Temă - șiruri de nr. reale

ex 1

a) $x_n = \dfrac{2^n + 3^n}{5^n}$

$$x_{n+1} - x_n = \frac{2^{n+1}}{5^{n+1}} + \frac{3^{n+1}}{5^{n+1}} - \frac{2^n}{5^n} - \frac{3^n}{5^n}$$

$$= \underbrace{\left(\frac{2}{5}\right)^{n+1} + \left(\frac{3}{5}\right)^{n+1}}_{x_{n+1}} - \left(\frac{2}{5}\right)^{n} - \left(\frac{3}{5}\right)^{n}$$

$x_{n+1} < x_n \Rightarrow$

$\Rightarrow$ șirul e $\downarrow$

$x_n \geq 0 \Rightarrow$ e mărginit

b) $x^n = \dfrac{(-1)^n}{n}$

nu e monoton, $x^n = -\dfrac{1}{n}$ când n impar

$\dfrac{1}{n}$ când n par

$x_n \to 0$.

$m = -1$

$M = 1$

c) $x_n = \frac{2^n}{n!}$

$\frac{x_{n+1}}{x_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1}$ | =>
$n \in N$

<=> $\frac{2}{n+1} \leq 1$ => șirul e ↘ =>

=> $M = 1$

$m = 0$ <= $\frac{2^n}{n!} \to 0$.

converge la 0.

d) $x_n = \frac{n}{n^2+1}$

$x_{n+1} - x_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1}$

Se observă că $n < n^2+1$ =>

=> șirul e ↘

$m = 0$

$M = 1$ converge la 0.

ex 2

a) $\lim\limits_{n\to\infty} \dfrac{n}{n^2+1} = 0$

$a_n \to a \iff \forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N} \ \forall n \geq n_\varepsilon$

$\Rightarrow |a_n - a| < \varepsilon$

Fie $\varepsilon > 0$ Alegem $n_\varepsilon =$ $\Rightarrow \forall n \geq n_\varepsilon \Rightarrow$

$\Rightarrow \dfrac{n}{n^2+1} < \varepsilon \ \Big/ (\cdot)^{-1} \iff$

$\dfrac{1}{\varepsilon} < \dfrac{n^2+1}{n} \leq 1+n$

$\dfrac{1}{\varepsilon} < n + \dfrac{1}{n} \leq 1+n$

$\dfrac{1}{\varepsilon} \leq 1+n$

$n \geq \boxed{\dfrac{1}{\varepsilon} - 1}$

$\Rightarrow$ alegem $n_\varepsilon = \max\left\{0, \left[\dfrac{1}{\varepsilon} - 1\right] + 1\right\}$

b) $\lim\limits_{n\to\infty} \dfrac{n^2}{-2n+4} = -\infty \iff$

$\iff \lim\limits_{n\to\infty} \dfrac{n^2}{2n-4} = \infty$

$a_n \to \infty \iff \forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N}$ a.î.

$\forall n \geq n_\varepsilon \Rightarrow x_n > \varepsilon$

Fie $\varepsilon > 0$ Alegem $n_\varepsilon =$ $\Rightarrow \forall n \geq n_\varepsilon \Rightarrow$

$$\frac{n^2}{2n-4} > \varepsilon$$

$$\varepsilon < \frac{1}{2} \cdot \frac{n^2}{n-2} < \frac{n^2}{n-2} < n^2$$

$$\Rightarrow \varepsilon < n^2$$

$$\sqrt{\varepsilon} < n$$

$$n_\varepsilon = [\sqrt{\varepsilon}] + 1$$

ex 3

a) $$\lim_{n\to\infty} \frac{5^n + 1}{7^n + 1} = \lim_{n\to\infty} \frac{7^n\left(\left(\frac{5}{7}\right)^n + \left(\frac{1}{7}\right)^n\right)}{7^n\left(1 + \left(\frac{1}{7}\right)^n\right)}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{5}{7}\right)^n + \left(\frac{1}{7}\right)^n}{1 + \left(\frac{1}{7}\right)^n} = \frac{0}{1} = 0.$$

b) $$\lim_{n\to\infty} \frac{4^n + (-2)^n}{4^{n-1} + 2} = \lim_{n\to\infty} \frac{2^{2n} + (-2)^n}{2^{2n-2} + 2}$$

$$= \lim_{n \to \infty} \frac{\cancel{2^{2n}}\left(1 + \frac{(-2)^n}{2^{2n}}\right)}{\cancel{2^{2n}}\left(\frac{2^{2n-2}}{2^{2n}} + \frac{2}{2^{2n}}\right)}$$

$$= \lim_{n \to \infty} \frac{1 + \frac{(-2)^n}{2^{2n}}}{\frac{1}{2^2} + \frac{2}{2^n}} = \frac{1}{\frac{1}{2^2}} = 4$$

$$\frac{(-2)^n}{2^{2n}} \to 0$$

$$\frac{2}{2^n} \to 0$$

c) $\lim_{n \to \infty} \left(\sin \frac{\pi}{10}\right)^n = 0.$

d) $\lim_{n \to \infty} \sqrt{9n^2 + 2n + 1} - 3n =$

$$= \lim_{n \to \infty} \frac{\left(\sqrt{9n^2 + 2n + 1} - 3n\right)\left(\sqrt{9n^2 + 2n + 1} + 3n\right)}{\sqrt{9n^2 + 2n + 1} + 3n}$$

$$= \lim_{n \to \infty} \frac{\cancel{9n^2} + 2n + 1 - \cancel{9n^2}}{\sqrt{9n^2 + 2n + 1} + 3n}$$

$$= \lim_{n \to \infty} \frac{2n + 1}{\sqrt{9n^2 + 2n + 1} + 3n}$$

$$= \lim_{n \to \infty} \frac{\cancel{n}\left(2 + \frac{1}{n}\right)}{\cancel{n}\left(\sqrt{9 + \frac{2}{n} + \frac{1}{n^2}} + 3\right)} = \frac{2}{6} = \frac{1}{3}$$

