$n \to \infty$

$$\sum_{n\geq 1} \frac{1}{4n^2-1}$$

$$\frac{1}{(2n-1)(2n+1)} = \frac{2n+1-(2n-1)}{2(2n-1)(2n+1)}$$

$$= \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2}\left(b_n - b_{n+1}\right)$$

Temă - serii de numere reale

Exercițiul 1

a) $$\sum_{n\geq 3} \frac{3}{5^n} = 3 \sum_{n\geq 3} \frac{1}{5^n}$$

$$= 3 \cdot \frac{1}{5^3} \sum_{n\geq 0} \frac{1}{5^n}$$

$$= \frac{3}{5^3} \cdot \frac{1}{1-\frac{1}{5}}$$

$$= \frac{3}{5^3} \cdot \frac{1}{\frac{4}{5}} = \frac{3}{5^{\cancel{3}2}} \cdot \frac{\cancel{5}}{4} = \frac{3}{100}$$

b) $$\sum_{n\geq 4} \frac{2^{n-3} + (-3)^{n+3}}{5^n}$$

$$= \sum_{n\geq 4} \frac{5^n\left(\frac{2^{n-3}}{5^n} + \frac{(-3)^{n+3}}{5^n}\right)}{5^n}$$

$$= \sum_{n\geq 4} \frac{2^n}{2^3 \cdot 5^n} + \frac{(-3)^n \cdot 3^3}{5^n}$$

$$= \frac{1}{2^3}\sum_{n\geq 4}\left(\frac{2}{5}\right)^n + 3^3 \sum_{n\geq 4} \frac{(-3)^n}{5^n}$$

$$= \cancel{\frac{1}{2^3}\left(\sum \left(\frac{2}{5}\right)^4 - \sum_{n\geq} \right.}$$

$$= \frac{1}{2^3} \cdot \left(\frac{2}{5}\right)^4 \sum_{n\geq 0}\left(\frac{2}{5}\right)^n + 3^3 \cdot \frac{(-3)^4}{5^4} \sum_{n\geq 0} \frac{(-3)^n}{5^n}$$

$$= \frac{1}{2^3} \cdot \frac{2^4}{5^4} \cdot \frac{1}{1 - \frac{2}{5}} + \frac{3^3 \cdot (-3)^4}{5^4} \cdot \frac{1}{1 + \frac{3}{5}}$$

$$= \frac{2}{5^3} \cdot \frac{5}{3} + \frac{5}{8} \cdot \frac{27 \cdot 81}{5^4}$$

$$= \frac{2}{3 \cdot 5^3} + \frac{(-3)^7}{5^3 \cdot 8}$$

c) $$\sum_{n\geq 5} e^n = e^5 \sum_{n\geq 0} e^n = \infty$$

$$e > 1 \Rightarrow \sum_{n\geq 0} e^n = \infty$$

d) $\sum_{n\geq 2} \left(-\frac{1}{\pi}\right)^n =$

$= \left(-\frac{1}{\pi}\right)^2 \sum_{n\geq 0} \left(-\frac{1}{\pi}\right)^n$

$= \frac{1}{\pi^2} \cdot \frac{1}{1+\frac{1}{\pi}}$

$= \frac{1}{\pi^2} \cdot \frac{\pi}{1+\pi} = \frac{1}{\pi(1+\pi)}$

e) $\sum_{n\geq 3} (-3)^n = (-3)^3 \sum_{n\geq 0} (-3)^n$

$\lim_{n\to\infty} a^n = \nexists$ când $a \leq -1$

Exercițiul 2.

a) $\sum_{n\geq 1} \frac{1}{4n^2-1} = \frac{1}{2}$

fie $a_n = \frac{1}{4n^2-1} = \frac{1}{(2n+1)(2n-1)}$

$= \frac{2n+1-(2n-1)}{2(2n+1)(2n-1)}$

$= \frac{1}{2}\left(\underbrace{\frac{1}{2n-1}}_{b_n} - \underbrace{\frac{1}{2n+1}}_{b_{n+1}}\right) \Leftrightarrow a_n = \frac{1}{2}(b_n - b_{n+1})$

