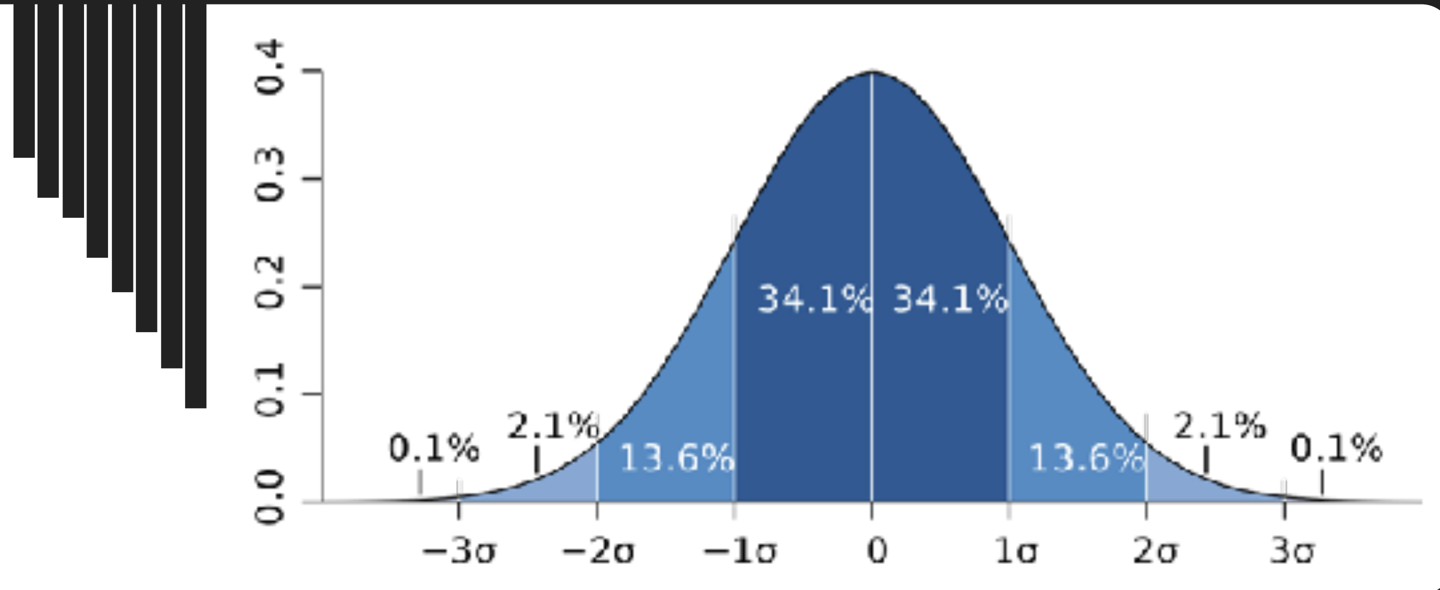
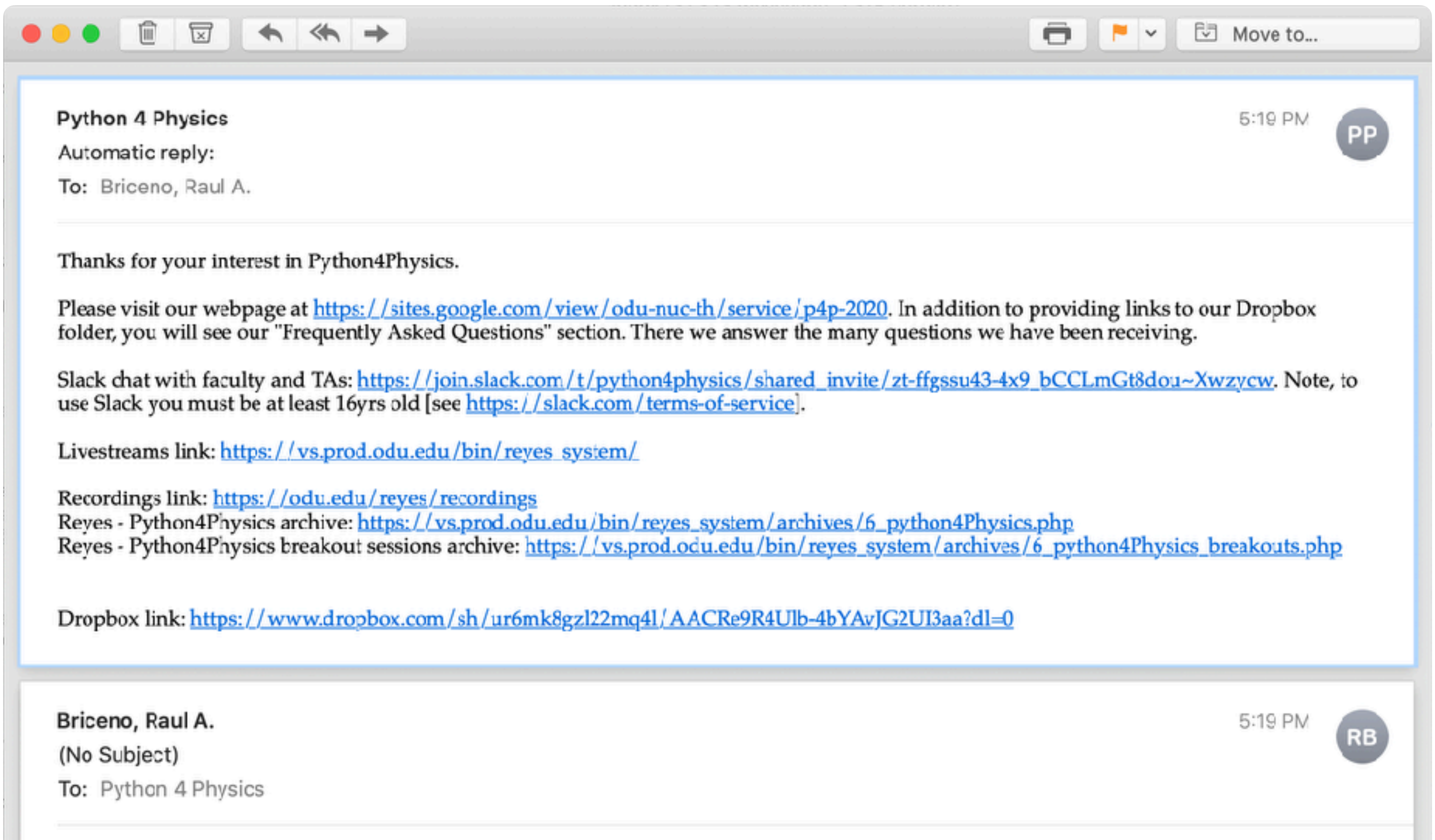


R. BRICEÑO, T. ROGERS

EM WAVES, INTEGRATION, AND FOURIER TRANSFORMS

ADMIN STUFF

Need links, just email python4physics@odu.edu



Move to...

Python 4 Physics 5:19 PM PP

Automatic reply:

To: Briceno, Raul A.

Thanks for your interest in Python4Physics.

Please visit our webpage at https://sites.google.com/view/odu-nuc-th/service/p4p-2020. In addition to providing links to our Dropbox folder, you will see our "Frequently Asked Questions" section. There we answer the many questions we have been receiving.

Slack chat with faculty and TAs: https://join.slack.com/t/python4physics/shared_invite/zt-ffgssu43-4x9_bCCLmGt8dou~Xwzycw. Note, to use Slack you must be at least 16yrs old [see https://slack.com/terms-of-service].

Livestreams link: https://vs.prod.odu.edu/bin/reyes_system/

Recordings link: https://odu.edu/reyes/recordings
Reyes - Python4Physics archive: https://vs.prod.odu.edu/bin/reyes_system/archives/6_python4Physics.php
Reyes - Python4Physics breakout sessions archive: https://vs.prod.odu.edu/bin/reyes_system/archives/6_python4Physics_breakouts.php

Dropbox link: https://www.dropbox.com/sh/ur6mk8gzl22mq41/AACRe9R4Ulb-4bYAvJG2UI3aa?dl=0

