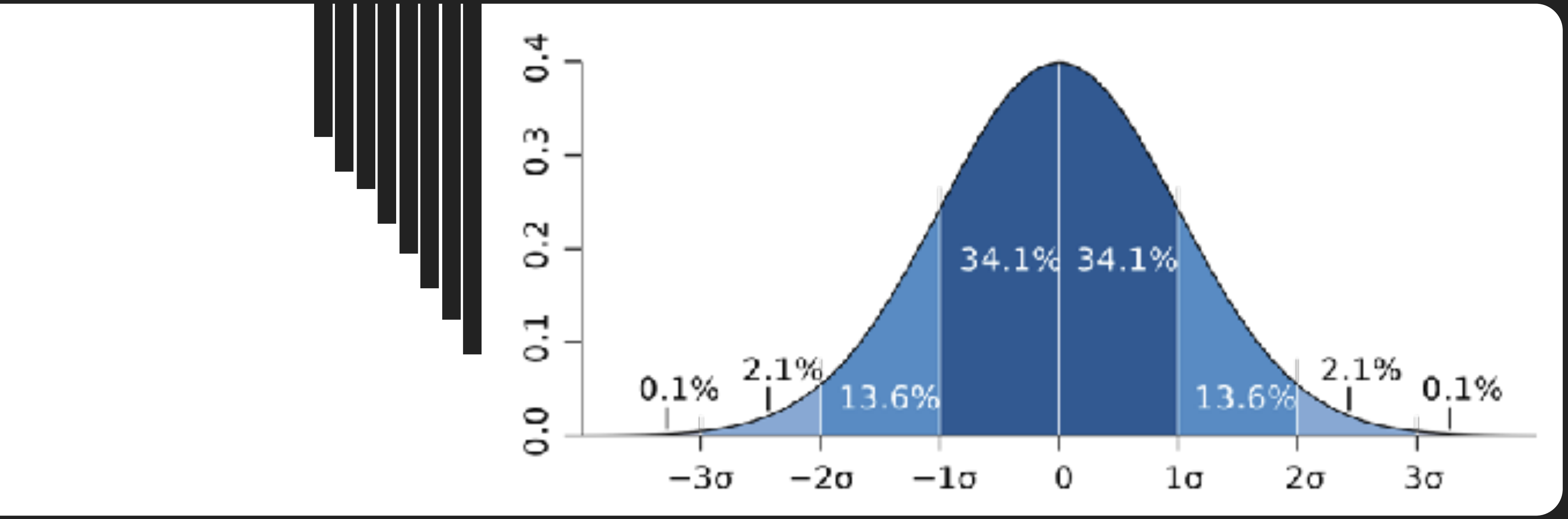

R. BRICEÑO, T. ROGERS

RATE OF CHANGE, CALCULUS, AND DIFFERENTIAL EQUATIONS

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Move to...

Python 4 Physics 5:19 PM PP

Automatic reply:

To: Briceno, Raul A.

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(No Subject)

To: Python 4 Physics

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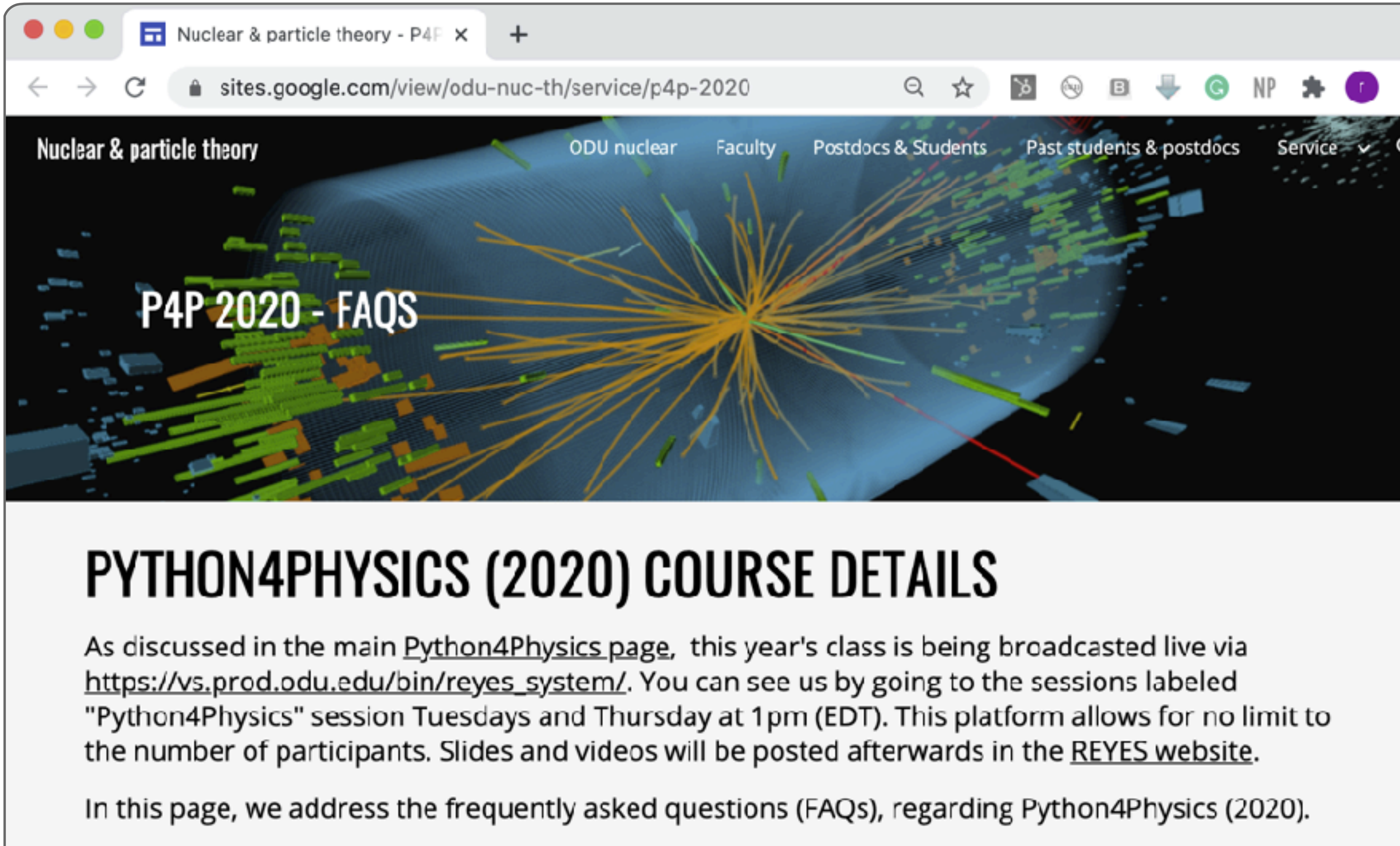

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REVIEW

To understand spins inside material, it is useful to construct a simple model. The most famous of which is the **Ising model [**https://en.wikipedia.org/wiki/Ising_model**]**, named after **Ernst Ising**.

Ising solved the simplest version of this model in 1-dimension (1D), which we consider here.

In the Ising model, spins are fixed. They can either **point up (+1)** or **down (-1)**. Spin inside material can either ***align*** or ***anti-align***.

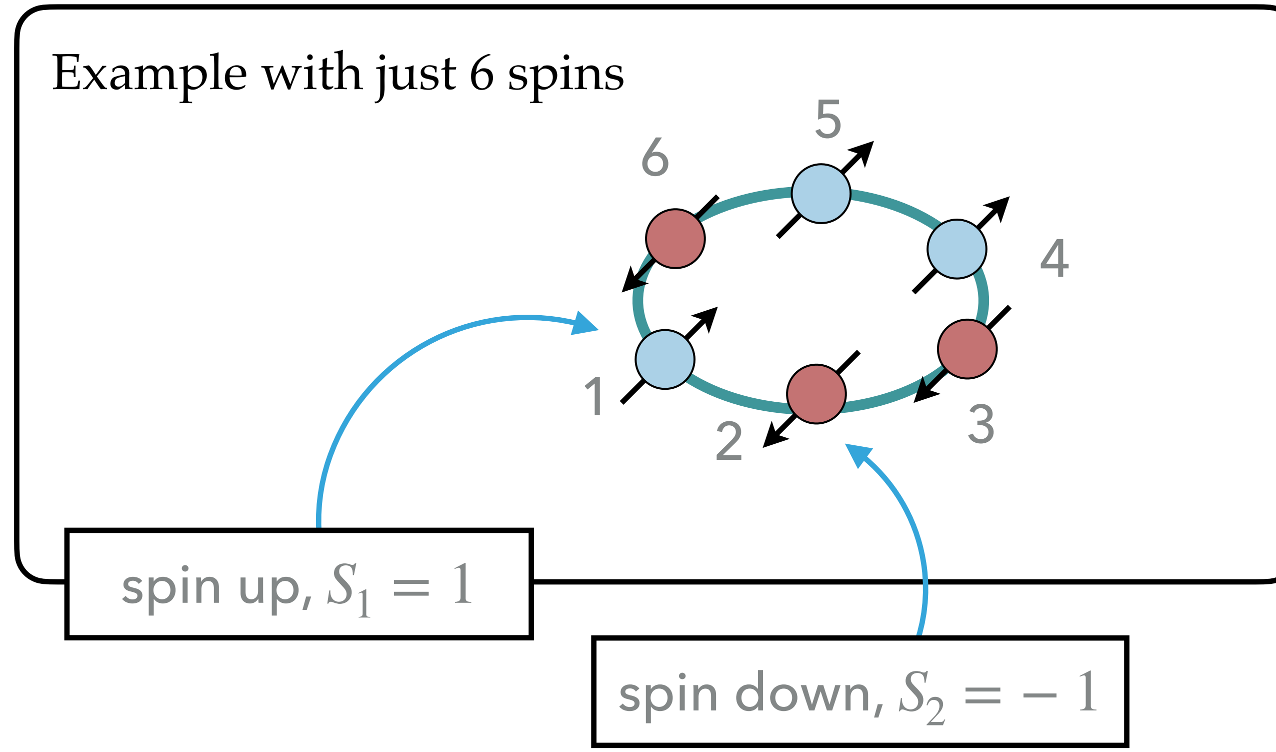
Depending on the material, certain alignment of spins, it might be favorable or dis-favorable for spins to align. This would manifest itself in the energy stored or the energy required to put the systems into a given configuration.

Ernst Ising [1900-1998]

ISING MODEL - PART 2

Consider N spins in a ring.

Enumerate the spin from 1 to N, as shown on the example to the right for N=6. Let i be a number satisfying $1 \leq i \leq N$. Then, we will label the value of the spin for that element in the ring as S_i.



In the example, we have N = 6, and the following values for the individual spins

$$S_1 = 1,\ \ S_2 = -1,\ \ S_3 = -1,\ \ S_4 = 1,\ \ S_5 = 1,\ \ S_6 = -1.$$

ISING MODEL - PART 3

With this, we can make a fairly general model for the energy of a given configuration of spins:

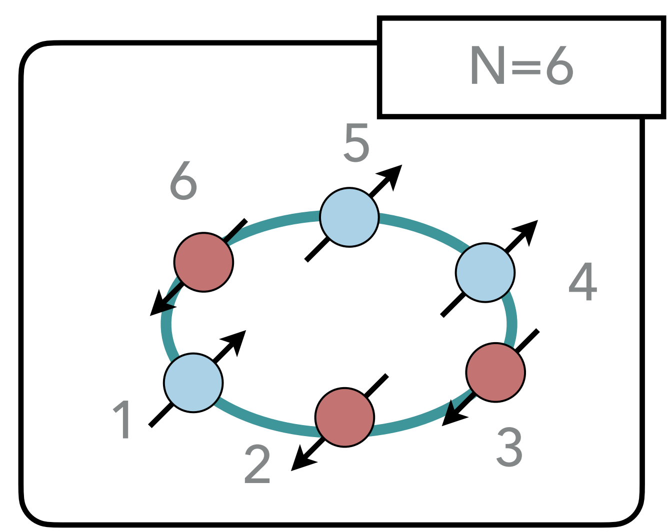
$$\boxed{E(a,b,c) = a\sum_{i=1}^{N} S_i\, S_{i+1} + b\sum_{i=1}^{N} S_i\, S_{i+2} + c\sum_{i=1}^{N} S_i}$$

a, b, c are constants that depend on the properties of the system. Here, we will consider different scenarios where we choose different values for these.

To understand this, we can consider a value for i, then the first term, S_iS_{i+1}, means that one must multiply the two values of the spin together. For this to make sense, we need to define $N+1=1,\ \ N+2=2$, This is referred to as ***"periodic boundary conditions"*.**

For example, here are the numbers for the example shown in the figure:

i	$i+1$	$i+2$	S_i	S_{i+1}	S_{i+2}	S_iS_{i+1}	S_iS_{i+2}
1	2	3	1	-1	-1	-1	-1
2	3	4	-1	-1	1	1	-1
3	4	5	-1	1	1	-1	-1
4	5	6	1	1	-1	1	-1
5	6	1	1	-1	1	-1	1
6	1	2	-1	1	-1	-1	1



EXERCISE #1

Write a function for the energy of N spins in a loop, as a function a, b, c.

$$\boxed{E(a,b,c) = a\sum_{i=1}^{N} S_i\, S_{i+1} + b\sum_{i=1}^{N} S_i\, S_{i+2} + c\sum_{i=1}^{N} S_i}$$

```
def rotate_vec(vec0):
    """
    this piece of code rotates an array (vec0)
    by one to the left
    """
    N=len(vec0)
    tmp = np.zeros(N)

    tmp[0:N-1]=vec0[1:N]
    tmp[N-1]=vec0[0]

    return tmp


def Eising(St,a,b,c):
    "rotate the spins by one"
    Stp1 = rotate_vec(St)
    "rotate the spins by two"
    Stp2 = rotate_vec(Stp1)

    "nearest neighbors"
    dEa = a * sum(St * Stp1)
    "next-to-nearest neighbors"
    dEb = b * sum(St * Stp2)
    "background field"
    dEc = c * sum(St)

    return dEa + dEb + dEc
```

look inside my code 😎

EXERCISE #2

Write code to list all possible configurations for N spins.

```python
def ising_lists(N):
    """
    if we N possible spins,
    we have 2^N possible states

    we will put these into lists,
    starting with a single spin that is
    either up or down
    """
    TOT=[[1],[-1]]

    for n in range(N-1):

        "tmp0 is a place holder for some of the states"

        Nt = len(TOT)
        for j0 in range(Nt):
            state1 = TOT[j0][:]
            state2 = TOT[j0][:]

            "we add 1 to the first one"
            state1.append(1)
            "we add -1 to the first one"
            state2.append(-1)

            TOT[j0]=state1
            TOT.append(state2)


    return TOT
```

look inside my code 😎

EXERCISE #3-7

Find the ***ground state energy*** and ***degeneracy*** for the following values of the parameters.

Exercise #3: $N = 3,\ a = 5,\ b = 1,\ c = 0.01$

Exercise #4: $N = 10,\ a = -5,\ b = 1,\ c = 0$

Exercise #5: $N = 10,\ a = -5,\ b = 1,\ c = 0.01$

Exercise #6: $N = 10,\ a = -5,\ b = 1,\ c = -0.01$

how do we make sense of these three results?

Exercise #7: $N = 15,\ a = 15,\ b = -0.2,\ c = 0.02$

EXERCISE #8

Consider a system with N=10 spins. This can have $2^{10} = 1{,}024$ distinct states.

Consider $a = 1, b = 0.1$.

Vary c continuously from *0* to -*3*.

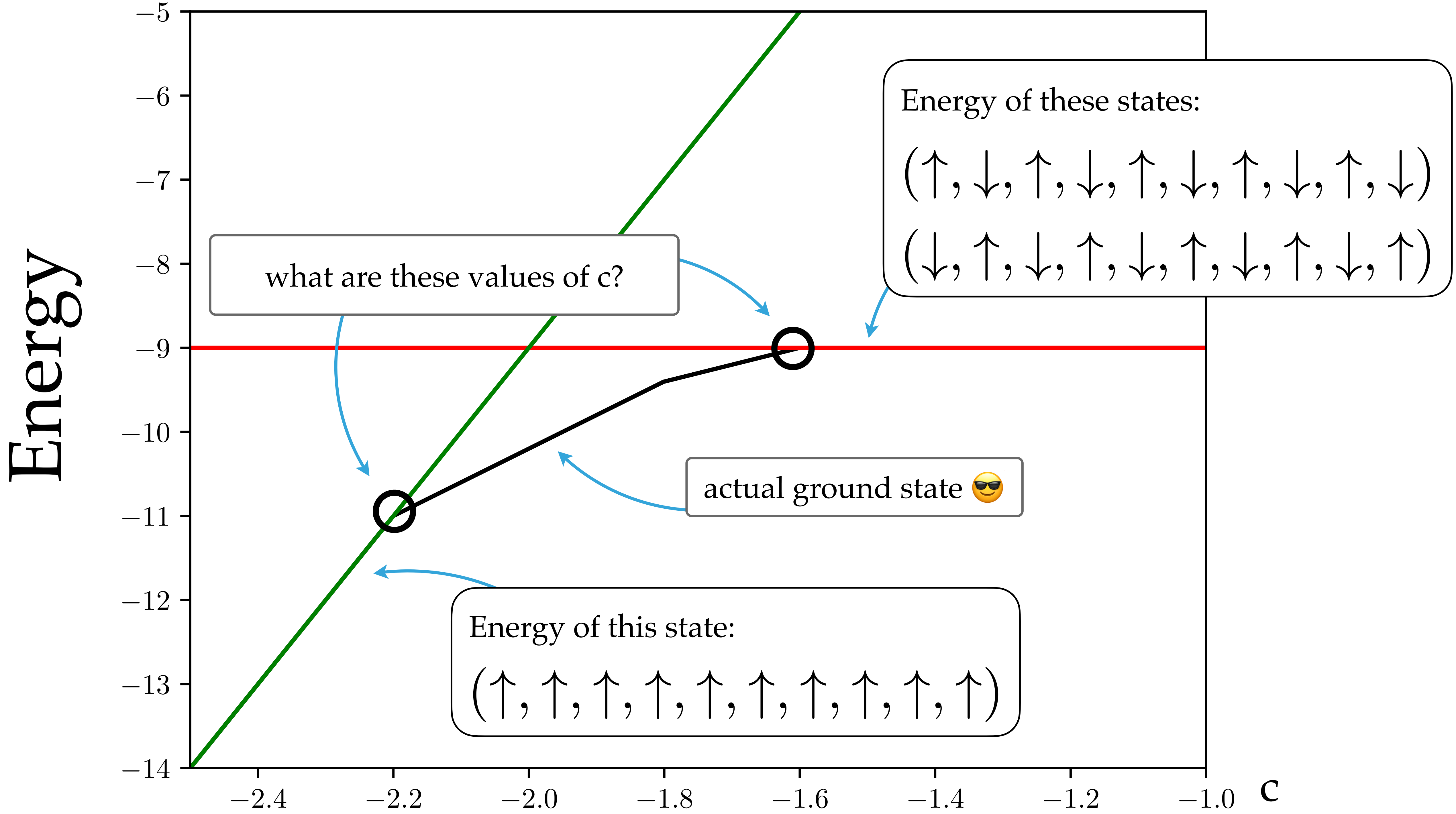
Q3: At what value of c are these no longer the ground state?

$$\boxed{\begin{array}{c}(\uparrow,\downarrow,\uparrow,\downarrow,\uparrow,\downarrow,\uparrow,\downarrow,\uparrow,\downarrow)\\(\downarrow,\uparrow,\downarrow,\uparrow,\downarrow,\uparrow,\downarrow,\uparrow,\downarrow,\uparrow)\end{array}}$$

Q4: At what value of c is this the ground state? [note, these two do not have have to be the same values for c]

$$\boxed{(\uparrow,\uparrow,\uparrow,\uparrow,\uparrow,\uparrow,\uparrow,\uparrow,\uparrow,\uparrow)}$$

EXERCISE #8



Energy
c
−5
−6
−7
−8
−9
−10
−11
−12
−13
−14
−2.4
−2.2
−2.0
−1.8
−1.6
−1.4
−1.2
−1.0
what are these values of c?
Energy of these states:
$(\uparrow, \downarrow, \uparrow, \downarrow, \uparrow, \downarrow, \uparrow, \downarrow, \uparrow, \downarrow)$
$(\downarrow, \uparrow, \downarrow, \uparrow, \downarrow, \uparrow, \downarrow, \uparrow, \downarrow, \uparrow)$
actual ground state 😎
Energy of this state:
$(\uparrow, \uparrow, \uparrow, \uparrow, \uparrow, \uparrow, \uparrow, \uparrow, \uparrow, \uparrow)$


WHAT'S NEXT?