$S_n = a_1 + a_2 + \dots + a_n$

$S_n = \frac{1}{2}\left(b_1 \cancel{-b_2} + \cancel{b_2} - \cancel{b_3} + \dots + b_n - b_{n+1}\right)$

$S_n = \frac{1}{2}\left(b_1 - b_{n+1}\right)$

$= \frac{1}{2}\left(1 - \underbrace{\frac{1}{2n+1}}_{\searrow 0}\right) \longrightarrow \frac{1}{2}$

b) $\sum_{n \geq 1} \frac{1}{\sqrt{n} + \sqrt{n+1}}$

$a_n = \frac{1}{\sqrt{n} + \sqrt{n+1}} = \frac{\sqrt{n+1} - \sqrt{n}}{n+1-n}$

$= \sqrt{n+1} - \sqrt{n}$

$S_n = \overbrace{\cancel{\sqrt{2}} - \sqrt{1}}^{a_1} + \overbrace{\cancel{\sqrt{3}} - \cancel{\sqrt{2}}}^{a_2} + \cancel{\sqrt{4}} - \cancel{\sqrt{3}} + \dots + \sqrt{n+1} - \cancel{\sqrt{n}}$

$= -1 + \sqrt{n+1}$

$= \sqrt{n+1} - 1$

$= \infty \Rightarrow \sum_{n \geq 1} \frac{1}{\sqrt{n} + \sqrt{n+1}} = \infty$

c) $\sum_{n \geq 5} \frac{1}{n(n+1)(n+2)}$

$= \sum_{n \geq 5} \frac{1}{n} \cdot \frac{n+2-(n+1)}{(n+1)(n+2)}$

$= \sum_{n \geq 5} \frac{1}{n}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

$$= \sum_{n \geq 5} \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}$$

$$S_n = a_5 + a_6 + \dots + a_n$$

$$= \frac{1}{5 \cdot 6} - \frac{1}{6 \cdot 7} + \frac{1}{6 \cdot 7} - \frac{1}{7 \cdot 8} + \dots + \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}$$

$$= \frac{1}{30} - \overset{\to 0}{\frac{1}{(n+1)(n+2)}} \to \frac{1}{30}$$

$$a_n = \frac{1}{n(n+1)(n+2)} = \frac{1}{2} \cdot \frac{n+2-n}{n(n+1)(n+2)}$$

$$= \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)$$

$$S_n = \frac{1}{2}\left(\frac{1}{5 \cdot 6} - \frac{1}{6 \cdot 7} + \frac{1}{6 \cdot 7} - \frac{1}{7 \cdot 8} + \dots + \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)$$

$$= \frac{1}{2} \cdot \left(\frac{1}{30} - \frac{1}{(n+1)(n+2)}\right)$$

$$\to \frac{1}{60}.$$

d) $\sum_{n\geq 1} \ln\left(1+\frac{1}{n}\right)$

$$a_n = \ln\left(1+\frac{1}{n}\right)$$

$$= \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln n.$$

$$S_n = a_1 + \dots + a_n = \cancel{\ln 2} - \ln 1 + \cancel{\ln 3} - \cancel{\ln 2} + \dots + \ln(n+1) - \cancel{\ln n}$$

$$= \ln n+1 \to \infty$$

e) $\sum_{n\geq 2} \frac{\ln\left(1+\frac{1}{n}\right)}{\ln\left(n^{\ln(n+1)}\right)}$

$$a_n = \frac{\ln(n+1) - \ln n}{\ln(n+1)\cdot \ln n} = \frac{1}{\ln n} - \frac{1}{\ln(n+1)}$$

$$S_n = a_2 + \dots + a_n$$

$$= \frac{1}{\ln 2} - \cancel{\frac{1}{\ln 3}} + \cancel{\frac{1}{\ln 3}} - \cancel{\frac{1}{\ln 4}} + \dots + \cancel{\frac{1}{\ln n}} - \frac{1}{\ln(n+1)}$$

$$= \frac{1}{\ln 2} - \overbrace{\frac{1}{\ln(n+1)}}^{\to 0}$$

$$= \frac{1}{\ln 2}$$