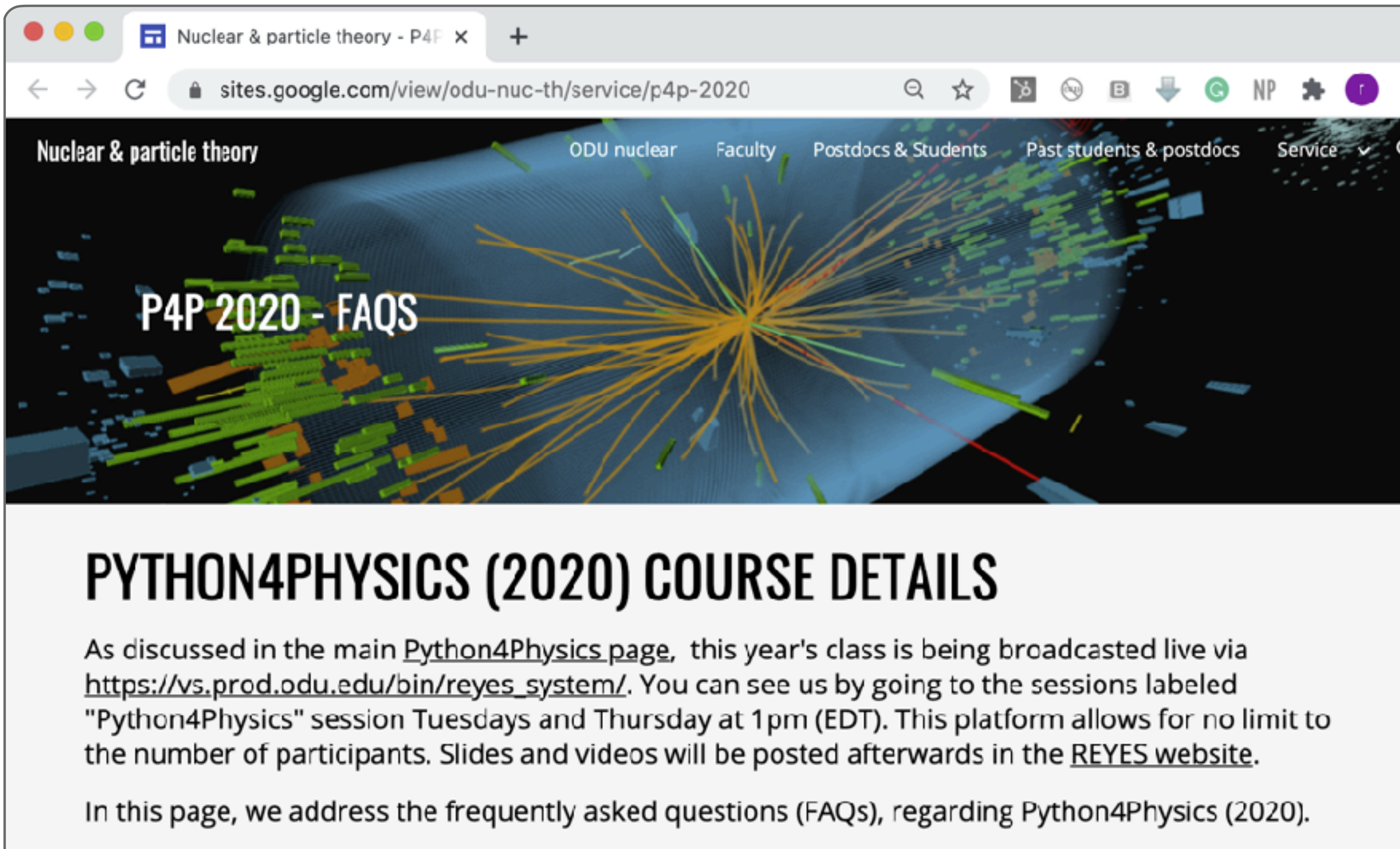
Briceno, Raul A. 5:19 PM RB

(No Subject)

To: Python 4 Physics

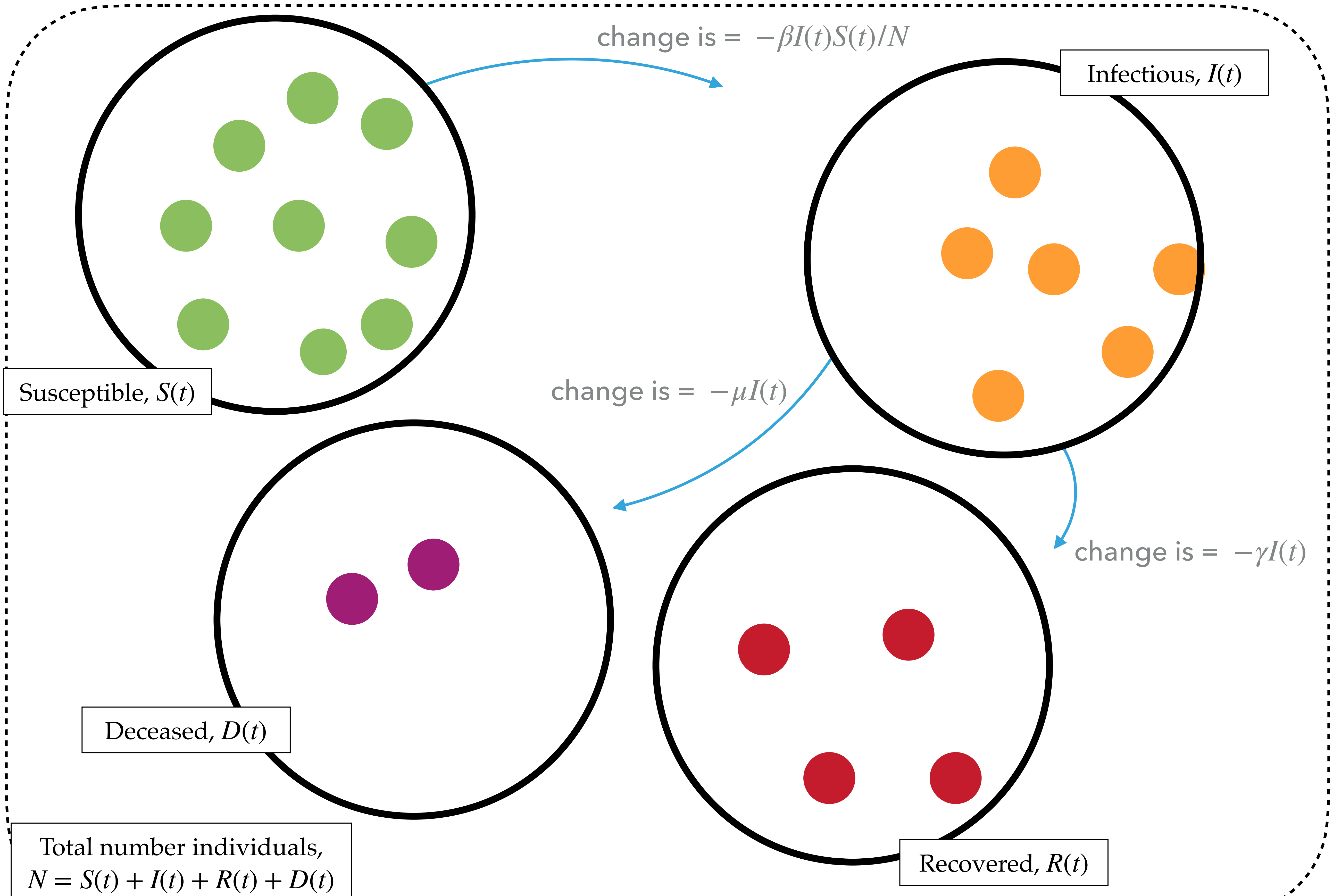
ADMIN STUFF

Need links, just email python4physics@odu.edu



REVIEW

SIRD - CARTOON FLOW



SIRD MODEL - SOLUTION

For this case, we have have additional parameter (μ) for describing the probability of an infectious individual to become deceased ($D(t)$).

$$\boxed{\frac{\Delta S(t)}{\Delta t} = -\beta \frac{S(t)}{N} I(t)} \quad \boxed{\frac{\Delta I(t)}{\Delta t} = \beta \frac{S(t)}{N} I(t) - \gamma I(t) - \mu I(t)}$$

$$\boxed{\frac{\Delta R(t)}{\Delta t} = \gamma I(t)} \quad \boxed{\frac{\Delta D(t)}{\Delta t} = \mu I(t)}$$

New equation for describing the change in the deceased population.

μ: the rate of mortality

SIRD MODEL - SOLUTION

For this case, we have have additional parameter (μ) for describing the probability of an infectious individual to become deceased ($D(t)$).

$$\boxed{\frac{\Delta S(t)}{\Delta t} = -\beta \frac{S(t)}{N} I(t)} \quad \boxed{\frac{\Delta I(t)}{\Delta t} = \beta \frac{S(t)}{N} I(t) - \gamma I(t) - \mu I(t)}$$

$$\boxed{\frac{\Delta R(t)}{\Delta t} = \gamma I(t)} \quad \boxed{\frac{\Delta D(t)}{\Delta t} = \mu I(t)}$$

Solutions satisfy

$$S(t+dt) = S(t) - dt\,\beta \frac{S(t)I(t)}{N}$$

$$I(t+dt) = I(t) + dt\,\left(\beta \frac{S(t)I(t)}{N} - (\gamma + \mu)\, I(t)\right)$$

$$R(t+dt) = R(t) + dt\gamma I(t)$$

$$D(t+dt) = D(t) + dt\mu I(t)$$

Total number individuals,
$N = S(t) + I(t) + R(t) + D(t)$

EXERCISE #1 - WRITE CODE FOR SOLVING THE DIFFERENT EQS.