OUTLOOK

This week: introduction to calculus and differential equations

Next Tuesday - Dr. F. Aslan : An introduction to Quantum Mechanics

Next Thursday - Models, Infection deceases, and more on differential equations

F. Aslan, Ph.D.

Postdoctoral fellow - JLab/UConn

CHANGE AND THE SYMBOL FOR IT

In what follows we are interested in evaluating the change of different quantities.

Consider a function of some independent variable x: $f(x)$

At some point x has a value x_1. At some later time it has the value x_2.

We can define the change in x using the Greek letter Δ: $\Delta x \equiv x_2 - x_1$

Similarly, we can define the change in $f(x)$: $\Delta f \equiv f(x_2) - f(x_1)$.

At times we will want to specify the shift in $f(x)$ in a particular way. We will define the initial and final point as $x_1 = x - a$ and $x_2 = x + a$ respectively where the distance from final and initial values of x is $x_2 - x1 = 2a$. The the difference labeled by a is defined as $\Delta_a f(x) \equiv f(x + a) - f(x - a)$

descriptive question	*descriptive quantities*
[*Where is the object?*]	[*position*]

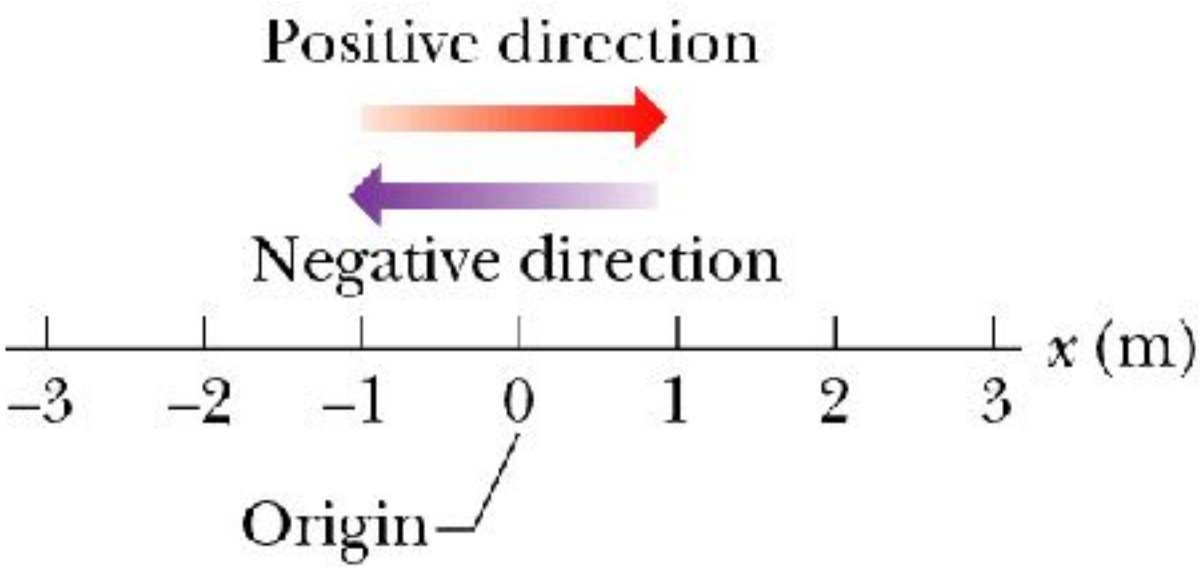
descriptive question	*descriptive quantities*
Where is the object?	*position*
[How far has it travelled?]	*[displacement, distance]*

DESCRIBING MOTION

descriptive question	*descriptive quantities*
Where is the object?	*position*
How far has it travelled?	*displacement, distance*
[*Where is it going?* *How fast is it going?*]	[*speed, velocity*]

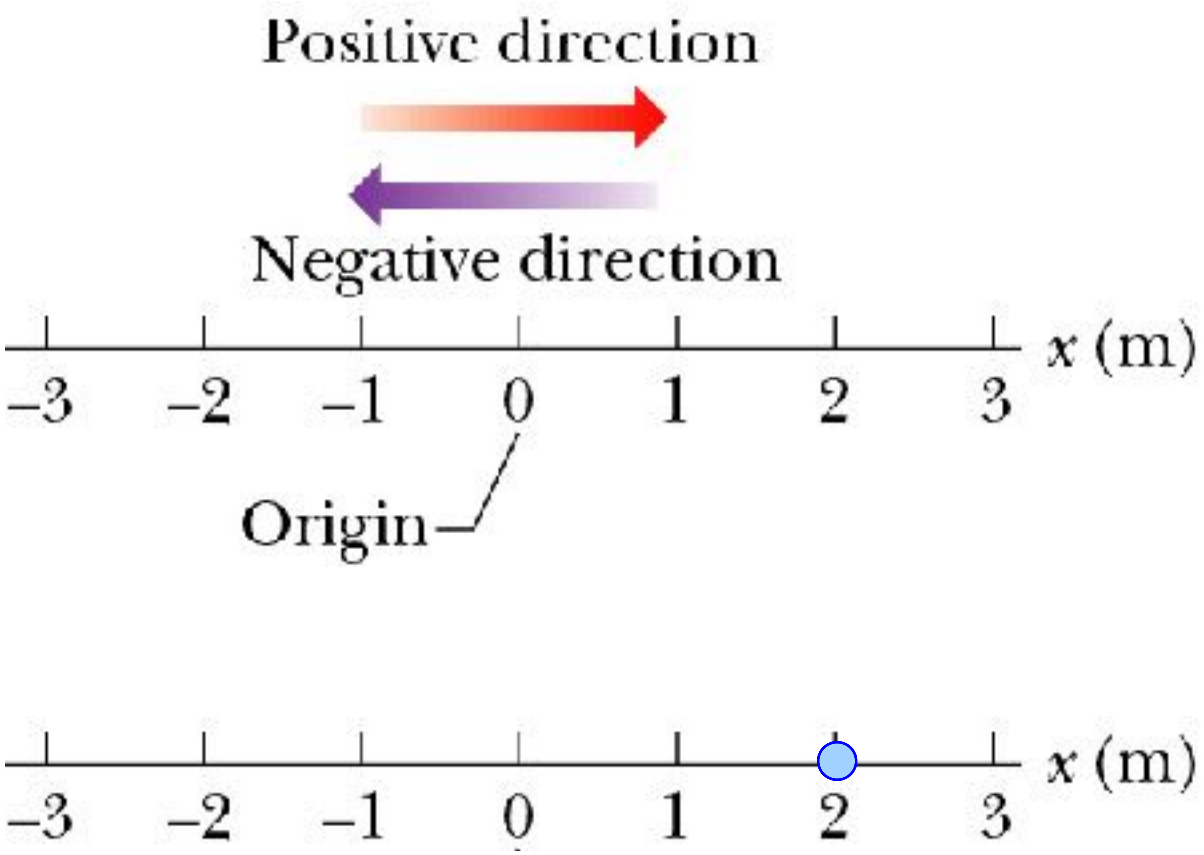
DESCRIBING MOTION

descriptive question	*descriptive quantities*
Where is the object?	*position*
How far has it travelled?	*displacement, distance*
Where is it going?	
How fast is it going?	*speed, velocity*
[*Is it speeding up, slowing down, or neither?*]	[*acceleration*]



Reference frames

To locate an object means to find its position relative to some **reference point,** often the origin (or zero point).



Reference frames

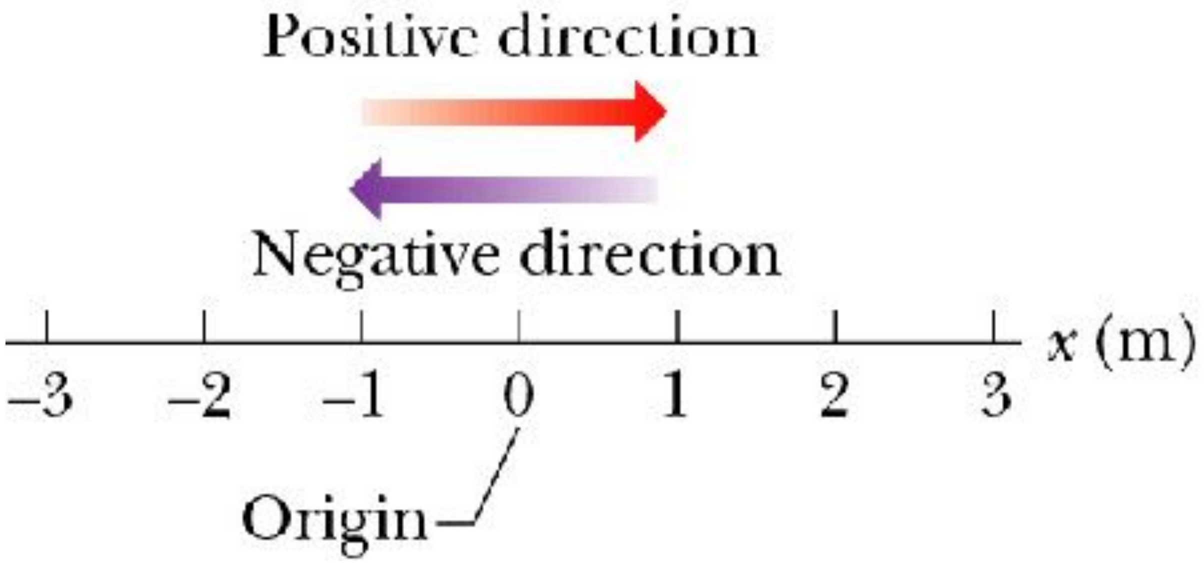
To locate an object means to find its position relative to some **reference point,** often the origin (or zero point).

The position of the ball is

$$\boxed{x = 2m}$$

The + indicates the direction to the right of the origin.

POSITION – WHERE IS AN OBJECT LOCATED?



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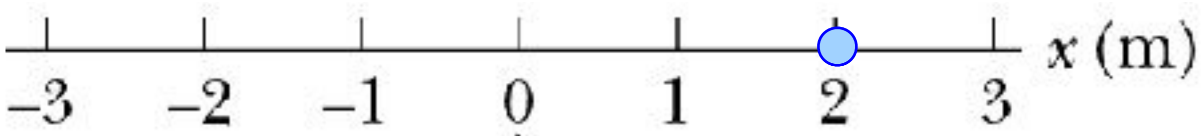
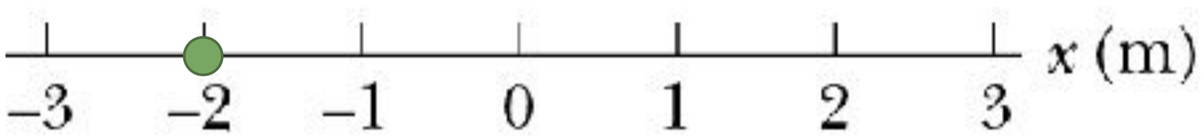


The position of the ball is

$$\boxed{x = 2m}$$

The + indicates the direction to the right of the origin.

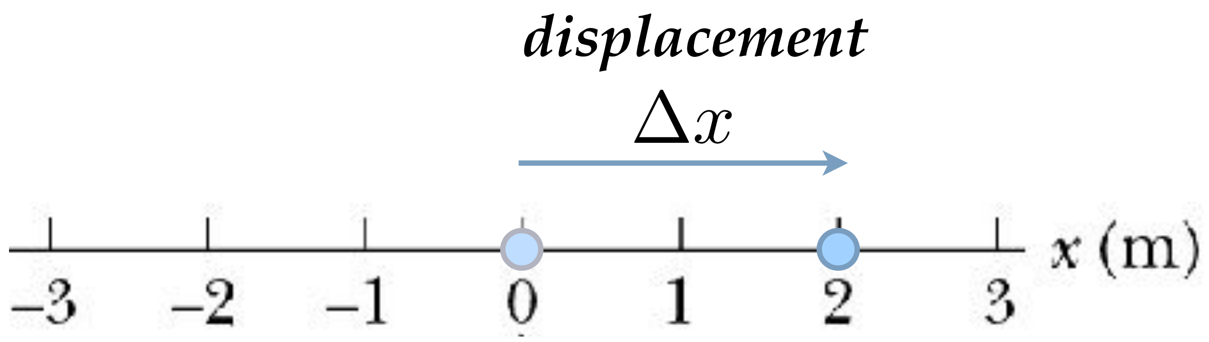


The position of the ball is

$$\boxed{x = -2m}$$

The - indicates the direction to the left of the origin.

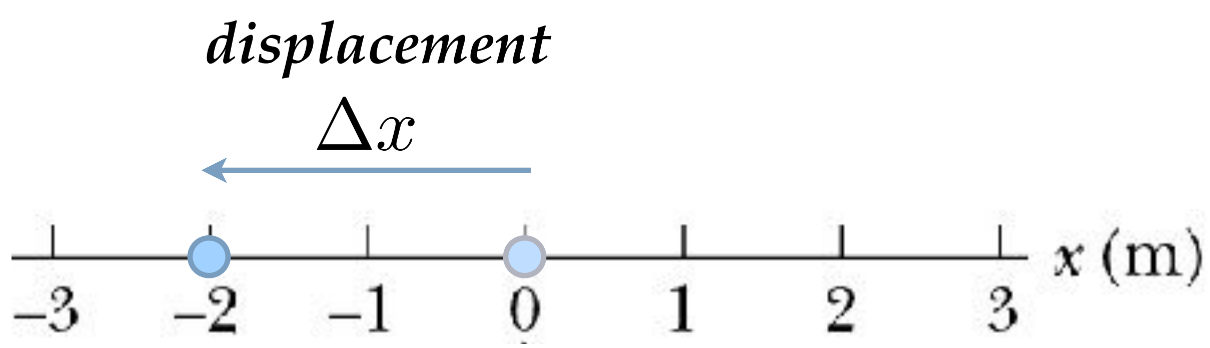
- ➜ 'distance' = total ground covered while traveling
- ➜ 'displacement' = ***vector*** from where you started to where you end up
- ➜ displacement and distance can be quite different



$$\Delta x = x_f - x_i$$
$$= 2m - 0m$$
$$= 2m$$

sign indicates the direction

& distance = 2m



$$\Delta x = x_f - x_i$$
$$= -2m - 0m$$
$$= -2m$$

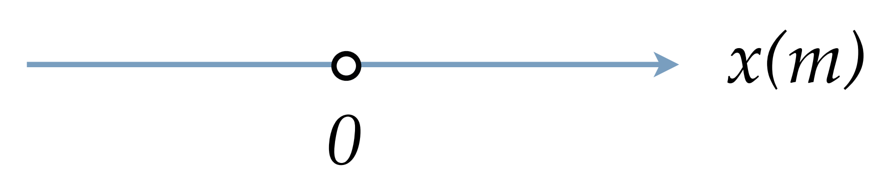
but distance = 2m

Question (3min)

A car moves 72 km to the right and then reverses direction and travels 120 km to the left.

1. What is the displacement of the car?

   a. 72 km
   b. -72 km
   c. 192 km
   d. -192 km
   e. 48 km
   f. -48 km



2. What is the distance travelled by the car?

   a. 72 km
   b. -72 km
   c. 192 km
   d. -192 km
   e. 48 km
   f. -48 km

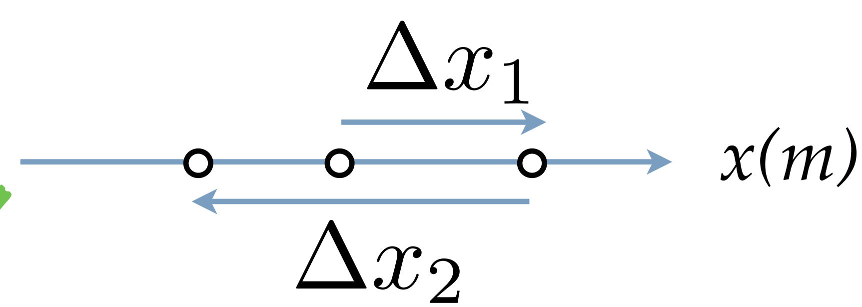
Question (3min)

A car moves 72 km to the right and than reverses direction and travels 120 km to the left.

1. What is the displacement of the car?

$$\begin{aligned}\Delta x &= x_f - x_i \\ &= \Delta x_1 + \Delta x_2 \\ &= (72\text{km} - 0\text{km}) + (-120\text{km}) \\ &= -48\text{km}\end{aligned}$$

a. 72 km
b. -72 km
c. 192 km
d. -192 km
e. 48 km
f. -48 km



2. What is the distance travelled by the car?

$$\begin{aligned}d &= |\Delta x_1| + |\Delta x_2| \\ &= |72\text{km} - 0\text{km}| + |-120\text{km}| \\ &= 192\text{km}\end{aligned}$$

a. 72 km
b. -72 km
c. 192 km
d. -192 km
e. 48 km
f. -48 km

Position - Where you are and **how far** you travelled

Speed – **How fast** you are moving

$$\boxed{\text{Average speed} = \frac{\text{total distance travelled}}{\text{time lapsed}}}$$

Velocity has magnitude and direction – It's a vector!

Examples:

- 30 mi/hr East
- 2 m/s up
- 25 km/s toward Richmond

Speed can be the magnitude of the velocity. It is a scalar.

Average velocity is the displacement divided by the time taken

$$\boxed{\bar{\nu} = \frac{\text{displacement}}{\text{time elapsed}} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}}$$

Dimensions of velocity – [m/s] in SI units (the same as speed!)

Velocity has magnitude and direction – It's a vector!

Examples:

- 30 mi/hr East
- 2 m/s up
- 25 km/s toward Richmond

Speed can be the magnitude of the velocity. It is a scalar.

