Examen Algebră 1 Băcinlescu Raul

(1)

a) $(R, +, \cdot)$ inel dacă

- $(R, +)$ grup abelian
- $(R, \cdot)$ semigrup
- „$\cdot$" distributivă față de „+"

Exemplu de inel cu unitate $(\mathbb{R}, +, \cdot)$

$(\mathbb{R}, +)$ grup abelian, + ASOCIATIVĂ, comutativă
el neutru $= 0$
toate el simetrizabile

$(\mathbb{R}, \cdot)$ semigrup, pt că $\cdot$ pe $\mathbb{R}$ e Asociativă
el unitate $= 1$

- distributivă față de + pe $\mathbb{R}$

$A = \{ 2 \cdot n \mid n \in \mathbb{Z} \}$ $(A, +, \cdot)$ inel comutativ care nu e unital

el. unitate pe $\mathbb{Z}$ e 1, dar $1 \notin A$

La fel ca mai sus $(A, +, \cdot)$ e inel
$(A, +)$ gr. ab.
$(A, \cdot)$ semigr.
- distrib. față de +

Examen Algebră 1 Băciulescu

(b) Definiția spațiului vectorial

k corp comutativ, $(V,+)$ grup abelian
$(k,+,\cdot)$ împreună cu operația externă

$\cdot : k \times V \to V$, V e k-sp vectorial dacă

$\alpha(x+y) = \alpha x + \alpha y$

$(\alpha+\beta)x = \alpha x + \beta x$ $\quad \forall \alpha,\beta \in k$

$(\alpha\beta)x = \alpha(\beta x)$ $\quad \forall x,y \in V$

$1 \cdot x = x$

$\mathbb{Q}$ subcorp a lui $\mathbb{R}$

Știm că orice corp comutativ poate fi privit ca și un sp vectorial peste orice subcorp al său. $\Rightarrow$ $\mathbb{R}$ e un $\mathbb{Q}$ sp vectorial

vectori: $2 \in \mathbb{R}$ Scalari: $5 \in \mathbb{Q}$

(c) $A = \{(0,a) \in \mathbb{R}^2 \mid a \in \mathbb{R}\}$ e sumand direct în ${}_{\mathbb{R}}\mathbb{R}^2$? $A \oplus B = \mathbb{R}^2 \Leftrightarrow$ $A + B = \mathbb{R}$

$A \cap B = \{0\}$

$a = 0 \Rightarrow (0,0)$
$b = 0 \Rightarrow (0,0)$

fie $B = \{(b,0) \in \mathbb{R}^2 \mid b \in \mathbb{R}\}$

$A + B = \{(a,b) \in \mathbb{R}^2 \mid a,b \in \mathbb{R}\} = \mathbb{R}^2$, $A \cap B = \{0\}$ evident

A este sumand direct în ${}_{\mathbb{R}}\mathbb{R}^2$, iar un complement direct de-al său este B.

(b) Definiția spațiului vectorial

(2)

a) $A = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - y + z = 0\}$

s.s.a. că $A \leq_{\mathbb{R}} \mathbb{R}^3$

$A \leq_{\mathbb{R}} \mathbb{R}^3 \Leftrightarrow$ $A \neq \emptyset$

$\forall \alpha, \beta \in \mathbb{R}$

$\forall x, y \in \cancel{\mathbb{R}^3}$ A $\quad \alpha x + \beta y \in A$

$(0,0,0) \in A$ pt că $2 \cdot 0 - 0 + 0 = 0 \Rightarrow$

$\Rightarrow A \neq \emptyset$

Fie $\alpha, \beta \in \mathbb{R}$ și $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \cancel{\mathbb{R}^3}$ A a.î.

