Tema 6 - limite de funcții

ex 1

a) $\lim\limits_{x\to\infty} x \cos^2 \frac{x+2}{x}$

$= \lim\limits_{x\to\infty} x \cdot \cos^2 \frac{\cancel{x}(1+\overset{\to 0}{\frac{2}{x}})}{\cancel{x}}$

$= \infty \cdot \cos 1 = \infty$

b) $\lim\limits_{x\to 1} \frac{x}{x^2+1} = \lim\limits_{x\to 1} \frac{x}{x^2(1+\frac{1}{x^2})} = \frac{1}{1+1} = \frac{1}{2}$

c) $\lim\limits_{x\to\infty} \frac{x^2+5}{x^3} = \lim\limits_{x\to\infty} \frac{\cancel{x^3}(\frac{1}{x}+\frac{5}{x^3})}{\cancel{x^3}} = \frac{0}{1} = 0$

d) $\lim\limits_{x\to\infty} \frac{(x+2)(2x+1)}{x^2+3x+5}$

$= \lim\limits_{x\to\infty} \frac{2x^2+x+4x+2}{x^2+3x+5}$

$= \lim\limits_{x\to\infty} \frac{2x^2+5x+2}{x^2+3x+5} = \lim\limits_{x\to\infty} \frac{\cancel{x^2}(2+\overset{\to 0}{\frac{5}{x}}+\overset{\to 0}{\frac{2}{x^2}})}{\cancel{x^2}(1+\underset{\to 0}{\frac{3}{x}}+\underset{\to 0}{\frac{5}{x^2}})}$

$= 2.$

e) $\lim\limits_{x\to 1} \frac{x^2-1}{x^3-1} = \cancel{\lim\limits_{x\to} \frac{x^3\left(\frac{1}{x}-\frac{1}{x^3}\right)}{x^3\left(1-\frac{1}{x^3}\right)}}$

$= \lim\limits_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x^2+x+1)}$

$= \frac{2}{3}$

f) $\lim\limits_{x\to 2} \left(\frac{1}{2-x} - \frac{2x}{4-x^2}\right)$

$= \lim\limits_{x\to 2} \frac{1}{2-x} - \frac{2x}{(2-x)(2+x)}$

$= \lim\limits_{x\to 2} \frac{2+x-2x}{(2-x)(2+x)}$

$= \lim\limits_{x\to 2} \frac{\cancel{2-x}}{(2+x)\cancel{(2-x)}} = \frac{1}{4}$

g) $\lim\limits_{x\to 1} \frac{1+x+x^2+\ldots+x^n-(n+1)}{x-1}$ $n\in\mathbb{N}$

~~$S_n = b_1 \frac{q^n-1}{q-1} = \frac{x^n-1}{x-1}$~~

~~Avem o p.g cu rația x $\Rightarrow S_n = \frac{x^n-1}{x-1}$~~

~~$\lim\limits_{x\to 1} \frac{\frac{x^n-1}{x-1}-(n+1)}{x-1} = \lim\limits_{x\to 1} \frac{(x^n-1)-(n+1)(x-1)}{(x-1)^2}$~~

h) $\lim\limits_{x \to 1} \dfrac{x + x^2 + \ldots + x^n - n}{x + x^2 + \ldots + x^m - m} \quad \forall m, n \in \mathbb{N}$

$$\overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x \to 1} \frac{1 + 2x + 3x^2 + \ldots + n x^{n-1}}{1 + 2x + 3x^2 + \ldots + m x^{m-1}}$$

$$= \frac{1 + 2 + \ldots + n}{1 + 2 + \ldots + m} = \frac{n(n+1)}{2} \cdot \frac{2}{m(m+1)}$$

$$= \frac{n(n+1)}{m(m+1)}$$

g) $\lim\limits_{x \to 1} \dfrac{1 + x + x^2 + \ldots + x^n - (n+1)}{x - 1}$

$$\overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x \to 1} \frac{1 + 2x + 3x^2 + \ldots + n x^{n-1}}{1}$$

$$= 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

i) $\lim\limits_{x \to 27} \dfrac{x - 27}{\sqrt[3]{x} - 3} \overset{\frac{0}{0}}{\underset{L'H}{=}}$

