Seminar 9

Subspațiu generat.

Fie K corp comutativ, V K-sp. vectorial, $A \subseteq V$, $\emptyset \neq X \subseteq V$.

- $A \leq_K V \iff \begin{cases} A \neq \emptyset \\ \forall x,y \in A,\ x+y \in A \\ \forall \alpha \in K,\ \forall x \in A,\ \alpha x \in A \end{cases} \iff \begin{cases} A \neq \emptyset \\ \forall x,y \in A,\ \forall \alpha, \beta \in K, \\ \alpha x + \beta y \in A. \end{cases}$

- subspațiul generat de X:

$v \in \langle X \rangle \iff \exists\, n \in \mathbb{N}^*,\ \exists\, \alpha_1, \dots, \alpha_n \in K,\ \exists\, x_1, \dots, x_n \in X:$

$v = \alpha_1 x_1 + \dots + \alpha_n x_n$. $\to$ combinație liniară

Obs: 1) Subspațiile lui ${}_{\mathbb{R}}\mathbb{R}^2$ sunt: $\{(0,0)\}$, $\mathbb{R}^2$, dreptele care trec prin origine.

2) Subspațiile lui ${}_{\mathbb{R}}\mathbb{R}^3$ sunt: $\{(0,0)\}$, $\mathbb{R}^3$, dreptele care trec prin origine, planele care trec prin origine.

(dacă luăm un punct care nu e în planul care trece prin origine $\Rightarrow \mathbb{R}^3$).

Lista 8

1. Fie V K-sp. vectorial, $S \leq_K V$, $x \in V \setminus S$, $y \in V$.

Notăm $\langle S \cup \{y\} \rangle = \langle S, y \rangle$. Să se arate că:

$$x \in \langle S, y \rangle \Rightarrow y \in \langle S, x \rangle$$

$x \in \langle S, y \rangle \iff \exists\, \alpha, \alpha_1, \dots, \alpha_n \in K,\ \exists\, s_1, \dots, s_n \in S$ a.î.

$$x = \alpha y + \alpha_1 s_1 + \dots + \alpha_n s_n \qquad (1)$$

Presupunem că $\alpha = 0$, $(1) \Rightarrow x = \alpha_1 s_1 + \dots + \alpha_n s_n \in S$ contradicție ($x \in V \setminus S$).

$\Rightarrow \alpha \neq 0 \overset{K \text{ corp}}{\Rightarrow} \exists\, \alpha^{-1} \in K$.

Din $(1) \Rightarrow \alpha y = x - \alpha_1 s_1 - \alpha_2 s_2 - \dots - \alpha_n s_n \quad | \cdot \alpha^{-1}$

$$y = \alpha^{-1} x - (\alpha^{-1}\alpha_1) s_1 - \dots - (\alpha^{-1}\alpha_n) s_n \in \langle S, x \rangle$$

(2.) Fie V K-sp. vect., $\alpha, \beta, \gamma \in K$; $x, y, z \in V$ a.î. $\alpha\gamma \neq 0$ și

$$\alpha x + \beta y + \gamma z = 0. \qquad (*)$$

Să se arate că $\langle x, y\rangle \overset{?}{=} \langle y, z\rangle$.

$\langle x, y\rangle \overset{?}{\subseteq} \langle y, z\rangle \quad \Leftarrow \quad x, y \overset{?}{\in} \langle y, z\rangle$

$\langle x, y\rangle \overset{?}{\supseteq} \langle y, z\rangle \quad \Leftarrow \quad y, z \overset{?}{\in} \langle x, y\rangle$

$\alpha \cdot \gamma \neq 0 \Rightarrow \alpha \neq 0 \Rightarrow \exists\, \alpha^{-1} \in K$

$(*) \Rightarrow \alpha^{-1} / \ \alpha x = -\beta y - \gamma z$

$$x = (-\alpha^{-1}\beta) y - (\alpha^{-1}\gamma) z \quad \Rightarrow x \in \langle y, z\rangle$$

