Tema 8 - funcții derivabile

ex 1: Derivata de ordin n.

a) $f:(1,\infty)\to\mathbb{R}$ $\quad f(x)=(1+x)^r,\ r\in\mathbb{R}$

$$f'(x) = (r)(1+x)^{r-1}\cdot(x)'$$
$$= r(1+x)^{r-1}$$

$$f''(x) = r\cdot(r-1)(1+x)^{r-2}$$

$$f'''(x) = r(r-1)(r-2)(1+x)^{r-3}$$

Pp: $f^{n}(x) = r(r-1)\dots(r-(n-1))(1+x)^{r-n}$

I $n=1 \Rightarrow f'(x) = (r-(1-1))(1+x)^{r-1}$
$$= r(1+x)^{r-1} \quad (A)$$

II $P(n)\to P(n+1)$

$\forall n\geq 1$

$P(n): f^{n}(x) = r(r-1)\dots(r-(n-1))(1+x)^{r-n}$

Vrem $P(n+1): f^{n+1}(x) = r(r-1)\dots(r(n-1)(r-n)(1+x)^{r-(n+1)}$

$$\cancel{f^{n+1}(x) = r(r-1)\dots(r-(n-1))(r-n)(1+x)^{r-n}\cdot\frac{1}{1+x}}$$
$$\cancel{= f^{n}(x)\frac{(r-n)}{1+x}}$$

$$f^{n+1}(x) = (f^{n}(x))' = r(r-1)\dots(r-(n-1))(r-n)(1+x)^{r-(n+1)} \quad (A)$$

$I, II \Rightarrow p(n)$ (A) pt $\forall n \geq 1$

b) $f: (-1, \infty) \to \mathbb{R}$ $\quad f(x) = x \cdot \ln(1+x)$

$$f'(x) = (x \cdot \ln(1+x))'$$
$$= \ln(1+x) + x \cdot \frac{1}{x+1}$$
$$= \ln(1+x) + \frac{x}{x+1}$$

$$f''(x) = \frac{1}{x+1} + \frac{x+1-x}{(x+1)^2}$$
$$= \frac{1}{x+1} + \frac{1}{(x+1)^2} = \frac{x+2}{(x+1)^2}$$

$$f'''(x) = \cancel{\frac{-1}{(x+1)^2} + \frac{-2(x+1)}{(x+1)^3}}$$
$$\cancel{= \frac{-1}{(x+1)^2} + \frac{-2}{(x+1)^2} =}$$

$$= \frac{x+2}{(x+1)^2} = \frac{(x+1)^2 - 2\cancel{(x+1)}(x+2)}{(x+1)^{3}}$$

$$= \cancel{x^2+2x+1} = \frac{x+1-2x-4}{(x+1)^3}$$

$$= (-1)\frac{x+3}{(x+1)^3}$$

Pp că $f^{(n)}(x) = (-1)^n \frac{x+n}{(x+1)^n}$ $\forall n \geq 1$

c) $f:(-\infty,-1)\to\mathbb{R} \quad f(x)=x\cdot\ln(1-x)$

$$f'(x)=\ln(1-x)+x\frac{1}{1-x}(-1)$$

$$f''(x)=\frac{-1}{1-x}+\left(\frac{-x}{1-x}\right)'$$

$$f''(x)=\frac{-1}{1-x}+\frac{-(1-x)\cancel{+}(-x)(-1)}{(1-x)^2}$$

$$=\frac{-1}{1-x}^{(1-x)}+\frac{x-1\cancel{+}x}{(1-x)^2}$$

$$=\frac{x-1+\cancel{2x}-1}{(1-x)^2}$$

$$=\frac{\cancel{2}x-2}{(1-x)^2}$$

$$f'''(x)=\cancel{3(1-x)2}$$

$$=\frac{(1-x)^{\cancel{2}}+2(\cancel{1-x})(x-2)}{(1-x)^{\cancel{4}\,3}}$$

$$=\cancel{\frac{1-2x+x^2+(2-2x)(x-2)}{(1-x)^4}}$$

$$=\cancel{\frac{1-2x+x^2+2x-4-2x^2+4x}{(1-}}$$

$$=\frac{1-x+2x-4}{(1-x)^3}=\frac{x-3}{(1-x)^3}$$

Pp că $f^{n}(x)=\frac{x-n}{(1-x)^n} \quad \forall n\geq 2$

I Et de vf: $f''(x)=\frac{x-2}{(1-x)^2}$ (A)

