = f(x) \Rightarrow f$ pară

$\int_{-1}^{1} \frac{x^2}{x^4+1}\,dx + \int_{-1}^{1} \frac{1}{x^4+1}\,dx$

d) $\int_{-1}^{1} \frac{x}{x^2+x+1}\,dx = \int_{-1}^{1} \frac{x\,dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} =$

$\cancel{u = x^2+x+1}$

$\cancel{du = (2x+1)\,dx}$

$x^2 + \frac{1}{2}\cdot x \cdot 2 + \frac{1}{4} - \frac{1}{4} + 1$

$u = x + \frac{1}{2} \Rightarrow x = u - \frac{1}{2}$

$du = dx$ $\qquad x = 1 \Rightarrow u = 1 + \frac{1}{2} = \frac{3}{2}$

$x = -1 \Rightarrow u = -1 + \frac{1}{2} = -\frac{1}{2}$

$= \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{\left(u-\frac{1}{2}\right)du}{u^2+\frac{3}{4}}$

$= \underbrace{\int \frac{u}{u^2+\frac{1}{2}}\,du}_{I_1} - \frac{1}{2}\int \frac{du}{u^2+\frac{3}{4}}$

$$I_1 = \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{u}{u^2+\frac{1}{2}}\,du = \int_{\frac{3}{4}}^{\frac{11}{4}} \frac{1}{2}\,\frac{dt}{t}$$

$t = u^2 + \frac{1}{2}$ $\qquad = \frac{1}{2}\ln t \Big|_{\frac{3}{4}}^{\frac{11}{4}} \; =$

$dt = 2u\,du$

$u = \frac{3}{2} \Rightarrow t = \frac{9}{4} + {}^{2)}\frac{1}{2} = \frac{11}{4}$

$u = -\frac{1}{2} \Rightarrow t = \frac{1}{4} + {}^{2)}\frac{1}{2} = \frac{3}{4}$

$= \frac{1}{2}\ln\frac{11}{4} - \frac{1}{2}\ln\frac{3}{4}$

$$I_2 = \frac{1}{2}\int \frac{du}{u^2+\frac{3}{4}} = \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{\sqrt{3}}\,\text{arctg}\,\frac{x}{\frac{\sqrt{3}}{2}} \Big|_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{1}{\sqrt{3}}\,\text{arctg}\,\frac{2x}{\sqrt{3}} \Big|_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$I = \frac{1}{2}\ln\frac{11}{4} - \frac{1}{2}\ln\frac{3}{4} - \frac{1}{\sqrt{3}}\,\text{arctg}\,\frac{2x}{\sqrt{3}} \Big|_{-\frac{1}{2}}^{\frac{3}{2}}$$

Exercițiul ~~4~~ 5

a) $\displaystyle\int_{-3}^{-2} \frac{x}{(x+1)(x^2+3)}\,dx$

$$= \int_{-3}^{-2} \left(\frac{A}{x+1} + \frac{Bx+C}{x^2+3}\right) dx$$

$$= \int_{-3}^{-2} \frac{Ax^2+3A+(Bx+C)(x+1)}{(x+1)(x^2+3)}\,dx$$

$$\int_{-3}^{-2} \frac{Ax^2+3A+Bx^2+Bx+Cx+C}{(x+1)(x^2+3)}\,dx$$

$$=\int_{-3}^{-2} \frac{x^2(A+B)+x(B+C)+3A+C}{(x+1)(x^2+3)}$$

$x^2:\ A+B=0 \qquad A=-B$

$x:\ B+C=1 \qquad B=1-C$

$3A+C=0 \qquad A=C-1$

$3C-3+C=0$

$4C=3 \Rightarrow C=\frac{3}{4} \qquad A=\frac{3}{4}-{}^{4)}1$

$A=-\frac{1}{4} \qquad B=\frac{1}{4}$

$$\int_{-3}^{-2}\left(\frac{-1}{4(x+1)}+\frac{\frac{1}{4}x+\frac{3}{4}}{x^2+3}\right)dx$$

$$=\frac{1}{4}\int_{-3}^{-2}\frac{1}{x+1}\,dx+\frac{1}{4}\int_{-3}^{-2}\frac{x+3}{x^2+3}\,dx$$

$$=\frac{1}{4}\int_{-3}^{-2}\frac{1}{x+1}\,dx+\frac{1}{4}\left(\int_{-3}^{-2}\frac{x}{x^2+3}\,dx+\int_{-3}^{-2}\frac{3}{x^2+3}\,dx\right)$$

