Seminar 10

Subspații

Fie K corp comutativ, V K-sp. vectorial, $A \subseteq V$

- $A \leq_K V \iff \begin{cases} A \neq \emptyset \\ \forall x,y \in A,\ x+y \in A \\ \forall \alpha \in K,\ \forall x \in A,\ \alpha x \in A \end{cases} \iff \begin{cases} A \neq \emptyset \\ \forall x,y \in A,\ \forall \alpha, \beta \in K, \\ \alpha x + \beta y \in A. \end{cases}$

Transformări liniare

Fie K corp comutativ, V, V' K-sp. vect., $f: V \to V'$

f transf. liniară $\begin{cases} f(x+y) = f(x) + f(y),\ \forall x,y \in V \\ f(\alpha x) = \alpha \cdot f(x),\ \forall x \in V, \forall \alpha \in K. \end{cases}$

$\iff f(\alpha x + \beta y) = \alpha f(x) + \beta f(y),\ \forall x,y \in V,\ \forall \alpha, \beta \in K.$

Lista 8

⑥ Fie V, V' K-spații vectoriale, $f: V \to V'$ o transf. liniară, $A \leq_K V$ și $A' \leq_K V'$. S.s.a.c:

a) $f(A) = \{ f(a) \in V' \mid a \in A \} \leq_K V'$;

b) $f^{-1}(A') = \{ x \in V \mid f(x) \in A' \} \leq_K V$.

Soluție: a) Clar $f(A) \subseteq V'$.

$A \leq_k V \Rightarrow 0_V \in A. \Rightarrow f(0_V) \in f(A) \Rightarrow f(A) \neq \emptyset$.

Obs. $f(0_V) = 0_{V'}$

Dem: $f(0_V) = f(0_V + 0_V) \xlongequal{f \text{ transf. lin.}} f(0_V) + f(0_V) \Rightarrow f(0_V) = 0_{V'}$.

Fie $a, b \in A$ $(f(a), f(b) \in f(A))$, $\alpha, \beta \in k$.

Vrem $\alpha f(a) + \beta f(b) \in f(A)$.

$$\left.\begin{array}{l} \left.\begin{array}{l} a, b \in A \\ \alpha, \beta \in k \\ A \leq_k V \end{array}\right| \Rightarrow \alpha a + \beta b \in A. \\ f(\alpha a + \beta b) \xlongequal{f \text{ transf. lin.}} \alpha f(a) + \beta f(b) \end{array}\right| \Rightarrow \alpha f(a) + \beta f(b) \in f(A).$$

Așadar $A \leq_k V$.

b) Clar $f^{-1}(A') \subset V$.

$$\left.\begin{array}{l} A' \leq_k V' \Rightarrow 0_{V'} \in A' \\ f(0_V) = 0_{V'} \end{array}\right| \Rightarrow 0_V \in f^{-1}(A') \Rightarrow f^{-1}(A') \neq \emptyset.$$

Fie $x, y \in f^{-1}(A')$, $\alpha, \beta \in k$. Vrem $\alpha x + \beta y \in f^{-1}(A')$

$\Updownarrow$ $\qquad\qquad\qquad\qquad$ $\Updownarrow$

$f(x), f(y) \in A'$ $\qquad\qquad$ $f(\alpha x + \beta y) \in A'$ $\cancel{(?)}$

$$f(\alpha x + \beta y) \xlongequal{f \text{ transf. lin}} \alpha \cdot \underbrace{f(x)}_{\in A'} + \beta \cdot \underbrace{f(y)}_{\in A'} \in A' \; \Uparrow$$

pt. că $A' \leq_k V'$.

Așadar $f^{-1}(A') \leq_k V$. $\square$.

Lista 9

① Fie funcțiile:

a) $f_1: \mathbb{R}^2 \to \mathbb{R}^2$, $f_1(x,y) = (-x,y)$ (simetria față de OY).

b) $f_2: \mathbb{R}^2 \to \mathbb{R}^2$, $f_2(x,y) = (x,-y)$ (simetria față de Ox)

c) $f_3: \mathbb{R}^2 \to \mathbb{R}^2$, $f_3(x,y) = (x\cos\varphi - y\sin\varphi,\ x\sin\varphi + y\cos\varphi)$

($\varphi \in \mathbb{R}$ fixat) (rotația de centru O, și unghi φ).

d) $f_4: \mathbb{R}^2 \to \mathbb{R}^3$, $f_4(x,y) = (x+y,\ 2x-y,\ 3x+2y)$.

