Tema 5 - serii de numere reale

ex 1:

a) $\sum\limits_{n\geq 1} \frac{n+7}{\sqrt{n^2+7}}$

$$\lim_{n\to\infty} \frac{n+7}{\sqrt{n^2+7}} = \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{7}{n}\right)}{\cancel{n}\sqrt{1+\frac{7}{n^2}}} = 1 \neq 0 \Rightarrow$$

$\Rightarrow$ serie divergentă.

b) $\sum\limits_{n\geq 1} \frac{1}{\sqrt[n]{n}}$

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}}$$

Calculăm $\ln \lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{n}} =$

$$= \lim_{n\to\infty} \ln \frac{1}{\sqrt[n]{n}} =$$

$$= \lim_{n\to\infty} \ln \frac{1}{n^{\frac{1}{n}}} = \lim_{n\to\infty} \ln n^{-\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(-\frac{1}{n}\right) \ln n = \lim_{n\to\infty} \frac{\ln n}{-n} \stackrel{L'H}{=}$$

$$= \lim_{n\to\infty} -\frac{1}{n} = \frac{1}{\infty} = 0$$

$\ln \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 0 \Rightarrow$

$\Rightarrow \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1 \Rightarrow$

$\Rightarrow$ serie divergentă.

c) $\sum_{n\geq 1} \frac{1}{\sqrt[n]{n!}}$

$\boxed{\sum \frac{1}{n} D, \sum \frac{1}{n^2} C}$

știm că $n^n > n! \,/\, ()^{-1}$

$$\frac{1}{n^n} < \frac{1}{n!} \,/\, \sqrt[n]{\ }$$

$$\frac{1}{n} < \frac{1}{\sqrt[n]{n!}} \,/\, \sum_{n\geq 1}$$

$$\sum_{n\geq 1} \frac{1}{n} < \sum_{n\geq 1} \frac{1}{\sqrt[n]{n!}}$$

fie $x_n = \frac{1}{n}$

$y_n = \frac{1}{\sqrt[n]{n!}}$

$\sum x_n \; D$

$\xrightarrow{C1C}$

$\Rightarrow \sum y_n = \sum \frac{1}{\sqrt[n]{n!}} \; D.$

d) $\sum_{n\geq 1}\left(1+\frac{1}{n}\right)^n$

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \neq 0 \Rightarrow$

$\Rightarrow$ serie divergentă

ex 2:

a) $\sum_{n\geq 1}\frac{2^n+3^n}{5^n}$

$\lim_{n\to\infty}\frac{2^n+3^n}{5^n} = \lim_{n\to\infty}\frac{\cancel{5^n}\left(\left(\frac{2}{5}\right)^n+\left(\frac{3}{5}\right)^n\right)}{\cancel{5^n}}$

$= \frac{0}{1} = 0 \Rightarrow$ nu știm natura seriei

$\sum_{n\geq 1}\frac{2^n}{5^n}+\sum_{n\geq 1}\frac{3^n}{5^n} =$

$= \sum_{n\geq 1}\left(\frac{2}{5}\right)^n+\sum_{n\geq 1}\left(\frac{3}{5}\right)^n \Rightarrow \sum\frac{2^n+3^n}{5^n}$ C

Seria se poate scrie ca sumă de 2 serii geometrice convergente cu rațiile $\frac{2}{5}$ și $\frac{3}{5}$

b) $\sum \frac{2^n}{3^n + 5^n}$

$$\lim_{n\to\infty} \frac{2^n}{3^n + 5^n} = \lim \frac{\cancel{5^n}\left(\overset{\nearrow 0}{\frac{2^n}{5^n}}\right)}{\cancel{5^n}\left(\left(\frac{3}{5}\right)^n + 1\right)} = \frac{0}{1} = 0$$

=> nu știm ce natură are.

