Tema 10

Exercițiul 1

a) $f:(-1,1)\to\mathbb{R}$, $f(x)=\dfrac{1}{\sqrt{1-x^2}}$

I. $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin\frac{x}{1} + \mathcal{C}$

$\displaystyle= \arcsin x + \mathcal{C}$

fie $F(x)=\arcsin x$

II. $\displaystyle\lim_{\substack{t\to b\\ t<b}} F(t) = \lim_{\substack{t\to 1\\ t<1}} \arcsin t = \frac{\pi}{2}\in\overline{\mathbb{R}}$

$\Rightarrow \exists \displaystyle\int_a^{b-} f(x)$ e convergentă

$\displaystyle\int_a^{b-} f(x) = \lim_{\substack{t\to b\\ t<b}} F(t) - F(a)$

$\displaystyle= \lim_{\substack{t\to 1\\ t<1}} \arcsin t - \arcsin -1$

$\displaystyle= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

b) $f:[1,\infty)\to\mathbb{R}$, $f(x)=\dfrac{1}{x(x+1)}$

tipul I

$$\frac{\quad}{n(n+1)}$$

$$\sum \frac{|\sin n|}{\quad}$$

$$\int \frac{1}{x(x+1)}\,dx = \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= \ln|x| - \ln|x+1| + C$$

$$x \in [1,\infty) \quad \Rightarrow$$

$$\Rightarrow \int \frac{1}{x(x+1)}\,dx = \ln x - \ln(x+1) + C$$

fie $F(x) = \ln x - \ln(x+1)$ o primitivă a lui f

$$\lim_{\substack{t \to b \\ t < b}} F(x) = \lim_{t\to\infty} \ln x - \ln(x+1)$$

$$= \lim_{t\to\infty} \ln \frac{x}{x+1}$$

$$= \lim_{t\to\infty} \ln \frac{x}{x\left(1+\frac{1}{x}\right)} = 1 \in \overline{\mathbb{R}} \Rightarrow$$

$\Rightarrow \exists$ lim și e convergentă

$$\int_1^{\infty} f(x) = \lim_{t\to\infty} F(t) - F(1)$$

$$= \lim_{t\to\infty} \ln\frac{t}{t+1} - \ln\frac{1}{2}$$

$$= \cancel{\lim_{t\to\infty}} \ln 1 - \ln\frac{1}{2}$$

$$= \ln 1 - \ln 1 + \ln 2$$

$$= \ln 2$$

$\varepsilon\ u_{m+p}| < \varepsilon$

Pas 1. AC

c) $f:(0,1] \to \mathbb{R}$ (tipul 2) $f(x) = \ln x$

$$\int \ln x\, dx = \int (x')\ln x\, dx =$$
$$= x\ln x - \int \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx$$
$$= x\ln x - x + \mathcal{C}$$

Fie $F(x) = x\ln x - x$ o primitivă

$$\lim_{\substack{t\to a\\ t>a}} F(x) = \lim_{t\searrow 0} t\ln t - t$$
$$= \lim_{t\searrow 0} \frac{\ln t}{\frac{1}{t}} \overset{\frac{\infty}{\infty}}{=} \lim_{t\searrow 0}\frac{\frac{1}{t}}{-\frac{1}{t^2}} =$$
$$= \lim_{t\searrow 0} \frac{1}{\cancel{t}}\cdot(-t^{\cancel{2}}) = 0 \in \overline{\mathbb{R}} \Rightarrow$$

$\Rightarrow$ ii e convergentă

$$\int_{a+}^{b} f(x)\,dx = F(b) - \lim_{\substack{t\to a\\ t>a}} F(t)$$

$$\int_{0+}^{1} f(x)\,dx = 1\cdot\ln 1 - 1 - 0$$
$$= -1$$

$\forall x \in (-1,1)$

$\cdot \varepsilon\, u_{n+p}| < \varepsilon$

Pas 1. AC

c) $f:(0,1] \to \mathbb{R}$ $\quad f(x) = \ln x$ (tipul 2)