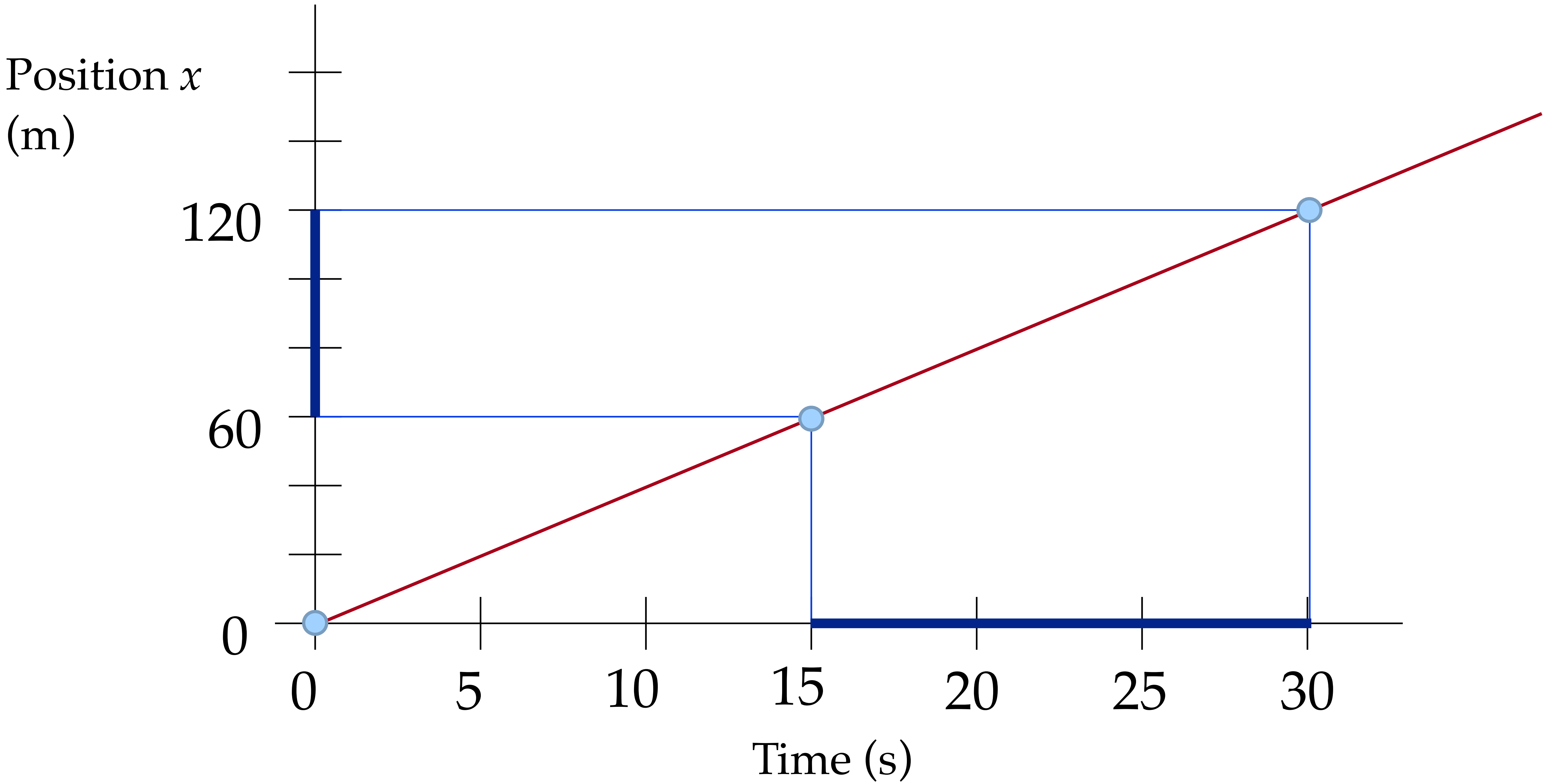
Average velocity is the displacement divided by the time taken

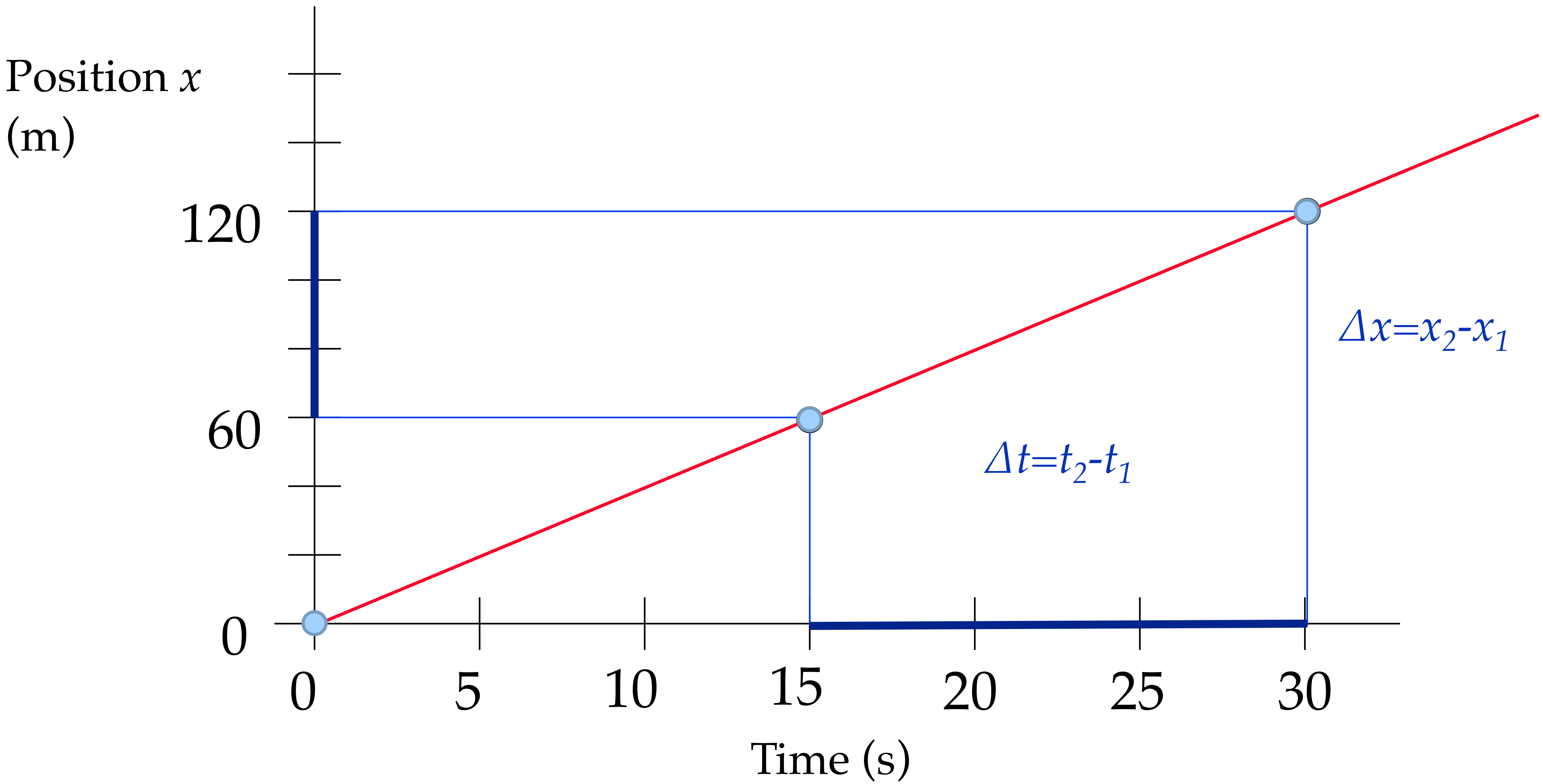
$$\boxed{\bar{\nu} = \frac{\text{displacement}}{\text{time elapsed}} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}}$$

the little bar on top means "average"

Imagine that at some point t_1=15 s your position is x_1=60 m and at t_2=30 s your position is x_2=120 m.

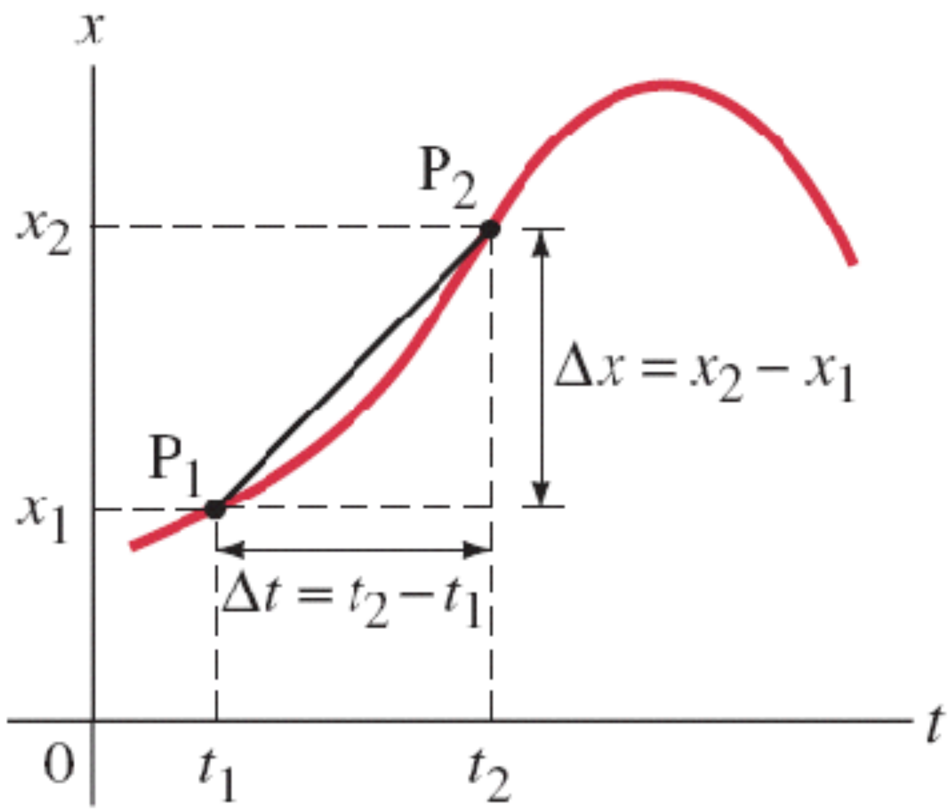
Let's plot position vs. time:

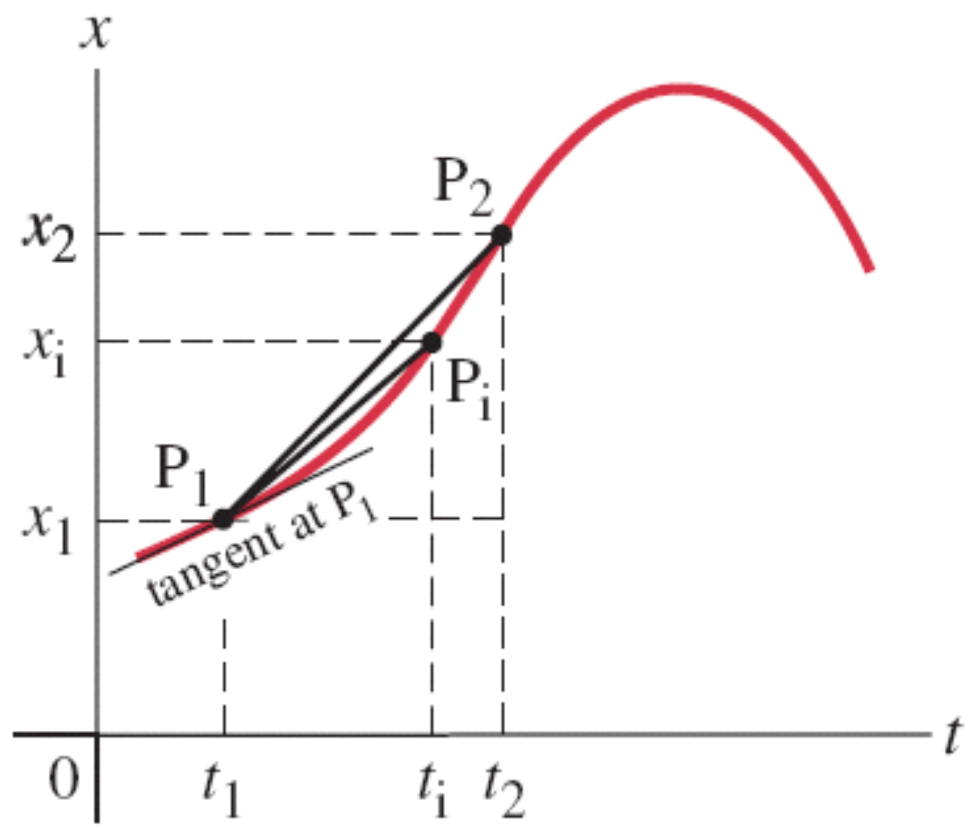




Slope gives velocity: $\bar{\nu} = \dfrac{x_2 - x_1}{t_2 - t_1}$ ➡ $\bar{\nu} = \dfrac{60\text{m}}{15\text{s}} = 4\dfrac{\text{m}}{\text{s}}$

Instantaneous velocity is a vector defined as "how fast" a particle is moving at a given instant.





$$\bar{\nu} = \frac{x_2 - x_1}{t_2 - t_1}$$

$$\boxed{\nu = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}}$$

Instantaneous velocity is the tangent to the curve at any point.

Acceleration is the rate of change of velocity

Average acceleration is a vector given as:

$$\boxed{\bar{a} = \frac{\Delta \nu}{\Delta t} = \frac{\nu_2 - \nu_1}{t_2 - t_1}}$$

Unit of acceleration: $\left[\frac{m}{s^2}\right]$

Acceleration is the rate of change of velocity

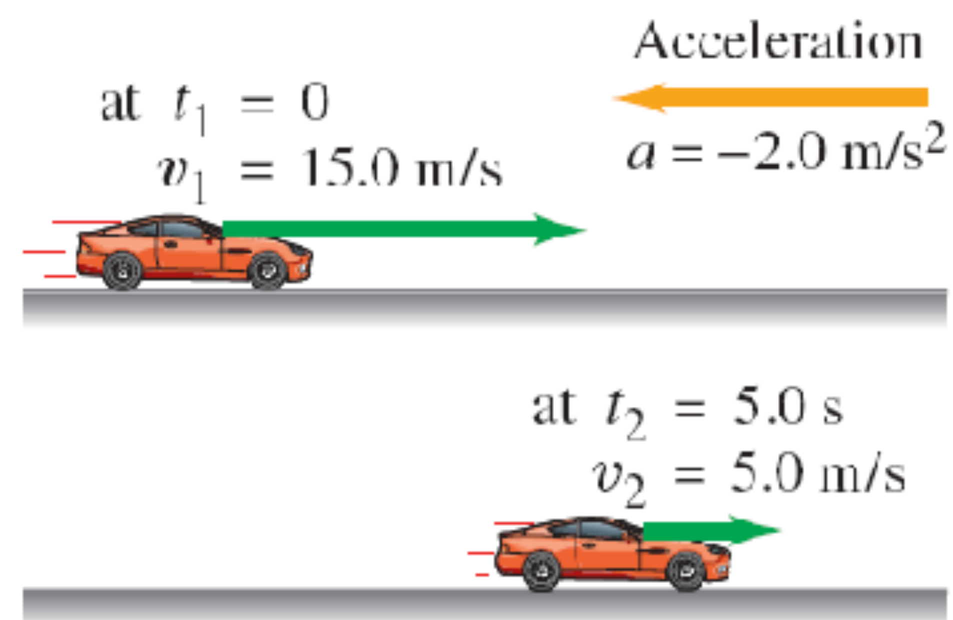
Average acceleration is a vector given as:

$$\boxed{\bar{a} = \frac{\Delta \nu}{\Delta t} = \frac{\nu_2 - \nu_1}{t_2 - t_1}}$$

Unit of acceleration: $\left[\frac{m}{s^2}\right]$

Example: A car moving at initial velocity of 15 m/s slows down to 5 m/s in 5 s. What is the car's average acceleration?

$$\bar{a} = \frac{\nu_2 - \nu_1}{t_2 - t_1} = \frac{5\frac{m}{s} - 15\frac{m}{s}}{5s - 0s} = -2\frac{m}{s^2}$$



Velocity is decreasing ➔ acceleration is negative
➔ the object is decelerating

CONCEPTUAL QUESTION

If the acceleration is zero, does it mean that the velocity is zero?

A. Yes.

B. No. ✓

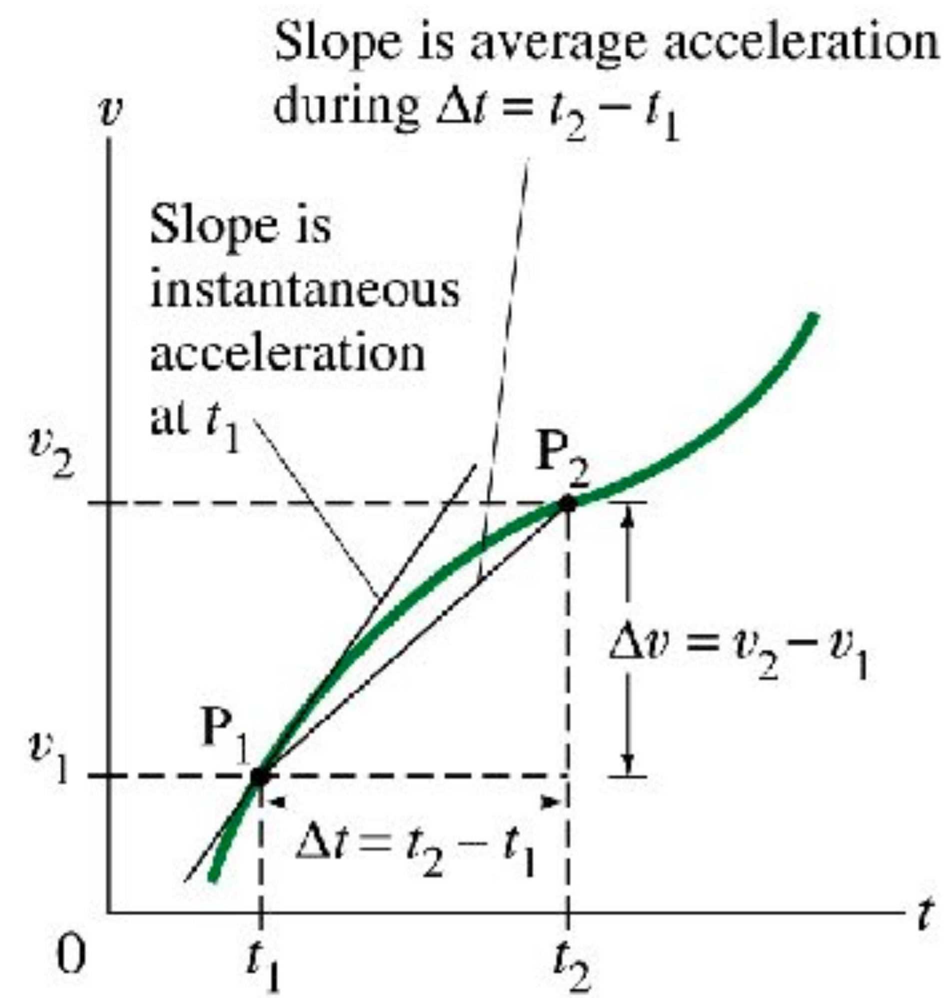
C. Maybe.

Acceleration is the rate of change of velocity.
When acceleration is zero ➔ change of velocity is zero ➔ velocity is constant (it can but it doesn't have to be zero)

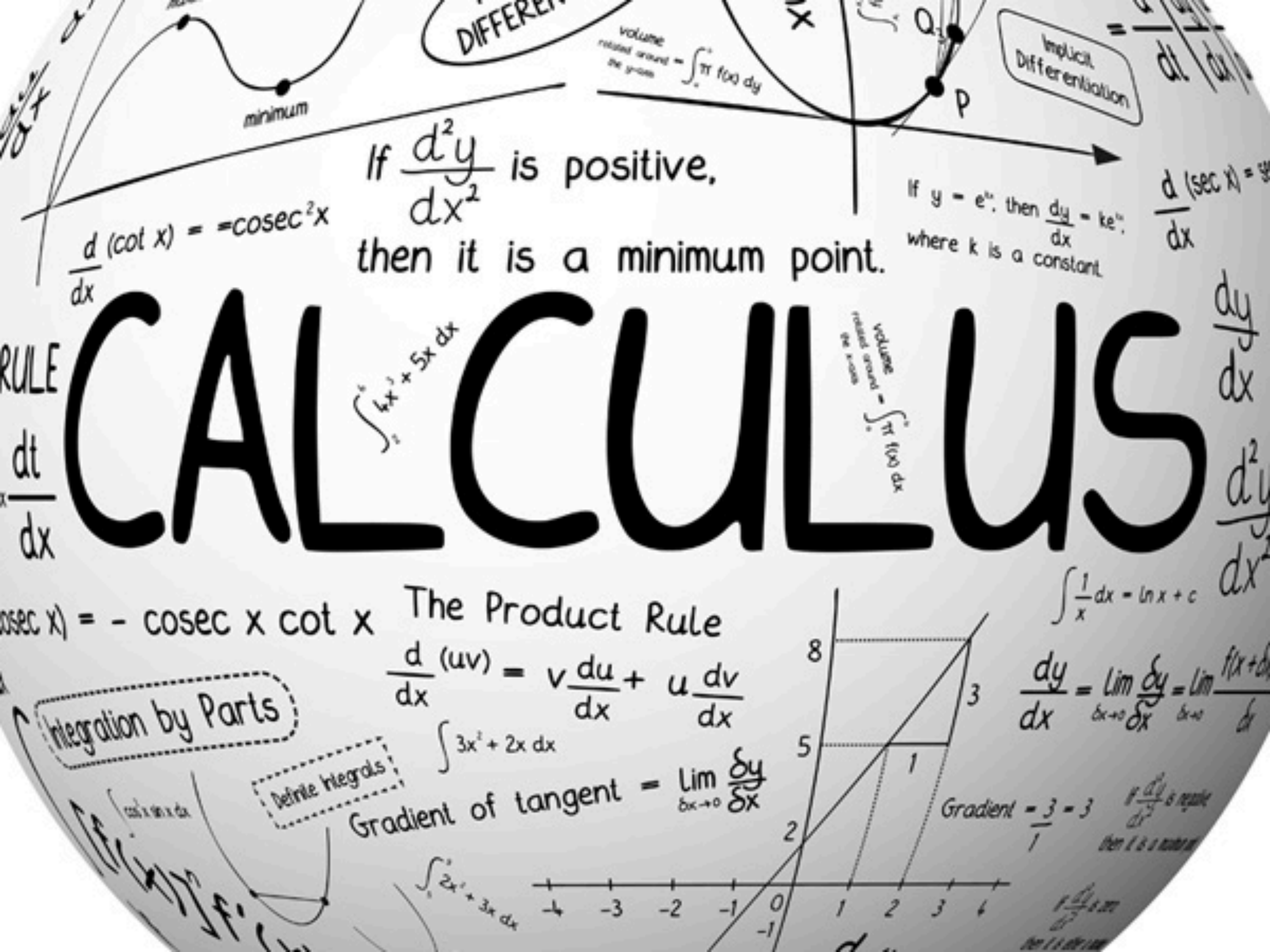
Instantaneous acceleration is the average acceleration in the limit as the time interval becomes infinitesimally short.

$$\boxed{a = \lim_{\Delta t \to 0} \frac{\Delta \nu}{\Delta t} = \frac{d\nu}{dt}}$$

Unit of acceleration: $\left[\frac{m}{s^2}\right]$



Instantaneous acceleration is the slope of the tangent to the v vs. t curve at that time.