$\alpha \cdot \gamma \neq 0 \Rightarrow \gamma \neq 0 \Rightarrow \exists\, \gamma^{-1} \in K$

$(*) \Rightarrow \gamma^{-1} / \ \gamma z = -\alpha x - \beta y$

$$z = -(\gamma^{-1}\alpha) x - (\gamma^{-1}\beta) y \quad \Rightarrow z \in \langle x, y\rangle.$$

(3.) Formează polinoamele

$$f_1 = 3X + 2, \quad f_2 = 4X^2 - X + 1, \quad f_3 = X^3 - X^2 + 3$$

un sistem de generatori pentru $\left(\langle f_1, f_2, f_3\rangle = \mathcal{P}_3(\mathbb{R})\right)$.

$\mathcal{P}_3(\mathbb{R}) = \{ f \in \mathbb{R}[X] \mid \text{grad } f \leq 3\}$?

<u>Sol 1</u>: Fie $\alpha, \beta, \gamma \in \mathbb{R}$

$$\alpha f_1 + \beta f_2 + \gamma f_3 = \alpha(3x + 2) + \beta(4x^2 - x + 1) + \gamma(x^3 - x^2 + 3)$$

este constant $\Leftrightarrow \alpha = \beta = \gamma = 0$, altfel gradul său ≥ 1.

Prin urmare, constantele nenule nu aparțin $\langle f_1, f_2, f_3\rangle \Rightarrow$

$\Rightarrow \mathcal{P}_3(\mathbb{R}) \neq \langle f_1, f_2, f_3\rangle$.

<u>Sol. 2</u>: Fie $f \in \mathcal{P}_3(\mathbb{R})$, $f = aX^3 + bX^2 + cX + d$. arbitrar

Pp. că există $\alpha, \beta, \gamma \in \mathbb{R}$ a.î.: $f = \alpha f_1 + \beta f_2 + \gamma f_3 \qquad (1)$

$(1) \Leftrightarrow aX^3 + bX^2 + cX + d = \gamma X^3 + (4\beta - \gamma) X^2 + (3\alpha - \beta) X + (2\alpha + \beta + 3\gamma)$

$$\Leftrightarrow \begin{cases} \gamma = a \\ 4\beta - \gamma = b \\ 3\alpha - \beta = c \\ 2\alpha + \beta + 3\gamma = d \end{cases} \quad (2)$$

Fie $a = b = c = 0,\ d = 1.$

Sistemul (2) devine: $\begin{cases} \gamma = 0 \\ 4\beta - \gamma = 0 \\ 3\alpha - \beta = 0 \\ 2\alpha + \beta + 3\gamma = 1 \end{cases}$ $\Rightarrow \alpha = \beta = \gamma = 0$ (contradicție cu ultima rel) $\to$ sistem incompatibil.

• Fie V K-sp. vect., $A, B \leq_K V$.

$$V = A \oplus B \quad \begin{cases} V = A + B \\ A \cap B = \{0\} \end{cases} \quad (= \{a+b \mid a \in A,\ b \in B\}).$$

sumă directă de subspații

(4.) Proprietatea unui subsp. de a fi sumand direct este tranzitivă.
Mai exact, dacă V K-sp. vect., $A, B, C, D \leq_K V$

$$\left.\begin{array}{l} V = A \oplus B \\ A = C \oplus D \end{array}\right\} \Rightarrow \exists\, S \leq_K V : V = C \oplus S.$$

$$\left.\begin{array}{l} V = A \oplus B \quad \begin{cases} V = A + B \\ A \cap B = \{0\} \end{cases} \\ A = C \oplus D \quad \begin{cases} A = C + D \\ C \cap D = \{0\} \end{cases} \end{array}\right\} \Rightarrow V = C + \underbrace{(D + B)}_{\overset{?}{=} S}$$

adunarea asoc.