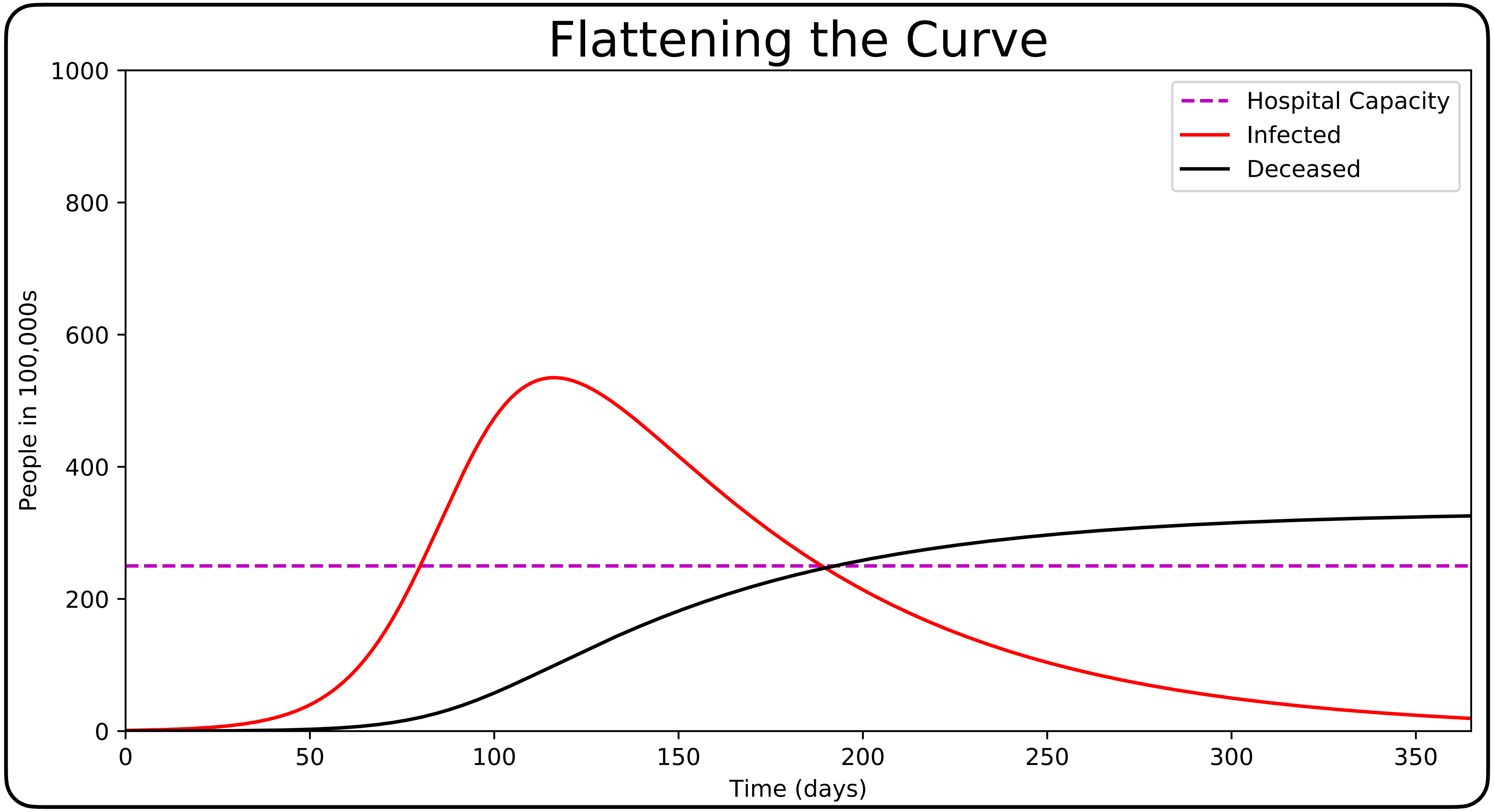
```
"this is the change in functions describing the SIRD model, up to the dt scale"
def SIRD(s,i,r,d, beta, gamma, mu):
    "population size"
    N = s+i+r+d

    "the shift in the subsceptibles"
    s_dot = -s*i*beta / N

    "the shift in the subsceptibles"
    i_dot = (s*i*beta/N) - gamma*i - mu*i

    "the shift in the subsceptibles"
    r_dot = gamma*i

    "the shift in the subsceptibles"
    d_dot = mu*i

    return s_dot, i_dot, r_dot, d_dot


"""
this code returns the solutions for the different equation.

We added an output option command, this is to just return either the

number of infected people or the deceased people
"""
def diff_eq_sird(model,s,i,r,d,beta,gamma,mu,steps,dt, output):
    for t in steps:
        if t < len(steps)-1:

            "note: here we call the model function"
            s_dot, i_dot, r_dot, d_dot = model(s[t], i[t], r[t], d[t], beta,gamma,mu)
            s[t+1] = s[t]+dt*s_dot
            i[t+1] = i[t]+dt*i_dot
            r[t+1] = r[t]+dt*r_dot
            d[t+1] = d[t]+dt*d_dot

    "only return number of infections"
    if output =='i':
        return i

    "only return number of deceased"
    if output =='d':
        return d
```

look inside our code 😎

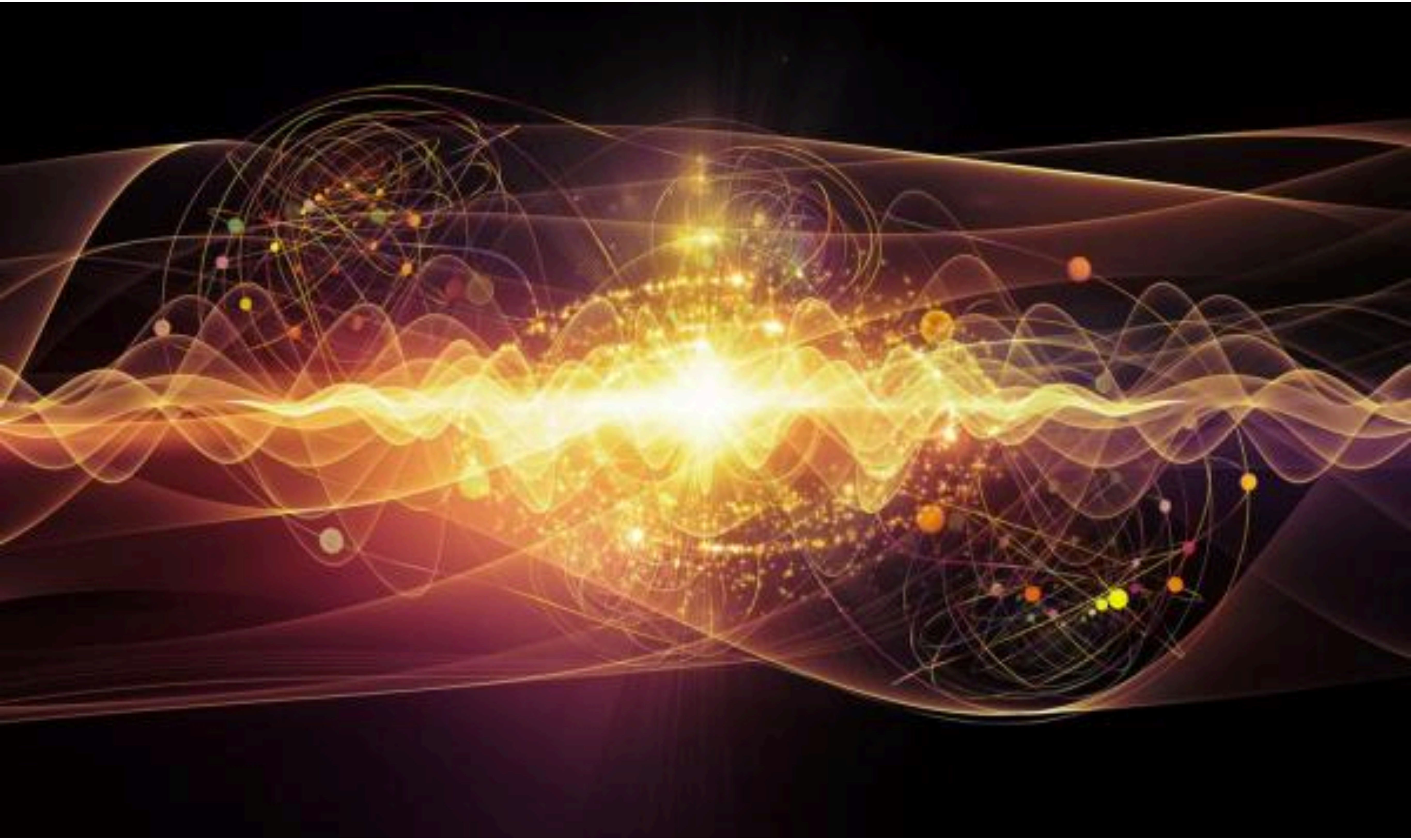
EXERCISE #2 - MAKE PLOTS!

Fix $\beta, \gamma, \mu = .09,\ .01,\ 0.005$, set the initial conditions of $S(0) = 998;\ \ I(0) = 1;\ \ R(0) = 0;\ \ D(0) = 0$, and calculate them as a function of time. Plot $I(t), D(t)$ vs. t.

If hospitals in your area only have capacity for 250 patients, will you have enough beds to support your sick? Which one of the variables above would you hope to reduce to assure that there are enough beds available?



ELECTROMAGNETIC WAVES!



MAXWELL'S EQUATIONS

Gauss's Law:

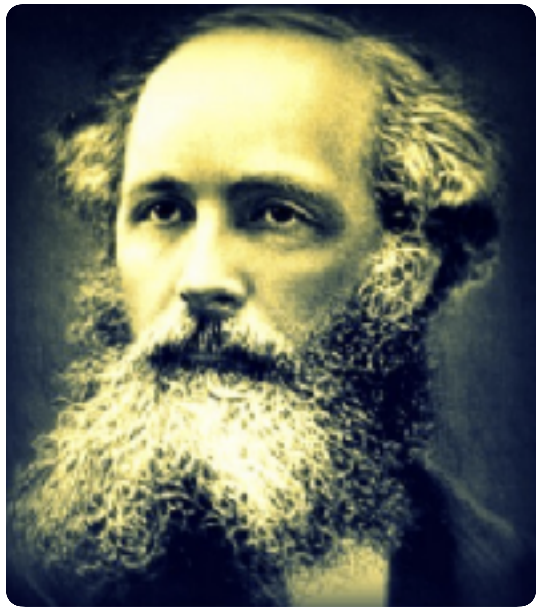
$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

➜ *electric fields from charges*

Gauss's Law for magnetic fields:

$$\vec{\nabla} \cdot \vec{B} = 0$$

➜ *no magnetic charges*



James Clerk Maxwell

[1831-1879]

Faraday's Law:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

➜ *electric field if a magnetic field changes (c.f. induction)*

Ampere-Maxwell Equation:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

➜ *magnetic fields from currents*
➜ *magnetic fields if electric fields change*

ELECTROMAGNETIC WAVES?