$$=\frac{1}{4}\int_{-3}^{-2}\frac{1}{x+1}\,dx+\frac{1}{4}\cdot\frac{1}{2}\ln|x^2+3|\Big|_{-3}^{-2}+3\frac{1}{\sqrt3}\operatorname{arctg}\frac{x}{\sqrt3}$$

$$=\frac{1}{4}\ln|x+1|\Big|_{-3}^{-2}+\frac{1}{8}(\ln(7)-\ln 12)+\frac{3}{\sqrt3}\left(\operatorname{arctg}\frac{-2}{\sqrt3}-\operatorname{arctg}\frac{-3}{\sqrt3}\right)$$

b) $\int_0^1 \frac{x+1}{(x^2+4x+5)^2} dx$

$u = x^2+4x+5$

$du = (2x+4)\,dx$

$= 2(x+2)\,dx$

~~$du = 2(u+1)\,dx$~~

$\int_0^1 \frac{x+2-1}{(x^2+4x+5)^2} dx$

$= \int_0^1 \frac{x+2}{(x^2+4x+5)^2} dx - \int_0^1 \frac{1\,dx}{(x^2+4x+5)^2}$

$= \int_{5}^{10} \frac{1}{2} \frac{du}{u^2} +$

$x=1 \Rightarrow u = 1^2+4+5 = 10$

$x=0 \Rightarrow u = 0+4\cdot 0+5 = 5$

$\int_0^1 \frac{1}{(x^2+4x+5)^2}$?

c) $\int_1^2 \frac{1}{x^3+x} dx = \int_1^2 \frac{1}{x(x^2+1)} dx$

$= \int_1^2 \left(\frac{A}{x} + \frac{Bx+C}{x^2+1}\right) dx$

$= \int_1^2 \frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)} dx$

$$= \int_1^2 \frac{x^2(A+B) + Cx + A}{x(x^2+1)} \, dx$$

$A + B = 0 \longrightarrow A = -B \Rightarrow B = -1$

$C = 0$

$A = 1$

$$= \int_1^2 \left(\frac{1}{x} + \frac{-x}{x^2+1} \right) dx$$

$$= \int_1^2 \frac{1}{x} dx + \int_1^2 \frac{-x}{x^2+1} dx$$

$$= \ln|x| \Big|_1^2 \cancel{+} - \frac{1}{2} \ln|u| \Big|_2^5$$

$$\int \frac{x}{x^2+1} dx = \int_2^5 \frac{1}{2} \frac{du}{u} = \frac{1}{2} \ln|u| \Big|_2^5$$

$u = x^2 + 1$

$du = 2x\,dx$

$x = 2 \Rightarrow u = 5$

$x = 1 \Rightarrow u = 2$

$$= \ln 2 - \frac{1}{2} \ln 5 + \frac{1}{2} \ln 2$$

$$= \frac{3}{2} \ln 2 - \frac{1}{2} \ln 5.$$

d) $$\int_0^2 \frac{x^3 + 2x^2 + x + 4}{(x+1)^2}$$

$$\begin{array}{r|l} x^3+2x^2+x+4 & x^2+2x+1 \\ \hline x^3+2x^2+x & x \\ \hline 4 & \end{array} \Rightarrow$$

$$\Rightarrow x^3+2x^2+x+4 = (x^2+2x+1)x+4$$
$$= (x+1)^2 \cdot x + 4$$

$$\int_0^2 \frac{(x+1)^2 \cdot x + 4}{(x+1)^2} dx$$

$$= \int_0^2 \frac{x \cdot \cancel{(x+1)^2}}{\cancel{(x+1)^2}} dx + \int_0^2 \frac{4}{(x+1)^2} dx$$

$$= \frac{x^2}{2}\Big|_0^2 + 4 \cdot \frac{2}{3} = \frac{4}{2} + \frac{8}{3} = \frac{14}{3}$$

$$\int_0^2 \frac{dx}{(x+1)^2} = \int_1^3 \frac{du}{u^2} = \frac{u^{-2+1}}{-2+1}\Big|_1^3$$