$\int \ln x \, dx = \int (x') \ln x \, dx =$

$= x \ln x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} dx$

$= x \ln x - x + \mathcal{C}$

Fie $F(x) = x \ln x - x$ o primitivă

$\lim\limits_{\substack{t \to a \\ t > a}} F(x) = \lim\limits_{t \searrow 0} t \ln t - t$

$= \lim\limits_{t \searrow 0} \frac{\ln t}{\frac{1}{t}} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{t \searrow 0} \frac{\frac{1}{t}}{-\frac{1}{t^2}} =$

$= \lim\limits_{t \searrow 0} \frac{1}{\cancel{t}} \cdot (-t^{\cancel{2}}) = 0 \in \overline{\mathbb{R}} \Rightarrow$

$\Rightarrow$ ii e convergentă

$\int_{a+}^{b} f(x)\, dx = F(b) - \lim\limits_{\substack{t \to a \\ t > a}} F(t)$

$\int_{0+}^{1} f(x)\, dx = 1 \cdot \ln 1 - 1 - 0$

$= -1$

$= , \forall x \in (-1, 1)$

$u_n = \frac{\sin n}{n(n+1)}$

1. AC ... $\leq \frac{|\sin n|}{\ldots}$

d) $f : [0,1) \to \mathbb{R}$ $\qquad f(x) = \frac{\arcsin x}{\sqrt{1-x^2}}$

$$\int \frac{\arcsin x}{\sqrt{1-x^2}}\,dx =$$

$$I = \int \arcsin x \cdot (\arcsin x)'\,dx$$

$$I = (\arcsin x)^2 - \int (\arcsin x)' \cdot \arcsin x\,dx$$

$$I = (\arcsin x)^2 - I$$

$$2I = (\arcsin x)^2$$

$$I = \frac{1}{2}\arcsin^2 x + \mathcal{C}$$

Fie $F(x) = \frac{1}{2}\arcsin^2 x + 0$

$$\lim_{\substack{t \to b \\ t < b}} \frac{1}{2}\arcsin^2 t = \lim_{\substack{t \to 1 \\ t < 1}} \frac{1}{2}\arcsin^2 t$$

$$= 1 \cdot \frac{1}{2} = \frac{1}{2} \in \overline{\mathbb{R}} \Rightarrow$$

$$\Rightarrow \int_0^{1-} f(x)\,dx = \cancel{F(t)} \lim_{t \uparrow b} F(t) - F(\overset{0}{a})$$

$$= \lim_{t \uparrow 1} \frac{1}{2}\arcsin^2 t - 0$$

$$= \frac{1}{2} - 0 = \frac{1}{2}$$

e) $f:(0,1]\to\mathbb{R}$ $\qquad f(x)=\dfrac{\ln x}{\sqrt{x}}$

$$\int \frac{\ln x}{\sqrt{x}}\,dx = \ln x\cdot 2\sqrt{x} - \int \frac{2}{\sqrt{x}}\,dx$$

$f=\ln x \qquad\qquad f'(x)=\dfrac{1}{x}$

$g'=\dfrac{1}{\sqrt{x}} \qquad\qquad g(x)=2\sqrt{x}$

$$= 2\sqrt{x}\ln x - 2\int\frac{dx}{\sqrt{x}}$$

$$= 2\sqrt{x}\ln x - 2\cdot\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

$$= 2\sqrt{x}\ln x - 2\cdot 2\cdot\sqrt{x}$$

$$= 2\sqrt{x}(\ln x - 2) + \mathcal{C}$$

Fie $F(x)=2\sqrt{x}(\ln x-2)$ o primitivă

$$\lim_{\substack{t\to 0\\ t>0}} 2\sqrt{t}(\ln t-2) = 2\lim_{t\searrow 0}\frac{\ln t-2}{\frac{1}{\sqrt{t}}} =$$

$$= 2\lim_{t\searrow 0}\frac{\frac{1}{t}}{-\frac{\frac{1}{2\sqrt{t}}}{t}} = 2\lim_{t\searrow 0}\frac{\frac{1}{t}}{-\frac{1}{2t\sqrt{t}}}$$

$$= 2\lim_{t\searrow 0}\frac{-2\cancel{t}\sqrt{t}}{\cancel{t}} = 2\cdot 0 = 0\in\mathbb{R} \Rightarrow$$

