Temă - șiruri numere reale - 2

ex 1

$x_n = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right), n\geq 2$

$\frac{x_{n+1}}{x_n} = \frac{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(n+1)^2}\right)}{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)}$

$= 1 - \frac{1}{(n+1)^2} < 1 \Rightarrow$ șirul e $\searrow$ (1)

$\frac{1}{(n+1)^2} < 1$, $n \in \cancel{N}$ $n\geq 2$

(x_n) descresc $\Rightarrow$ $M = x_2 = \left(1-\frac{1}{2^2}\right)$

(2) $= 1 - \frac{1}{4} = \frac{3}{4}$

$m = 0.$

(1), (2) $\Rightarrow$ (x_n) convergent

$\lim_{n\to\infty}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)$

$= \lim_{n\to\infty}\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)$

$$\lim_{n\to\infty} \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{n}{n-1}\cdot\frac{n-1}{n}\left(1+\frac{1}{n}\right)$$

$$\lim_{n\to\infty} \frac{1}{2}\cdot\left(1+\frac{1}{n}\right) = \frac{1}{2}\left(1+0\right)$$
$$= \frac{1}{2}$$

ex 2

a) $\lim\limits_{n\to\infty} \frac{3^n}{4^n} = \lim\limits_{n\to\infty} \left(\frac{3}{4}\right)^n = 0$

$\frac{3}{4} < 1$

b) $\lim\limits_{n\to\infty} \frac{2^n+(-2)^n}{3^n} = \lim\limits_{n\to\infty} \frac{\cancel{3^n}\left(\left(\frac{2}{3}\right)^n + \frac{(-2)^n}{3^n}\right)}{\cancel{3^n}}$

$= \lim\limits_{n\to\infty} \left(\frac{2}{3}\right)^n + \left(\frac{-2}{3}\right)^n = 0$

c) $\lim\limits_{n\to\infty} \frac{5-n^3}{n^2+1} = \lim\limits_{n\to\infty} \cancel{\frac{n^3\left(\overset{\to 0}{\frac{5}{n^3}}-1\right)}{n^3\left(\underset{\to 0}{\frac{1}{n}}+\underset{\to 0}{\frac{1}{n^3}}\right)}}$

$\cancel{= \frac{-1}{0} = -\infty}$

$= \lim \frac{n^2\left(\overset{\to 0}{\frac{5}{n^2}} - n\right)}{n^2\left(1+\frac{1}{n^2}\right)} = \frac{-\infty}{1} = -\infty$

d) $\lim_{n\to\infty}\left(2+\frac{4^n+(-5)^n}{7^n+1}\right)^{2n^3-n^2}$

$\lim_{n\to\infty}\left(2+\frac{7^n\left(\left(\frac{4}{7}\right)^n+\left(\frac{-5}{7}\right)^n\right)}{7^n\left(1+\frac{1}{7^n}\right)}\right)^{2n^3-n^2}$

$=\lim_{n\to\infty} 2^{2n^3-n^2} = 2^\infty = \infty$

e) $\lim_{n\to\infty}\frac{1+2+\dots+n}{n^2}=\lim_{n\to\infty}\frac{\frac{n(n+1)}{2}}{n^2}$

$=\lim_{n\to\infty}\frac{n+1}{2n}=\lim_{n\to\infty}\frac{n(1+\frac{1}{n})}{2n}$

$=\frac{1}{2}$

f) $\lim_{n\to\infty}\left(\frac{n^3+4n+1}{2n^3+5}\right)^{\frac{-2n^4+1}{n^4+3n+1}}$

$=\lim_{n\to\infty}\left(\frac{n^3\left(1+\frac{4}{n^2}+\frac{1}{n^3}\right)}{n^3\left(2+\frac{5}{n^3}\right)}\right)^{\frac{-2n^4+1}{n^4+3n+1}}$

$=\lim_{n\to\infty}\left(\frac{1}{2}\right)^{\frac{n^4\left(-2+\frac{1}{n^4}\right)}{n^4\left(1+\frac{3}{n^3}+\frac{1}{n^4}\right)}}$

$=\left(\frac{1}{2}\right)^{-2}=2^2=4$

g) $\lim (\cos(-2013))^n = \lim_{n\to\infty} (\cos(2013))^n = 0$

$\nexists k \in \mathbb{Z}$ aî $2013 = k\pi \Rightarrow |\cos 2013| < 1$

h) $\lim_{n\to\infty} \left(\frac{n^5 + 3n + 1}{2n^5 - n^4 + 3}\right)^{\frac{3n - n^4}{n^3+1}}$

$= \lim_{n\to\infty} \left(\frac{\cancel{n^5}\left(1 + \frac{3}{n^4} + \frac{1}{n^5}\right)}{\cancel{n^5}\left(2 - \frac{1}{n} + \frac{3}{n^5}\right)}\right)^{\frac{3n-n^4}{n^3+1}}$

$= \lim_{n\to\infty} \left(\frac{1}{2}\right)^{\frac{\cancel{n^3}\left(\frac{3}{n^2} - n\right)}{\cancel{n^3}\left(1 + \frac{1}{n^3}\right)}}$