$S \overset{?}{=} D + B$

$(D+B) \cap C \overset{?}{=} \{0\}$

Fie $c \in C \cap (D+B) \Rightarrow c \in C$ și $c \in D+B \Rightarrow \exists\, d \in D, \exists\, b \in B : c = d + b \Rightarrow$

$\Rightarrow \underset{B}{\overset{\cap}{b}} = \underbrace{c - d}_{\in C + D = A} \in A \cap B = \{0\} \Rightarrow b = 0$ și $\underset{C}{\overset{\cap}{c}} = \underset{D}{\overset{\cap}{d}} \in C \cap D = \{0\} \Rightarrow c = 0$

⑤ Fie $\mathbb{R}$-sp. vectorial $\mathbb{R}^{\mathbb{R}} = \{ f \mid f : \mathbb{R} \to \mathbb{R} \}$. În acest spațiu considerăm:

$$\mathbb{R}_i^{\mathbb{R}} = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ impară} \};$$

$$\mathbb{R}_p^{\mathbb{R}} = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ pară} \}.$$

Să se arate că:

1) $\mathbb{R}_i^{\mathbb{R}}, \mathbb{R}_p^{\mathbb{R}} \leq_{\mathbb{R}} \mathbb{R}^{\mathbb{R}}$

2) $\mathbb{R}^{\mathbb{R}} = \mathbb{R}_i^{\mathbb{R}} \oplus \mathbb{R}_p^{\mathbb{R}}$.

Fie $f : \mathbb{R} \to \mathbb{R}$, f - impară : $\forall x \in \mathbb{R}, f(-x) = -f(x)$ (sim față de origine)

f - pară : $\forall x \in \mathbb{R}, f(-x) = f(x)$ (graf. simetric față de Oy)

1) $\mathbb{R}_i^{\mathbb{R}} \overset{?}{\leq}_{\mathbb{R}} \mathbb{R}^{\mathbb{R}}$

$\mathbb{R}_i^{\mathbb{R}} \neq \emptyset$ $\left(\theta : \mathbb{R} \to \mathbb{R},\ \theta(x) = 0,\ \theta \in \mathbb{R}_i^{\mathbb{R}} \right)$.

Fie $f, g \in \mathbb{R}_i^{\mathbb{R}}$, $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ impară $(\in \mathbb{R}_i^{\mathbb{R}})$

$$\begin{aligned}
\forall x \in \mathbb{R},\ \underline{(\alpha f + \beta g)(-x)} &= (\alpha f)(-x) + (\beta g)(-x) \\
&= \alpha f(-x) + \beta g(-x) \\
&= -\alpha f(x) - \beta g(x) \\
&= -(\alpha f(x) + \beta g(x)) \\
&= -((\alpha f)(x) + (\beta g)(x)) \\
&= \underline{-(\alpha f + \beta g)(x)} \quad \Rightarrow \text{impară}.
\end{aligned}$$

$\mathbb{R}_p^{\mathbb{R}} \leq_{\mathbb{R}} \mathbb{R}^{\mathbb{R}} \to$ temă.

2) $\mathbb{R}^{\mathbb{R}} \overset{?}{=} \mathbb{R}_i^{\mathbb{R}} + \mathbb{R}_p^{\mathbb{R}}$

$\mathbb{R}_i^{\mathbb{R}} \cap \mathbb{R}_p^{\mathbb{R}} \overset{?}{=} \{\theta\}$

$\forall f : \mathbb{R} \to \mathbb{R}$, $\overset{?}{\exists}$ $g, h : \mathbb{R} \to \mathbb{R}$: g pară și h impară a.î. $g + h = f$ (1)