$\Rightarrow$ i conv.

$$m = \frac{\sin n}{n(n+1)}$$

AC Studeiem $\sum |u_n| = \sum \frac{|\sin n|}{n(n+1)}$

$$\int_{0+}^{1} f(x)\,dx = 2\sqrt{1}(\ln 1 - 2) - 0$$

$$= 2\cdot(-2) = -4$$

f) $f : [e, \infty) \to \mathbb{R}$ $\quad f(x) = \frac{1}{x\cdot(\ln x)^3}$

$$\int \frac{1}{x(\ln x)^3}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx \Rightarrow J = \int \frac{du}{u^3} = \frac{u^{-3+1}}{-3+1} + C$

$$= -\frac{1}{2u^2} = -\frac{1}{2(\ln x)^2} + C$$

Fie $F(x) = -\frac{1}{2(\ln x)^2}$

$$\lim_{t\to\infty} -\frac{1}{2(\ln x)^2} = -\frac{1}{\infty} = 0 \in \overline{\mathbb{R}} \Rightarrow$$

$\Rightarrow$ îi C.

$$\Rightarrow \int_{e}^{\infty} f(x)\,dx = 0 - F(e)$$

$$= 0 - \frac{1}{2(\ln e)^2} = -\frac{1}{2}$$

g) $f:\left(\frac{1+\sqrt{3}}{2}, 2\right] \to \mathbb{R}$ $\quad f(x) = \frac{1}{\cancel{x}\sqrt{2x^2-2x-1}}$

Substituția lui Euler:

$$\left.\begin{array}{l}\sqrt{2x^2-2x-1} \\ 2>0\end{array}\right\} \Leftrightarrow$$

$$\Rightarrow \sqrt{2x^2-2x-1} = \sqrt{2}\cdot x + t \quad /()^2$$

$$2x^2-2x-1 = (x\sqrt{2}+t)^2$$

$$\cancel{2x^2} - 2x - 1 = \cancel{x^2\cdot 2} + 2x\sqrt{2}t + t^2$$

$$-2x - 2x\sqrt{2}\cdot t = t^2+1$$

$$-2x(1+t\sqrt{2}) = t^2+1$$

$$x = \frac{t^2+1}{-2(1+t\sqrt{2})}$$

$$dx = \left(\frac{t^2+1}{-2(1+t\sqrt{2})}\right)' dt$$

$$dx = \frac{-2t\cdot 2(1+t\sqrt{2}) + 2\sqrt{2}(t^2+1)}{4(1+t\sqrt{2})^2}$$

$$= \cancel{\frac{-4-4t\sqrt{2}+2t^2\sqrt{2}+2\sqrt{2}}{4(1+t\sqrt{2})^2}}$$

$$= \cancel{\frac{2(\sqrt{2}+t^2\sqrt{2}-2t\sqrt{2})}{\cancel{4}_2(1+t\sqrt{2})^2}}$$

$$m = \frac{\sin n}{n(n+1)}$$

AC Studeire $\sum |u_n| = \sum \frac{|\sin n|}{n(n+1)}$

$$= -\frac{2t + t^2\sqrt{2} - \sqrt{2}}{2(1+t\sqrt{2})^2} = -\frac{\sqrt{2}t + t^2 - 1}{\sqrt{2}(1+\sqrt{2}t)^2}$$

$$\sqrt{2x^2-2x-1} = \sqrt{2} \cdot \frac{t^2+1}{-2(1+t\sqrt{2})} + t$$

$$= \cancel{-\sqrt{2}} \frac{\sqrt{2}t + t^2 - 1}{\cancel{\sqrt{2}}(1+\sqrt{2}t)} + t$$