If $\frac{d^2y}{dx^2}$ is positive,
then it is a minimum point.
CALCULUS
minimum
Implicit Differentiation
If $y = e^{kx}$, then $\frac{dy}{dx} = ke^{kx}$, where k is a constant
$\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$
RULE
The Product Rule
$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$
Integration by Parts
Definite Integrals
$\int 3x^2 + 2x\,dx$
Gradient of tangent $= \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$
$\int \frac{1}{x}dx = \ln x + c$
$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$
Gradient $= \frac{3}{1} = 3$

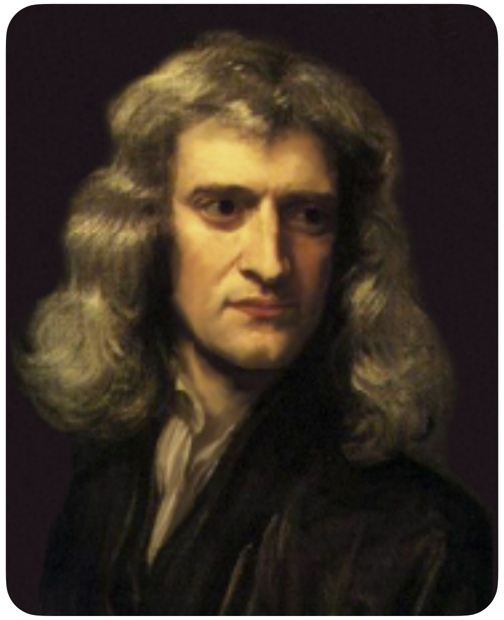
Calculus is a remarkably powerful field of mathematics. Although its history dates back to ancient Greece, the formalization of the field is largely attributed to **Newton** and **Leibniz**, who made important independent contributions.

Leibniz's interest lied primarily in setting foundational notation and mathematical rules. While **Newton's** interest was always describing the motion of physical objects, which is our interest as well.

Calculus can be divided into two fields:

<u>differential calculus:</u> interested in the rate of change of quantities, or equivalently the slopes of the corresponding curves

<u>integral calculus:</u> interested in the continuous sum of function, or equivalent the area under the curve of a function or between different functions.



Isaac Newton
England, 1643 - 1727

Gottfried Wilhelm Leibniz
France, 1646 - 1716

DIFFERENTIAL EQUATIONS

Differential calculus focuses on making sense and defining this limit:

$$\boxed{\frac{d}{dx}f(x) \equiv \lim_{a\to 0}\frac{\Delta_a f(x)}{2a} = \lim_{a\to 0}\frac{f(x+a)-f(x-a)}{2a}}$$

derivative of $f(x)$

This is of course the instantaneous rate of change of the $f(x)$.

Determining this for different functions analytically is not always easy…

…numerically it is much easier

EXERCISE #1 - DEFINE DERIVATIVE

Write code for a "*derivative*" of any function, f, at x for a non-zero value of a.

Note, in Python, you can define functions which take function as inputs. Here is an example of this.

```
def fin_deriv(f, x, a):
    """
    this defines the rate of change of the function f
    as a function of x

    the final point is at xf = x + a
    the initial point is at xi = x - a
    the difference between these two is xf - xi = 2*a
    """

    "the numerator"
    num = f(x+a) - f(x-a)
    "the denominator"
    denum = 2.0* a

    return num / denum
```

look inside my code 😎

A **derivative** is a function the rate of change of that function.

Here is a list of derivatives of some of the most common functions, where c = constant:

$$\frac{dc}{dx} = 0 \qquad \frac{d}{dx}\sin(cx) = c\cos(cx) \qquad \frac{d}{dx}\ln(cx) = \frac{1}{x}$$

$$\frac{d}{dx}(cx^n) = cnx^{n-1} \qquad \frac{d}{dx}\cos(cx) = -c\sin(cx) \qquad \frac{d}{dx}e^{cx} = ce^{cx}$$

if not yet, you should memorize them 🤓

TESTING CONVERGENCE - PART1

Remember, the derivative is defined in the limit that $a = 0$. So the first thing we should do is test the "*convergence*" of our code. In particular, we want to plot the resulting function at some value of x as a function of a .

WARNING: In practice, we cannot take a to be exactly zero. This is not well defined numerically. Instead, we need to verify that if a is sufficiently small, our answer no longer depends on the value chosen for a. If we reach a value of a, and our answer no longer varies we would say that the answer has converged.

We can visualize the answer converging to a desired answer, but we would also like to introduce a quantity that measures the error of our result. If the derivative is independent of a, then the derivative of this with respect to a must be zero.

To define this, let us follow these steps.

1) Calculate the derivative first at $a + \delta_a$: $\dfrac{\Delta_{a+\delta} f(x)}{2(a+\delta)}$

2) Then you calculate at a smaller value, a: $\dfrac{\Delta_a f(x)}{2a}$

3) Take the difference between these two values and divide by δ.

4) In order to see this better, we will multiply this derivative by 100:

$$\boxed{\sigma_a = \frac{100}{\delta} \times \left| \frac{\Delta_a f(x)}{2a} - \frac{\Delta_{a+\delta} f(x)}{2(a+\delta)} \right|}$$

EXERCISE #2 - TEST CONVERGENCE

In what follows we will vary $a = [10^{-5}, .1)$ and fix x = 1.

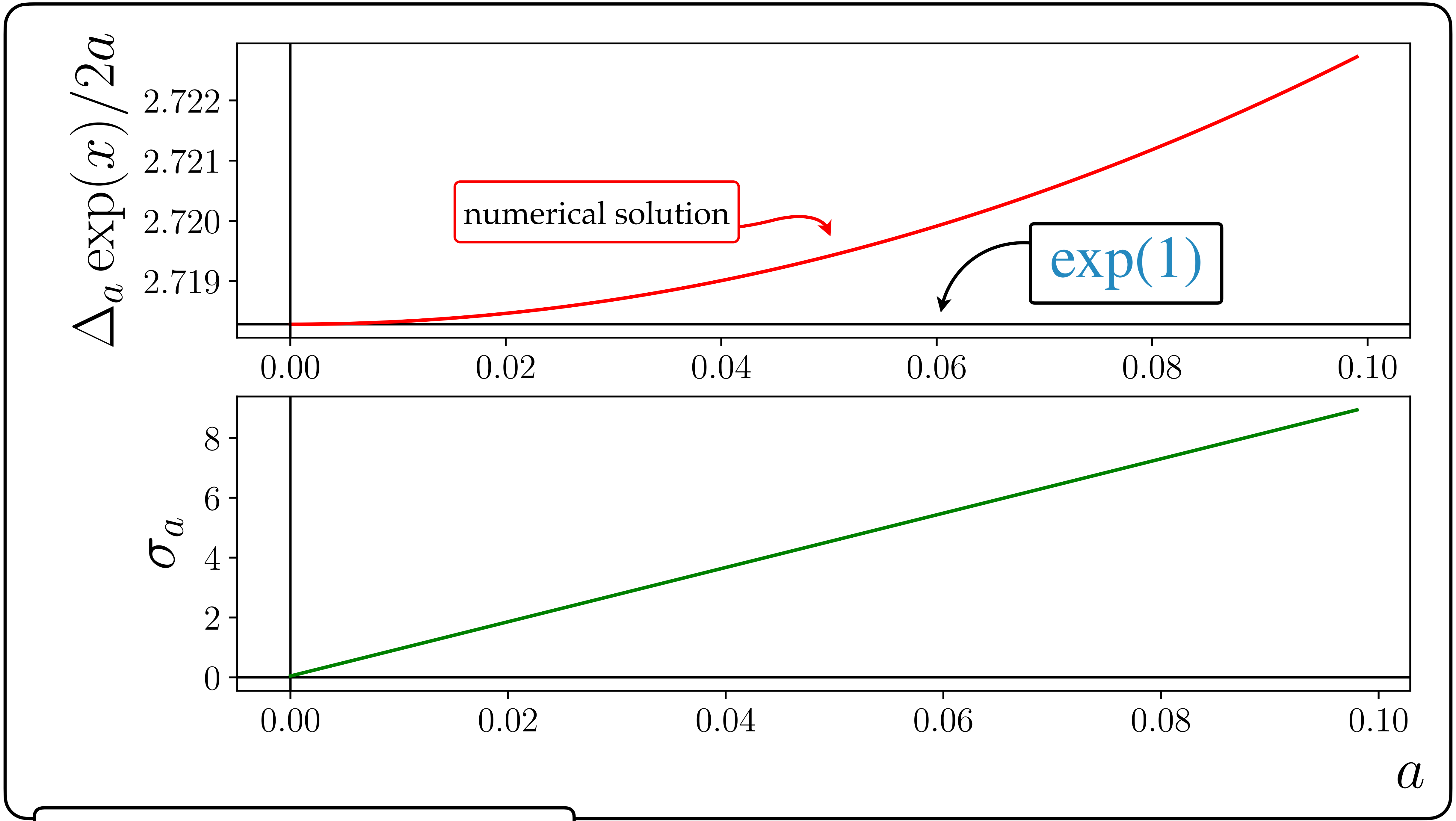
```
x = 1
a=np.arange(pow(10,-5),.1,pow(10,-3))
```

Note, I chose to vary a in steps of 10^{-3}, but you are welcomed to try a different step size.

Let us test your code by taking the numerical derivative of the following functions:

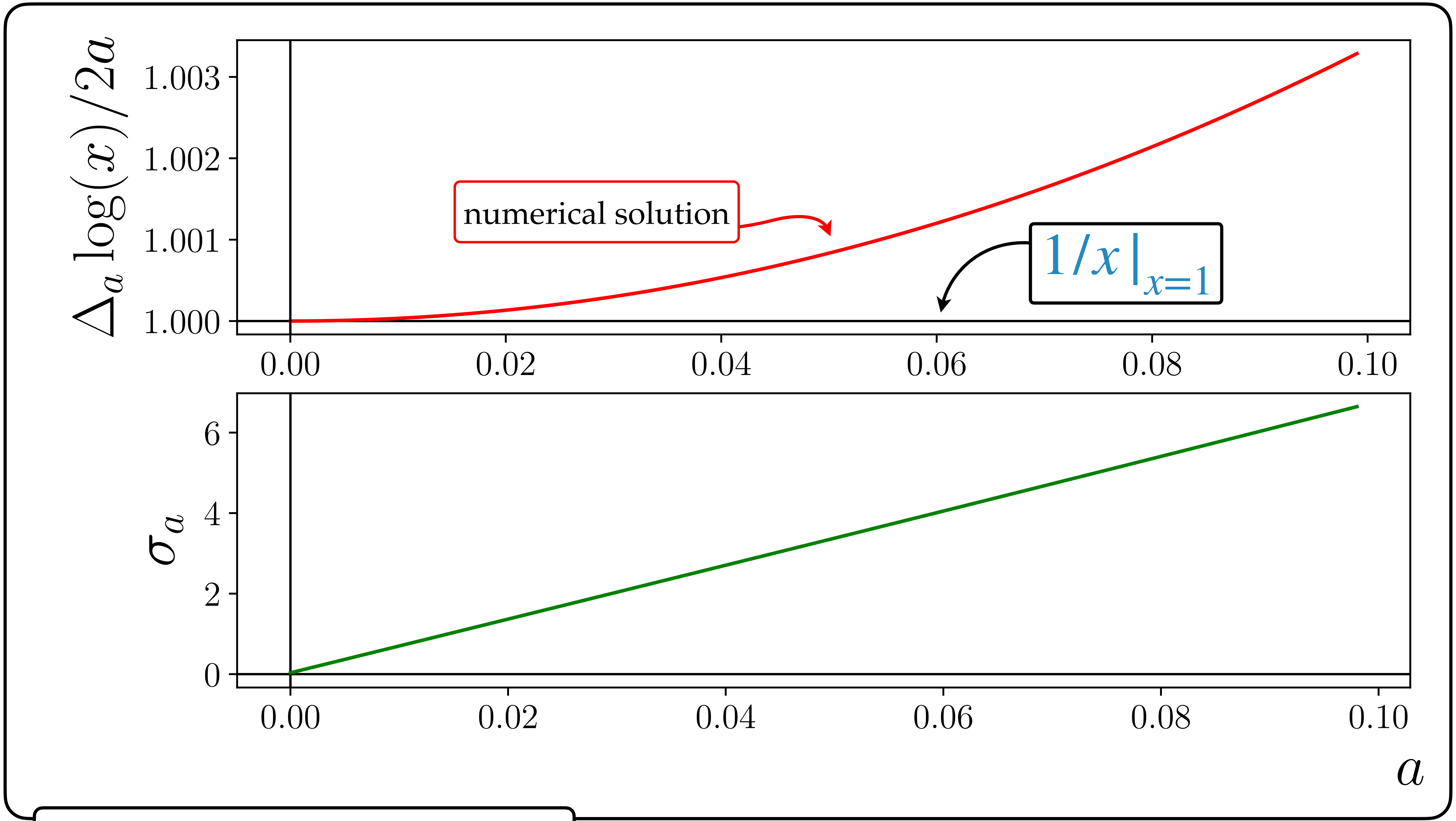
$$\exp(x),\ \ \log(x),\ \ \sin(x),\ \ \cos(x),\ \frac{x^3}{6} + \frac{x^4}{8}$$

EXERCISE #2 - TEST CONVERGENCE: $\exp(x)$



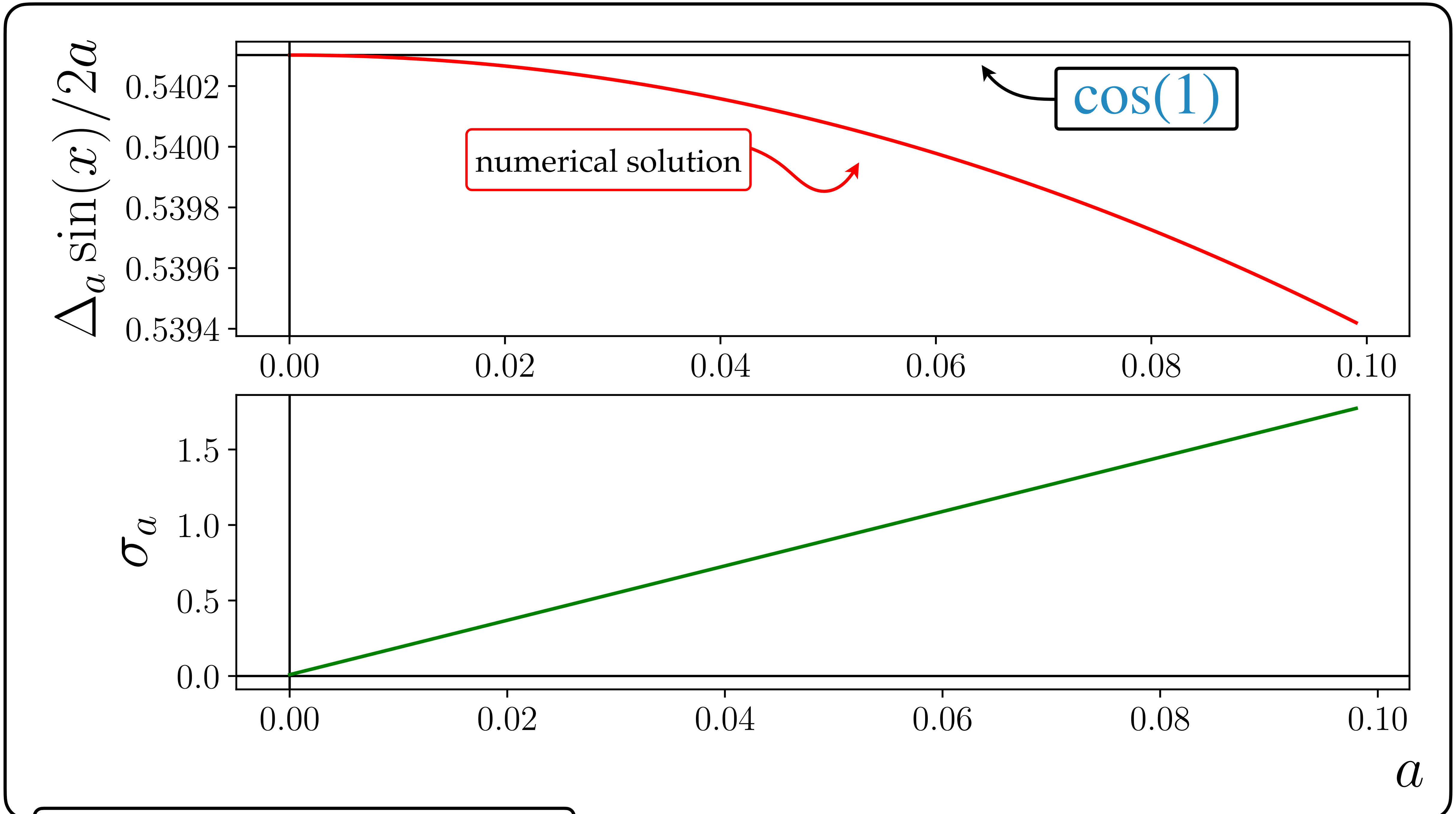
$$\sigma_a = \frac{100}{\delta} \times \left| \frac{\Delta_a f(x)}{2a} - \frac{\Delta_{a+\delta} f(x)}{2(a+\delta)} \right|$$

EXERCISE #2 - TEST CONVERGENCE: *log*(x)



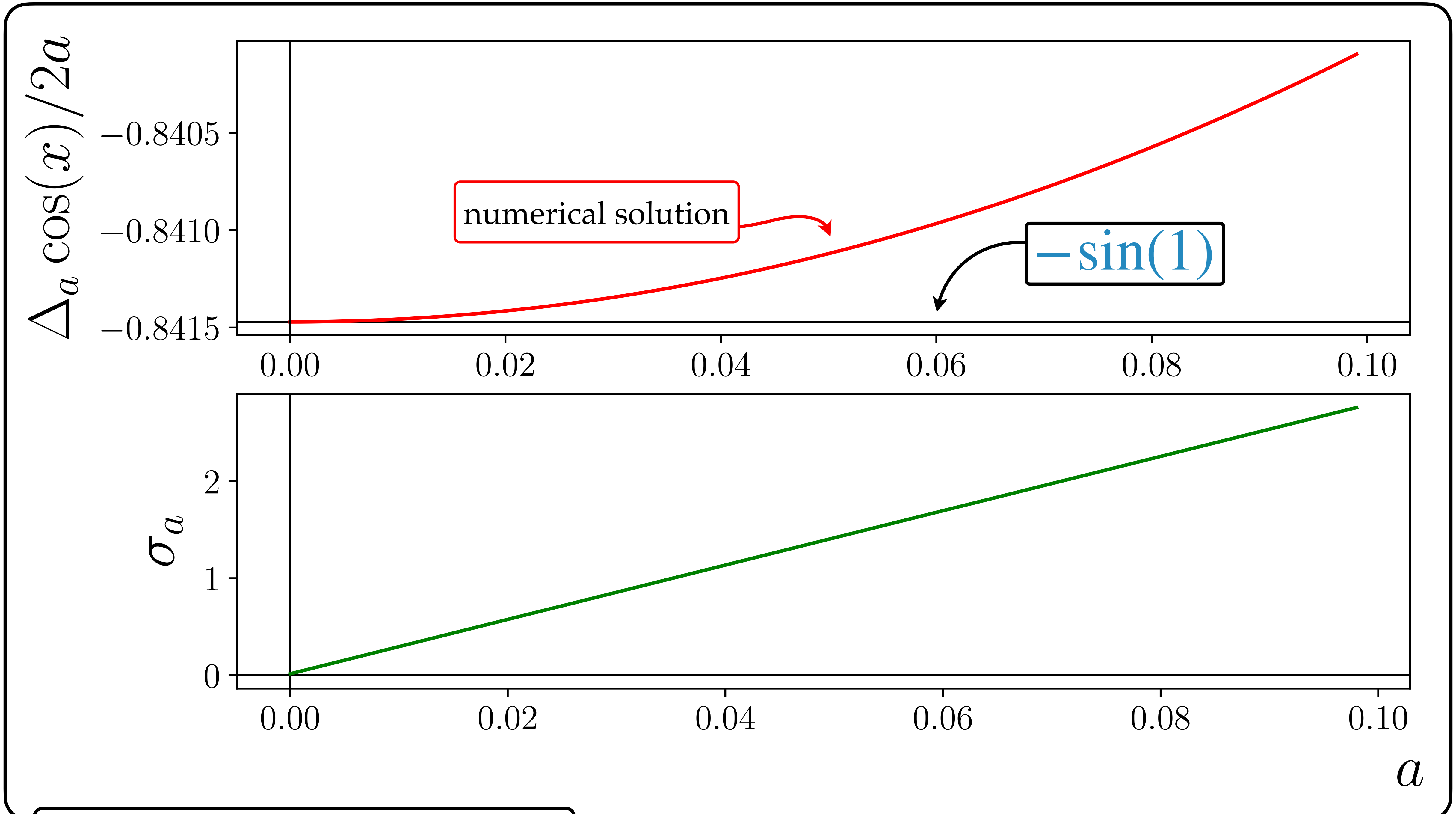
$$\sigma_a = \frac{100}{\delta} \times \left| \frac{\Delta_a f(x)}{2a} - \frac{\Delta_{a+\delta} f(x)}{2(a+\delta)} \right|$$