$u = x+1$

$du = dx$

$x=2 \Rightarrow u=3$

$x=0 \Rightarrow u=1$

$$= -\frac{1}{u}\Big|_1^3$$
$$= -\frac{1}{3} + 1$$
$$= \frac{2}{3}$$

e) $\int_0^1 \frac{dx}{(x+1)(x^2+4)}$

$$= \int_0^1 \left(\frac{A}{x+1} + \frac{Bx+C}{x^2+4}\right) dx$$

$$= \int_0^1 \frac{Ax^2+4A+Bx^2+Cx+Bx+C}{(x^2+4)(x+1)} dx$$

$$\int_0^1 \frac{x^2(A+B)+x(B+C)+4A+C}{(x^2+4)(x+1)}$$

$A+B=0 \Rightarrow A=-B$

$B+C=0 \Rightarrow C=A$

$4A+C=1$

$4A+A=1$

$5A=1$

$A=\frac{1}{5}=C \quad \cancel{A}\ B=-\frac{1}{5}$

$$\int_0^1 \frac{dx}{5(x+1)}+\int_0^1 \frac{-\frac{1}{5}x+\frac{1}{5}}{x^2+4}\,dx$$

$$\int_0^1 \frac{dx}{5(x+1)}-\int_0^1 \frac{x-1}{5(x^2+4)}\,dx$$

$$\frac{1}{5}\int_0^1 \frac{dx}{x+1}-\frac{1}{5}\left(\int_0^1 \frac{x}{x^2+4}\,dx-\int_0^1 \frac{1}{x^2+4}\,dx\right)$$

$$\frac{1}{5}\int_0^1 \frac{dx}{x+1}-\cancel{\tfrac{1}{5}\int}\ \frac{1}{5}\cdot\frac{1}{2}(\ln 5-\ln 4)-\frac{1}{2}\operatorname{arctg}\frac{x}{2}\Big|_0^1$$

$$I_1=\int_0^1 \frac{x}{x^2+4}\,dx=\int_4^5 \frac{1}{2}\frac{du}{u}=\frac{1}{2}\ln|u|\Big|_4^5$$

$u=x^2+4$

$du=2x\,dx$

$$=\frac{1}{2}(\ln 5-\ln 4)$$

$x=1 \Rightarrow u=5$

$x=0 \Rightarrow u=4$

$$=\frac{1}{5}\ln 2-\frac{1}{10}(\ln 5-\ln 4)-\frac{1}{2}\operatorname{arctg}\frac{\pi}{8}$$

f) $\int_2^3 \frac{2x^3+x^2+2x-1}{x^4-1}\,dx$

	x^3	x^2	x^1	x^0
	2	1	2	-1
-1	2	-1	-1	0

$2x^2-x-1=$

$=2x^2-2x+x-1=$

$=2x(x-1)+(x-1)$

$=(2x-1)(x-1)$

$\Rightarrow$

$\Rightarrow 2x^3+x^2+2x-1=(x+1)(2x^2-x-1)$

$=\int_2^3 \frac{(x+1)(2x^2-x-1)}{(x^2-1)(x^2+1)}\,dx=$

$=\int_2^3 \frac{\cancel{(x+1)}(2x-1)\cancel{(x-1)}}{\cancel{(x-1)}\cancel{(x+1)}(x^2+1)}\,dx$

$=\int_2^3 \frac{2x-1}{x^2+1}\,dx$

$=2\int_2^3 \frac{x}{x^2+1}\,dx-\int_2^3 \frac{1}{x^2+1}\,dx$

$=\int_5^{10} \frac{du}{u}-\operatorname{arctg} x\Big|_2^3$

$=\ln 10-\ln 5-\operatorname{arctg} 3+\operatorname{arctg} 2$

Exercițiul 6

a) $\int_{-1}^{1} \frac{1}{\sqrt{4-x^2}}\,dx =$

$= \cancel{\ln|x+} \arcsin \frac{x}{2} \Big|_{-1}^{1}$

$= \arcsin \frac{1}{2} - \arcsin -\frac{1}{2}$

$= \frac{\pi}{6} + \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$

b) $\int_{0}^{1} \frac{1}{\sqrt{x^2+x+1}}\,dx$

$= \int_{0}^{1} \frac{1}{\sqrt{x^2+2\cdot\frac{1}{2}x+\frac{1}{4}-\frac{1}{4}+1}}\,dx$

$= \int_{0}^{1} \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}} = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{du}{\sqrt{u^2+\frac{3}{4}}}$

$u = x+\frac{1}{2}$

$du = dx$

$x = 1 \Rightarrow u = \frac{3}{2}$

$x = 0 \Rightarrow u = \frac{1}{2}$

$= \cancel{\frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2u}{\sqrt{3}}} \cancel{\frac{\sqrt{3}u}{2}}$