$$= \frac{-\sqrt{2}t - t^2 + 1 + t(1+\sqrt{2}t)}{(1+\sqrt{2}t)}$$

$$\int \frac{-\sqrt{2}t - t^2 + 1}{\sqrt{2}\cancel{(1+\sqrt{2}t)}^2} \cdot \frac{-2\cancel{(1+t\sqrt{2})}}{t^2+1} \cdot \frac{\cancel{(1+\sqrt{2}t)}}{-\sqrt{2}t - t^2 + 1 + t(1+\sqrt{2}t)}$$

$$= \int \frac{(-\sqrt{2}t - t^2 + 1)(-2)}{\sqrt{2}(t^2+1)(-\sqrt{2}t - t^2 + 1 + t + \sqrt{2}t^2)}$$

4) $f:[0,\infty)\to\mathbb{R}$ $\quad f(x)=\frac{\pi}{2}-\operatorname{arctg}x$

$$\int\left(\frac{\pi}{2}-\operatorname{arctg}x\right)dx =$$

$$= \int\frac{\pi}{2}\,dx - \int \operatorname{arctg}x\,dx \;\boxed{=}$$

$$\int \operatorname{arctg}x\,dx = \int (x)' \operatorname{arctg} x\,dx =$$

$$= x\operatorname{arctg}x - \int\frac{x}{x^2+1}\,dx \;=$$

$$\int \frac{x}{x^2+1}\,dx = \int \frac{1}{2}\cdot\frac{du}{u} = \frac{1}{2}\ln|u|$$

$u=x^2+1$

$du=2x\,dx$

$$= \frac{1}{2}\ln|x^2+1|$$

$$= x\operatorname{arctg}x - \frac{1}{2}\ln(x^2+1)+\mathcal{C}$$

$$\boxed{=}\;\frac{\pi}{2}\cdot x - x\operatorname{arctg}x+\frac{1}{2}\ln(x^2+1)+\mathcal{C}$$

Fie $F(x)=x\frac{\pi}{2}-x\operatorname{arctg}x+\frac{1}{2}\ln(x^2+1)$

$$\lim_{t\to\infty} \overset{\infty\cdot\infty}{t\frac{\pi}{2}} - \overset{\infty}{t}\,\overset{\frac{\pi}{2}}{\operatorname{arctg}t}+\frac{1}{2}\overset{\infty}{\ln}(t^2+1)$$

$$=\lim_{t\to\infty} t\left(\frac{\pi}{2}-\operatorname{arctg}t\right)+\frac{1}{2}\ln(t^2+1)$$

$$=\lim_{t\to\infty} \frac{\frac{\pi}{2}-\operatorname{arctg}t}{\frac{1}{t}}+\frac{1}{2}\ln(t^2+1) \;\; \frac{\frac{0}{0}}{L'H}$$

$n(n+1)$

AC ... $= \sum \frac{|\sin n|}{n(n+1)}$

$$= \lim_{t\to\infty} \frac{-\frac{1}{1+t^2}}{-\frac{1}{t^2}} + \frac{1}{2}\ln(t^2+1)$$

$$= \lim_{t\to\infty} \overset{\nearrow 1}{\frac{t^2}{1+t^2}} + \frac{1}{2}\overset{\nearrow \infty}{\ln(t^2+1)}$$

$= \infty \Rightarrow$ ii e divergentă

i) $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x)=\frac{1}{1+x^2}$

$$\int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x + \mathcal{C}$$

Fie $F(x) = \operatorname{arctg} x$ o primitivă

tipul 3

$$\left.\begin{aligned} &\lim_{t\to\infty} \operatorname{arctg} t = \frac{\pi}{2} \\ &\lim_{t\to-\infty} \operatorname{arctg} t = -\frac{\pi}{2} \end{aligned}\right\} \Rightarrow \text{ii e conv.}$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

j) $f:\left(\frac{1}{3}, 3\right] \to \mathbb{R}$ $\quad f(x) = \frac{1}{\sqrt[3]{3x-1}}$