➜ Faraday’s law told us that

“time-varying magnetic fields generate electric fields”

➜ James Clerk Maxwell found that

“time-varying electric fields generate magnetic fields”

➜ taken together then

electric field → magnetic field → electric field → …

➜ with this we can have disturbances in the electric and magnetic fields that propagate across space

➜ *ELECTROMAGNETIC WAVES*

unlike others waves you might be more familiar with, such as sound waves or waves on a string, no medium is required - it is not atoms moving around, but instead the electric and magnetic fields and these can exist even in a vacuum

ELECTROMAGNETIC WAVES?

Consider a region of space where there are no charges or currents. This means that in this region

$$\rho(\vec{r},t) = 0 \qquad \text{and} \qquad \vec{J}(\vec{r},t) = 0$$

Furthermore, if we restrict out attention to one-dimensional space, Faraday's Law and Ampere-Maxwell equation simplify to…

$$\boxed{\frac{\partial^2 E(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E(x,t)}{\partial t^2} = 0} \qquad \text{and} \qquad \boxed{\frac{\partial^2 B(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 B(x,t)}{\partial t^2} = 0.}$$

The speed of light! $c = 3 \times 10^8\, m/s$

ELECTROMAGNETIC WAVES?

Consider a region of space where there are no charges or currents. This means that in this region

$$\rho(\vec{r},t) = 0 \qquad \text{and} \qquad \vec{J}(\vec{r},t) = 0$$

Furthermore, if we restrict out attention to one-dimensional space, Faraday's Law and Ampere-Maxwell equation simplify to…

$$\boxed{\frac{\partial^2 E(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E(x,t)}{\partial t^2} = 0} \qquad \text{and} \qquad \boxed{\frac{\partial^2 B(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 B(x,t)}{\partial t^2} = 0.}$$

Wave equation:

The wave equation is one of the more important differential equations in physics. It is a second-order partial differential equation.

A ***partial differential equation*** is differential equation involving derivatives of two more variables.

For a nice introduction to waves by Dr. Aslan, look at our previous class. 🤓

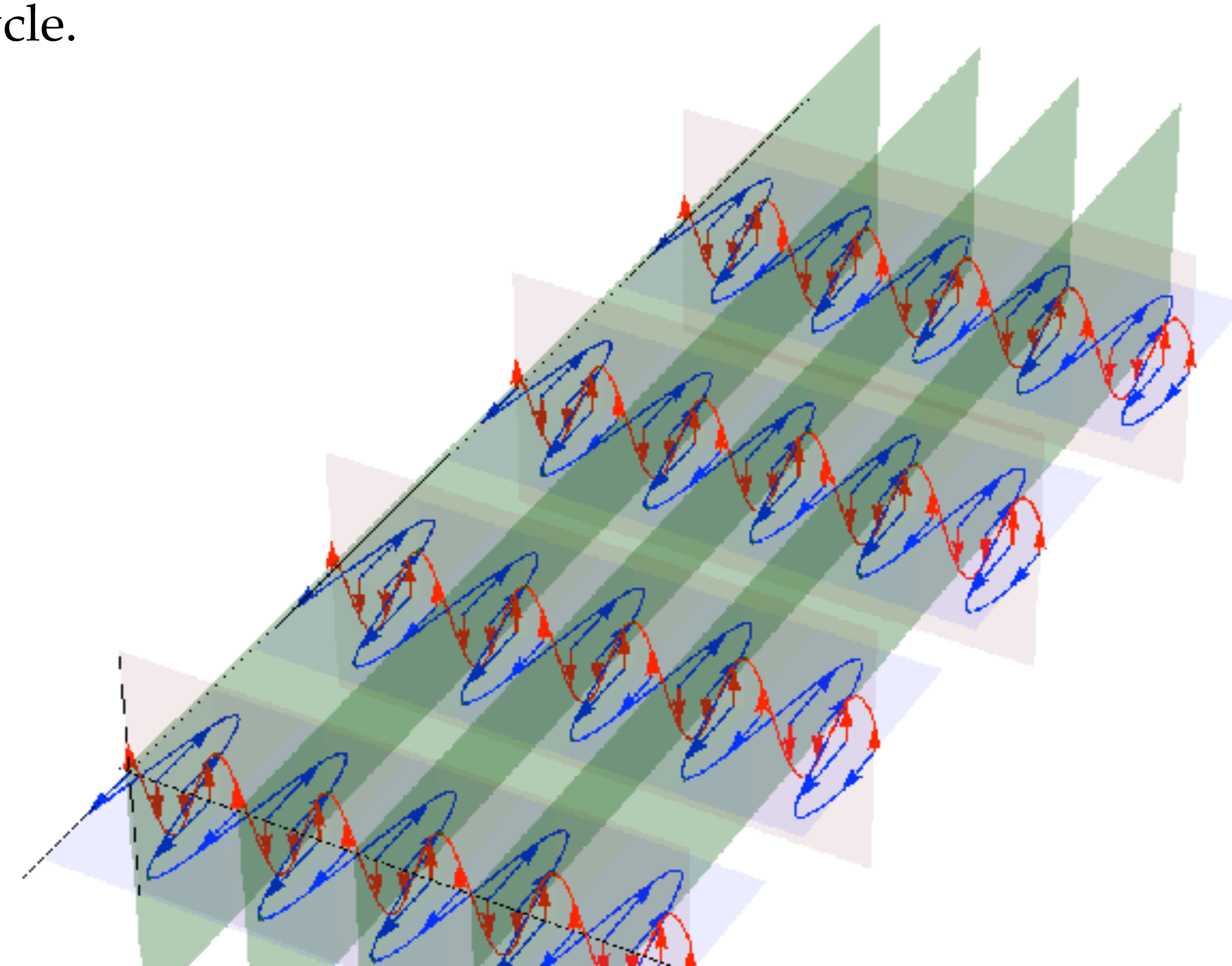
Solution to these differential equations, can be written as

$$E(x,t) = E_0 \cos\left(\frac{2\pi}{\lambda}x - 2\pi f\, t\right), \qquad B(x,t) = B_0 \cos\left(\frac{2\pi}{\lambda}x - 2\pi f\, t\right).$$

where E_0 is the amplitude, λ **is the wavelength** and f **is the frequency**.

The frequency is equal to the inverse of the period (T), which is the defined as the total time it takes to do a full cycle.

In other words $f = \dfrac{1}{T}$.