EXERCISE #2 - TEST CONVERGENCE: $\sin(x)$



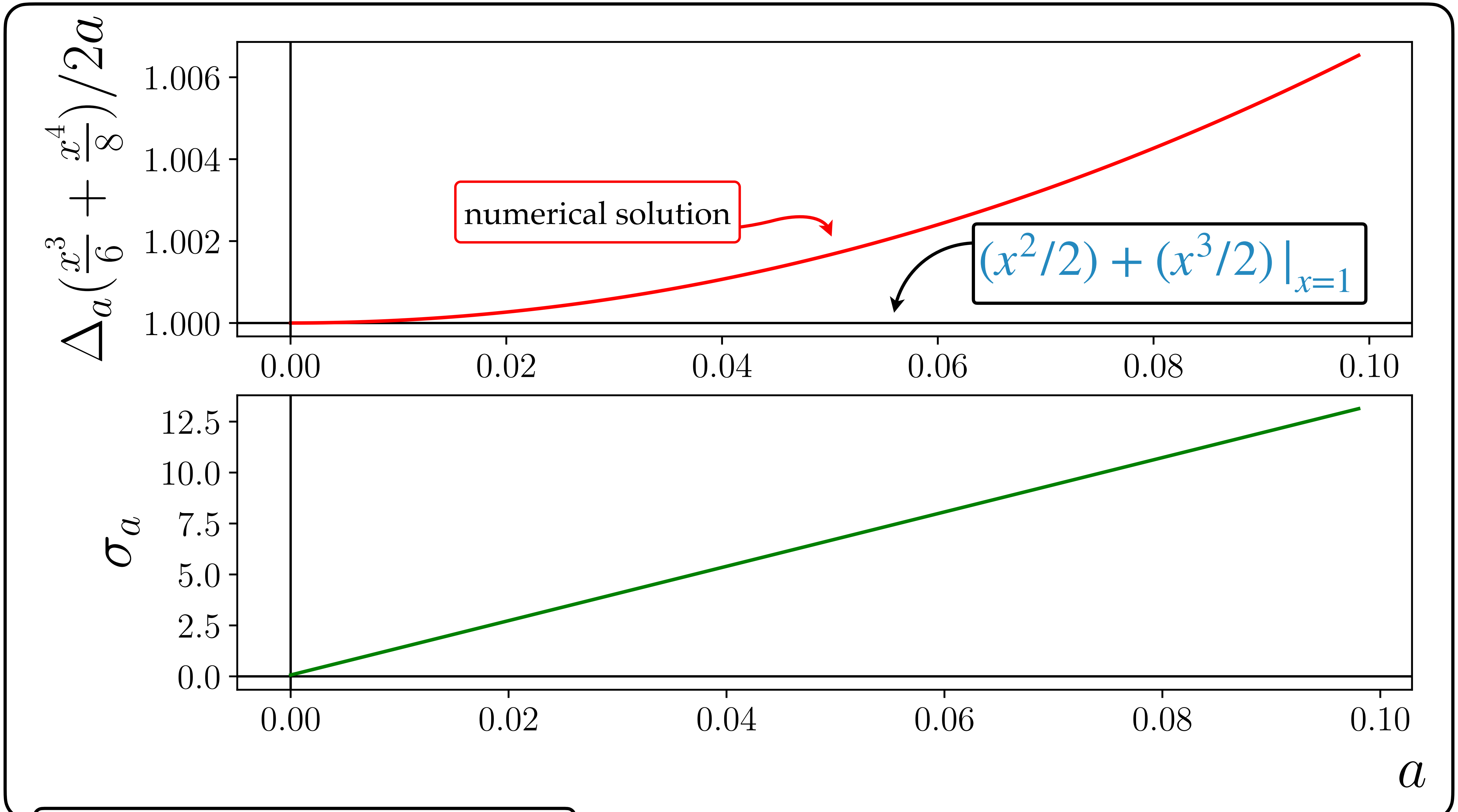
$$\sigma_a = \frac{100}{\delta} \times \left| \frac{\Delta_a f(x)}{2a} - \frac{\Delta_{a+\delta} f(x)}{2(a+\delta)} \right|$$

EXERCISE #2 - TEST CONVERGENCE: $\cos(x)$



$$\sigma_a = \frac{100}{\delta} \times \left| \frac{\Delta_a f(x)}{2a} - \frac{\Delta_{a+\delta} f(x)}{2(a+\delta)} \right|$$

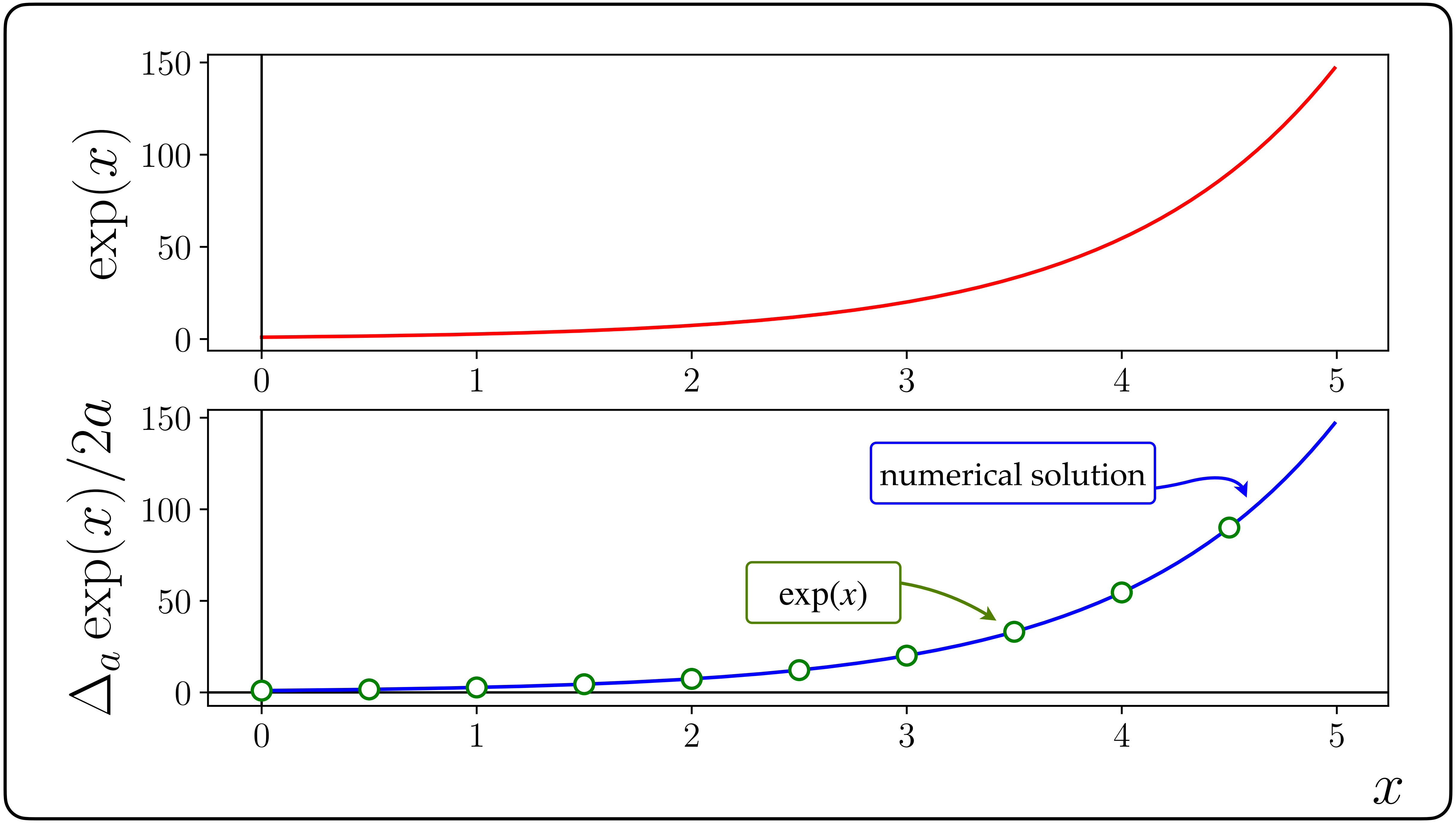
EXERCISE #2 - TEST CONVERGENCE: $(x^3/6) + (x^4/8)$



$$\sigma_a = \frac{100}{\delta} \times \left| \frac{\Delta_a f(x)}{2a} - \frac{\Delta_{a+\delta} f(x)}{2(a+\delta)} \right|$$

EXERCISE #3 - PLOT DERIVATIVES AS A FUNCTION OF X

Having tested the convergence of your code, you are now in a place to plot the derivative of the same functions as a function of x. We will fix $a = 10^{-3}$.



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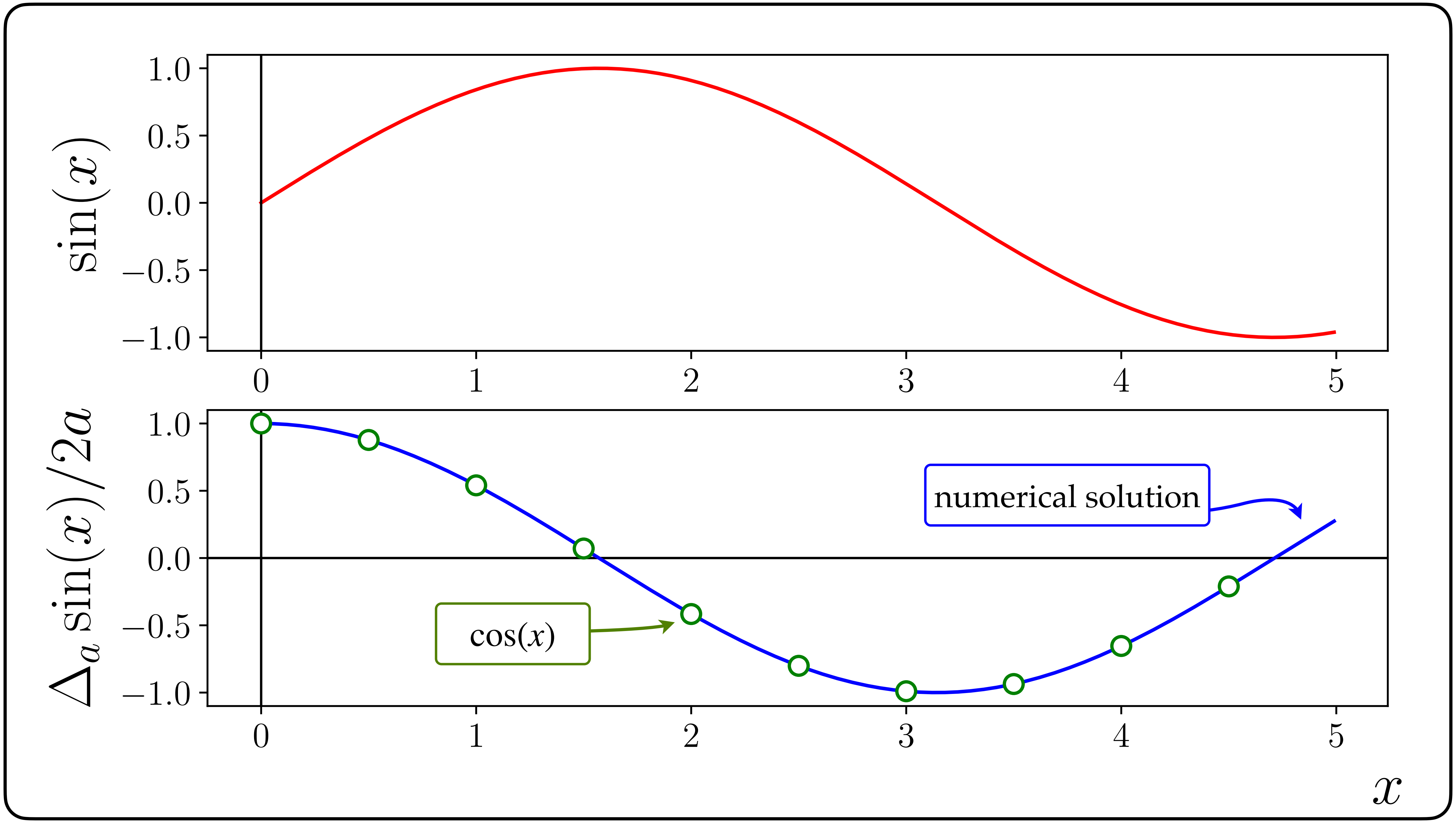

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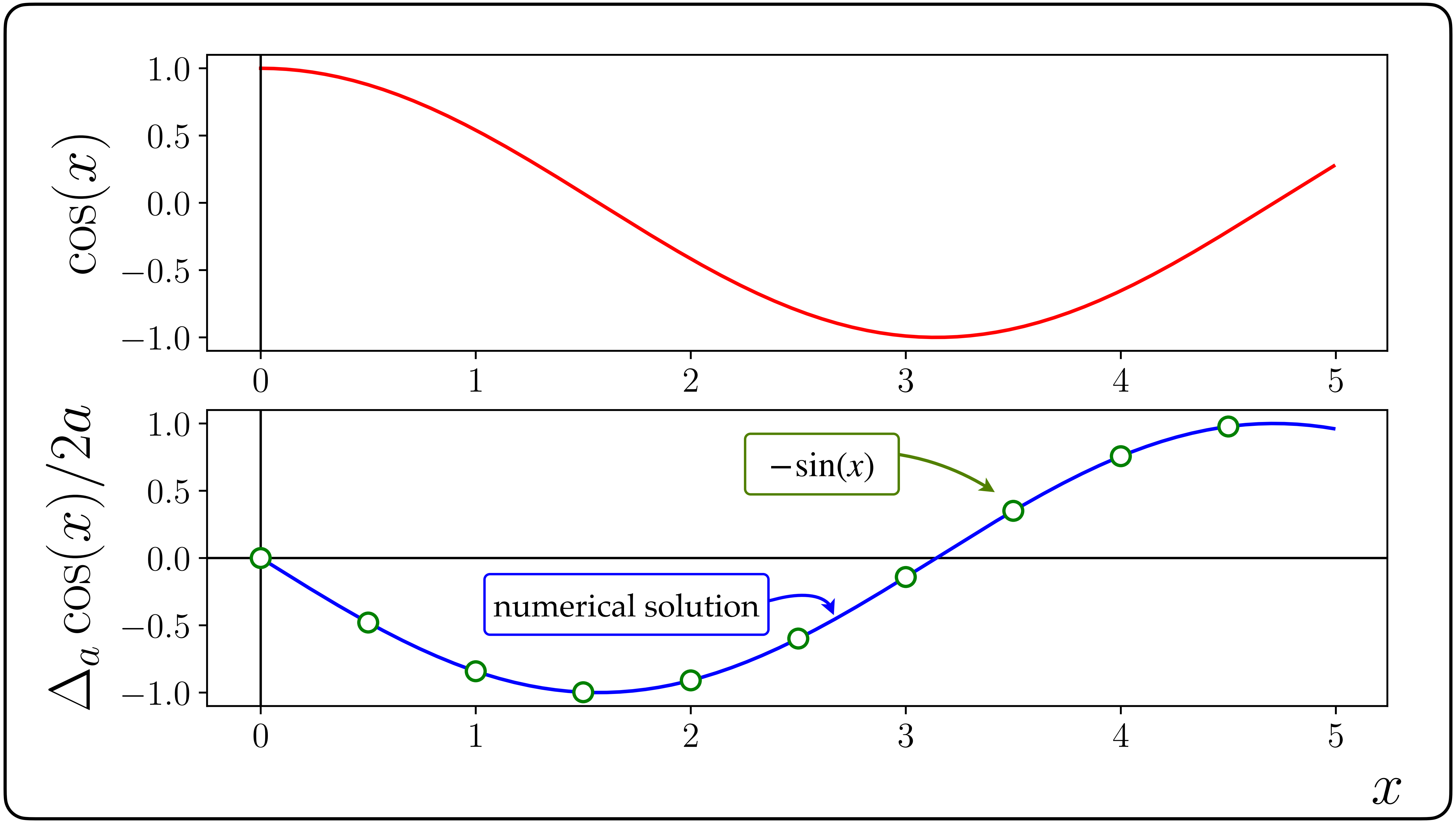
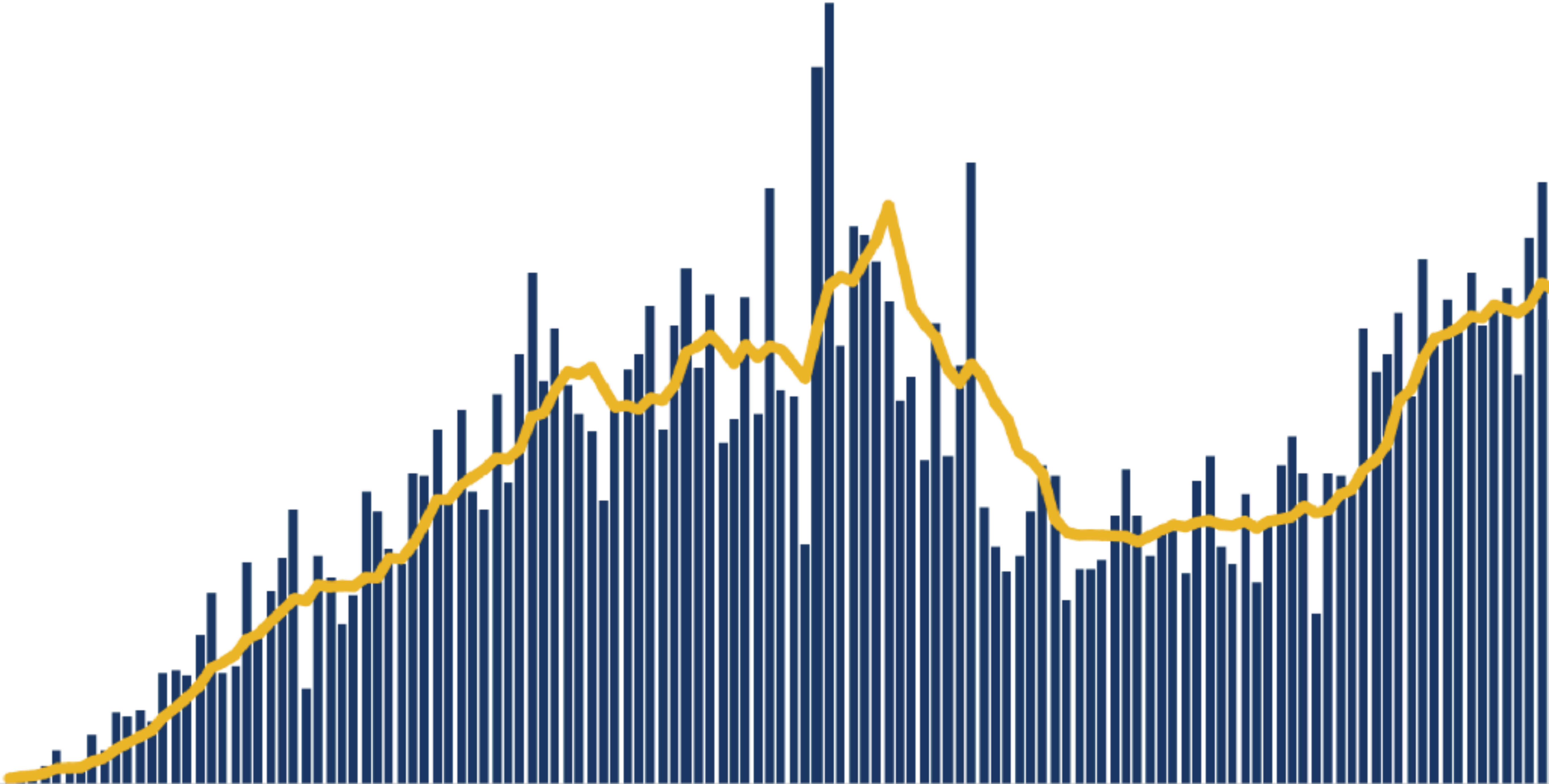


FIRST ORDER DIFFERENTIAL EQUATION



Differential equations are equations that relate one or more functions and their derivations.

