# Seminar 04

## Inversa unei matrici

Fie $k \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, $A \in \mathcal{M}_n(k)$ $(n \in \mathbb{N}, n \geq 2)$

$A$ inversabilă în $(\mathcal{M}_n(k), \cdot) \Longleftrightarrow \det A \neq 0$.

Dc. $A$ este inv., atunci $A^{-1} = \frac{1}{\det A} \cdot A^*$, unde

$$A^* = \begin{pmatrix} \Gamma_{11} & \Gamma_{21} & \dots & \Gamma_{n1} \\ \Gamma_{12} & \Gamma_{22} & \dots & \Gamma_{n2} \\ \vdots & \vdots & & \vdots \\ \Gamma_{1n} & \Gamma_{2n} & \dots & \Gamma_{nn} \end{pmatrix} = {}^T(\Gamma_{ij})$$

complemente algebrice

④ Sunt inversabile următoarele matrici? În caz afirmativ, să se determine inversele lor.

a) $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 4 & 1 & 4 \end{pmatrix}$

$$\det A = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 4 & 1 & 4 \end{vmatrix} \xrightarrow[C_3 - 2C_1]{C_2 - C_1} \begin{vmatrix} 1 & 0 & 0 \\ 2 & -3 & -3 \\ 4 & -3 & -4 \end{vmatrix} = (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} -3 & -3 \\ -3 & -4 \end{vmatrix} = 12 - 9 = 3 \neq 0$$

$\Rightarrow$ A este inversabilă.

$\Gamma_{11} = (-1)^{1+1} \cdot \begin{vmatrix} -1 & 1 \\ 1 & 4 \end{vmatrix} = -5$; $\quad \Gamma_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 2 & 1 \\ 4 & 4 \end{vmatrix} = -(8-4) = -4$

$\Gamma_{13} = (-1)^{1+3} \cdot \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = 6$; $\quad \Gamma_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = -(4-2) = -2$

$$\Gamma_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 1 & 2 \\ 4 & 4 \end{vmatrix} = -4; \qquad \Gamma_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} = -(1-4) = 3$$

$$\Gamma_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3; \qquad \Gamma_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -(1-4) = 3$$

$$\Gamma_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = (-1-2) = -3.$$

Așadar $A^{-1} = \frac{1}{3} \cdot \begin{pmatrix} -5 & -2 & 3 \\ -4 & -4 & 3 \\ 6 & 3 & -3 \end{pmatrix} = \begin{pmatrix} -5/3 & -2/3 & 1 \\ -4/3 & -4/3 & 1 \\ 2 & 1 & -1 \end{pmatrix}$

b) $B = \begin{pmatrix} 3 & 4 & 2 \\ 6 & 8 & 5 \\ 9 & 12 & 10 \end{pmatrix}$ $C_1$ și $C_2$ sunt proporționale

$\det B = 0$ $\left(C_2 = \frac{4}{3} \cdot C_1\right) \Rightarrow B$ NU e inversabilă.

c) $C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$ temă

d) $D = \begin{pmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{pmatrix}$ $(\lambda \in \mathbb{C})$.

$$\det D = \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} \xlongequal{L_1+L_2+L_3} \begin{vmatrix} \lambda+2 & 1 & 1 \\ \lambda+2 & \lambda & 1 \\ \lambda+2 & 1 & \lambda \end{vmatrix} = (\lambda+2) \begin{vmatrix} 1 & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} \xlongequal[L_3-L_1]{L_2-L_1}$$

$$= (\lambda+2) \begin{vmatrix} 1 & 1 & 1 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda+2) \cdot (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda+2)(\lambda-1)^2.$$

$(\lambda+2)(\lambda-1)^2=0 \Rightarrow \lambda_1=-2,\ \lambda_{2,3}=1.$

I Dacă $\lambda=-2$ sau $\lambda=1 \Rightarrow \det \Delta=0 \Rightarrow \Delta$ NU e inversabilă

II Dacă $\lambda\in\mathbb{C}\setminus\{-2,1\} \Rightarrow \det\Delta\neq 0 \Rightarrow \Delta$ este inversabilă:

$\Gamma_{11}=(-1)^{1+1}\cdot\begin{vmatrix}\lambda & 1\\ 1 & \lambda\end{vmatrix}=\lambda^2-1$ $\qquad \Gamma_{12}=(-1)^{1+2}\begin{vmatrix}1 & 1\\ 1 & \lambda\end{vmatrix}=1-\lambda$

$\Gamma_{13}=(-1)^{1+3}\cdot\begin{vmatrix}1 & \lambda\\ 1 & 1\end{vmatrix}=1-\lambda$ $\qquad \Gamma_{21}=(-1)^{2+1}\begin{vmatrix}1 & 1\\ 1 & \lambda\end{vmatrix}=1-\lambda$

$\Gamma_{22}=(-1)^{2+2}\cdot\begin{vmatrix}\lambda & 1\\ 1 & \lambda\end{vmatrix}=\lambda^2-1$ $\qquad \Gamma_{23}=(-1)^{2+3}\begin{vmatrix}\lambda & 1\\ 1 & 1\end{vmatrix}=1-\lambda$

$\Gamma_{31}=(-1)^{3+1}\cdot\begin{vmatrix}1 & 1\\ \lambda & 1\end{vmatrix}=1-\lambda$ $\qquad \Gamma_{32}=(-1)^{3+2}\begin{vmatrix}\lambda & 1\\ 1 & 1\end{vmatrix}=1-\lambda$

$\Gamma_{33}=(-1)^{3+3}\cdot\begin{vmatrix}\lambda & 1\\ 1 & \lambda\end{vmatrix}=\lambda^2-1.$

Așadar $\Delta^{-1}=\dfrac{1}{(\lambda+2)(\lambda-1)^2}\begin{pmatrix}\lambda^2-1 & 1-\lambda & 1-\lambda\\ 1-\lambda & \lambda^2-1 & 1-\lambda\\ 1-\lambda & 1-\lambda & \lambda^2-1\end{pmatrix}.$

5 Să se rezolve urm. ec. matriceale:

a) $\underbrace{\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}}_{}X=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}.$ $\Rightarrow$

este inversabilă deoarece are $\det = 1-4=-3\neq 0.$

$X=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}^{-1}\cdot\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}=$ temă

b) $\underbrace{\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}}$ $X = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

are det $= 0$, deci folosim altă metodă.

Clar $X \in \mathcal{M}_2(\mathbb{R})$. Fie $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} a+2c & b+2d \\ a+2c & b+2d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

$\Rightarrow \begin{cases} a+2c = 1 \\ b+2d = 2. \end{cases} \Rightarrow \begin{cases} a = 1-2c \\ b = 2-2d. \end{cases}$

Așadar $X \in \left\{ \begin{pmatrix} 1-2c & 2-2d \\ c & d \end{pmatrix} \middle| \ c, d \in \mathbb{R} \right\}$.

c) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ <u>temă</u>

d) $X \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{pmatrix}$ <u>temă</u>

e) $\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{E} X = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$

## Metoda I:

$\det E = 1\cdot 1\cdot 1\cdot 1 = 1 \neq 0 \Rightarrow E$ inversabilă.

$$\Rightarrow X = E^{-1}\cdot \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} = \ldots$$ <u>temă</u>

## Metoda II:

Clar $X \in \mathcal{M}_{4,1}(\mathbb{R})$. Fie $X = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$.

$$E\cdot\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a+b+c+d \\ b+c+d \\ c+d \\ d \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} a+b+c+d = 4 \\ b+c+d = 3 \\ c+d = 2 \\ d = 1 \end{cases}$$

$$\Rightarrow a = b = c = d = 1 \Rightarrow X = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

# Rangul unei matrici

Fie $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, $m, n \in \mathbb{N}^*$, $A \in \mathcal{M}_{mn}(K)$.

Dacă $A = O_{mn}$ at. $\text{rang}\, A = 0$.

Dacă $A \neq O_{mn}$, at. $\text{rang}\, A = r \ (\in \mathbb{N}^*) \iff$ A are un minor de ordinul r nenul și toți minorii de ordinul r+1 obținuți prin bordarea sa sunt nuli.