II $\quad p(n) \rightarrow p(n+1)$

$$P(n): f^n(x) = \frac{x-n}{(1-x)^n} \quad \forall n \geq 2$$

$$P(n+1): f^{n+1}(x) = \frac{x-(n+1)}{(1-x)^{n+1}}$$

$$(f^n(x))' = \frac{(1-x)^n - (x-n)\,n\,(1-x)^{n-1}\cdot(-1)}{(1-x)^{2(n+1)}}$$

$$= \frac{(1-x)^{n-1}\cdot\left(1-x+n(x-n)\right)}{(1-x)^{2n+2}}$$

$$= \frac{(1-x+n(x-n))(1-x)^{n-1}}{(1-x)^2\cdot\left((1-x)^{n-1}\cdot(1-x)\right)^2}$$

d) $f:(-1,1)\to\mathbb{R}$ $\quad f(x)=\sqrt{3x+4}$

$$f'(x)=\frac{3}{2\sqrt{3x+4}}$$

$$f''(x)=\cancel{\frac{-3-\frac{3}{\sqrt{3x+4}}}{2\sqrt{3x+4}}}$$

$$=\cancel{\frac{-3\sqrt{3x+4}-3}{2\sqrt{3x+4}\cdot\sqrt{3x+4}}}$$

$$=\left(\frac{3}{2\sqrt{3x+4}}\right)'=\frac{-3\cdot\left(2\sqrt{3x+4}\right)'}{4\sqrt{(3x+4)^2}}$$

$$=\frac{-3\cdot\frac{\cancel{2}\cdot 3}{\cancel{2}\sqrt{3x+4}}}{4\sqrt{(3x+4)^2}}=\frac{-9}{4\,\cancel{\sqrt{}}(3x+4)^{\frac{3}{2}}}$$

$$f'''(x)=\left(\frac{-9}{4(3x+4)^{\frac{3}{2}}}\right)'$$

$$=\frac{9\left(4(3x+4)^{\frac{3}{2}}\right)'}{16(3x+4)^3}$$

$$=\frac{9\cdot\overset{3}{\cancel{6}}(3x+4)^{\frac{1}{2}}}{\underset{8}{\cancel{16}}\cdot(3x+4)^3}=\frac{27}{8\cdot(3x+4)^3\,(3x+4)^{-\frac{1}{2}}}$$

$$=\frac{27}{8\cdot(3x+4)^{\frac{5}{2}}}$$

Pp că $f^{n}(x) = \dfrac{(-1)^{n+1} \cdot 3^{n}}{2^{n} \cdot (3x+4)^{\frac{2n-1}{2}}}$ $n \geq 2$

e) $f : \left(-\frac{1}{2}, \infty\right) \to \mathbb{R}$, $f(x) = \dfrac{1}{\sqrt{2x+1}}$

$$f'(x) = \cancel{\left(\frac{1}{\sqrt{2x+1}}\right)' = }$$

$$= \left((2x+1)^{-\frac{1}{2}}\right)'$$

$$= -\frac{1}{2}(2x+1)^{-\frac{1}{2}-1}$$

$$= -\frac{1}{2}(2x+1)^{-\frac{3}{2}}$$

$$f''(x) = -\frac{1}{2}\cdot\left(-\frac{3}{2}\right)(2x+1)^{-\frac{3}{2}-1}$$

$$= \frac{3}{4}(2x+1)^{-\frac{5}{2}}$$

$$f'''(x) = \frac{3}{4}\cdot\left(-\frac{5}{2}\right)(2x+1)^{-\frac{5}{2}-1}$$

$$= -\frac{15}{8}(2x+1)^{-\frac{7}{2}}$$

$$f^{iv}(x) = -\frac{15}{8}\cdot\left(-\frac{7}{2}\right)(2x+1)^{-\frac{7}{2}-1}$$