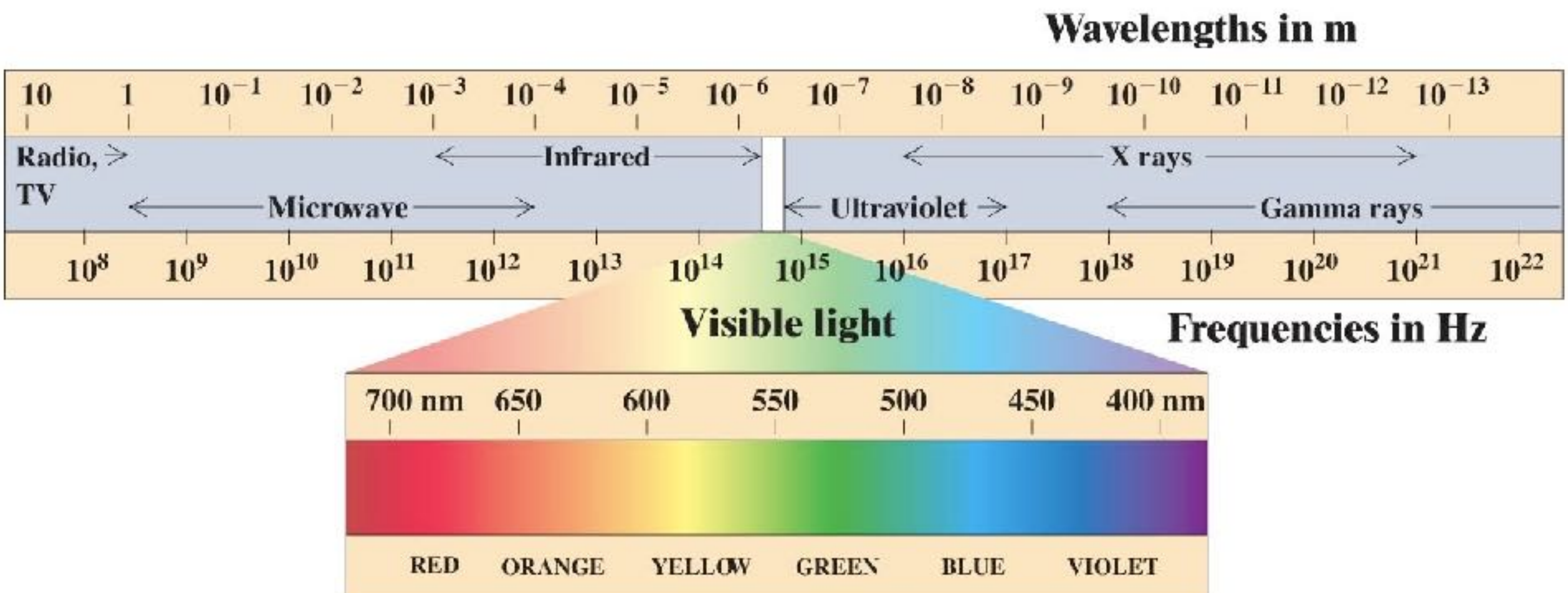


THE EM SPECTRUM

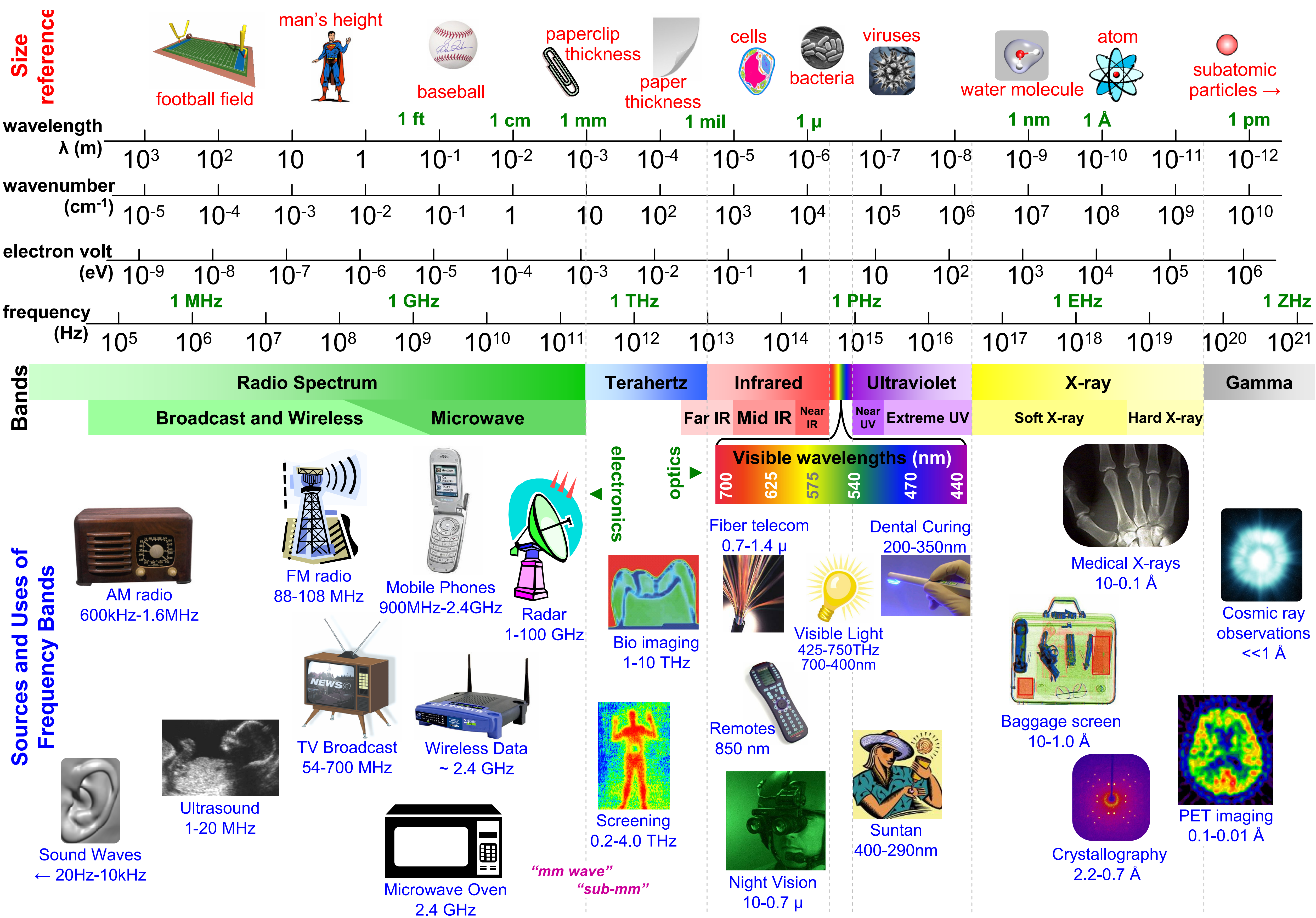
➜ We saw that sinusoidal waves are described by a wavelength, λ, and a frequency, *f*

➜ they are related by $c = f\lambda$

➜ e/m waves of different wavelengths have different names



THE EM SPECTRUM



CONCEPTUAL QUESTIONS

➜ ultraviolet waves have a shorter wavelength than infrared waves

which waves have the higher frequency

1. ultraviolet
2. infrared
3. same frequency

➜ radio waves have a longer wavelength than microwaves

which waves have the higher speed

1. radio waves
2. microwaves
3. same speed

➜ Light having a certain frequency, wavelength (λ) and speed is traveling through empty space. If the frequency of the light were doubled, then its new wavelength would be:

1. 4λ
2. 2λ
3. λ
4. $\lambda/2$
5. $\lambda/4$

CONCEPTUAL QUESTIONS

➜ ultraviolet waves have a shorter wavelength than infrared waves

which waves have the higher frequency

1. ultraviolet ✓
2. infrared
3. same frequency

➜ radio waves have a longer wavelength than microwaves

which waves have the higher speed

1. radio waves
2. microwaves
3. same speed ✓

➜ Light having a certain frequency, wavelength (λ) and speed is traveling through empty space. If the frequency of the light were doubled, then its new wavelength would be:

1. 4λ
2. 2λ
3. λ
4. $\lambda/2$ ✓
5. $\lambda/4$

EXERCISE #1 - CHECK WAVE EQUATION

Use the finite difference definition of the derivative [look back project#6]:

$$\frac{d}{dx} f(x) \equiv \lim_{a \to 0} \frac{\Delta_a f(x)}{2a} = \lim_{a \to 0} \frac{f(x+a) - f(x-a)}{2a}$$

To re-write the wave equation,

$$\frac{\partial^2 E(x,t)}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E(x,t)}{\partial t^2} = 0$$

in a discrete wave.

Having done that, test that $E_0 \cos\left(\frac{2\pi}{\lambda} x - 2\pi f\, t\right)$ is indeed a solution to the wave equation for any values of E_0 and λ, remembering that $f = c/\lambda$.

To this do, you can calculation $\frac{\partial^2 E(x,t)}{\partial x^2}$ and $\frac{1}{c^2} \frac{\partial^2 E(x,t)}{\partial t^2}$ separately, and check that they equal to each other for small value of a.

EXERCISE #1 - CHECK WAVE EQUATION

note, I set $c = 1$

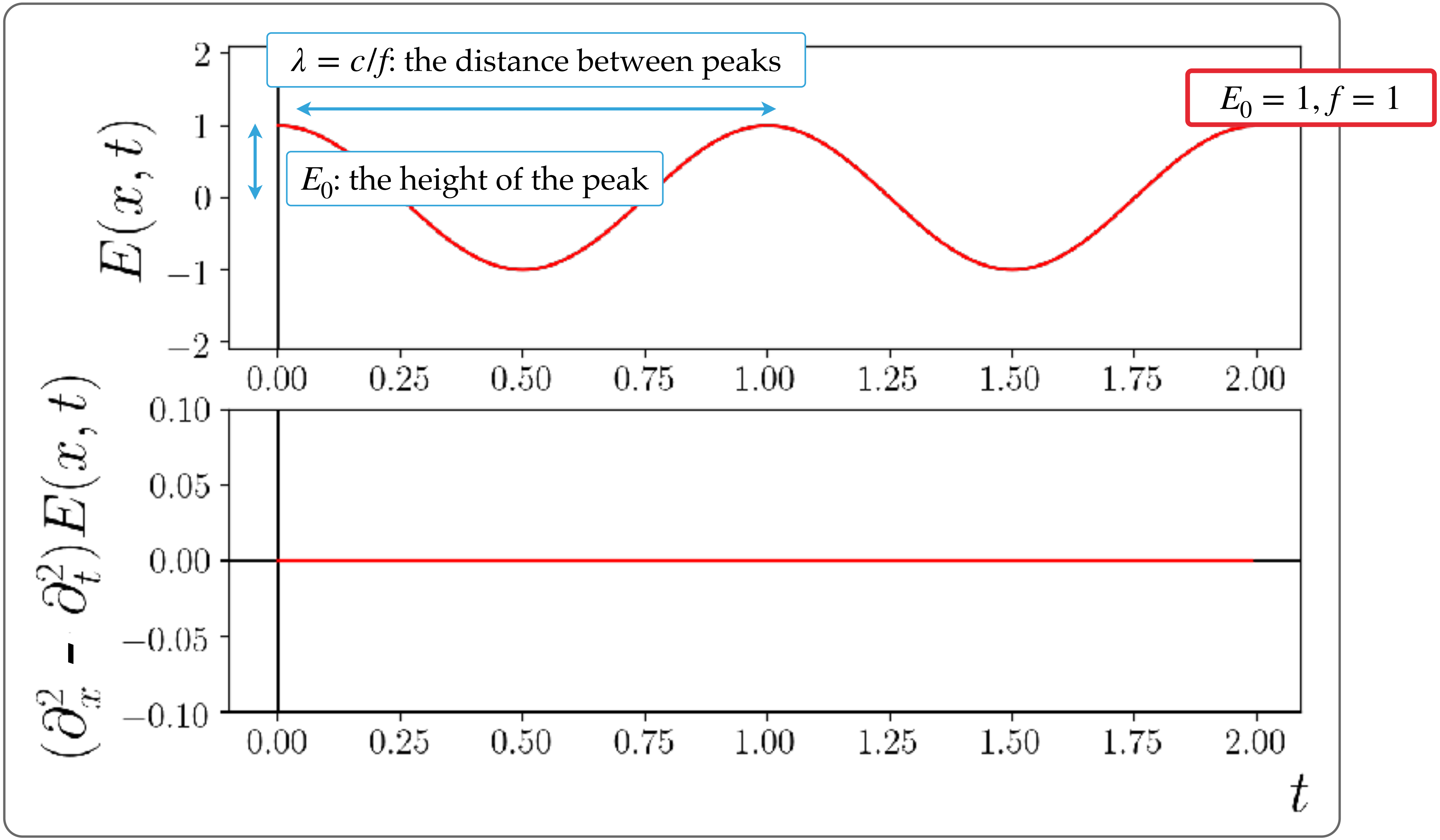
```
def E_field(E0, x,t,f):
    "I am setting c =1 "
    c=1
    lamd = c/f

    darg0 = 2 * pi * x / lamd
    darg1 = - 2 * pi * f * t

    return E0 * cos(darg0 + darg1)

def dE(E0, x,t,f, var):

    "the numerator"
    if var =='t':
        num = E_field(E0, x,t+a,f) - E_field(E0, x,t-a ,f)
    elif var =='x':
        num = E_field(E0, x+a,t,f) - E_field(E0, x-a,t ,f)

    "the denominator"
    denum = 2.0 * a

    return num / denum

def dE2(E0, x,t,f, var1, var2):

    "the numerator"
    if var2 =='t':
        num = dE(E0, x,t+a,f,var1) - dE(E0, x,t-a ,f,var1)
    elif var2 =='x':
        num = dE(E0, x+a,t,f,var1) - dE(E0, x-a,t ,f,var1)

    "the denominator"
    denum = 2.0 * a

    return num / denum


def test_wave_Eq(E0, x,t,f):

    "derivative with respect to x"

    dEtt = dE2(E0, x,t,f, 't', 't')
    dExx = dE2(E0, x,t,f, 'x', 'x')

    return dExx - dEtt
```

look inside my code 😎

For defining the E-field, its derivative, second derivative and testing the wave equation.