~~$\alpha(x_1, y_1, z_1$~~

$(*) \begin{cases} 2x_1 - y_1 + z_1 = 0 \\ 2x_2 - y_2 + z_2 = 0 \end{cases} \Rightarrow$

$\Rightarrow$ Vrem $\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) \in A$

$(\alpha x_1, \alpha y_1, \alpha z_1) + (\beta x_2, \beta y_2, \beta z_2)$

$(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$

Vrem $2(\alpha x_1 + \beta x_2) - (\alpha y_1 + \beta y_2) + \alpha z_1 + \beta z_2 = 0$

$2\alpha x_1 + 2\beta x_2 - \alpha y_1 - \beta y_2 + \alpha z_1 + \beta z_2 = 0$

$\alpha(2x_1 - y_1 + z_1) + \beta(2x_2 - y_2 + z_2) = 0$

Știm din $(*)$ că $2x_1 - y_1 + z_1 = 0$, $2x_2 - y_2 + z_2 = 0 \Rightarrow$

$\Rightarrow \alpha(2x_1 - y_1 + z_1) + \beta(2x_2 - y_2 + z_2) = 0 \Rightarrow$

$\Rightarrow \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) \in A \Rightarrow$

$\Rightarrow A \leq_{\mathbb{R}} \mathbb{R}^3$

b) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x, 2x - y)$

i) S.s.a că f e endomorfism.

f transformare liniară $\Leftrightarrow f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$
$\forall x, y \in \mathbb{R}^2$, $\forall \alpha, \beta \in \mathbb{R}$

Fie $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, $\alpha, \beta \in \mathbb{R}$

$f(\alpha(x_1, y_1) + \beta(x_2, y_2)) = \alpha f(x_1, y_1) + \beta f(x_2, y_2)$

$f(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) = (\alpha x_1 + \beta x_2, 2(\alpha x_1 + \beta x_2) -$
$- \alpha y_1 - \beta y_2)$

$\alpha f(x_1, y_1) + \beta f(x_2, y_2) = \alpha(x_1, 2x_1 - y_1) + \beta(x_2, 2x_2 - y_2)$
$= (\alpha x_1, 2x_1\alpha - y_1\alpha) + (\beta x_2, 2x_2\beta - y_2\beta)$
$= (\alpha x_1 + \beta x_2, 2x_1\alpha - y_1\alpha + 2x_2\beta - y_2\beta)$
$= (\alpha x_1 + \beta x_2, 2(\alpha x_1 + \beta x_2) - \alpha y_1 - \beta y_2)$ $\Rightarrow$

$\Rightarrow f(\alpha(x_1, y_1) + \beta(x_2, y_2)) = \alpha f(x_1, y_1) + \beta(f(x_2, y_2))$ $\forall \alpha, \beta \in \mathbb{R}$
$\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 \Rightarrow f$ transf. lin.

$\mathbb{R}^2 = \mathbb{R}^2$ (domeniul egal cu codomeniul)
f transf. lin. $\Big\} \Rightarrow$

$\Rightarrow f$ endomorfism.

ii) Este f bijectiv.

I f inj. Fie $(x_1, y_1), (x_2, y_2)$ Dacă $f(x_1, y_1) = f(x_2, y_2)$
vrem $(x_1, y_1) = (x_2, y_2)$

$f(x_1, y_1) = f(x_2, y_2)$

$(x_1, 2x_1 - y_1) = (x_2, 2x_2 - y_2) \Rightarrow$

$\Rightarrow x_1 = x_2 \quad 2x_1 - y_1 = 2x_2 - y_2$

$\Rightarrow 2x_1 - y_1 = 2x_1 - y_2 \Rightarrow$

$\Rightarrow y_1 = y_2 \Rightarrow (x_1, y_1) = (x_2, y_2)$

$\Rightarrow f$ inj.

II f surj. $\Leftrightarrow \forall (y_1, y_2) \in \mathbb{R}^2 \; \exists (x_1, \cancel{y_2}\, x_2) \in \mathbb{R}^2$ a.î.

$f(x_1, x_2) = (y_1, y_2) \Rightarrow$

$\Rightarrow \cancel{(2x_1} \; \cancel{(x_1, 2x_1 - y_1)}$

$(x_1, 2x_1 - x_2) = (y_1, y_2)$

$\Rightarrow$ pt fiecare $y_1, y_2 \in \mathbb{R} \; \exists \, x_1, 2x_1 - x_2 \in \mathbb{R} \Rightarrow$

$\Rightarrow f$ surj. I, II $\Rightarrow f$ bij.