$$\frac{2^n}{3^n + 5^n} \leq \frac{2^n}{5^n} = \left(\frac{2}{5}\right)^n \Big/ \sum$$

$$\sum_{n\geq 1} \frac{2^n}{3^n + 5^n} \leq \sum_{n\geq 1} \frac{2^n}{5^n}$$

seria $\sum_{n\geq 1} \frac{2^n}{5^n} = \frac{2}{5} \cdot \sum_{n\geq 1} \left(\frac{2}{5}\right)^{n-1}$

e convergentă fiind seria geometrică de rație $\frac{2}{5} < 1$

ex 3:

a) $\sum_{n\geq 1} \frac{1}{2n-1}$

$\lim_{n\to\infty} \frac{1}{2n-1} = \frac{1}{\infty} = 0$ => nu știm natura

$$\sum_{n\geq 1} \frac{1}{2n-1} > \sum_{n\geq 1} \frac{1}{2n}$$

$$\sum_{n\geq 1} \frac{1}{2n-1} > \frac{1}{2} \sum_{n\geq 1} \frac{1}{n}$$

Știm că $\sum_{n\geq 1} \frac{1}{n}$ D

~~fie x~~, Am găsit o serie D mai mică decât seria $\sum \frac{1}{2n-1}$ $\Rightarrow$

$$\Rightarrow \sum_{n\geq 1} \frac{1}{2n-1} \text{ D.}$$

b) $\sum_{n\geq 1} \frac{1}{(2n-1)^2}$

$\lim_{n\to\infty} \frac{1}{(2n-1)^2} = 0 \Rightarrow$ nu știu natura.

$\cancel{\sum_{n\geq 1} \frac{1}{(2n-1)^2}}$ fie $x_n = \frac{1}{(2n-1)^2)}$ $y_n = \frac{1}{n^\alpha}$

Calculăm

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{1}{(2n-1)^2} \cdot n^\alpha$$

$$= \lim_{n\to\infty} \frac{n^\alpha}{(2n-1)^2} = \lim_{n\to\infty} \frac{n^\alpha}{4n^2 - 4n + 1}$$

Fixăm $\alpha = 2 \Rightarrow$

$$\Rightarrow \lim_{n\to\infty} \frac{n^2}{n^2\left(4 - \frac{4}{n} + \frac{1}{n^2}\right)} = \frac{1}{4} \in (0, \infty) \Rightarrow$$

$$\Rightarrow \sum x_n \sim \sum y_n$$

$$\sum \frac{1}{(2n-1)^2} \sim \sum \frac{1}{n^2} \; C \Rightarrow$$

$$\Rightarrow \sum \frac{1}{(2n-1)^2} \; C.$$

c) $\sum_{n\geq 1} \frac{1}{\sqrt{4n^2-1}}$

$\lim \frac{1}{\sqrt{4n^2-1}} = 0 \Rightarrow$ nu știm natura

fie $x_n = \frac{1}{\sqrt{4n^2-1}}$ $y_n = \frac{1}{n^\alpha}$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{n^\alpha}{\sqrt{4n^2-1}}$$

$$= \lim_{n\to\infty} \frac{n^\alpha}{n\sqrt{4 - \frac{1}{n^2}}}$$

fixăm $x = 1 \Rightarrow$

$$\Rightarrow \lim_{n\to\infty} \frac{n}{n\sqrt{4-\frac{1}{n^2}}} = \frac{1}{2} \in (0,\infty) \Rightarrow$$

$$\Rightarrow \sum x_n \sim \sum y_n$$

$$\sum x_n \sim \sum \frac{1}{n} \; D \Rightarrow$$

$$\Rightarrow \sum \frac{1}{\sqrt{4n^2-1}} \; D.$$

d) $$\sum_{n\geq 1} \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}}$$

$$\lim_{n\to\infty} \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}} = \lim_{n\to\infty} \frac{n\sqrt{1+\frac{1}{n}}}{n\sqrt[3]{n^2-\frac{1}{n^2}}}$$

$= 0 \Rightarrow$ nu știm natura.

fie $x_n = \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}}$ $\quad y_n = \frac{1}{n^\alpha}$

$$\cancel{\lim_{n\to\infty} \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}} \cdot n^\alpha}$$

$$= \cancel{\lim_{n\to\infty} \frac{n\sqrt{1+\frac{1}{n}}}{n\sqrt[3]{n^2-1/n^2}} \cdot n^\alpha}$$