Here is a simple example of a differential equation: $\frac{d}{dx}f(x) = g(x)$.

In general, differential equations can take a more complicated form.

A significant fraction of the most important equations in Physics are differential equations or can be written as such. Examples include:

- $F = ma$, which can be written as $F = m\frac{d^2}{dt^2}x(t)$
- Wave equations
- Schrödinger Equation
- Maxwells Equations

Although finding an analytical solution to differential equations may seem like a daunting task, obtaining a numerical solution is more straightforward. In fact, all we need is skills we have developed and a bit of algebra.

To see this, let us consider the simple example previously mentioned: $\frac{d}{dx}f(x) = g(x)$.

Numerically we know we need to replace the derivative with a finite difference by introducing a finite value of a

$$\frac{d}{dx}f(x) = \lim_{a\to 0}\frac{\Delta_a f(x)}{2a} = \lim_{a\to 0}\frac{f(x+a) - f(x-a)}{2a}$$

Just as before, we will keep a fixed and check that our results as independent of it. By replacing the derivative with this expression, we are able to rewrite this equations as an algebraic one

$$\frac{f(x+a) - f(x-a)}{2a} = g(x).$$

We can rewrite this by solving for $f(x+a)$

$$f(x+a) = f(x-a) + g(x)2a.$$

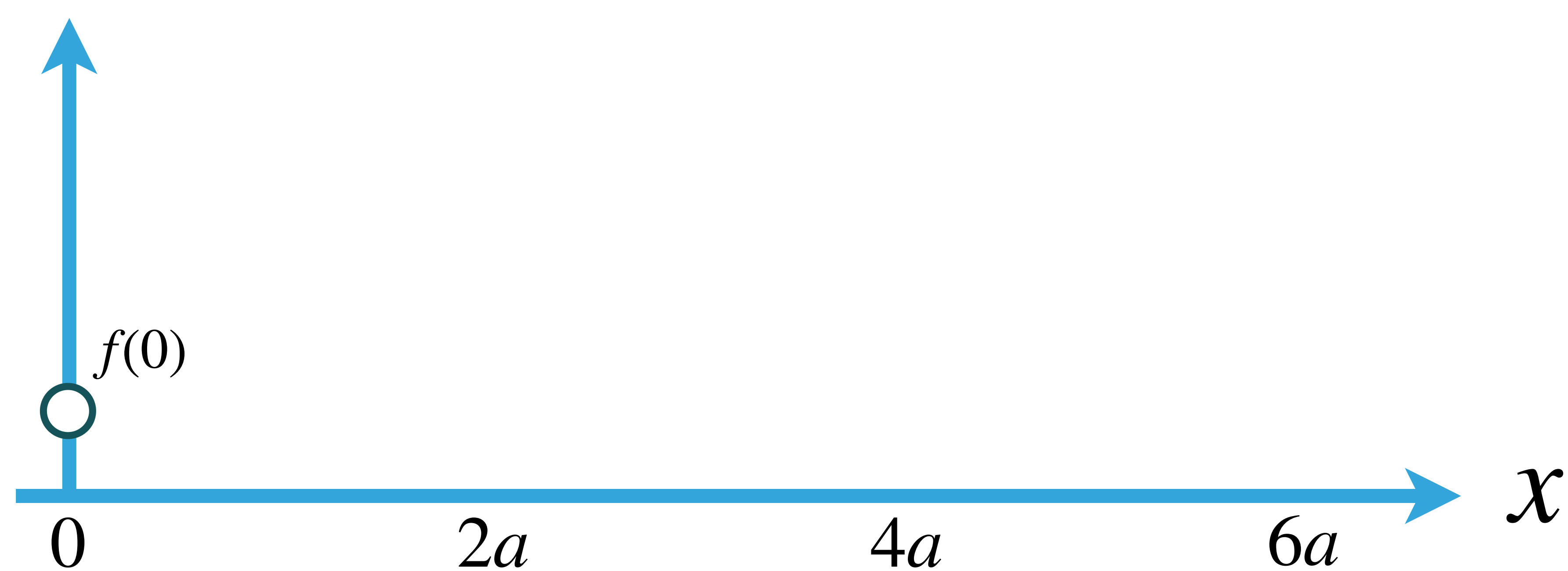
We can rewrite this by replacing $x = x_0 + a$: $f(x_0 + 2a) = f(x_0) + g(x_0 + a)2a$.

Since x_0 is the name of a generic variable, we can just leave it as x

$$f(x + 2a) = f(x) + g(x + a)2a.$$

Notice, if we know the first value of our function, we can determine all the other points using this equation.

Let the first point be a $x = 0$ with value $f(0)$.



We can rewrite this by replacing $x = x_0 + a$: $f(x_0 + 2a) = f(x_0) + g(x_0 + a)2a$.

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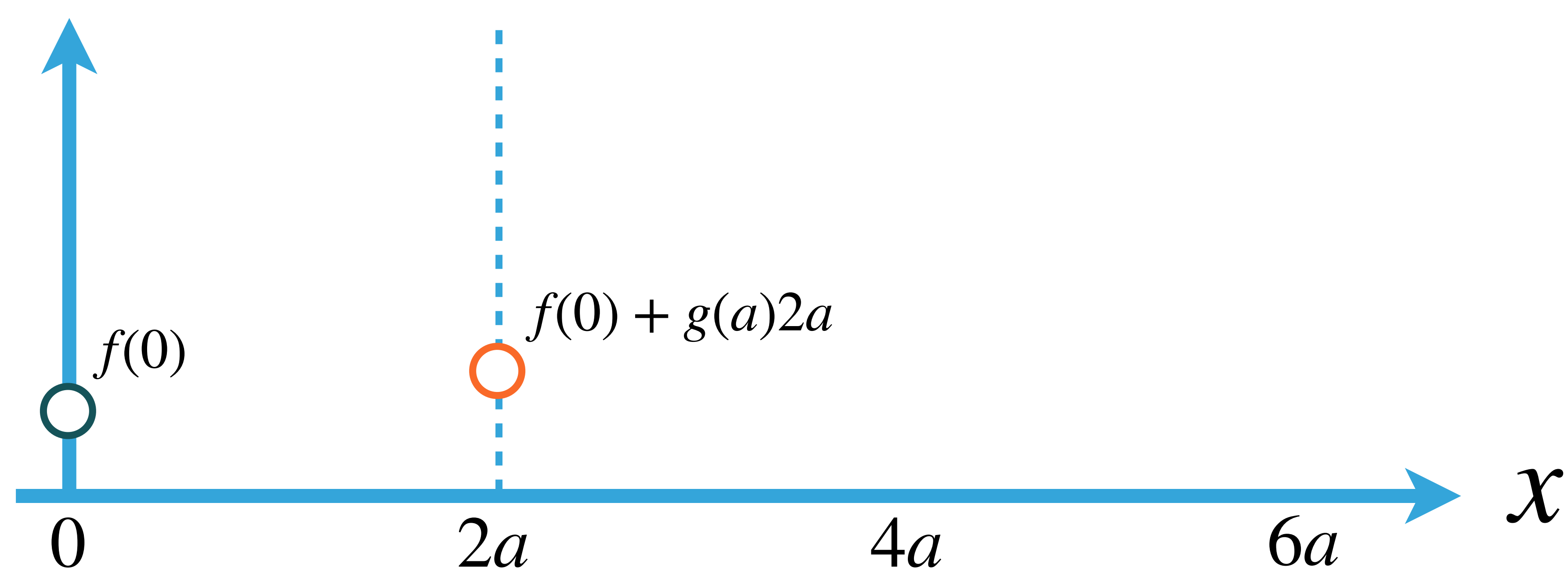
$$f(x + 2a) = f(x) + g(x + a)2a.$$

Notice, if we know the first value of our function, we can determine all the other points using this equation.

Let the first point be a $x = 0$ with value $f(0)$.
Then, the next point will be at $x = 2a$ with value $f(2a) = f(0) + g(a)2a$.



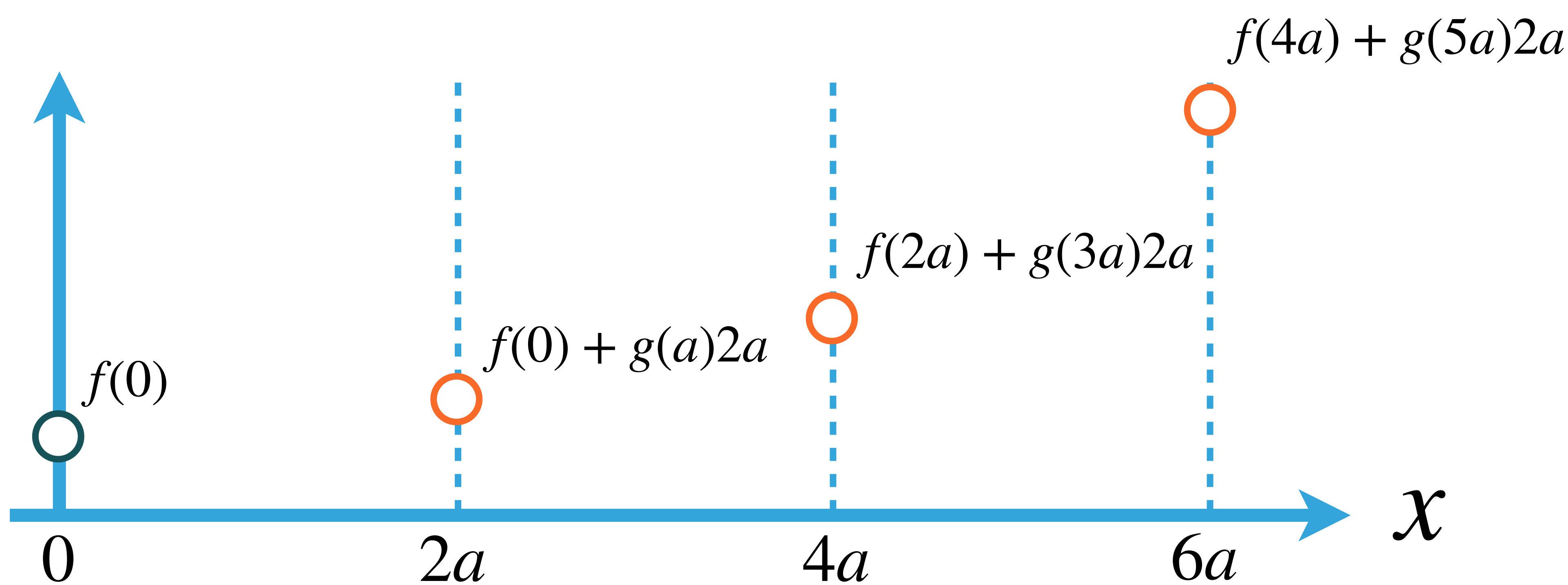
We can rewrite this by replacing $x = x_0 + a$: $f(x_0 + 2a) = f(x_0) + g(x_0 + a)2a$.

Since x_0 is the name of a generic variable, we can just leave it as x

$$f(x + 2a) = f(x) + g(x + a)2a.$$

Notice, if we know the first value of our function, we can determine all the other points using this equation.

Let the first point be a $x = 0$ with value $f(0)$.
Then, the next point will be at $x = 2a$ with value $f(2a) = f(0) + g(a)2a$.
Follow by $f(4a) = f(2a) + g(3a)2a = f(0) + g(3a)2a + g(a)2a$
and $f(6a) = f(4a) + g(5a)2a = f(0) + g(5a)2a + g(3a)2a + g(a)2a$.



EXERCISE #4 - WRITE CODE THAT SOLVES DIFF. EQ. FOR ANY $g(x)$

Write code for obtaining $f(x)$ for a range of values of x given any function of $g(x)$ and some initial value of $f(0)$.

To do this, you can write a “for loop” that reiterated the equation we obtained in the previous slide:

$$f(x + 2a) = f(x) + g(x + a)2a.$$

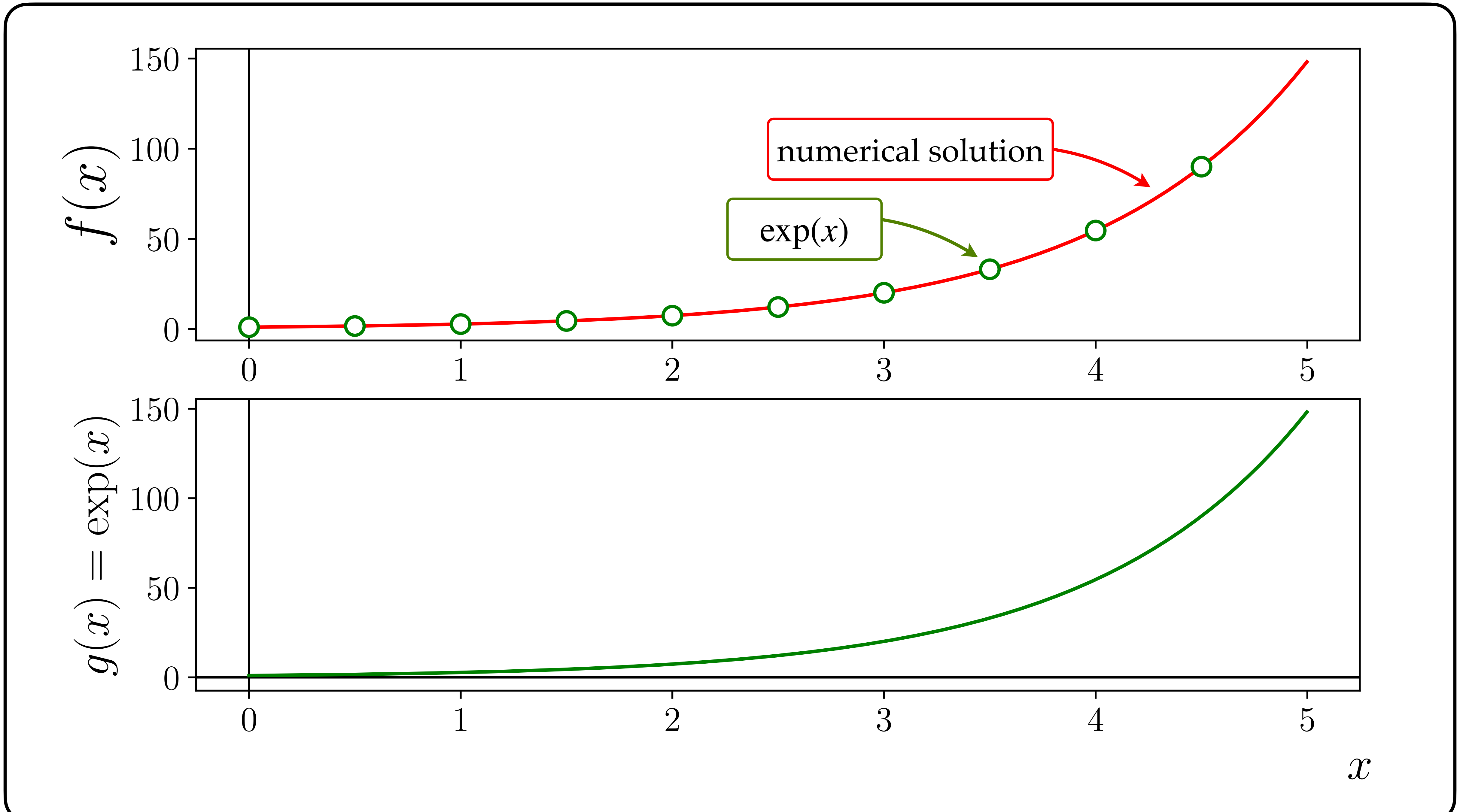
```python
def diff_eq(g, f0, xs, a):

    """
    this solves the diff. eq. df/dx = g(x) by discretizing it
    (f(x+a) - f(x-a))/2a = g(x)
    f(x+a) = f(x-a) + 2*a*g(x)

    or equivalently

    f(x+2 a) = f(x) + 2*a*g(x+a)

    we put the list of points into fs

    we initiate this with
    the initial value of f, which is f0 = f(xs[0])

    """

    fs=[f0]

    "note in this loop, we skip over the first element"
    for i0 in range(1,len(xs)):

        "we grab the previous term in the list"
        f0 = fs[-1]

        """
        we need the derivative at x+a,
        since xs[i0] =  x+2*a
        we can use the fact that x+a = xs[i0]-a
        """
        df = g(xs[i0]-a)*2*a

        "we add these together"
        fs.append(f0+ df)

    return fs
```

look inside my code 😎

Don’t be scared! The code is very short, I just wrote lots of comment to explain its logic 🤓

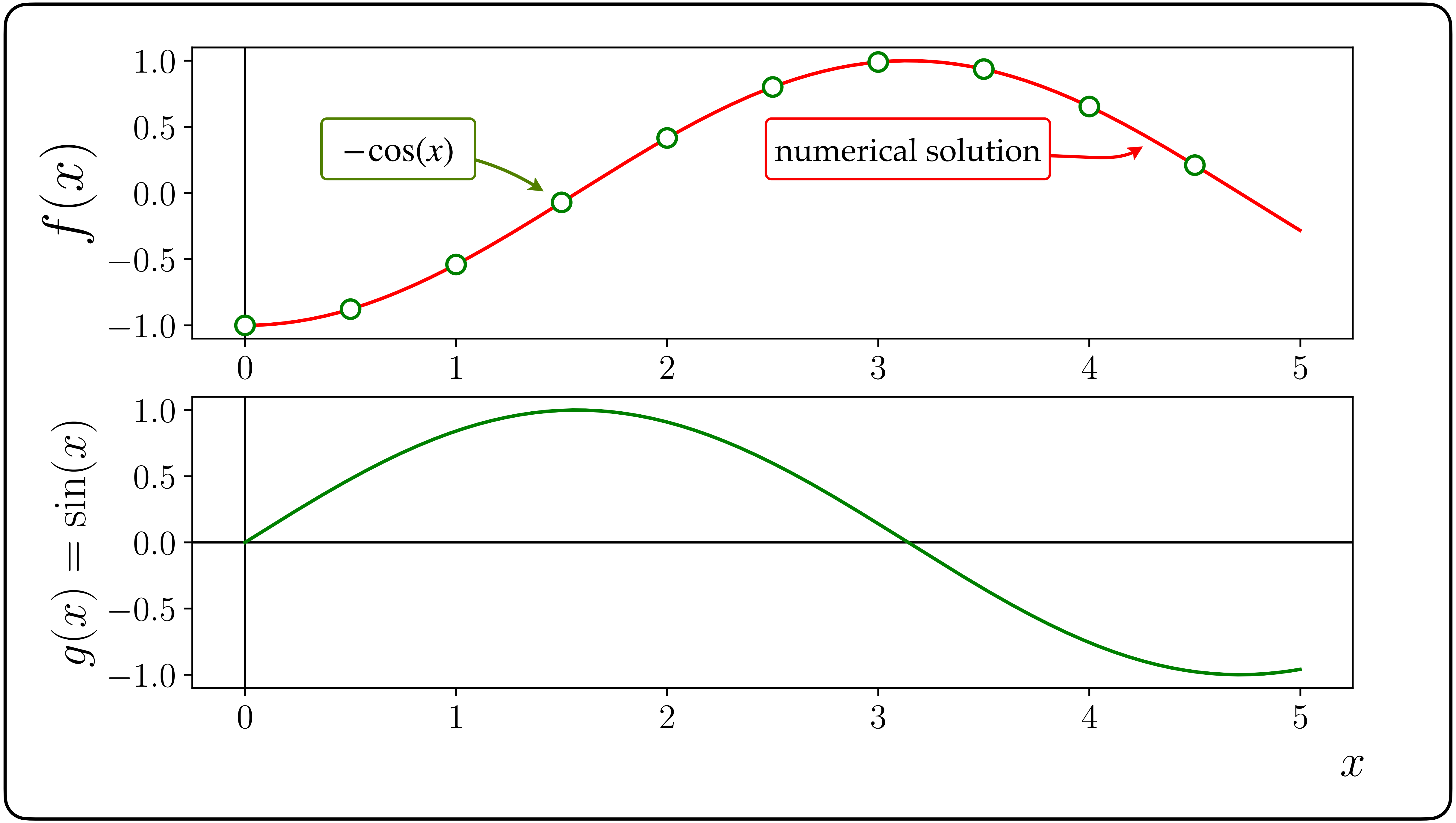
EXERCISE #5 - PUT YOUR CODE TO A TEST BY SOLVING FOR $f(x)$

Solve for $f(x)$ in this differential equation $\dfrac{df(x)}{dx} = g(x)$, by fixing $a = 10^{-3}$ and $x = [0,5]$, when $f(0) = 1$ and $g(x) = \exp(x)$.