$\int \frac{1}{\sqrt[3]{3x-1}}\,dx$ $\quad u = 3x-1$

$du = 3\,dx$

$\ldots + u_{n+p}| < \varepsilon$

Pas I. AC

$$= \int \frac{1}{3}\frac{du}{\sqrt[3]{u}} = \frac{1}{3}\frac{u^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}$$

$$= \frac{1}{3}\cdot\frac{3}{2}u^{\frac{2}{3}} = \frac{1}{2}u^{\frac{2}{3}} + \mathcal{C}$$

$$= \frac{1}{2}\sqrt[3]{(3x-1)^2} + \mathcal{C}$$

Fie $F(x) = \frac{1}{2}\sqrt[3]{(3x-1)^2}$

$$\lim_{\substack{x\to\frac{1}{3}\\ x>\frac{1}{3}}} \frac{1}{2}\sqrt[3]{(3x-1)^2} = \frac{1}{2}\sqrt[3]{\left(3\cdot\frac{1}{3}-1\right)^2}$$

$$=0$$

$$\int_{\frac{1}{3}}^{3} = F(3) - 0$$

$$= \frac{1}{2}\sqrt[3]{(3\cdot 3-1)^2} = \frac{1}{3}\sqrt[3]{8^2}$$

$$= \frac{1}{3}\sqrt[3]{64}$$

$$= \frac{1}{3}2^2 = \frac{4}{3}$$

k) $f:[1,\infty)\to\mathbb{R}$ $\quad f(x) = \frac{x}{(1+x^2)^2}$

$$\int \frac{x}{(1+x^2)^2}dx \qquad u = 1+x^2$$

$$du = 2x\,dx$$

$\ldots, \forall x \in (-1, 1)$

$$= \int \frac{1}{2} \frac{du}{u^2} = \frac{1}{2} \cdot \frac{u^{-2+1}}{-1}$$

$$= -\frac{1}{2} \frac{1}{u}$$

$$= -\frac{1}{2(1+x^2)} + C$$

$$F(x) = -\frac{1}{2(1+x^2)}$$

$$\lim_{t \to \infty} -\frac{1}{2(1+t^2)} = \frac{-1}{\infty} = 0 \in \overline{\mathbb{R}}$$

$$\Rightarrow \int_1^\infty f(x)\,dx = 0 - F(1)$$

$$= 0 + \frac{1}{2(1+1)} = \frac{1}{4}$$

Exercițiul 2

a) $f : [1, \infty) \to \mathbb{R}$ $\quad f(x) = \frac{1}{\Delta\sqrt{1+x^2}}$

$x \in [1, \infty) \Rightarrow$ ple $= \infty$

$g : [1, \infty) \to \mathbb{R}$ $\quad g(x) = \frac{1}{x^p}$

$$L = \lim \frac{f(x)}{g(x)} = \lim_{x \to \infty} x^p \cdot \frac{1}{\Delta\sqrt{1+x^2}} \Bigg\} \Rightarrow$$

~~Aleg p = 1~~

$$\Rightarrow \cancel{L = \frac{1}{\Delta} \in (0, \infty)} \quad \cancel{\Delta \in (0, \infty)}$$

$\cdot\, \varepsilon\, u_{m+p}| < \varepsilon$

Pas 1. A C

$\cancel{\text{Aleg } p = 2 \Rightarrow \lim_{x \to \infty} \frac{x^2}{s \cdot \sqrt{x^4\left(\frac{1}{x^4} + \frac{1}{x^2}\right)}}}$

$\cancel{= \lim_{x \to \infty} \frac{x^2}{s \cdot x^2 \sqrt{\frac{1}{x^4} + \frac{1}{x^2}}} = \frac{1}{0} = \infty}$