Să se arate că f_1, f_2, f_3, f_4 sunt transformări liniare de $\mathbb{R}$ spații vectoriale. Care dintre acestea sunt izomorfisme? Care sunt endomorfisme? Care sunt automorfisme?

a) Fie $\alpha, \beta \in \mathbb{R}$, $(x_1,y_1), (x_2,y_2) \in \mathbb{R}^2$

$$f_1(\alpha(x_1,y_1) + \beta(x_2,y_2)) \overset{?}{=} \alpha \cdot f_1(x_1,y_1) + \beta \cdot f_1(x_2,y_2)$$

$$\begin{aligned} f_1(\alpha(x_1,y_1) + \beta(x_2,y_2)) &= f_1((\alpha x_1, \alpha y_1) + (\beta x_2, \beta y_2)) \\ &= f_1(\alpha x_1 + \beta x_2,\ \alpha y_1 + \beta y_2) \\ &= (-\alpha x_1 - \beta x_2,\ \alpha y_1 + \beta y_2) \quad ① \end{aligned}$$

$$\begin{aligned} \alpha f_1(x_1,y_1) + \beta f_1(x_2,y_2) &= \alpha(-x_1,y_1) + \beta(-x_2,y_2) = (-\alpha x_1, \alpha y_1) + \\ &\quad + (-\beta x_2, \beta y_2) \\ &= (-\alpha x_1 - \beta x_2,\ \alpha y_1 + \beta y_2) \quad ② \end{aligned}$$

Din ① și ② $\Rightarrow f_1$ transf. liniară.

- f_1 endomorfism. • Fie $(x_1,y_1), (x_2,y_2) \in \mathbb{R}^2$

$$f_1(x_1,y_1) = f_1(x_2,y_2) \Leftrightarrow (-x_1,y_1) = (-x_2,y_2) \Rightarrow \left.\begin{array}{l} x_1 = x_2 \\ y_1 = y_2 \end{array}\right\} \Rightarrow$$

$\Rightarrow (x_1,y_1) = (x_2,y_2)$

$\Rightarrow f_1$ inj.

• $\forall\ (x,y) \in \mathbb{R}^2,\ \exists\ (-x,y) \in \mathbb{R}^2 :\ f_1(-x,y) = (-(-x),\ y) = (x,y).$

$\Rightarrow f$ surj.

$\Rightarrow f_1$ bij. $\Rightarrow f_1$ izom. $\Rightarrow f_1$ autom.

d) $f_4(\alpha(x_1,y_1)+\beta(x_2,y_2)) \overset{?}{=} \alpha f_4(x_1,y_1) + \beta f_4(x_2,y_2)$

$$f_4(\alpha(x_1,y_1)+\beta(x_2,y_2)) = f_4((\alpha x_1, \alpha y_1)+(\beta x_2, \beta y_2))$$
$$= f_4(\alpha x_1+\beta x_2,\ \alpha y_1+\beta y_2)$$
$$= (\alpha x_1+\beta x_2+\alpha y_1+\beta y_2,\ 2\alpha x_1+2\beta x_2-\alpha y_1-\beta y_2,$$
$$3\alpha x_1+3\beta x_2+2\alpha y_1+2\beta y_2)$$

$$\alpha f_4(x_1,y_1)+\beta(x_2,y_2) = \alpha(x_1+y_1,\ 2x_1-y_1,\ 3x_1+2y_1)+\beta(x_2+y_2,$$
$$2x_2-y_2,\ 3x_2+2y_2)$$
$$=(\alpha x_1+\alpha y_1,\ 2\alpha x_1-\alpha y_1,\ 3\alpha x_1+2\alpha y_1)+(\beta x_2+\beta y_2,\ 2\beta x_2-\beta y_2,\ 3\beta x_2+$$
$$+2\beta y_2)$$
$$=(\alpha x_1+\alpha y_1+\beta x_2+\beta y_2,\ 2\alpha x_1-\alpha y_1+2\beta x_2-\beta y_2,\ 3\alpha x_1+2\alpha y_1+3\beta x_2+$$
$$+2\beta y_2)$$

f_4 - transf. liniară. $\qquad f_4: \mathbb{R}^2 \to \mathbb{R}^3$

- ~~endomorfism~~
- ~~izomorfism~~ ?
- ~~automorfism~~

$\forall\ (x,y)\in\mathbb{R}^2,\ \exists\ (a,b,c)\in\mathbb{R}^3$ a.î.