$\frac{1}{n} \to 0$

e) $\lim\limits_{n\to\infty}\left(5+\dfrac{1-2n^3}{3n^4+2}\right)^2$

$= \lim\limits_{n\to\infty}\left(5+\dfrac{n^4\left(\frac{1}{n^4}-\frac{2n^3}{n^4}\right)}{n^4\left(3+\frac{2}{n^4}\right)}\right)^2$

$= \lim\limits_{n\to\infty}\left(5+\dfrac{\underbrace{\frac{1}{n^4}}_{\to 0}-\underbrace{\frac{2}{n}}_{\to 0}}{3+\underbrace{\frac{2}{n^4}}_{\to 0}}\right)^2$

$= \left(5+\frac{0}{3}\right)^2 = 5^2 = 25$

f) $\lim\limits_{n\to\infty}\sqrt[3]{n^3+n+3}-\sqrt[3]{n^3+1}$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$\lim\limits_{n\to\infty}\dfrac{\left(\sqrt[3]{n^3+n+3}-\sqrt[3]{n^3+1}\right)(\;---\;)}{\left(\sqrt[3]{n^3+n+3}\right)^2+\left(\sqrt[3]{(n^3+n+3)(n^3+1)}\right)+\left(\sqrt[3]{n^3+1}\right)^2}$

$=\lim\limits_{n\to\infty}\dfrac{n^3+n+3-n^3-1}{\left(n\sqrt[3]{1+\frac{1}{n^2}+\frac{3}{n^2}}\right)^2+\;---}$

$=\lim \dfrac{n+2}{n^2(\quad)}\to 0$

g) $\lim\limits_{n\to\infty}\left(\dfrac{n^3+5n+1}{n^2-1}\right)^{\frac{1-5n^4}{6n^4+1}}$

$\lim\limits_{n\to\infty}\left(1+\dfrac{n^3+5n+1-n^2+1}{n^2-1}\right)^{\frac{1-5n^4}{6n^4+1}}$

$=\lim\limits_{n\to\infty}\left(1+\dfrac{n^3-n^2+5n+2}{n^2-1}\right)^{\frac{n^2-1}{n^3-n^2+5n+2}\cdots\frac{1-5n^4}{6n^4+1}}$

$= e^{\lim\limits_{n\to\infty}\frac{n^3-n^2+5n+2}{n^2-1}\cdot\frac{1-5n^4}{6n^4+1}}$

$= e^{\lim\limits_{n\to\infty}\frac{(n^3-5n^7-n^2+5n^6+5n-25n^5+2-10n^4)}{6n^6+n^2-6n^4-1}}$

$= e^{-\infty} = 0$

h) $\lim\limits_{n\to\infty}\underbrace{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)}_{A_n}$

$A_n = \dfrac{1}{2}\cdot\dfrac{2}{3}\cdot\dfrac{3}{4}\cdot\ldots\cdot\dfrac{n-1}{n} = \dfrac{1}{n}\to 0$

ex 4

a) S.s.d că $\exists (x_n) \nearrow$ $x_n \in G$

$x_n \to t$

Fie $x_1 \in B(t, 1)$

Fie $x_2 \in B(t, \frac{1}{2})$ aî $x_1 < x_2$ $\Big\} \Rightarrow$

Analog $x_3 \dots$

$\Rightarrow x_n \in B(t, \frac{1}{n}) \cap G$ un șir $\nearrow$ de nr [illegible]

$|x_n - t| \leq \frac{1}{n} \Rightarrow x_n \to t$

$x_n = t - \frac{1}{n}$

b) S.s.d că $\exists (y_n) \nearrow$ $y_n \in G$

$y_n \to t$

Fie $y_n \in$

ex 5

$0 < x_0 < \frac{1}{a}$ $\quad x_{n+1} = 2x_n - a x_n^2$

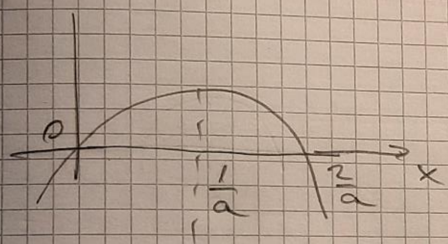
a)

Dacă $x_n < \frac{1}{a}$

atunci $x_{n+1} > 0 < \frac{1}{2}$ ...

Fie $f : \mathbb{R} \to \mathbb{R}$ $\quad f(t) = -at^2 + 2t$

f fct. de gr. II



$V\left(-\frac{b}{2a}, \frac{-\Delta}{4a}\right)$

$-\frac{b}{2a} = \frac{-2}{2(-a)} = \frac{1}{a}$

$f(t) \in \left(0, \frac{1}{2}\right) \forall t \in \left(0, \frac{1}{2}\right)$

$\frac{-\Delta}{4a} = \frac{-2}{-4a} = \frac{1}{2a}$

Cum $x_{n+1} = f(x_n)$

$x_n \in \left(0, \frac{1}{2}\right) \Rightarrow x_{n+1} \in \left(0, \frac{1}{2}\right)$

b)