**Obs.** $\text{rang}\, A \leq \min\{m, n\}$.

1. Să se determine rangurile următoarelor matrice:

a) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & 0 & -3 \end{pmatrix}$ Clar $\text{rang}\, A \leq 3$.

Metoda 1:

Minori de ordinul 1:

$|1| = 1 \neq 0 \Rightarrow \text{rang}\, A \geq 1$.

Minori de ordinul 2:

$\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1\cdot 0 - 1\cdot 1 = -1 \neq 0 \Rightarrow \text{rang}\, A \geq 2$.

Minori de ordinul 3:

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 3 & 0 \end{vmatrix} \xrightarrow{L_1 - L_2} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & 0 \end{vmatrix} = (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0 \Rightarrow \text{rang}\, A = 3$

## Metoda 2:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & 0 & -3 \end{pmatrix} \underset{L_3-L_1}{\overset{L_2-L_1}{\sim}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -2 \\ 0 & 2 & -1 & -4 \end{pmatrix} \underset{C_4-C_1}{\overset{C_2-C_1,\ C_3-C_1}{\sim}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 2 & -1 & -4 \end{pmatrix} \overset{L_3+2L_2}{\sim}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & -8 \end{pmatrix} \overset{C_4-2C_2}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -8 \end{pmatrix} \overset{C_4-8C_3}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \Rightarrow$$

$\Rightarrow$ rang $A = 3$.

b) $B = \begin{pmatrix} 1 & 2 & 1 & -2 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 2 & -3 \end{pmatrix}$ la curs

c) $C = \begin{pmatrix} 3 & 0 & 3 & 0 & 3 \\ 0 & 2 & 0 & 2 & 0 \\ 3 & 2 & 0 & 3 & 2 \\ 0 & 2 & 0 & 2 & 0 \end{pmatrix}$ Clar rang $C \leq 4$.

## Metoda 1:

Minori de ordinul 1:

$|3| = 3 \neq 0 \Rightarrow$ rang $C \geq 1$.

Minori de ordinul 2:

$\begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6 \neq 0 \Rightarrow$ rang $C \geq 2$.

Minori de ordinul 3:

$\begin{vmatrix} 3 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 2 & 0 \end{vmatrix} = 3 \begin{vmatrix} 0 & 2 \\ 3 & 2 \end{vmatrix} = -18 \neq 0 \Rightarrow$ rang $C \geq 3$.

Minori de ordinul 4: ($\exists$ 2 moduri de a-l borda pe cel anterior)

$$\begin{vmatrix} 3 & 0 & 3 & 0 \\ 0 & 2 & 0 & 2 \\ 3 & 2 & 0 & 3 \\ 0 & 2 & 0 & 2 \end{vmatrix} \overset{L_2=L_4}{=\!=} 0.$$

$$\begin{vmatrix} 3 & 0 & 3 & 3 \\ 0 & 2 & 0 & 0 \\ 3 & 2 & 0 & 2 \\ 0 & 2 & 0 & 0 \end{vmatrix} \overset{L_2=L_4}{=\!=} 0.$$

Așadar rang $C=3$.

## Metoda 2:

$$\begin{pmatrix} 3 & 0 & 3 & 0 & 3 \\ 0 & 2 & 0 & 2 & 0 \\ 3 & 2 & 0 & 3 & 2 \\ 0 & 2 & 0 & 2 & 0 \end{pmatrix} \overset{L_3-L_1}{\sim} \begin{pmatrix} 3 & 0 & 3 & 0 & 3 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 2 & -3 & 3 & -1 \\ 0 & 2 & 0 & 2 & 0 \end{pmatrix} \underset{C_5-C_1}{\overset{C_3-C_1}{\sim}} \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 2 & -3 & 3 & -1 \\ 0 & 2 & 0 & 2 & 0 \end{pmatrix} \underset{L_4-L_2}{\overset{L_3-L_2}{\sim}}$$

$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{C_4-C_2}{\sim} \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \underset{C_5-\frac{1}{3}C_3}{\overset{C_4+\frac{1}{3}C_3}{\sim}} \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

$\Rightarrow$ rang $C=3$.

d) $D = \begin{pmatrix} 2 & \alpha & -2 & 2 \\ 4 & -1 & 2\alpha & 5 \\ 2 & 10 & -12 & 1 \end{pmatrix}$ $(\alpha \in \mathbb{C})$.