$= \frac{1}{2}^{(-\infty)} = 2^{\infty} = \infty$

ex 3

a) $\lim_{n\to\infty} \left(1 + \frac{1}{-n^3 + 3n}\right)^{n^2 - n^3} \overset{1^{\infty}}{=}$

$= e^{\lim_{n\to\infty} \frac{n^2 - n^3}{-n^3 + 3n}}$

$= e^{\lim_{n\to\infty} \frac{\cancel{n^3}\left(\frac{1}{n} - 1\right)}{\cancel{n^3}\left(-1 + \frac{3}{n^2}\right)}}$

$= e$

b) $\lim\limits_{n\to\infty} (3n^2+5)\ln\left(1+\frac{1}{n^2}\right)$

$= \lim\limits_{n\to\infty} \ln\left(1+\frac{1}{n^2}\right)^{3n^2+5}$

$= \ln \lim\limits_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{3n^2+5}$

$= \ln e^{\lim\limits_{n\to\infty}\frac{3n^2+5}{n^2}}$

$= \ln e^{\lim\limits_{n\to\infty}\frac{3n^2+5}{n^2}}$

$= \lim\limits_{n\to\infty} \frac{3n^2+5}{n^2}$

$= \lim\limits_{n\to\infty} \frac{\cancel{n^2}\left(3+\frac{5}{n^2}\right)}{\cancel{n^2}} = 3$

c) $\lim\limits_{n\to\infty} \frac{n^n}{1^1+2^2+\dots+n^n}$

Stolz - Cesaro

(x_n) (y_n) a.î. (y_n) s.î. și $\lim y_n=\infty$

$\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \lim\limits_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$

$= \lim\limits_{n\to\infty} \frac{(n+1)^{n+1}-n^n}{(n+1)^{n+1}}$

$= \lim\limits_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)^{n+1}} - \frac{n^n}{(n+1)^n}\cdot\frac{1}{n+1}$

$$= \lim_{n\to\infty} 1 - \left[\frac{n}{n+1}\right]^n \cdot \frac{1}{n+1}$$

$$= 1 - \frac{1}{e} \cdot \frac{1}{\infty} = 1 - 0 = 1$$

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e$$

d) $$\lim_{n\to\infty} \frac{x_1 + 2x_2 + \dots + nx_n}{n^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)x_{n+1}}{(n+1)^2 - n^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)x_{n+1}}{(\cancel{n}+1-\cancel{n})(n+1+n)}$$

$$= \lim_{n\to\infty} \frac{(n+1)\,x_{n+1}}{2n+1} = \frac{x}{2}$$

ex 4

a) $$x_n = \frac{a^n - a^{-n}}{a^n + a^{-n}}$$

$$x_n = \frac{a^n - \frac{1}{a^n}}{a^n + \frac{1}{a^n}} = \frac{a^{2n}-1}{a^n} \cdot \frac{a^n}{a^{2n}+1}$$

$$x_n = \frac{a^{2n}-1}{a^{2n}+1}$$

$\underline{I}$ $|a| < 1 \Rightarrow x_n \to -1$

$\underline{II}$ $|a| = 1 \Rightarrow x_n = 0$

$\underline{III}$ $|a| > 1 \Rightarrow a^{2n} \to \infty \Rightarrow x_n \to 1$

(factor comun a^{2n}...)

b) $y_n = \dfrac{a^n + b^n}{a^{n+1} + b^{n+1}}$ $\Rightarrow a \neq -b$
$a + b \neq 0$

$\underline{C_I}$ $a = b \neq 0 \Rightarrow y_n = \dfrac{2a^n}{2a^{n+1}} = \dfrac{1}{a}$

$\underline{C_{II}}$ $a \neq b$. Fie $a = \max\{a, b\} \Rightarrow a > b$

subcazul $\underline{I}$ $a = 0 \Rightarrow y_n = \dfrac{1}{b}$

subcazul $\underline{II}$ $a \neq 0 \Rightarrow$

$$y_n = \frac{1 + \left(\frac{b}{a}\right)^n}{a + \left(\frac{b}{a}\right)^n \cdot b} \to \frac{1}{a} \quad \left(\left|\frac{b}{a}\right| < 1\right)$$

$$y_n = \frac{\cancel{b^n}\left(\overset{\to 0}{\left(\frac{a}{b}\right)^n} + 1\right)}{\cancel{b^n}\left(\underset{\to 0}{\left(\frac{a^n}{b^n}\right)} \cdot a + b\right)} \to \frac{1}{b}$$

c) $z_n = \dfrac{1+a+\dots+a^n}{1+b+\dots+b^n}$, $a, b > 0$

I $a = b > 0$

$z_n \to 1$

II $a \neq b$

$a, b > 0 \Rightarrow$ (x_n): $1 + a + \dots + a^n \to \infty$

(y_n): $1 + b + \dots + b^n \to \infty$

$$z_n = \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$$

$$= \frac{a^{n+1}}{b^{n+1}} = \left(\frac{a}{b}\right)^{n+1} = \begin{cases} 0, & a < b \\ \infty, & a > b \end{cases}$$