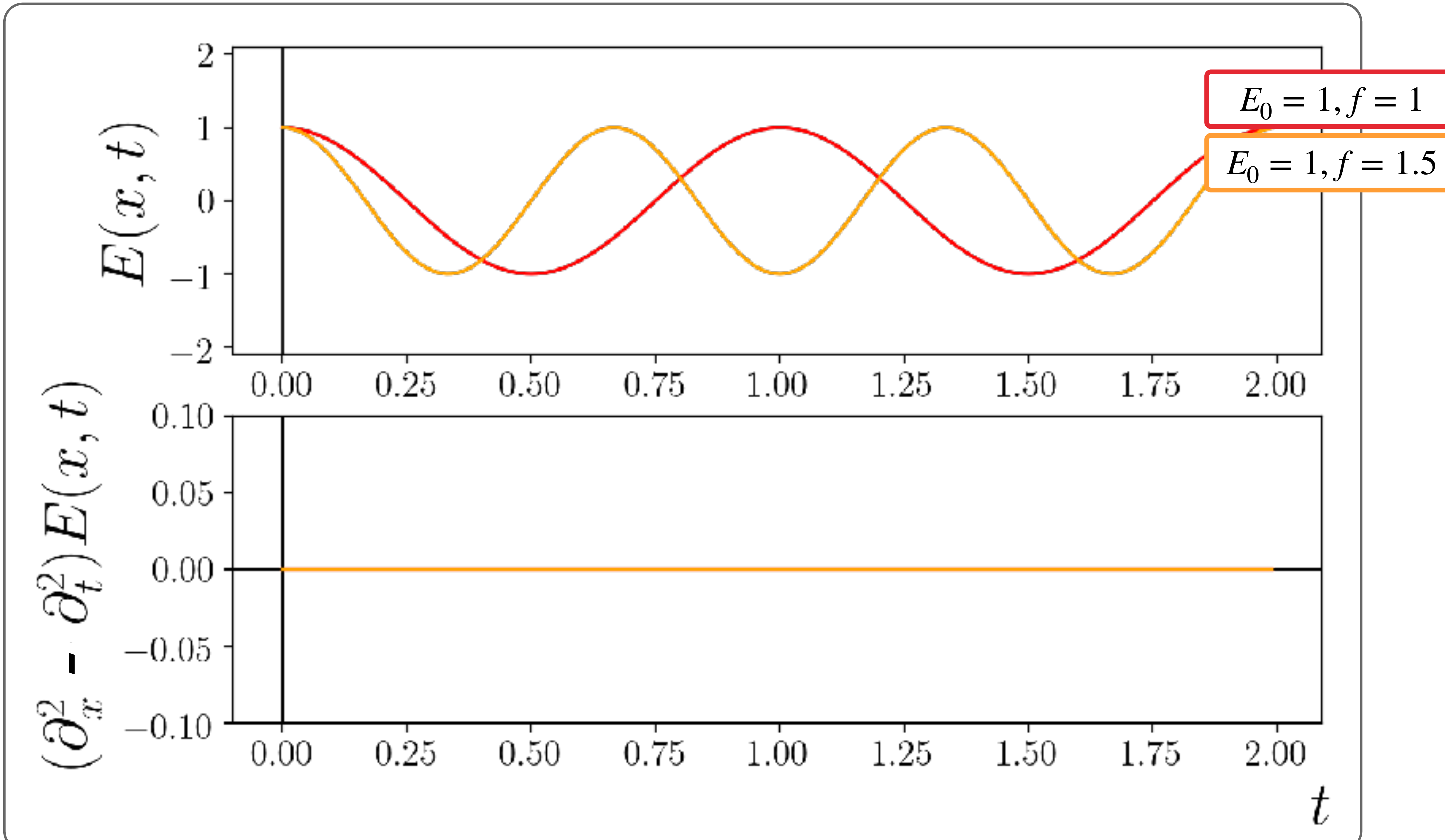
EXERCISE #1 - CHECK WAVE EQUATION

Here I fix $x = 0$ and $c = 1$ and vary the other parameters



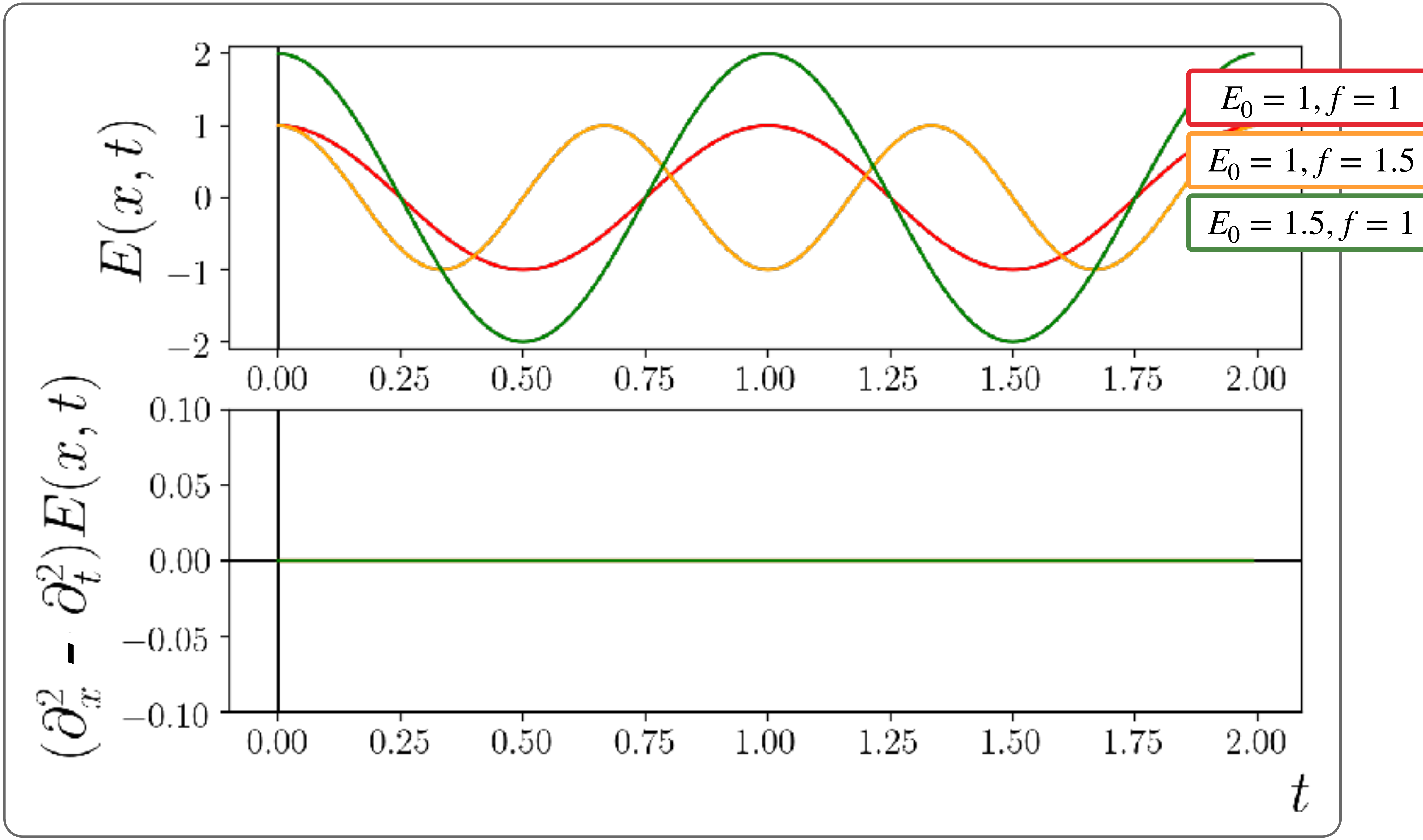
EXERCISE #1 - CHECK WAVE EQUATION

Here I fix $x = 0$ and $c = 1$ and vary the other parameters