Clar rang $D \leq 3$.

## Metoda 1:

Minori de ordinul 1:

$|2| = 2 \neq 0 \Rightarrow$ rang $D \geq 1$.

Minori de ordinul 2:

$$\begin{vmatrix} 2 & 2 \\ 4 & 5 \end{vmatrix} = 10-8 = 2 \neq 0 \Rightarrow \text{rang } D \geq 2.$$

Minori de ordinul 3: ($\exists$ doar 2 minori care îl bordează pe cel anterior)

$$\begin{vmatrix} 2 & \alpha & 2 \\ 4 & -1 & 5 \\ 2 & 10 & 1 \end{vmatrix} \xrightarrow[L_3-L_1]{L_2-2L_1} \begin{vmatrix} 2 & \alpha & 2 \\ 0 & -1-2\alpha & 1 \\ 0 & 10-\alpha & -1 \end{vmatrix} = 2\begin{vmatrix} -1-2\alpha & 1 \\ 10-\alpha & -1 \end{vmatrix} =$$

$$= 2(1+2\alpha - 10+\alpha) = 2(3\alpha - 9).$$

Cazul I. Dacă $2(3\alpha-9)=0 \Leftrightarrow 3\alpha - 9 = 0 \Leftrightarrow \alpha = 3$.

$$\begin{vmatrix} 2 & -2 & 2 \\ 4 & 2\cdot 3 & 5 \\ 2 & -12 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -2 & 2 \\ 4 & 6 & 5 \\ 2 & -12 & 1 \end{vmatrix} \xrightarrow[L_3-L_1]{L_2-2L_1} \begin{vmatrix} 2 & -2 & 2 \\ 0 & 10 & 1 \\ 0 & -10 & -1 \end{vmatrix} \xrightarrow{L_2=-L_1} 0.$$

$\Rightarrow$ rang $D = 2$.

Cazul II. Dacă $\alpha \neq 3 \Leftrightarrow 2(3\alpha-9) \neq 0 \Rightarrow$ rang $D = 3$.

Așadar rang $D = \begin{cases} 2, & \text{dc. } \alpha = 3 \\ 3, & \text{dc. } \alpha \neq 3. \end{cases}$

Metoda 2:

$$\begin{pmatrix} 2 & \alpha & -2 & 2 \\ 4 & -1 & 2\alpha & 5 \\ 2 & 10 & -12 & 1 \end{pmatrix} \underset{L_3-L_1}{\overset{L_2-2L_1}{\sim}} \begin{pmatrix} 2 & \alpha & -2 & 2 \\ 0 & -1-2\alpha & 2\alpha+4 & 1 \\ 0 & 10-\alpha & -10 & -1 \end{pmatrix} \overset{\dots}{\sim}$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1-2\alpha & 2\alpha+4 & 1 \\ 0 & 10-\alpha & -10 & -1 \end{pmatrix} \overset{C_2 \leftrightarrow C_4}{\sim} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 2\alpha+4 & -1-2\alpha \\ 0 & -1 & -10 & 10-\alpha \end{pmatrix} \overset{L_3+L_2}{\sim}$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 2\alpha+4 & -1-2\alpha \\ 0 & 0 & 2\alpha-6 & 9-3\alpha \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2\alpha-6 & 9-3\alpha \end{pmatrix}$$

Cazul I: Dacă $2\alpha-6=0 \Leftrightarrow \alpha=3$

Atunci $D \sim \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \operatorname{rang} D = 2.$

Cazul II: Dacă $\alpha \neq 3 \Leftrightarrow 2\alpha-6 \neq 0.$

Atunci $D \sim \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2\alpha-6 & 0 \end{pmatrix} \Rightarrow \operatorname{rang} D = 3.$

Așadar $\operatorname{rang} D = \begin{cases} 2, & \text{dc. } \alpha=3 \\ 3, & \text{altfel } (\alpha \neq 3). \end{cases}$