$\cancel{\alpha = 1}$

$\cancel{\lim \frac{x_n}{y_n} = 0 \Rightarrow}$

$\cancel{\sum \frac{1}{n^0}}$

$\cancel{\text{fie } \alpha = 1.}$

$$\cancel{\lim_{n\to\infty} \frac{n\sqrt{1+\frac{1}{n}}}{n\sqrt[3]{\frac{1}{n} - \frac{1}{n^5}}} = \frac{1}{0} = \infty}$$

fie $y_n = \frac{\sqrt{n^2}}{\sqrt[3]{n^5}} = \frac{n}{n^{\frac{5}{3}}} =$

$= \frac{1}{n^{\frac{5}{3}} \cdot n^{-1}} = \frac{1}{n^{\frac{5}{3}-1}} = \frac{1}{n^{\frac{2}{3}}} \Rightarrow$

$\Rightarrow \alpha = \frac{2}{3}$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}} \cdot n^{\frac{2}{3}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^2+n}}{\sqrt[3]{(n^5-n)\, n^{-\frac{2}{3}\cdot 3}}}$$

$$= \lim_{n\to\infty} \frac{n\sqrt{1+\frac{1}{n}}}{\sqrt[3]{n^5\cdot n^{-2}+n\cdot n^{-2}}}$$

$$= \lim_{n\to\infty} \frac{n\sqrt{1+\frac{1}{n}}}{\sqrt[3]{n^3+n^{-1}}} = \lim_{n\to\infty} \frac{\cancel{n}\sqrt{1+\frac{1}{n}}}{\cancel{n}\sqrt[3]{1+\frac{1}{n^4}}}$$

$$= 1 \in (0,\infty) \Rightarrow$$

$$\Rightarrow \sum x_n \sim \sum y_n$$

$$\sum y_n = \sum \frac{1}{n^{\frac{2}{3}}}$$

$$\alpha \leq 1 \Rightarrow \sum y_n \; D.$$

$$\Rightarrow$$

$$\Rightarrow \sum x_n \; D.$$

ex 4:

a) $\sum\limits_{n\geq 1} \frac{100^n}{n!}$

$x_n = \frac{100^n}{n!}$

$\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim\limits_{n\to\infty} \frac{\cancel{100^{n+1}}^{\,100}}{\cancel{(n+1)!}_{\,n+1}} \cdot \frac{\cancel{n!}}{\cancel{100^n}} \overset{\lim\limits_{n\to\infty}}{=} \frac{100}{n+1}$

$= 0 < 1 \Rightarrow \sum x_n$ convergentă

b) $\sum \frac{2^n n!}{n^n}$

$\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim\limits_{n\to\infty} \frac{\cancel{2^{n+1}}^{\,2}\,\cancel{(n+1)!}^{\,n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{\cancel{2^n}\,\cancel{n!}}$

$= \lim\limits_{n\to\infty} \frac{2 \cdot n^n \cdot (n+1)}{(n+1)^{n+1}}$

$= \lim\limits_{n\to\infty} \frac{2 \cdot (n^{n+1} + n^n)}{(n+1)^{n+1}}$

$= 2 \lim\limits_{n\to\infty} \frac{n^n \cancel{(n+1)}}{(n+1)^{\cancel{n+1}}} =$

$= 2 \lim\limits_{n\to\infty} \frac{n^n}{(n+1)^n} = \frac{2}{e} < 1 \Rightarrow C$

c) $\sum_{n\geq 1} \frac{3^n n!}{n^n}$

$$\lim_{n\to\infty} \frac{\overset{3}{\cancel{3^{n+1}}}\, \overset{n+1}{\cancel{(n+1)}}!}{(n+1)^{n+1}} \cdot \frac{n^n}{\cancel{3^n}\cdot \cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{3 \cdot n^n \cdot \cancel{(n+1)}}{(n+1)^{n\cancel{+1}}}$$

$$= \lim_{n\to\infty} 3 \cdot \left(\frac{n}{n+1}\right)^n = \frac{3}{e} \quad \Big\} \lim \ldots > 1$$

$e \approx 2{,}4$

$\Rightarrow \sum x_n$ D.