Aleg $p = 1 \Rightarrow \lim_{x \to \infty} \frac{x}{s\sqrt{x^2\left(\frac{1}{x^2} + 1\right)}} = \frac{1}{s} \in (0, \infty)$

$$\left.\begin{array}{l} L = \frac{1}{s} \in (0, \infty) \\ p = 1 \\ f(x) \geq 0 \end{array}\right\} \Rightarrow \int_1^\infty f(x)\,dx \text{ e } D.$$

b) $f : [0, \frac{\pi}{2}) \to \mathbb{R}$ $\quad f(x) = \frac{1}{\cos x}$

$pb = \frac{\pi}{2}$ $\quad g(x) = \frac{1}{\left(\frac{\pi}{2} - x\right)^p}$

$L = \lim_{x \to \frac{\pi}{2}} \frac{f(x)}{g(x)} = \lim_{x \to \frac{\pi}{2}} \frac{1}{\cos x} \left(\frac{\pi}{2} - x\right)^p \overset{L'H}{=}$

$\cancel{= \lim_{x \to \frac{\pi}{2}} \frac{p\left(\frac{\pi}{2} - x\right)^{p-1}}{-\sin x}}$ $\quad$ Alegem $p = 1$

$\lim_{x \to \frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)}{\cos x} \overset{L'H}{=} \lim_{x \to \frac{\pi}{2}} \frac{-1}{-\sin x} = 1$

$, \forall x \in (-1, 1)$

$$\left.\begin{array}{l} f(x) \geq 0 \\ L \in (0, \infty) \\ p = 1 \end{array}\right\} \Rightarrow \int_0^{\frac{\pi}{2}} f(x)\,dx \ D.$$

c) $f : (0, \infty) \to \mathbb{R} \qquad f(x) = \left(\frac{\operatorname{arctg} x}{x}\right)^2$

Alegem $c = 1$ punct intermediar

$\boxed{I \quad x \in (0, 1]}$ $\qquad g(x) = \frac{1}{(x-0)^p}$

$$\cancel{L = \lim_{x \to 1} \left(\frac{\operatorname{arctg} x}{x}\right)^2 \cdot x^p \qquad p = 1}$$

$$\cancel{= \left(\frac{\pi}{4}\right)^2 = \frac{\pi^2}{16}}$$

$p_{le} = 0$

$$L = \lim_{x \searrow 0} \left(\frac{\operatorname{arctg} x}{x}\right)^2 \cdot x^p \qquad \begin{array}{l} p = 1 \\ \cancel{p = 0} \Rightarrow \end{array}$$

$$\Rightarrow \lim_{x \searrow 0} x^p \cdot x^{-2} \cdot \operatorname{arctg}^2 x =$$

$$\cancel{\lim_{x \searrow 0} \frac{\operatorname{arctg}^2 x}{x^2} =}$$

$$= \lim_{x \searrow 0} x^{-1} \operatorname{arctg}^2 x = \lim_{x \searrow 0} \frac{2 \operatorname{arctg} x \cdot \frac{1}{1+x^2}}{1} = 0$$

$\cancel{d)\ f:(1,\infty)\to\mathbb{R} \qquad f(x)=\left(\frac{\ln x}{x\sqrt{x^2-1}}\right)^2}$

II $x\in[1,\infty)$

$p_c=\infty \qquad g:[1,\infty)\to\mathbb{R} \qquad g(x)=\frac{1}{x^p}$

$$L=\lim_{x\to\infty}\left(\frac{\operatorname{arctg} x}{x}\right)^2\cdot x^p$$

$$=\lim_{x\to\infty}\frac{\operatorname{arctg}^2 x}{\cancel{x^2}}\cdot\cancel{x^2} \qquad p=2$$

$$=\left(\frac{\pi}{2}\right)^2=\frac{\pi^2}{4}$$

$$\left.\begin{array}{l} f(x)\geq 0 \\ L\in(0,\infty) \\ p=2\geq 1 \end{array}\right\} \Rightarrow \int_1^{\infty} f(x)\,dx \quad C$$

d) $f:(1,\infty)\to\mathbb{R} \qquad f(x)=\left(\frac{\ln x}{x\sqrt{x^2-1}}\right)^2$

Alegem $c=e$ punct intermediar

I $x\in(1,e]$

$p_c=1 \qquad g:(1,e]\to\mathbb{R} \qquad g(x)=\frac{1}{(x-1)^p}$

$\lim\limits_{x \to \infty} 2^{\frac{1}{x}} = 0 \Rightarrow$ nu avem concluzie