(iii) Determinați $\ker f$, $\operatorname{Im} f$

$\operatorname{Im} f = \{ f(x) \mid x \in \mathbb{R}^2 \}$ $\quad f$. bijectivă $\Rightarrow$

$\Rightarrow \operatorname{Im} f = \mathbb{R}^2$

$\ker f = \{ x \in \mathbb{R}^2 \mid f(x) = 0 \}$

$f(x_1, x_2) = 0 \Leftrightarrow (x_1, 2x_1 - x_2) = 0$

$x_1 = 0 \quad 2x_1 - x_2 = 0 \Rightarrow$

$\Rightarrow x_2 = 0 \Rightarrow$

$\Rightarrow \left.\begin{matrix} (0,0) \in \ker f \\ f \text{ bijectivă} \end{matrix}\right\} \Rightarrow$

$\Rightarrow \ker f = \{(0,0)\}$

iv) matricea lui f în baza canonică.

$f(e_1) = f(1,0) = (1, 2 \cdot 1 - 0) = (1,2)$

$f(e_2) = f(0,1) = (0, 2 \cdot 0 - 1) = (0,-1)$

$$[f]_e = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

v) Fie $u_1 = (1,0)$ $\quad u_2 = (1,1)$

u baza a lui $\mathbb{R}$ $\mathbb{R}^2 \Leftrightarrow \forall v \in \mathbb{R}^2, \exists \alpha_1, \alpha_2 \in \mathbb{R}$ unici a.î.

(u_1, u_2)

$v = \alpha_1 u_1 + \alpha_2 u_2$

(u_1, u_2) formează bază a lui $\mathbb{R}^2 \iff$

$$\iff \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \neq 0 \qquad \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \neq 0 \Rightarrow$$

$$\begin{matrix} \uparrow & \uparrow \\ u_1 & u_2 \end{matrix}$$

$\Rightarrow (u_1, u_2)$ formează bază a lui $\mathbb{R}^2$.

$[f]_{u,e} = ?$

$f(u_1) = f(1,0) = (1, 2 \cdot 1 - 0) = (1,2) = \alpha_1 \cdot e_1 + \alpha_2 \cdot e_2$

$f(u_2) = f(1,1) = (1, 2 \cdot 1 - 1) = (1,1) = \alpha_3 e_1 + \alpha_4 e_2$

$$\begin{cases} (1,2) = \alpha_1 (1,0) + \alpha_2 (0,1) \\ (1,1) = \alpha_3 (1,0) + \alpha_4 (0,1) \end{cases}$$

$$\left.\begin{matrix} (1,2) = (\alpha_1, \alpha_2) \Rightarrow \alpha_1 = 1 \quad \alpha_2 = 2 \\ (1,1) = (\alpha_3, \alpha_4) \Rightarrow \alpha_3 = 1 \quad \alpha_4 = 1 \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow [f]_{u,e} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

(3) S, T subspații le $\mathbb{R}$ sp vectorial $\mathbb{R}^4$

$S = \langle u_1, u_2 \rangle \qquad T = \langle v_1, v_2 \rangle$

$$\dim S = \text{rang} \begin{array}{cl} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix} & \begin{array}{l} \to u_1 \\ \to u_2 \end{array} \end{array} = 2$$

$S \leq_{\mathbb{R}} \mathbb{R}^4$

avem $\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1 - 0 = -1 \neq 0 \Rightarrow$

$\Rightarrow \text{rang} = 2 \Rightarrow$

$\Rightarrow \dim S = 2 \Rightarrow$

$\Rightarrow (u_1, u_2)$ bază în S

$$\dim T = \text{rang} \begin{array}{cl} \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix} & \begin{array}{l} \to v_1 \\ \to v_2 \end{array} \end{array} = 2$$

Avem $\begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} = 2 \cdot 0 - 2 \cdot 1 = -2 \neq 0 \Rightarrow \text{rang}\, T = 2 \Rightarrow$