$$= \frac{105}{16}$$

Pp că $f^{n}(x) = (2x+1)^{-\left(\frac{2n+1}{2}\right)} (-1)^{n} \cdot$
$\cdot 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

ex 2:

a) $f : \mathbb{R} - \left\{\frac{-b}{a}\right\} \to \mathbb{R}$ $\qquad f(x) = \frac{1}{ax+b}$

$$f'(x) = \left(\frac{1}{ax+b}\right)' = \frac{-(ax+b)'}{(ax+b)^2}$$

$$= \frac{-a}{(ax+b)^2}$$

$$f''(x) = \left(\frac{-a}{(ax+b)^2}\right)' = \frac{a \cdot 2 \cdot \cancel{(ax+b)}\,\overbrace{(ax+b)'}^{a}}{(ax+b)^{\cancel{4}\,3}}$$

$$= \frac{2a^2}{(ax+b)^3}$$

$$f'''(x) = \frac{-2a^2 \cdot 3\cancel{(ax+b)^2} \cdot a}{(ax+b)^{\cancel{6}\,4}}$$

$$= \frac{-6a^3}{(ax+b)^4}$$

Pp că $f^{n}(x) = \frac{(-1)^{n} \cdot 1 \cdot 2 \cdot \ldots \cdot \overset{n}{\cancel{(n+1)}}\, a^{n}}{(ax+b)^{n+1}}$

$\forall n \geq 1$

I ✓ $n=1 \Rightarrow f'(x) = \frac{-a}{(ax+b)^2}$

II $P(n) \to P(n+1)$

$$P(n) = f^n(x) = \frac{(-1)^n \cdot 1 \cdot \ldots \cdot n \cdot a^n}{(ax+b)^{n+1}}$$

$$P(n+1) = f^{n+1}(x) = \frac{(-1)^{n+1} \cdot 1 \cdot \ldots \cdot (n+1) \cdot a^{n+1}}{(ax+b)^{n+2}}$$

$$(f^n(x))' = \frac{-((-1)^n \cdot 1 \cdot \ldots \cdot n \cdot a^n)(n+1)\cancel{(ax+b)}^n \cdot a}{(ax+b)^{2n} \cdot (ax+b)}$$

$$= \frac{(-1)^{n+1} \cdot 1 \cdot \ldots \cdot (n+1) \cdot a^{n+1}}{(ax+b)^{n+1}} \quad (A)$$

$\Rightarrow$ I, II adv.

b) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \sin(ax+b)$

$f'(x) = \cos(ax+b) \cdot a = a \cdot \sin\left(ax+b+\frac{\pi}{2}\right)$

~~$f''(x) = -a^2 \sin(ax+b)$~~

~~$f'''(x) = -a^3 \cos(ax+b)$~~

~~$f^{IV}(x) = a^4 \sin(ax+b)$~~

$f''(x) = a^2 \sin\left(ax+b+\frac{2\pi}{2}\right)$

P_p că $f^{n}(x) = a^{n} \sin\left(ax + b + \frac{n\pi}{2}\right)$ $\forall n \geq 1$

$\underline{I}$ $n = 1 \Rightarrow f'(x) = a \cdot \sin\left(ax + b + \frac{\pi}{2}\right)$ (A)

$\overline{II}$ $P(n) \Rightarrow P(n+1)$

$P(n): f^{n}(x) = a^{n} \sin\left(ax + b + \frac{n\pi}{2}\right)$ $\forall n \geq 1$

$P(n+1): f^{n+1}(x) = a^{n+1} \sin\left(ax + b + \frac{(n+1)\pi}{2}\right)$, $\forall n \geq 1$

$\left(f^{n}(x)\right)' = a^{n} \cdot \cos\left(ax + b + \frac{n\pi}{2}\right) \cdot a$

$= a^{n+1} \cdot \sin\left(ax + b + \frac{(n+1)\pi}{2}\right)$ (A)

$\underline{I}, \underline{II} \Rightarrow P(n)$ e (A)

c) $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \cos(ax + b)$

$f'(x) = -\sin(ax + b) \cdot a = a \cdot \cos\left(ax + b + \frac{\pi}{2}\right)$