$f_4(x,y) = (a,b,c)$, $(a,b,c)\in\mathbb{R}^3$ arbitrar. $\to$ Întrebarea este asupra existenței

$$\Leftrightarrow (1) \begin{cases} x+y = a \\ 2x-y = b \\ 3x+2y = c \end{cases}$$

$\to$ sistemul trebuie să fie compatibil

Luând $(0,0,1)\in\mathbb{R}^3 \Rightarrow (1)$ sist. incompatibil $\Rightarrow f_4$ nu e surjectivă

$\Rightarrow f_4$ nu e izomorfism.

(2.) Există o transformare liniară $f: \mathbb{R}^3 \to \mathbb{R}^2$ a.î.

$f(1,0,3) = (1,1)$ și $f(-2,0,-6) = (2,1)$?

Pp. că există f cu prop. din enunț. Atunci

$$(2,1) = f(-2,0,-6) = f((-2)(1,0,3)) = (-2) f(1,0,3) = (-2)(1,1) = (-2,-2)$$

contrad.

Răsp: **NU**.

(3.) Fie V, V_1, V_2 K-sp. vectoriale, $f: V \to V_1$, $g: V \to V_2$ și $h: V \to V_1 \times V_2$, $h(x) = (f(x), g(x))$ $\forall x \in V$.

Să se arate că h este o transf. liniară $\Leftrightarrow$ f și g sunt transf. liniare. Generalizare.

$V_1 \times V_2$ $\underbrace{K\text{-sp. vect.}}_{\text{temă}}$ produs direct.

$\forall (x_1, x_2), (y_1, y_2) \in V_1 \times V_2$, $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$

$\forall \alpha \in K$, $\forall (x_1, x_2) \in V_1 \times V_2$, $\alpha (x_1, x_2) = (\alpha x_1, \alpha x_2)$.

h - transf. liniară $\Leftrightarrow \forall \alpha, \beta \in K, \forall x, y \in V, h(\alpha x + \beta y) = \alpha h(x) + \beta h(y)$

$\Leftrightarrow$ — // — $(f(\alpha x + \beta y), g(\alpha x + \beta y)) = \alpha (f(x), g(x)) + \beta (f(y), g(y))$

$\Leftrightarrow$ — // — $(f(\alpha x + \beta y), g(\alpha x + \beta y)) = (\alpha f(x), \alpha g(x)) + (\beta f(y), \beta g(y))$

$\Leftrightarrow$ — // — $(f(\alpha x + \beta y), g(\alpha x + \beta y)) = (\alpha f(x) + \beta f(y), \alpha g(x) + \beta g(y))$

$\Leftrightarrow \forall \alpha, \beta \in K, \forall x, y \in V,$

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

și $\Leftrightarrow$ f și g

$g(\alpha x + \beta y) = \alpha g(x) + \beta g(y)$ transf. liniare.

Generalizare:

Fie $V, V_1, \dots, V_n$ K - sp. vect.

$V_1 \times \dots \times V_n$ K - sp. vect. produs direct.

$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$

$\alpha(x_1, \dots, x_n) = (\alpha x_1, \dots, \alpha x_n)$

$f_1 : V \to V_1, \dots, f_n : V \to V_n$ funcții

$h : V \to V_1 \times \dots \times V_n$, $h(x) = (f_1(x), \dots, f_n(x))$

h transf. liniară $\Leftrightarrow$ $f_1, \dots, f_n$ transf. liniare.

4) a) Fie $m \in \mathbb{N}^*$ și $f : \mathbb{R}^m \to \mathbb{R}$. Să se arate că f este o transf. liniară de $\mathbb{R}$ - sp. vect. $\Leftrightarrow$ $\exists\, a_1, \dots, a_m \in \mathbb{R}$ unic det. a.î.

$$f(x_1, \dots, x_m) = a_1 x_1 + \dots + a_m x_m, \ \forall\, x_1, \dots, x_m \in \mathbb{R}. \qquad (1)$$

b) Să se determine transf. liniară $f : \mathbb{R}^m \to \mathbb{R}^n$ $(m, n \in \mathbb{N}^*)$.

a) Soluția I:

$\boxed{\Leftarrow}$ $f : \mathbb{R}^m \to \mathbb{R}$, $f(x_1, \dots, x_m) = a_1 x_1 + \dots + a_m x_m$, f - transf. lin. ?