$= \ln\left|u+\sqrt{u^2+\frac{3}{4}}\right|$

$= \cancel{\frac{2}{\sqrt{3}}\left(\operatorname{arctg}\frac{\sqrt{3}}{2}\cdot\frac{3}{2} - \operatorname{arctg}\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\right)}$

c) $\int_{-1}^{1} \frac{1}{\sqrt{4x^2+x+1}}\, dx$

$$4x^2+x+1 = 4x^2 + 2\cdot 2x \cdot \frac{1}{4} + \frac{1}{16} - \frac{1}{16} + 1$$

$$16 \qquad = \frac{15}{16}$$

$$= \left(2x + \frac{1}{4}\right)^2$$

$$= \int_{-1}^{1} \frac{dx}{\sqrt{\left(2x+\frac{1}{4}\right)^2 + \frac{15}{16}}}$$

$$u = 2x + \frac{1}{4}$$

$$du = 2\,dx$$

$$x = 1 \Rightarrow u = 2\cdot 1 + \frac{1}{4} = \frac{9}{4}$$

$$x = -1 \Rightarrow u = 2(-1) + \frac{1}{4} = -\frac{7}{4}$$

$$= \int_{-\frac{7}{4}}^{\frac{9}{4}} \frac{1}{2} \frac{du}{\sqrt{u^2+\frac{15}{16}}} = \cancel{\frac{1}{2}\cdot\frac{4}{\sqrt{15}}\,\mathrm{arctg}}$$

$$= \frac{1}{2} \ln\left(u + \sqrt{u^2+\frac{15}{16}}\right) \Bigg|_{-\frac{7}{4}}^{\frac{9}{4}}$$

$$\cancel{= \frac{1}{2} \ln\left(\frac{9}{4} + \frac{3}{2} + \sqrt{9}\right.}$$

$$= \frac{1}{2} \ln\left(\frac{9}{4} + \sqrt{\frac{81}{16}+\frac{15}{16}}\right) - \frac{1}{2} \ln\left(\cancel{\frac{49}{16}}\,-\frac{7}{4} + \sqrt{\frac{49}{16}+\frac{15}{16}}\right)$$

d) $\int_2^3 \frac{x^2}{(x^2-1)\sqrt{x^2-1}}dx$

$\int_2^3 \frac{\cancel{x^2-1}}{\cancel{(x^2-1)}\sqrt{x^2-1}}dx + \int_2^3 \frac{1}{(x^2-1)\sqrt{x^2-1}}dx$

$\int_2^3 \frac{dx}{\sqrt{x^2-1}} + \int_2^3 \frac{1}{(x^2-1)\sqrt{x^2-1}}dx$

Ex 7

a) $\int_2^3 \sqrt{x^2+2x-7}\,dx$

$= x\sqrt{x^2+2x-7} - \int_2^3 \frac{x(\cancel{2}x+\cancel{2}^{1})}{\cancel{2}\sqrt{x^2+2x-7}}dx$

$= x\sqrt{x^2+2x-7} - \int_2^3 \frac{x(x+1)}{\sqrt{x^2+2x-7}}dx$

c) $\int_0^{3/4} \frac{1-x^2+x^2}{(x+1)\sqrt{x^2+1}}dx$

$\int_0^{3/4} \frac{(1-x)(1+x)}{(x+1)\sqrt{x^2+1}} + \int_0^{3/4} \frac{x^2}{(x+1)\sqrt{x^2+1}}$

$\sqrt{x^2+1} = x+t \quad /()^2$

$\cancel{x^2}+1 = \cancel{x^2}+2xt+t^2$

$\cancel{x = t(2x+t)}$ $\qquad 2xt = t^2-1$

$\cancel{t = \frac{1}{2x+1}}$ $\qquad x = \frac{t^2-1}{2t}$

$$dt = \left(\frac{1}{2x+1}\right)' dx$$

$$dt = \frac{-2}{(2x+1)^2} dx$$

$$= \frac{-2}{(2\cdot x+1)^2}$$

d) $\displaystyle\int_2^3 \frac{dx}{x\sqrt{x^2-1}}$

$$\sqrt{x^2-1} = t(x-1) \quad / ()^2$$

$$x^2-1 = t^2$$

$$dx = \left(\frac{t^2-1}{2t}\right)' dt$$

$$dx = \frac{2t\cdot 2t - (t^2-1)\cdot 2}{4t^2} dt$$

$$dx = \frac{\overset{2}{\cancel{4}}t^2 - \cancel{2}t^2 + \overset{1}{\cancel{2}}}{\underset{2}{\cancel{4}}t^2} dt$$

$$\int \frac{\dfrac{2t^2 - t^2 + 1}{2t^2} dt}{\left(\dfrac{t^2-1}{2t}+1\right)\left(t + \dfrac{t^2-1}{2t}\right)}$$