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$$\cancel{= \lim_{x \to 27} \frac{1}{\frac{1}{3\sqrt[3]{(x-3)^2}}} = \lim_{x \to 27} 3\sqrt[3]{(x-3)^2}}$$

$$= \lim_{x \to 27} \frac{1}{1/(3\sqrt[3]{x^2})} = \lim_{x \to 27} 3 \cdot \sqrt[3]{x^2}$$

$$= 27$$

j) $\lim\limits_{x \to 1} \dfrac{\sqrt[3]{x} - 1}{\sqrt[4]{x} - 1} \overset{\frac{0}{0}}{\underset{L'H}{=}}$

$= \lim\limits_{x \to 1} \dfrac{\dfrac{1}{3\sqrt[3]{x^2}}}{\dfrac{1}{4\sqrt[4]{x^3}}} = \lim\limits_{x \to 1} \dfrac{4\sqrt[4]{x^3}}{3\sqrt[3]{x^2}}$

$= \dfrac{4}{3}$

k) $\lim\limits_{x \to \infty} \sqrt{ax^3 + x^2 + bx + c} - (bx + c), \forall a, b, c > 0$

ex 2

a) $\lim\limits_{x \to \infty} \left(\dfrac{1}{x}\right)^{\frac{5x+1}{2x+4}}$

$= \lim\limits_{x \to \infty} \left(\dfrac{1}{x}\right)^{\frac{x(5 + \frac{1}{x})}{x(2 + \frac{4}{x})}}$

$= 0^{\frac{5}{2}} = 0.$

b) $\lim\limits_{x \to 0} \left(\dfrac{3\sin x - tg\, x}{x}\right)^{\frac{\sin x + 2x}{x}}$

$= \lim\limits_{x \to 0} \left(3 \underset{\to 1}{\dfrac{\sin x}{x}} - \underset{\to 1}{\dfrac{tg\, x}{x}}\right)^{\frac{\sin x + \cancel{2x}}{\cancel{x}}} \underset{\to 1}{\dfrac{\sin x}{x}} + 2$

$= 2^3 = 8$

c) $\lim_{x \to 0} (1 + \cos x)^{\frac{1}{x^2}}$

$= (1+1)^{\infty} = 2^{\infty} = \infty$

d) $\lim_{x \to 0} (e^x - x + 1)^{\frac{1}{1-\cos x} \cdot \frac{1+\cos x}{1+\cos x}}$

$= \lim_{x \to 0} (e^x - x + 1)^{\frac{1+\cos x}{1-\cos^2 x}}$

$= \lim_{x \to 0} (e^x - x + 1)^{\frac{1+\cos x}{\sin^2 x}}$

$= (1 - 1 + 1)^{\frac{2}{0}} = (1^{\infty})$

$= \lim_{x \to 0} (1 + e^x - x)^{\frac{1+\cos x}{\sin^2 x} \cdot \frac{1}{e^x - x} (e^x - x)}$

$= e^{\lim_{x \to 0} \frac{(1+\cos x)(e^x - x)}{\sin^2 x}}$

$= e^{\infty} = \infty.$

e) $\lim_{x \to 0} (1 + \sin x)^{\frac{1}{x}} \overset{1^{\infty}}{=}$

$= \lim_{x \to 0} (1 + \sin x)^{\frac{1}{\sin x} \cdot \sin x \cdot \frac{1}{x}}$

$= e^{\lim_{x \to 0} \frac{\sin x}{x}} = e$

f) $\lim\limits_{x\to\infty}\left(\frac{x+7}{x}\right)^{x} =$

$$= \lim_{x\to\infty}\left(\frac{\cancel{x}\left(1+\frac{7}{x}\right)}{\cancel{x}}\right)^{x}$$

$$= \lim_{x\to\infty}\left(1+\frac{7}{x}\right)^{x\cdot\frac{x}{7}\cdot\frac{7}{x}}$$

$$= e^{\lim\limits_{x\to\infty}\cancel{x}\cdot\frac{7}{\cancel{x}}} = e^{7}$$

ex 3 :