$\Rightarrow (v_1, v_2)$ bază în T

$$\dim S+T = \text{rang} \underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}}_{A}$$

Calculez

$$\text{rang}\, A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix} \underset{\sim}{C_2 - C_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix} \underset{\sim}{L_4 + L_2}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix} \underset{\substack{C_3+C_2 \\ C_4+C_2}}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix} \overset{C_4-C_3}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix} \Rightarrow \text{rang } A = 4$$

$\Rightarrow \dim S+T = 4 \Rightarrow (u_1, u_2, v_1, v_2)$ formează bază în $S+T$.

$$\dim S + \dim T = \dim(S+T) + \dim(S \cap T)$$

$$2+2 = 4 + \dim S\cap T \Rightarrow$$

$$\Rightarrow \dim S \cap T = 0 \Rightarrow \emptyset \text{ bază în } S \cap T$$

(h) $\begin{cases} x_1 + 2x_2 - 2x_3 = 1 \\ x_1 + x_2 + x_3 = 3 \\ 2x_1 + 6x_2 - 2x_3 = 6 \\ -x_1 + \alpha x_2 + x_3 = 2 \end{cases}$

Avem $\bar{A} = \left(\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 1 & 1 & 1 & 3 \\ 2 & 6 & -2 & 6 \\ -1 & \alpha & 1 & 2 \end{array}\right)$

$$\bar{A} = \left(\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 1 & 1 & 1 & 3 \\ 2 & 6 & -2 & 6 \\ -1 & \alpha & 1 & 2 \end{array}\right) \begin{array}{l} L_2 - L_1 \\ \sim \\ L_3 - 2L_1 \\ L_4 + L_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & \alpha+2 & -1 & 3 \end{array}\right) \underset{\sim}{L_3 \leftrightarrow L_4} \left(\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & \alpha+2 & -1 & 3 \\ 0 & 2 & 2 & 4 \end{array}\right)$$

$$\begin{array}{l} L_4 + 2L_2 \\ \sim \\ L_3 + (\alpha+2)L_2 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & -1+3(\alpha+2) & 3+(\alpha+2)\cdot 2 \\ 0 & 0 & 8 & 8 \end{array}\right) \begin{array}{l} -1+3(\alpha+2) \\ = 3\alpha+5 \\ 3+2\alpha+4 \\ 2\alpha+7 \end{array}$$

$$\underset{L_4 - \frac{8}{3\alpha+5}L_3}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 3\alpha+5 & 2\alpha+7 \\ 0 & 0 & 0 & \cancel{2\alpha+7} \\ & & & 8 - \frac{8}{3\alpha+5}(2\alpha+7) \end{array}\right) \Rightarrow$$

$\Rightarrow$ ~~S.C.N~~ dacă $8 - \dfrac{8(2\alpha+7)}{3\alpha+5} \neq 0$

Sistem compatibil $\quad 8 = \dfrac{8(2\alpha+7)}{3\alpha+5}$

$$24\alpha + 40 = 16\alpha + 56$$

$$8\alpha = 16 \Rightarrow \alpha = 2 \Rightarrow$$

Cu $\alpha = 2$ Avem sistemul:

$$\begin{cases} x_1 + 2x_2 - 2x_3 = 1 \\ -x_2 + 3x_3 = 2 \\ x_3(\underbrace{3\cdot 2+5}_{11}) = \underbrace{2\cdot 2+7}_{11} \Rightarrow \end{cases}$$

$\Rightarrow 11 x_3 = 11 \Rightarrow x_3 = 1$

$-x_2 + 3 = 2$

$-x_2 = -1 \Rightarrow x_2 = 1$

$x_1 + 2 - 2 = 1$

$x_1 = 1 \Rightarrow$

$\Rightarrow$ Pt $\alpha = 2$ $S = \{(1,1,1)\}$ compatibil determinat

Dacă $\alpha \neq 2 \Rightarrow$ Sistem Incompatibil, pt.

că $0 \neq 8 - \frac{8}{3\alpha+5}(2\alpha+7)$