Fie $x \in \mathbb{R}$: (1) $\Rightarrow$

$$\begin{array}{l} f(x) = g(x) + h(x) \\ f(-x) = g(-x) + h(-x) = g(x) - h(x) \\ \hline \end{array} +$$

$$g(x) = \frac{1}{2}(f(x) + f(-x))$$

$$h(x) = \frac{1}{2}(f(x) - f(-x))$$

verif.: g pară ; h impară (temă)

$g(-x) = \frac{1}{2}(f(-x) + f(x)) = g(x)$

$h(-x) = \frac{1}{2}(f(-x) - f(x)) = -\frac{1}{2}(f(x) - f(-x)) = -h(x)$

Fie $f \in \mathbb{R}^{\mathbb{R}}$, f pară $\Leftrightarrow \forall x \in \mathbb{R}$, $f(-x) = f(x)$

și f impară $\Leftrightarrow \forall x \in \mathbb{R}$, $f(-x) = -f(x)$

$\forall x \in \mathbb{R}$, $f(x) = 0 \Leftrightarrow f = \theta$.

Transformări liniare

Fie K corp comutativ, V, V' K-sp. vect., $f : V \to V'$

f transf. liniară $\begin{cases} f(x+y) = f(x) + f(y), & \forall x, y \in V \\ f(\alpha x) = \alpha \cdot f(x), & \forall x \in V, \forall \alpha \in K. \end{cases}$

$\Leftrightarrow f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$, $\forall x, y \in V$, $\forall \alpha, \beta \in K$.

Lista 8

⑥ Fie V, V' K-spații vectoriale, $f : V \to V'$ o transf. liniară, $A \leq_K V$ și $A' \leq_K V'$. S.s.a.c:

a) $f(A) = \{ f(a) \in V' \mid a \in A \} \leq_K V'$;

b) $f^{-1}(A') = \{ x \in V \mid f(x) \in A' \} \leq_K V$.

Soluție: a) Clar $f(A) \subseteq V'$.

$A \leq_k V \Rightarrow 0_V \in A. \Rightarrow f(0_V) \in f(A) \Rightarrow f(A) \neq \emptyset$.

Obs. $f(0_V) = 0_{V'}$

Dem: $f(0_V) = f(0_V + 0_V) \xlongequal{f \text{ transf. lin.}} f(0_V) + f(0_V) \Rightarrow f(0_V) = 0_{V'}$.

Fie $a, b \in A$ $(f(a), f(b) \in f(A))$, $\alpha, \beta \in k$.

Vrem $\alpha f(a) + \beta f(b) \in f(A)$.

$$\left.\begin{array}{l} \left.\begin{array}{l} a, b \in A \\ \alpha, \beta \in k \\ A \leq_k V \end{array}\right| \Rightarrow \alpha a + \beta b \in A. \\ f(\alpha a + \beta b) \xlongequal{f \text{ transf. lin.}} \alpha f(a) + \beta f(b) \end{array}\right| \Rightarrow \alpha f(a) + \beta f(b) \in f(A).$$

Așadar $A \leq_k V$.

b) Clar $f^{-1}(A') \subset V$.

$$\left.\begin{array}{l} A' \leq_k V' \Rightarrow 0_{V'} \in A' \\ f(0_V) = 0_{V'} \end{array}\right| \Rightarrow 0_V \in f^{-1}(A') \Rightarrow f^{-1}(A') \neq \emptyset.$$

Fie $x, y \in f^{-1}(A')$, $\alpha, \beta \in k$. Vrem $\alpha x + \beta y \in f^{-1}(A')$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$f(x), f(y) \in A'$ $\qquad\qquad$ $f(\alpha x + \beta y) \in A'$ ~~(?)~~

$$\Uparrow$$

$$f(\alpha x + \beta y) \xlongequal{f \text{ transf. lin}} \alpha \cdot \underbrace{f(x)}_{\in A'} + \beta \cdot \underbrace{f(y)}_{\in A'} \in A'$$

$\uparrow$ pt. că $A' \leq_k V'$.

Așadar $f^{-1}(A') \leq_k V$. $\square$.