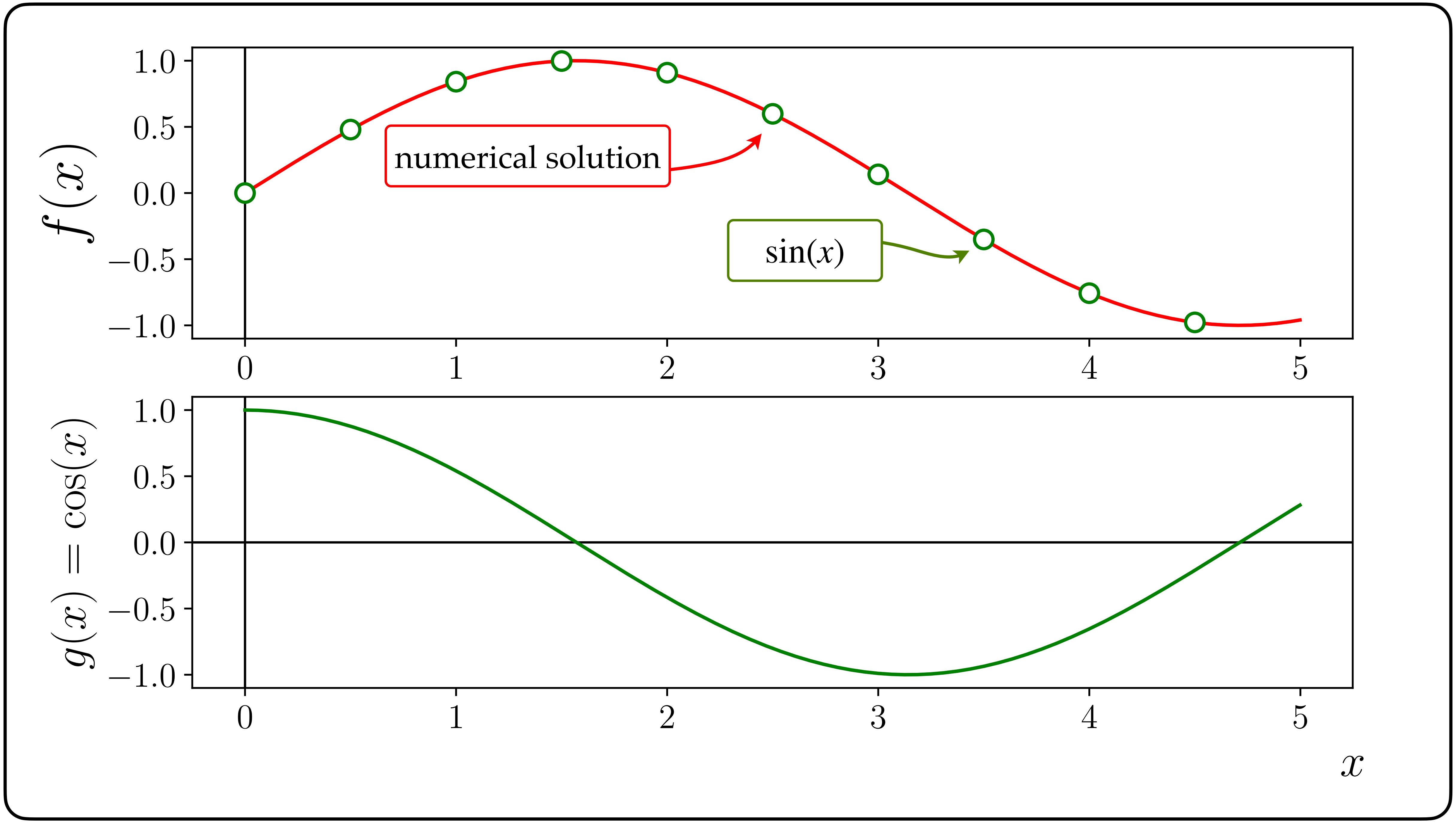
EXERCISE #5 - PUT YOUR CODE TO A TEST BY SOLVING FOR $f(x)$

Solve for $f(x)$ in this differential equation $\dfrac{df(x)}{dx} = g(x)$, by fixing $a = 10^{-3}$ and $x = [0,5)$, when $f(0) = -1$ and $g(x) = \sin(x)$.



EXERCISE #5 - PUT YOUR CODE TO A TEST BY SOLVING FOR $f(x)$

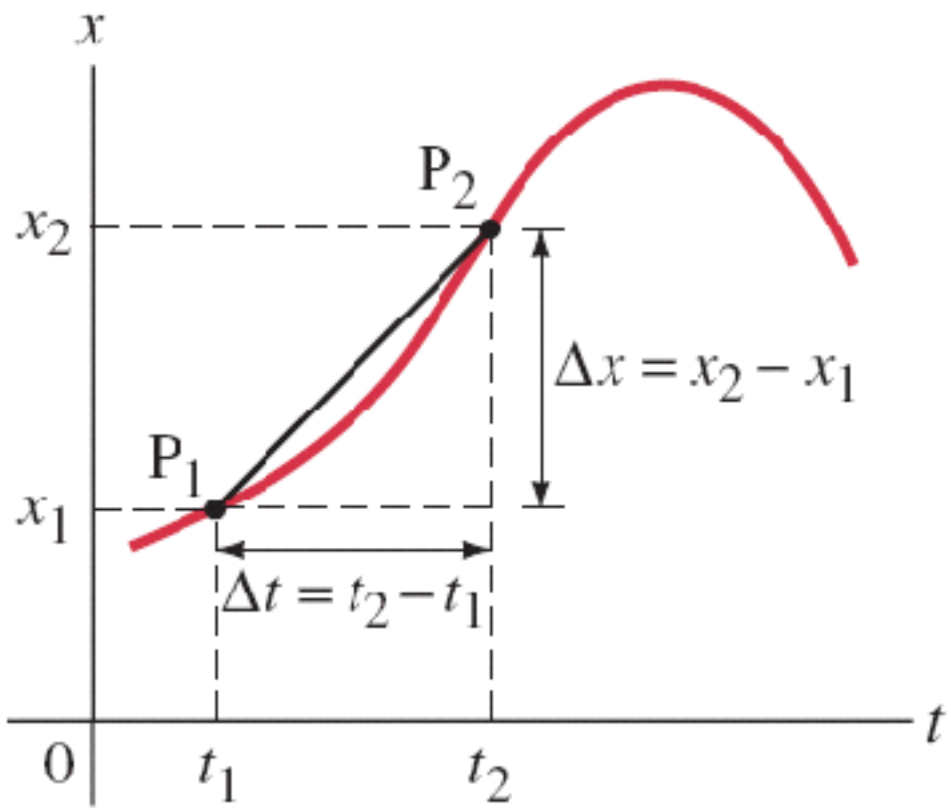
Solve for $f(x)$ in this differential equation $\dfrac{df(x)}{dx} = g(x)$, by fixing $a = 10^{-3}$ and $x = [0,5)$, when $f(0) = 0$ and $g(x) = \cos(x)$.

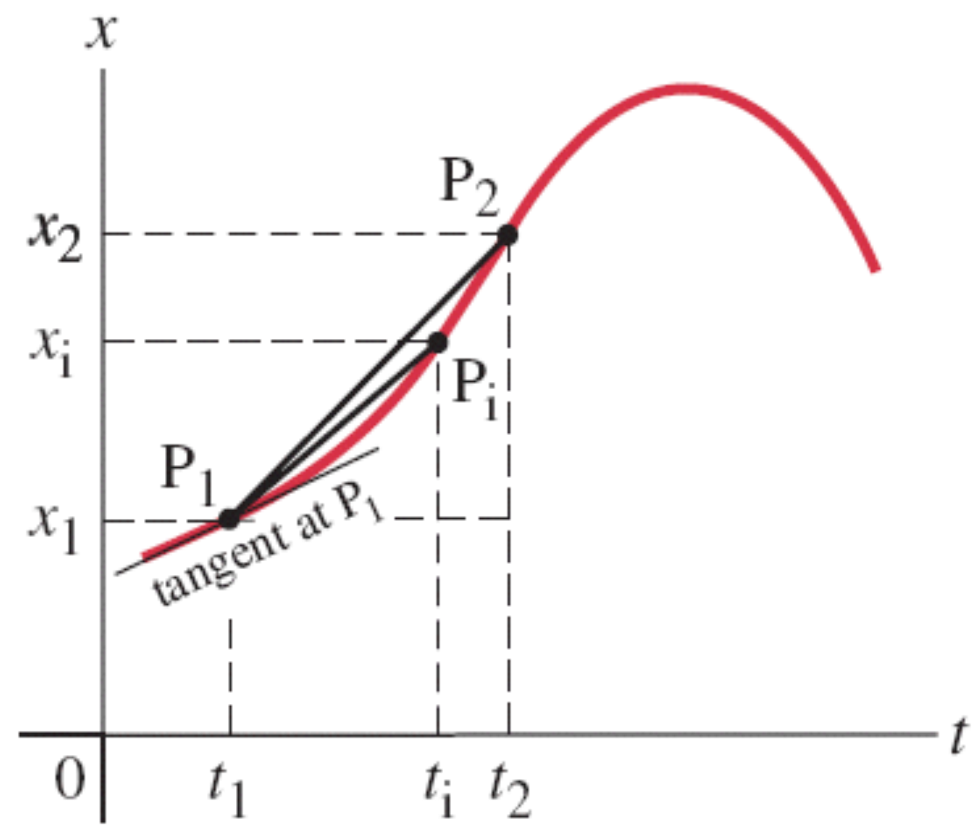


SECOND ORDER DIFFERENTIAL EQUATION

VELOCITY - DERIVATIVE OF POSITION

Instantaneous velocity is a vector defined as "how fast" a particle is moving at a given instant.





$$\bar{\nu} = \frac{x_2 - x_1}{t_2 - t_1}$$

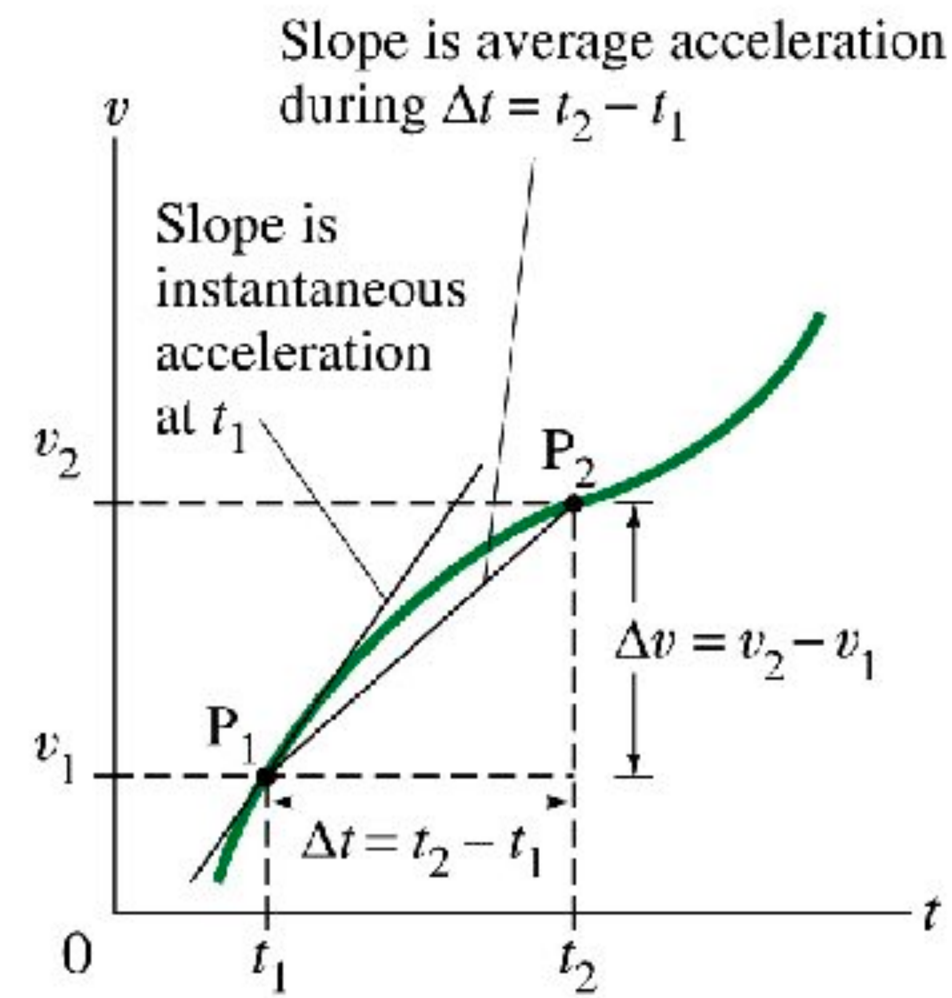
$$\boxed{\nu = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}}$$

Instantaneous velocity is the tangent to the curve at any point.

Instantaneous acceleration is the average acceleration in the limit as the time interval becomes infinitesimally short.

$$\boxed{a = \lim_{\Delta t \to 0} \frac{\Delta \nu}{\Delta t} = \frac{d\nu}{dt} = \frac{d^2 x}{dt^2}}$$

Instantaneous acceleration is the slope of the tangent to the v vs. t curve at that time.

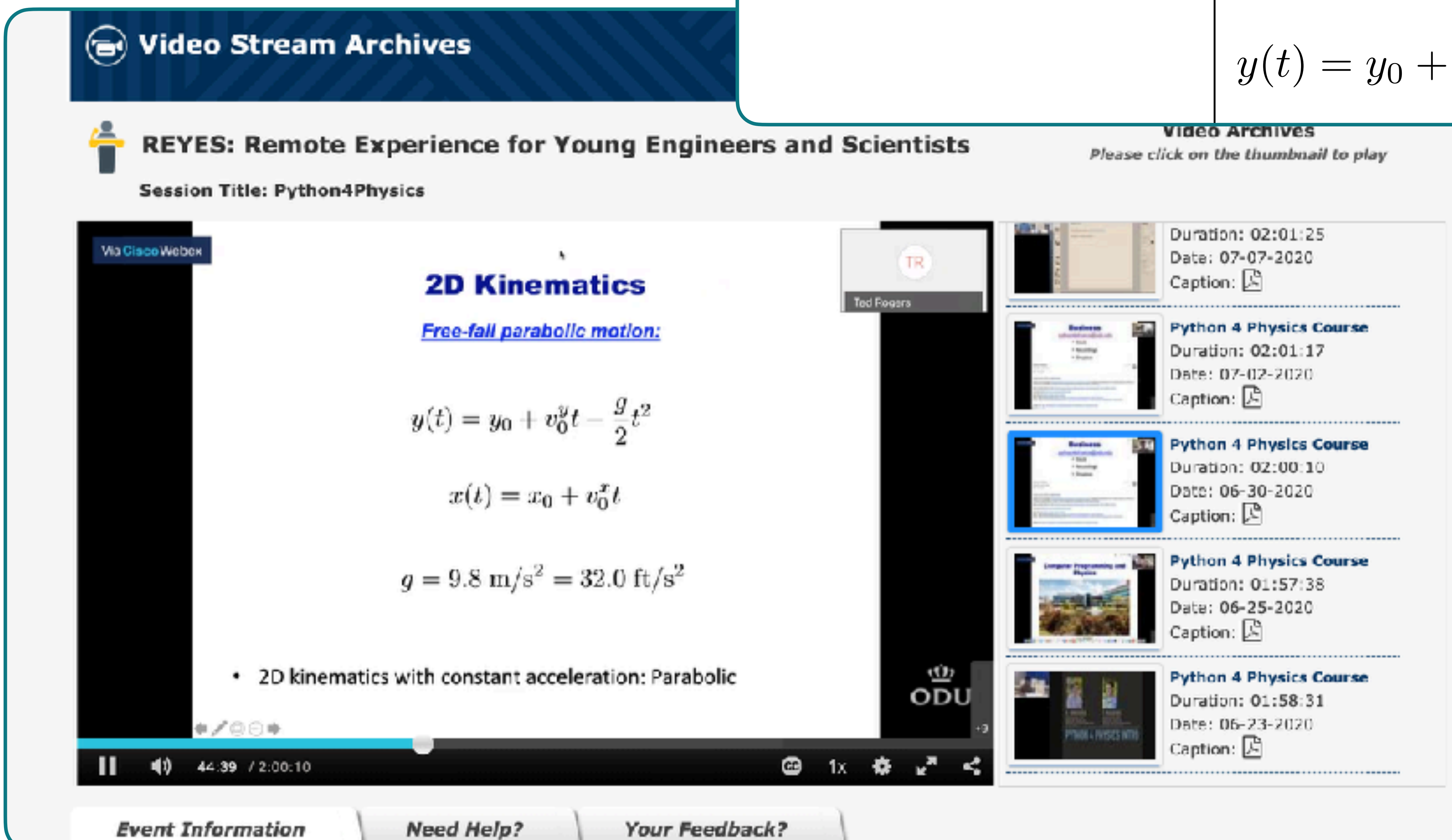


PROJECTILE MOTION

Remember the projectile motion problem Prof. Rogers introduced?

[Where did these equations come from?]

Position as a function of time	$x(t) = x_0 + v_0^x t + \dfrac{a_x}{2} t^2$ $y(t) = y_0 + v_0^y t + \dfrac{a_y}{2} t^2$



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Position as a function of time	$x(t) = x_0 + v_0^x t + \frac{a_x}{2} t^2$ $y(t) = y_0 + v_0^y t + \frac{a_y}{2} t^2$

They are solutions to the second order differential equation for the each coordinate:

$$a_x(t) = \frac{d^2}{dt^2} x(t),$$
$$a_y(t) = \frac{d^2}{dt^2} y(t).$$

You get the solutions above, if both $a_x(t)$ and $a_y(t)$ are both time independent. In which case the equations simplify

$$a_x = \frac{d^2}{dt^2} x(t),$$
$$a_y = \frac{d^2}{dt^2} y(t).$$

Having solved the simplest class of differential equations, let us consider the second easiest. For convenience we will change the independent variable to t, but we can still use the previous finding by replacing t with x.

We consider the second derivative of the function $f(t)$, and we set it equal to another function $h(t)$.

$$\frac{d^2}{dt^2} f(t) = h(t).$$

This might seen challenging, but it because relatively easy if we define firs the single derivative of $f(t)$ to be equal to $g(t)$

$$\frac{d}{dt} f(t) = g(t).$$

Then the first equation can be rewritten as

$$\frac{d}{dt}\left(\frac{d}{dt} f(t)\right) = \frac{d}{dt} g(t) = h(t).$$

Then we see that we have replaced the second order differential equation, with two single order differential equations

$$\frac{d}{dt} f(t) = g(t), \qquad \text{and} \qquad \frac{d}{dt} g(t) = h(t).$$

We have simplified the problem, at the cost of having two unknown functions: $f(t)$ and $g(t)$. Fortunately, we have two equations and two unknowns, so we can indeed solve for the unknowns!