$\forall\, \alpha, \beta \in K$, $\forall\, (x_1, \dots, x_m), (y_1, \dots, y_m) \in \mathbb{R}^m$

$$f(\alpha(x_1, \dots, x_m) + \beta(y_1, \dots, y_m)) = \alpha f(x_1, \dots, x_m) + \beta f(y_1, \dots, y_m)$$

$$f(\alpha(x_1, \dots, x_m) + \beta(y_1, \dots, y_m)) = f((\alpha x_1, \dots, \alpha x_m) + (\beta y_1, \dots, \beta y_m))$$

$$= f(\alpha x_1 + \beta y_1, \dots, \alpha x_m + \beta y_m)$$

$$\overset{(1)}{=} a_1(\alpha x_1 + \beta y_1) + \dots + a_m(\alpha x_m + \beta y_m)$$

$$= \alpha(a_1 x_1 + \dots + a_m x_m) + \beta(a_1 y_1 + \dots + a_1 y_m)$$

$$= \alpha f(x_1, \dots, x_m) + \beta f(y_1, \dots, y_m).$$

$\boxed{\Rightarrow}$ • <u>Existența $a_1, \dots, a_m$:</u> Fie $x_1, \dots, x_m \in \mathbb{R}^m$

$e_1 = (1, 0, \dots, 0)$
$e_2 = (0, 1, \dots, 0)$
$\vdots$
$e_m = (0, 0, \dots, 1)$

$$f(x_1, \dots, x_m) = f\big((x_1, 0, \dots, 0) + (0, x_2, \dots, 0) + \dots + (0, \dots, 0, x_m)\big)$$
$$= f\Big(x_1 \underbrace{(1, 0, \dots, 0)}_{e_1} + x_2 \underbrace{(0, 1, \dots, 0)}_{e_2} + \dots + x_m \underbrace{(0, \dots, 0, 1)}_{e_m}\Big)$$
$$= f(x_1 \cdot e_1 + x_2 \cdot e_2 + \dots + x_m \cdot e_m)$$
$$= x_1 \cdot f(e_1) + \dots + x_m \cdot f(e_m)$$

Luăm $a_i = f(e_i) \in \mathbb{R}$, $i = \overline{1,m}$ și obținem (1).

• <u>Unicitatea $a_1, \dots, a_m$:</u> Fie $b_1, \dots, b_m \in \mathbb{R}$: $f(x_1, \dots, x_m) = b_1 x_1 + \dots + b_m x_m$
$\forall (x_1, \dots, x_m) \in \mathbb{R}^m$

$$a_i = f(e_i) = b_1 \cdot 0 + \dots + 1 \cdot b_i + \dots + b_m \cdot 0 = b_i, \ \forall i = \overline{1,m}.$$
$$= f(0, 0, \dots, 0, 1, 0, \dots, 0)$$

b) $f : \mathbb{R}^m \to \mathbb{R}^n$, $f(x_1, \dots, x_m) = \big(f_1(x_1, \dots, x_m), f_2(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)\big)$

$$\mathbb{R}^m \xrightarrow{f} \mathbb{R}^n \xrightarrow{e_i} \mathbb{R}, \quad f_i : \mathbb{R}^m \to \mathbb{R}, \ i = \overline{1,m}.$$

$f_i = e_i \circ f$ $\qquad e_i(y_1, \dots, y_n) = y_i$, $\qquad e_i$ proiecția (a i-a) canonică.
e_i transf. liniară

f transf. liniară $\Leftrightarrow$ $f_1, \dots, f_n$ transf. liniare $\Leftrightarrow$
$\Leftrightarrow \exists\, a_1^1, \dots, a_m^1, a_1^2, \dots, a_m^2, \dots, a_1^m, \dots, a_m^m \in \mathbb{R}$ unic det. a. î.

$$f(x_1, \dots, x_m) = (a_1^1 x_1 + \dots + a_m^1 x_m, \ a_1^2 x_1 + \dots + a_m^2 x_m, \dots, a_1^m x_1 + \dots + a_m^m x_m)$$

5. Să se arate că există o transformare liniară $f : \mathbb{R}^2 \to \mathbb{R}^2$ a.î.
$f(1,1) = (2,5)$ și $f(1,0) = (1,4)$.
Să se determine $f(2,3)$. Este f izomorfism?

$f : \mathbb{R}^2 \to \mathbb{R}^2$ transf. liniară $\Rightarrow \exists\, a, b, c, d \in \mathbb{R}$ a. î.

$$f(x,y) = (ax + by, cx + dy)$$

a, b, c, d ? (temă)

Să se determine $f(2,3)$. Este f izomorfism? (temă)