e) $\sum_{n\geq 1} \frac{n^2}{\left(2+\frac{1}{n}\right)^n}$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)^2}{\left(2+\frac{1}{n+1}\right)^{n+1}} \cdot \frac{\left(2+\frac{1}{n}\right)^n}{n^2}$$

$$= \lim_{n\to\infty} \cancel{(n+1)^2}\ \frac{\left(\frac{2n+1}{n}\right)^n}{\left(\frac{2n+3}{n+1}\right)^{n+1}} \cdot \frac{(n+1)^2}{n^2}$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n}\right)^n}_{e} \underbrace{(n+1)}_{\infty} \underbrace{\frac{(2n+1)^n}{(2n+3)^{n+1}}}_{\frac{1}{4}} \cdot \underbrace{\left(\frac{n+1}{n}\right)^2}_{1} = \infty \Rightarrow D$$

ex 5.

a) $\sum_{n\geq 1} \frac{a^n}{n^n}$ $\quad x_n = \frac{a^n}{n^n}$

Criteriul radicalului

$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{\frac{a^n}{n^n}} = \lim_{n\to\infty} \frac{a}{n}$ $\Big\}$ =>

$a > 0$

$\Rightarrow \lim \frac{a}{n} = 0 < 1 \Rightarrow \sum x_n$ C.

b) $\sum_{n\geq 1} \left(\frac{n^2+n+1}{n^2} a\right)^n$

Crit radicalului:

$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{n^2+n+1}{n^2} a$

$= \lim_{n\to\infty} \frac{\cancel{n^2}\left(1 + \overset{\to 0}{\frac{1}{n}} + \overset{\to 0}{\frac{1}{n^2}}\right) a}{\cancel{n^2}}$

$= a$

$\underline{\text{I}}$ $a < 1 \Rightarrow \sum x_n$ C.

$\underline{II}$ $a > 1 \Rightarrow \sum x_n$ D.

$\underline{III}$ $a = 1 \Rightarrow ?$

Avem $\displaystyle\sum_{n \geq 1} \left(\frac{n^2+n+1}{n^2}\right)^n$

$$\lim_{n \to \infty} \left(\frac{n^2+n+1}{n^2}\right)^n =$$

$$= \lim_{n \to \infty} \left(\frac{\cancel{n^2}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)}{\cancel{n^2}}\right)^n$$

$$= \lim_{n \to \infty} \left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n \text{ "}1^\infty\text{"}$$

$$= \lim_{n \to \infty} \left(1 + 1 + \cancel{\frac{1}{n}} + \frac{1}{n} + \frac{1}{n^2} - 1\right)^n$$

$$= \lim_{n \to \infty} \left(1 + \frac{n+1}{n^2}\right)^{n \cdot \frac{n^2}{n+1} \cdot \frac{n+1}{n^2}}$$

$$= e^{\lim n \cdot \frac{n+1}{n^2}}$$

$$= e^1 \neq 0 \Rightarrow \sum x_n \text{ D.}$$

c) $\sum_{n\geq 1} \frac{3^n}{2^n+a^n}$

~~$\lim_{n\to\infty} \frac{3^n}{2^n+a^n} = \lim 3$~~

~~Se observă $\frac{3^n}{2^n+a^n} \leq \frac{3^n}{2^n}$~~

$\lim_{n\to\infty} \frac{3^n}{2^n+a^n}$

I $a<3 \Rightarrow \lim_{n\to\infty} \frac{\cancel{3^n}\ 1}{\cancel{3^n}\left(\left(\frac{2}{3}\right)^n+\left(\frac{a}{3}\right)^n\right)}$

$\left(\frac{2}{3}\right)^n \to 0$, $\left(\frac{a}{3}\right)^n \to 0$

$= \frac{1}{0} = \infty \Rightarrow \sum \frac{3^n}{2^n+a^n}$ D.

II $a>3$.

$\lim_{n\to\infty} \frac{\cancel{a^n}\cdot\left(\frac{3}{a}\right)^n}{\cancel{a^n}\left(\left(\frac{2}{a}\right)^n+1\right)} = \frac{0}{1} = 0 \Rightarrow$

$\left(\frac{2}{a}\right)^n \to 0$

$\Rightarrow$ nu știm natura.

Se observă că :

$$\frac{3^n}{2^n+a^n} \le \frac{3^n}{a^n} \quad , a > 3$$

$\sum y_n = \sum \left(\frac{3}{a}\right)^n$ serie geometrică

rația $q = \frac{3}{a}$ C

$\sum x_n = \frac{3^n}{2^n+a^n}$

C1C $\Rightarrow$

$\Rightarrow \sum x_n$ e C.