We can solve these two differential equations numerically by introducing a finite values of a as before. We get two coupled equations:

$$\frac{g(t+a)-g(t-a)}{2a} = h(t),$$
$$\frac{f(t+a)-f(t-a)}{2a} = g(t).$$

Solving for $g(t+a)$ and $f(t+a)$

$$g(t+a) = g(t-a) + h(t)2a,$$
$$f(t+a) = f(t-a) + g(t)2a.$$

As before, we can shift the argument by a to rewrite these solutions as

$$g(t+2a) = g(t) + h(t+a)2a,$$
$$f(t+2a) = f(t) + g(t+a)2a.$$

Note, in the equation we need to calculate $g(t+a)$, but we never quite calculate this. Instead, we calculate $g(t+2a)$ and $g(t)$. For sufficiently small values of a, you can approximate this by $g(t+2a)$, and the error you would introduce would be linear in a. This means that the error would be vanish as you take the $a \to 0$ limit. Alternatively you can use the following average, and the error would be quadratic in a:

$$g(t+a) = \frac{g(t+2a)+g(t)}{2} + \mathcal{O}(a^2),$$

Big O notation: it tells you how the error behaves in the asymptotic limit, which in this case refers to the $a \to 0$ limit. 🤓 Proving that the approximation scales like $\mathcal{O}(a^2)$ can be done by Taylor expanding both sides of the equality.

Now you can write similar code to before to loop over all values of both $f(t)$ and $g(t)$. Note, for this we need two initial values at $x = 0$: $f(0)$ and $g(0)$.

EXERCISE #6 - WRITE CODE FOR $x(t)$

First, write code for the expected behavior of $x(t)$ and $v(t)$ as a function of time.

Position as a function of time	$x(t) = x_0 + v_0^x t + \frac{a_x}{2} t^2$ $y(t) = y_0 + v_0^y t + \frac{a_y}{2} t^2$
Velocity as a function of time	$v_x(t) = x'(t) = v_0^x + a_x t$ $v_y(t) = y'(t) = v_0^y + a_y t$

```
def x_func(t, x0, v0, a0):
    x1 = v0 * t
    x2 = a0 * pow(t,2)/2.0
    return x0 + x1 + x2

def v_func(t, v0, a0):
    return v0 + a0 * t
```

look inside my code 😎

EXERCISE #7 - WRITE CODE FOR $f(t), g(t)$ GIVEN $h(t)$

Use these iterative equations

$$g(t+2a) = g(t) + h(t+a)2a,$$
$$f(t+2a) = f(t) + g(t+a)2a.$$

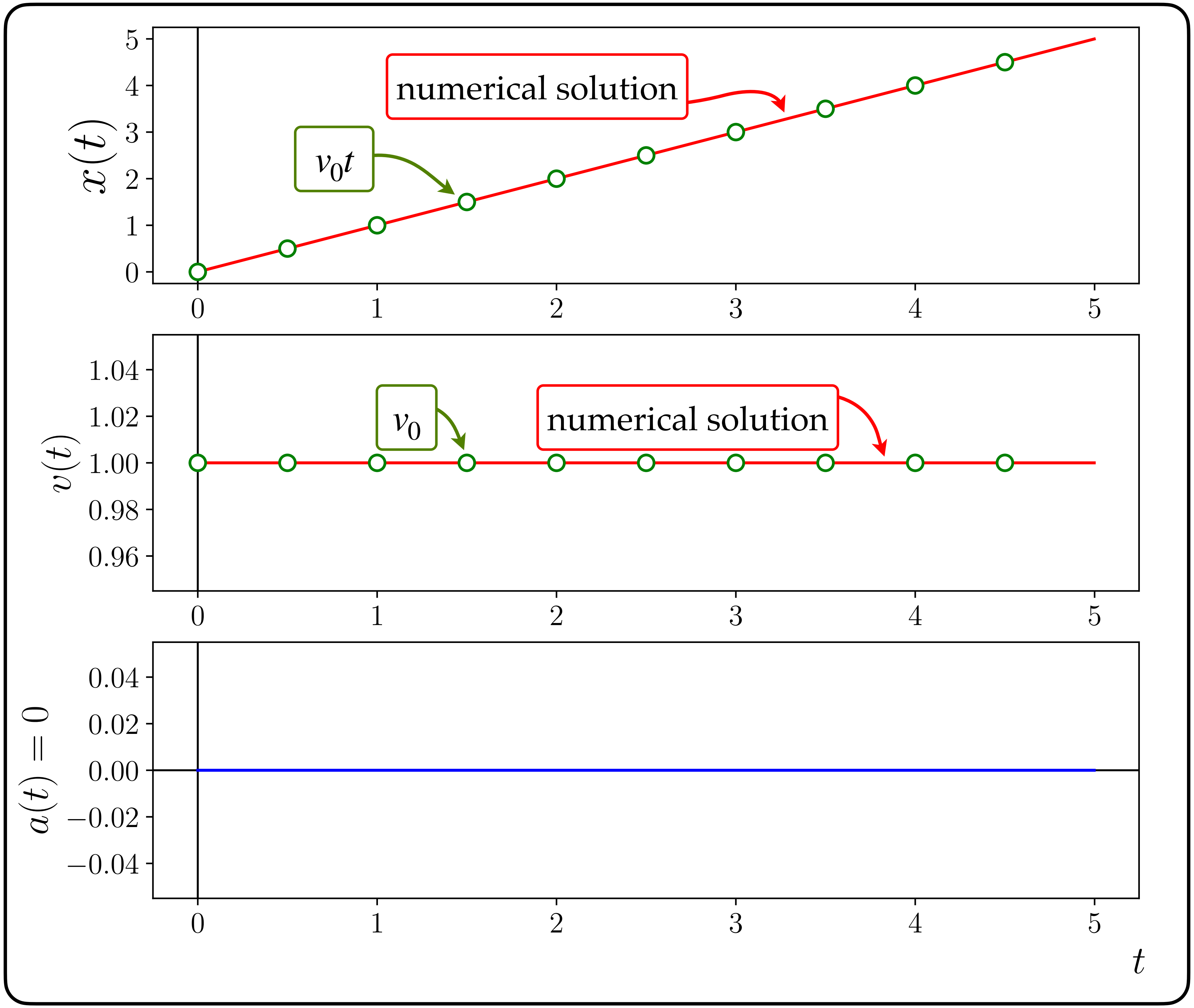
```python
def second_diff_eq(h, f0, g0, ts, a):

    """
    this solves the diff. eq. d^2f/dx^2 = h(x) by discretizing
    re-writting it as a coupled linear equations

    dg/dt = h(t)
    df/dt = g(t)

    g(t + 2*a) = g(t) + 2 * a * h(t + a)
    f(t + 2*a) = f(t) + 2 * a * g(t + a)

    we put the list of points into fs, gs

    we initiate this with
    the initial value of f, which is f0 = f(ts[0])
    the initial value of g, which is g0 = g(ts[0])

    note, we need to calculate g(t + a), but we have g(t + 2*a) and g(t)
    we will estimate this with the average of these two, which we will call

    gbar = ( g(t + 2*a) + g(t) ) / 2
    """

    fs=[f0]
    gs=[g0]

    "note in this loop, we skip over the first element"
    for i0 in range(1,len(ts)):
        "we start with g(t): we grab the previous term in the list"
        g0 = gs[-1]
        "the shift in g"
        dg = h(ts[i0] + a) * 2*a

        gs.append(  g0 + dg )

        "next we calculate f(t): we grab the previous term in the list"
        f0 = fs[-1]
        "we estimate g(t+a) with the average of the last two point in gs"
        gbar = (gs[-1] + gs[-2]) /2.0
        df = gbar * 2 * a

        "we add these together"
        fs.append(f0 + df)

    return fs, gs
```

look inside my code 😎

Don’t be scared! The code is very short, I just wrote lots of comment to explain its logic 🤓

EXERCISE #8 - REPRODUCE THESE PLOTS

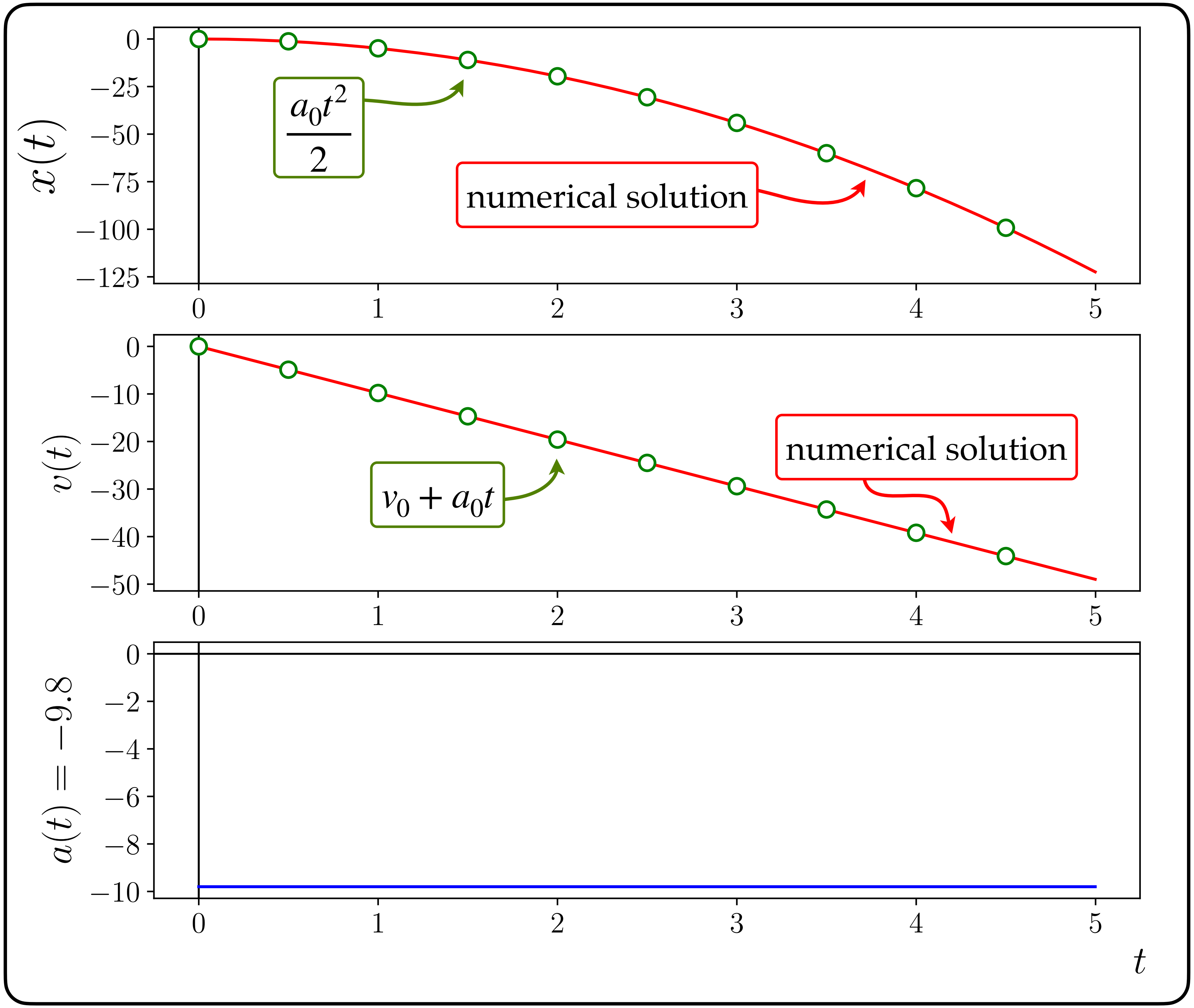
Test your code and test the kinematic equations obtained assuming a constant acceleration. For $(x_0, v_0) = (0{,}1)$ and $a(t) = 0$.



$$x(t) = x_0 + v_0^x t + \frac{a_x}{2} t^2$$

EXERCISE #8 - REPRODUCE THESE PLOTS

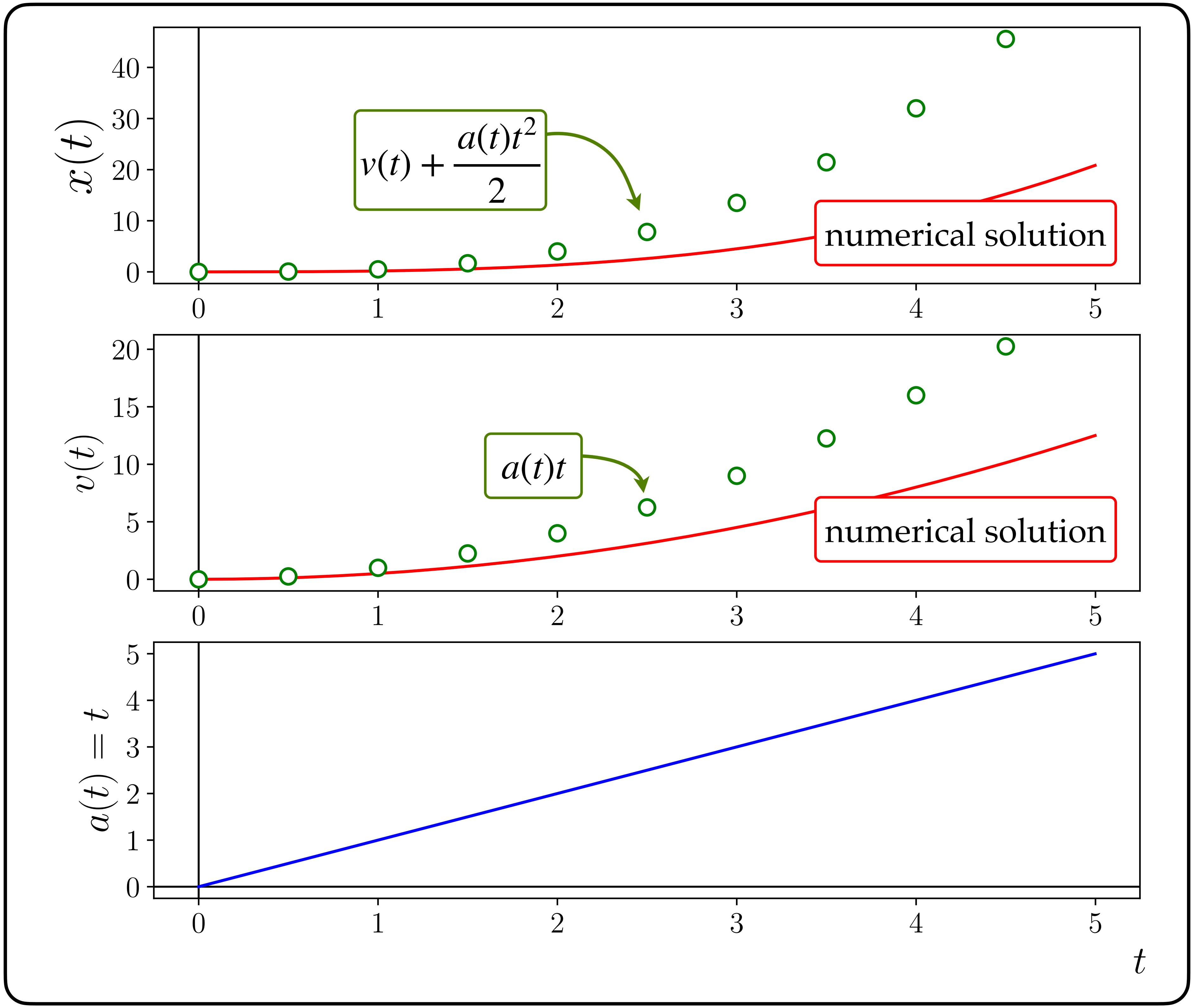
Test your code and test the kinematic equations obtained assuming a constant acceleration. For $(x_0, v_0) = (0{,}0)$ and $a(t) = -9.8$.



$$x(t) = x_0 + v_0^x t + \frac{a_x}{2} t^2$$

EXERCISE #8 - REPRODUCE THESE PLOTS

Test your code and test the kinematic equations obtained assuming a constant acceleration. For $(x_0, v_0) = (0,0)$ and $a(t) = t$.



$$x(t) = x_0 + v_0^x t + \frac{a_x}{2} t^2$$

SUMMARY - CONCEPTS COVERED

Physics concepts/equations

Position

Velocity : $v = \lim_{\Delta t \to 0} \dfrac{\Delta x}{\Delta t}$

Acceleration : $a = \lim_{\Delta t \to 0} \dfrac{\Delta v}{\Delta t}$

Calculus

Derivatives: $\dfrac{d}{dx} f(x) = \lim_{\Delta x \to 0} \dfrac{\Delta f}{\Delta x}$

Numerical solution to differential equations

Equation: $\dfrac{d}{dx} f(x) = g(x)$ Solution: $f(x + 2a) = f(x) + g(x + a)2a$

Equation: $\dfrac{d^2}{dt^2} f(t) = h(t)$

Solution: $g(t + 2a) = g(t) + h(t + a)2a, \; f(t + 2a) = f(t) + g(t + a)2a.$

Python

Functions of function: def $f(g, x)$: return $g(x)$

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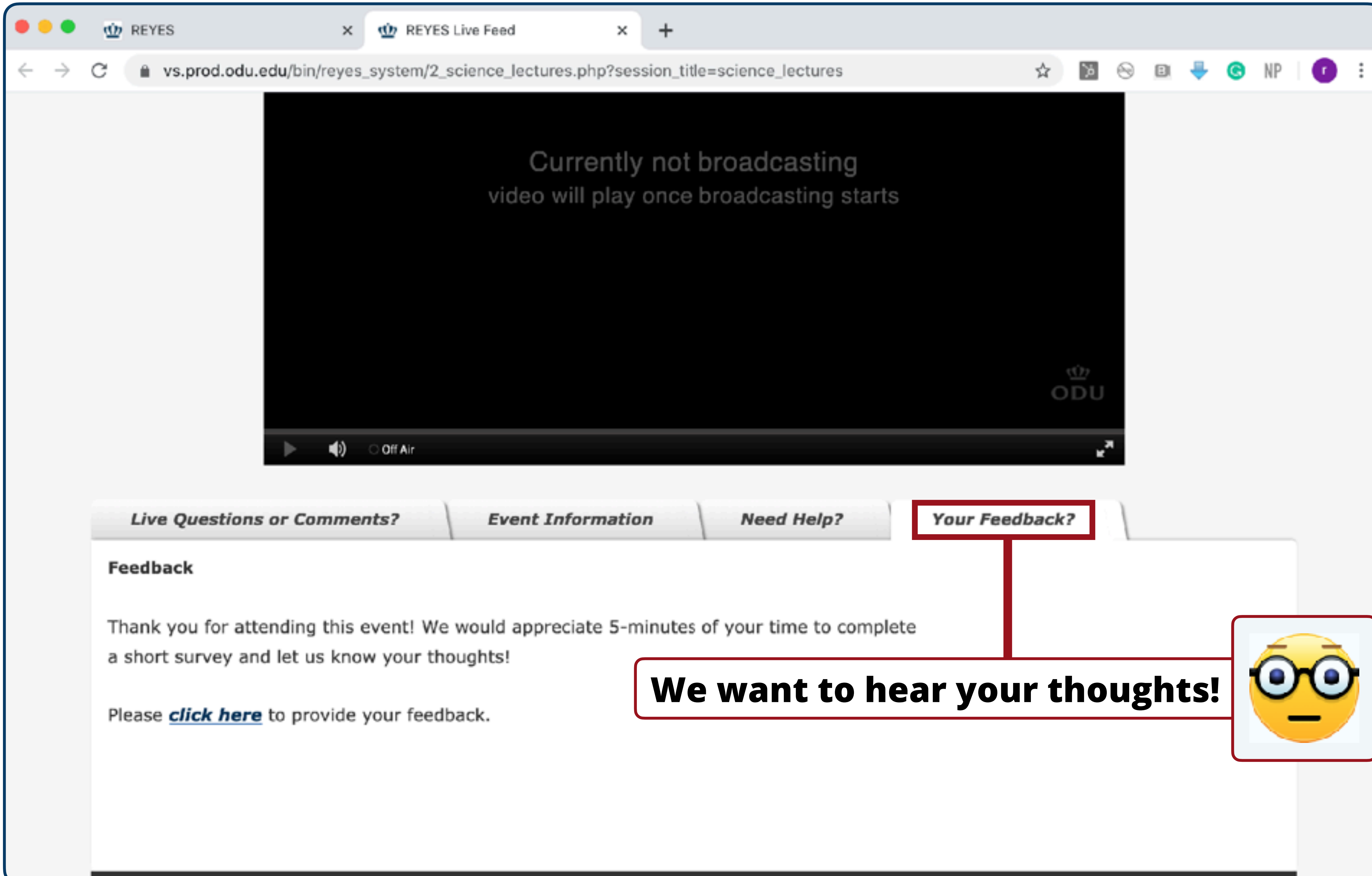

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