$f''(x) = a^{2} \cdot \cos\left(ax + b + \frac{2\pi}{2}\right)$

P_p că $f^{n}(x) = a^{n} \cdot \cos\left(ax + b + \frac{n\pi}{2}\right)$

d) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=e^{ax+b}$

$f'(x) = e^{ax+b}\cdot a$

$f''(x) = a^2\cdot e^{ax+b}$

$f^{n}(x) = a^{n}\cdot e^{ax+b}$ $\forall n \geq 1$

I $n=1 \Rightarrow f'(x) = a^1\cdot e^{ax+b}$

II $P(n)\to P(n+1)$

$(f^{n}(x))' = (a^{n}\, e^{ax+b})'$

$= a^{n+1}\cdot e^{ax+b}$ (A)

I și II $\Rightarrow P(n)$ e adevărat.

$a^{b} = e^{b\ln a}$

ex 3

a) $f:(0,\infty)\to\mathbb{R}$ $f(x)=x^{x} = e^{x\ln x}$

$f'(x) = (e^{x\ln x})' = (x\ln x)'\, e^{x\ln x}$

$= (\ln x + 1)\, e^{x\ln x} = x^{x}(\ln x + 1)$

b) $f:(0,\infty)\to\mathbb{R}$ $f(x) = x^{\frac{1}{x}} = e^{\frac{1}{x}\ln x}$

$$f'(x) = \left(e^{\frac{1}{x}\ln x}\right)' = \left(\frac{1}{x}\ln x\right)' e^{\frac{1}{x}\ln x}$$

$$= \left(-\frac{1}{x^2}\cdot \ln x + \frac{1}{x^2}\right) e^{\frac{1}{x}\ln x}$$

$$= \frac{1-\ln x}{x^2} \cdot x^{\frac{1}{x}}$$

c) $f : (0, \pi) \to \mathbb{R} \quad f(x) = \sin x^x$

$$f'(x) = (\sin x^x)' = \cos x^x \cdot (x^x)'$$

$$= \cos x^x \cdot x^x(\ln x + 1)$$

d) $f : (0, \infty) \to \mathbb{R} \quad f(x) = x^{\sin x} = e^{\sin x \ln x}$

$$f'(x) = e^{\sin x \cdot \ln x} \cdot (\sin x \cdot \ln x)'$$

$$= \left(\cos x \cdot \ln x + \frac{\sin x}{x}\right) e^{\sin x \cdot \ln x}$$

ex 4: $\frac{1}{x+1} < \ln(x+1) - \ln x < \frac{1}{x}, \forall x > 0$

$$(\ln(x+1))' < \ln(x+1) - \ln(x) < (\ln x)'$$

fie $f(x) = \ln(x+1)$ $\quad f, g : (0, \infty) \to \mathbb{R}$

f, g cresc.

$g(x) = \ln(x)$ $\quad f > g$

$\cancel{f'(x) < g'(x)}$ $\quad f'(x) < f(x) - g(x) < g'(x)$

ex 6: $f: \mathbb{R} \to \mathbb{R} \quad f(x) = x + |x-1|$

a) $f_s(1) = \lim\limits_{x \nearrow 1} \frac{f(x) - f(1)}{x-1}$

$= \lim\limits_{x \nearrow 1} \frac{x + |x-1| - 1 - 0}{x-1}$ $x - 1 < 0$

$= \lim\limits_{x \nearrow 1} \frac{\cancel{x} + \cancel{x} - \cancel{x} - 1}{x-1} = 0$

$f_d(1) = \lim\limits_{x \searrow 1} \frac{f(x) - f(1)}{x-1}$ $x - 1 > 0$

$= \lim \frac{x + x - 1 - 1}{x-1} = \lim\limits_{x \searrow 1} \frac{2x-2}{x-1}$

$= 2$.

$\exists \lim\limits_{x \nearrow 1} \frac{f(x) - f(1)}{x-1}$, $\exists \lim\limits_{x \searrow 1} \frac{f(x) - f(1)}{x-1} \Rightarrow$

$\Rightarrow$ funcția f are derivate laterale în 1

$f_s(1) \neq f_d(1) \Rightarrow f$ nu este derivabilă în 1