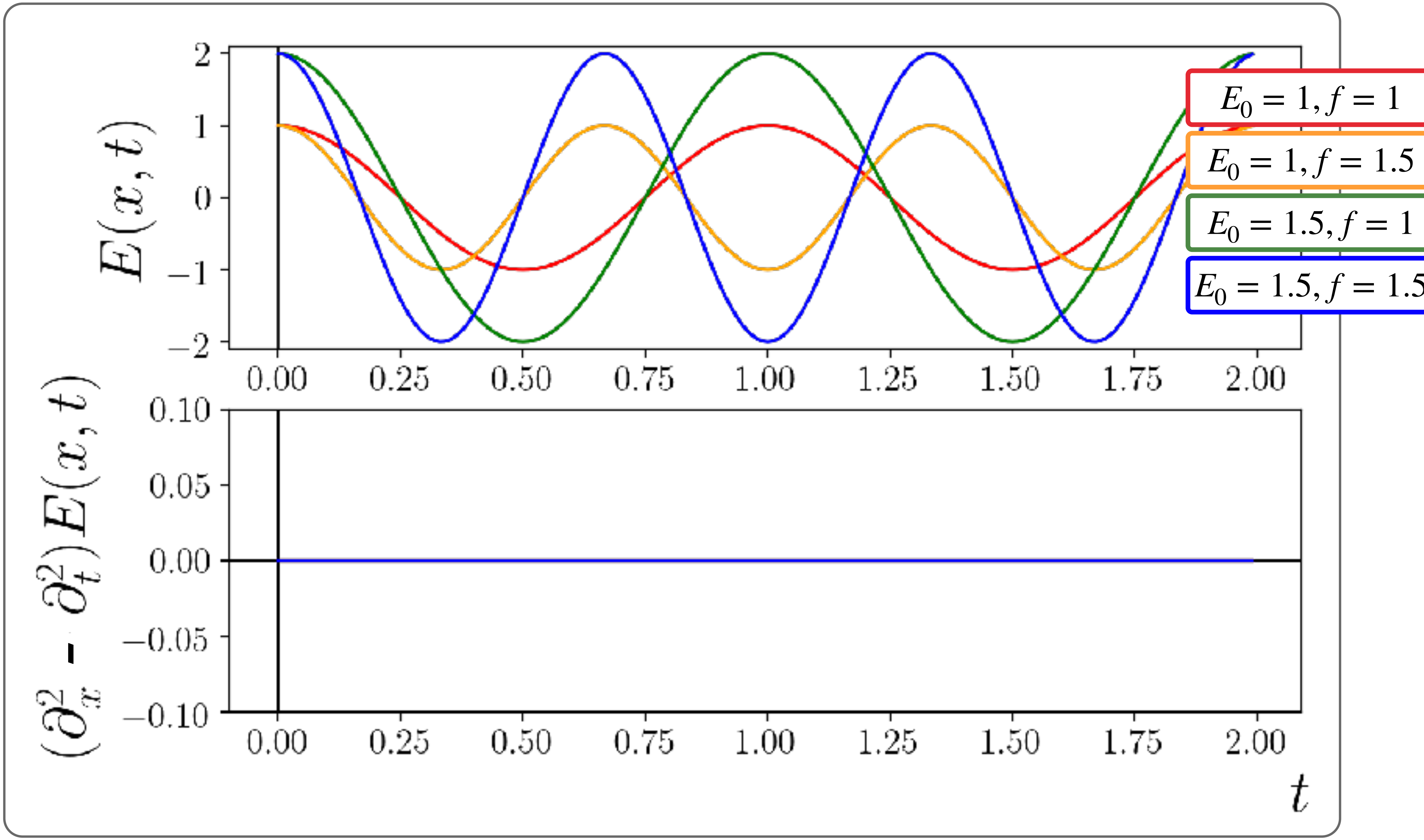
EXERCISE #1 - CHECK WAVE EQUATION

Here I fix $x = 0$ and $c = 1$ and vary the other parameters



EXERCISE #1 - CHECK WAVE EQUATION

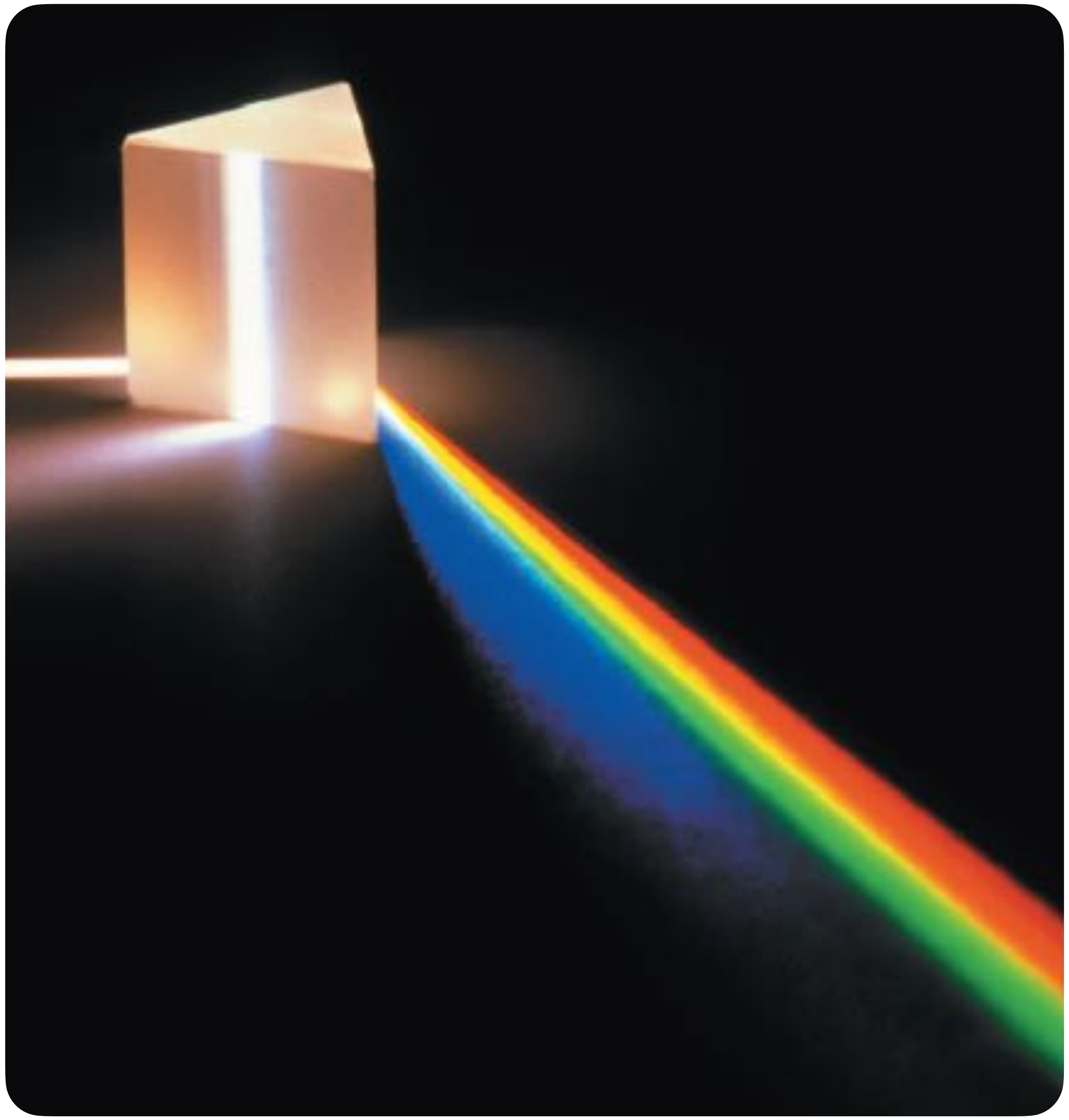
Here I fix $x = 0$ and $c = 1$ and vary the other parameters