a) $\lim\limits_{n\to\infty}\left[\lim\limits_{x\to 0}\left(1+\sin^2 x+\sin^2 2x+\dots+\sin^2 nx\right)^{\frac{1}{n^3x^2}}\right]$

$$l = \lim_{x\to 0}\left(1+\sin^2 x+\dots+\sin^2 nx\right)^{\frac{1}{n^3x^2}}$$

$$l = \lim_{x\to 0}\left(1+\sin^2 x+\dots+\sin^2 nx\right)^{\frac{\sin^2 x+\dots+\sin^2 nx}{\sin^2 x+\dots+\sin^2 nx}\cdot\frac{1}{n^3x^2}}$$

$$= e^{\lim\limits_{x\to 0}\frac{\sin^2 x+\dots+\sin^2 nx}{n^3x^2}} = \frac{0}{0}$$

$$\cancel{= e^{\lim\limits_{x\to 0}\cos^2}}$$

$$= e^{\lim\frac{1}{n^3}\left(\frac{\sin^2 x}{x^2}+\frac{\sin^2 2x}{x^2}+\dots\frac{\sin^2 nx}{x^2}\right)}$$

$$= e^{\lim\limits_{x\to 0}\frac{1}{n^3}\left(\left(\overset{\to 1}{\frac{\sin x}{x}}\right)^2+4\left(\overset{\to 1}{\frac{\sin 2x}{2x}}\right)^2+3^2\left(\overset{\to 1}{\frac{\sin 3x}{3x}}\right)^2+\dots+n^2\left(\frac{\sin nx}{nx}\right)^2\right)}$$

$$= e^{\lim \frac{1}{n^3}(1+2^2+3^2+\dots+n^2)}$$

$$= e^{\frac{1}{n^3}(1+2^2+3^2+\dots+n^2)}$$

$$\lim_{n\to\infty} \ell = e^{\lim_{n\to\infty} \frac{1+2^2+3^2+\dots+n^2}{n^3}}$$

$$= e^{\lim_{n\to\infty} \frac{\cancel{n}(n+1)(2n+1)}{6n^{\cancel{3}\,2}}}$$

$$= e^{\lim \frac{2n^2+3n+1}{6n^2}}$$

$$= e^{\lim_{n\to\infty} \frac{\cancel{n^2}\left(2+\frac{3}{n}\xrightarrow{}0+\frac{1}{n^2}\xrightarrow{}0\right)}{6\cancel{n^2}}}$$

$$= e^{\frac{1}{3}}$$

b) $\lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+\ln(1+x)+\ln(1+2x)+\dots+\ln(1+nx)\right)^{\frac{1}{n^2x}}\right]$

$$\ell = \lim_{x\to 0}\left(1+\ln(1+x)+\dots+\ln(1+nx)\right)^{\frac{\ln(1+x)+\dots}{\ln(1+x)+\dots}\cdot\frac{1}{n^2x}}$$

$$\ell = e^{\lim_{x\to 0}\frac{\ln(1+x)+\dots+\ln(1+nx)}{n^2x}}$$

$$\ell = e^{\lim_{x\to 0}\frac{1}{n^2}\left(\frac{\ln(1+x)}{x}+\frac{2\ln(1+2x)}{2x}+\dots+\frac{n\ln(1+nx)}{nx}\right)}$$

$$\ell = e^{\frac{1+2+\dots+n}{n^2}}$$

$$\lim_{n\to\infty} e^{\frac{n(n+1)}{2n^2}}$$

$$= \lim_{n\to\infty} e^{\frac{n(1+\frac{1}{n})}{2n}} = e^{\frac{1}{2}}$$

ex 4 :

a) $$\lim_{x\to 0} \frac{e^{2x}-1}{3x} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0} \frac{2e^{2x}}{3}$$

$$= \frac{2e^0}{3} = \frac{2}{3}$$

b) $$\lim_{x\to 0} \frac{e^x - \cos x}{3x} = \lim_{x\to 0} \frac{e^x + \sin x}{3}$$

$$= \frac{1}{3}$$