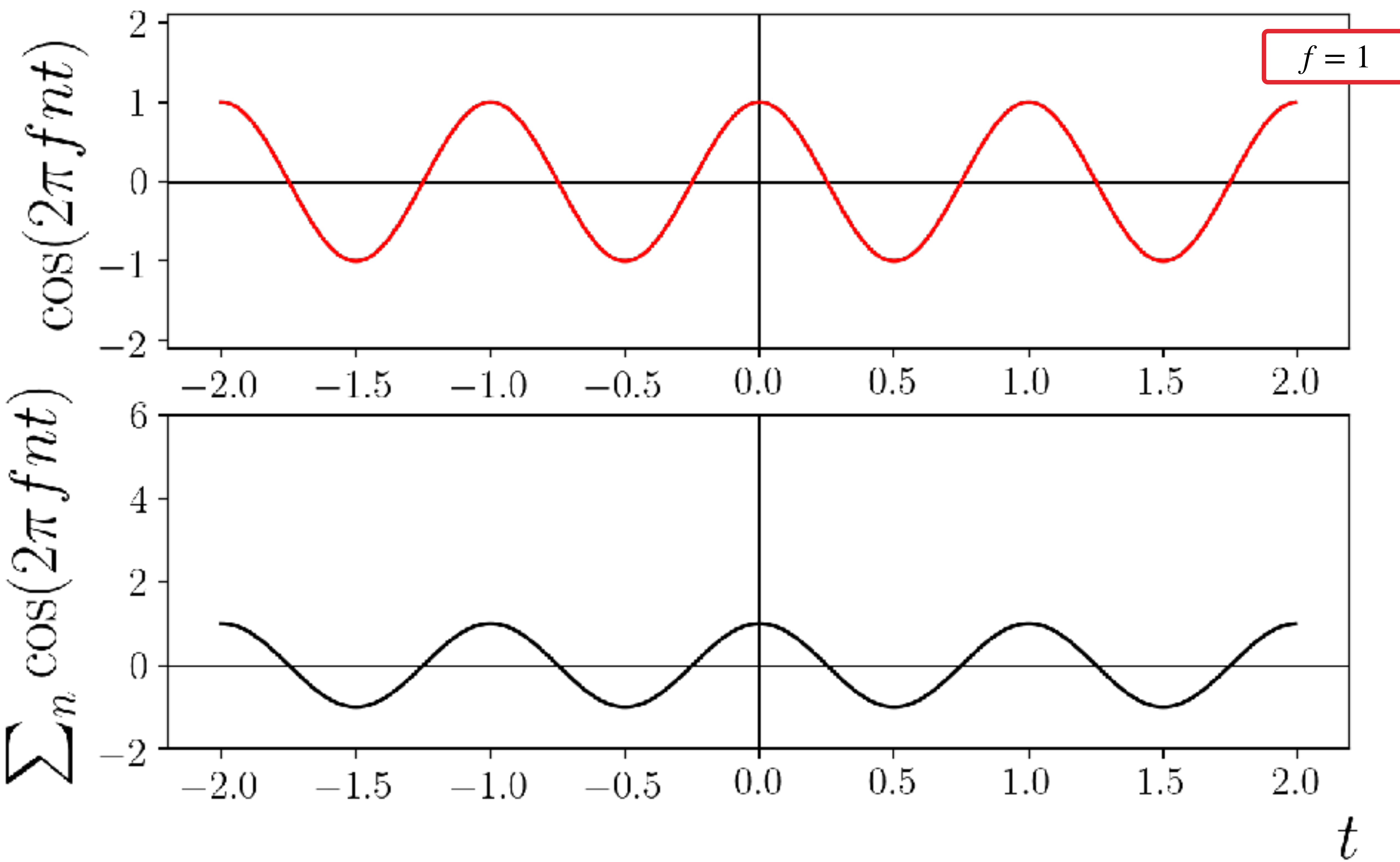
ADDING WAVES TOGETHER

In general, the light we see will a sum of waves with different frequencies.

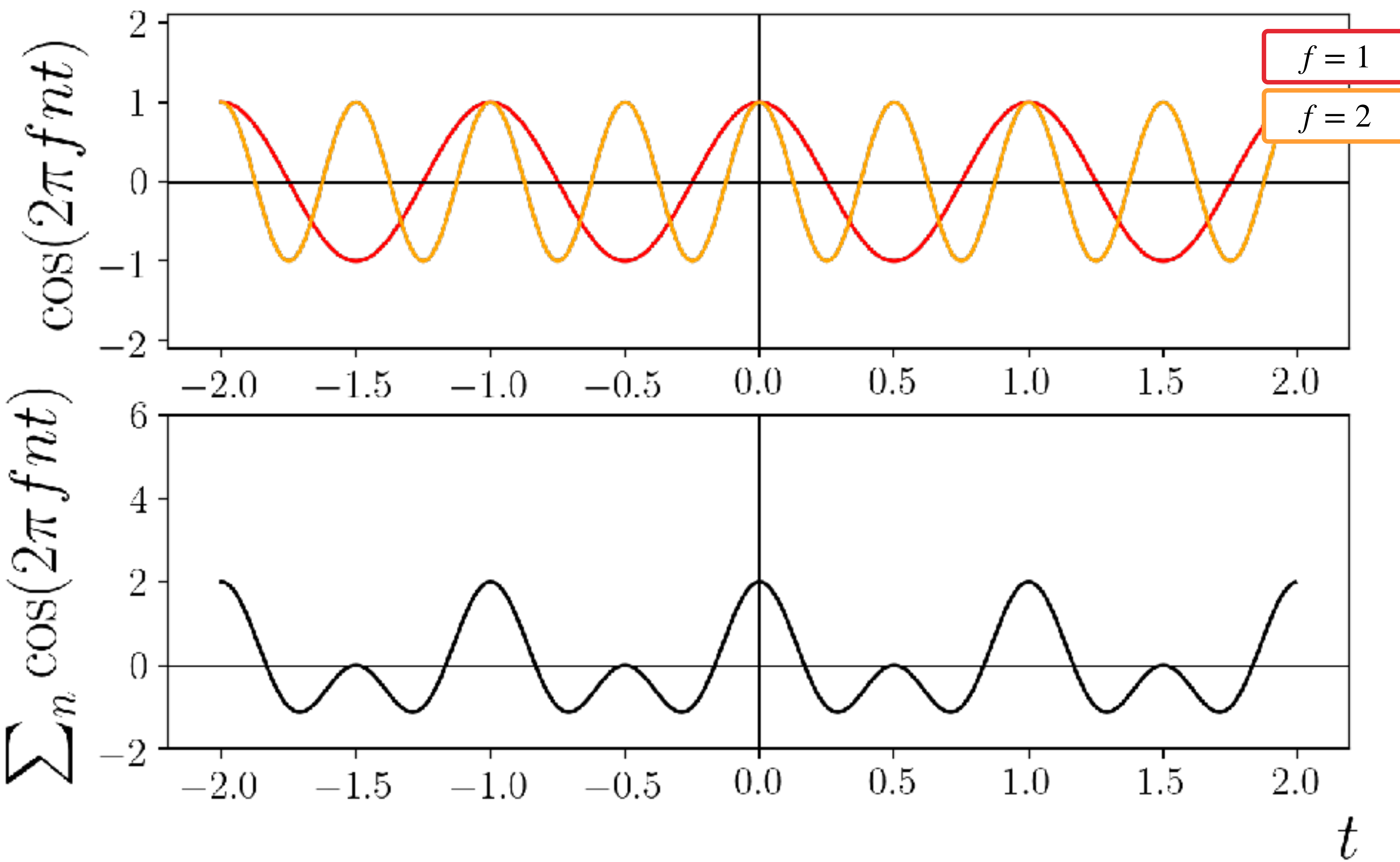
Adding waves is relatively straightforward to do numerically.



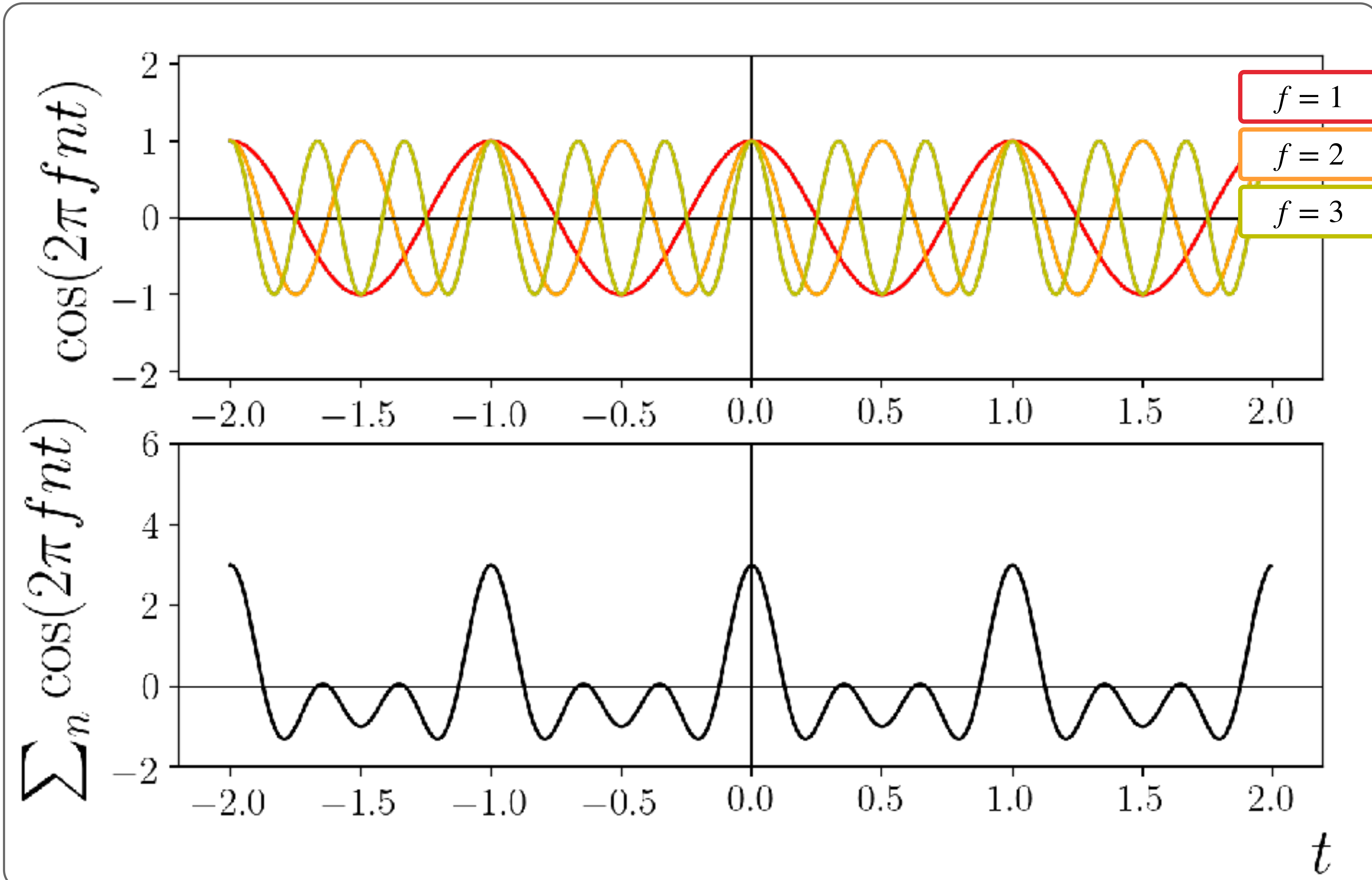
EXERCISE #2 - PLOT THE SUM OF WAVES



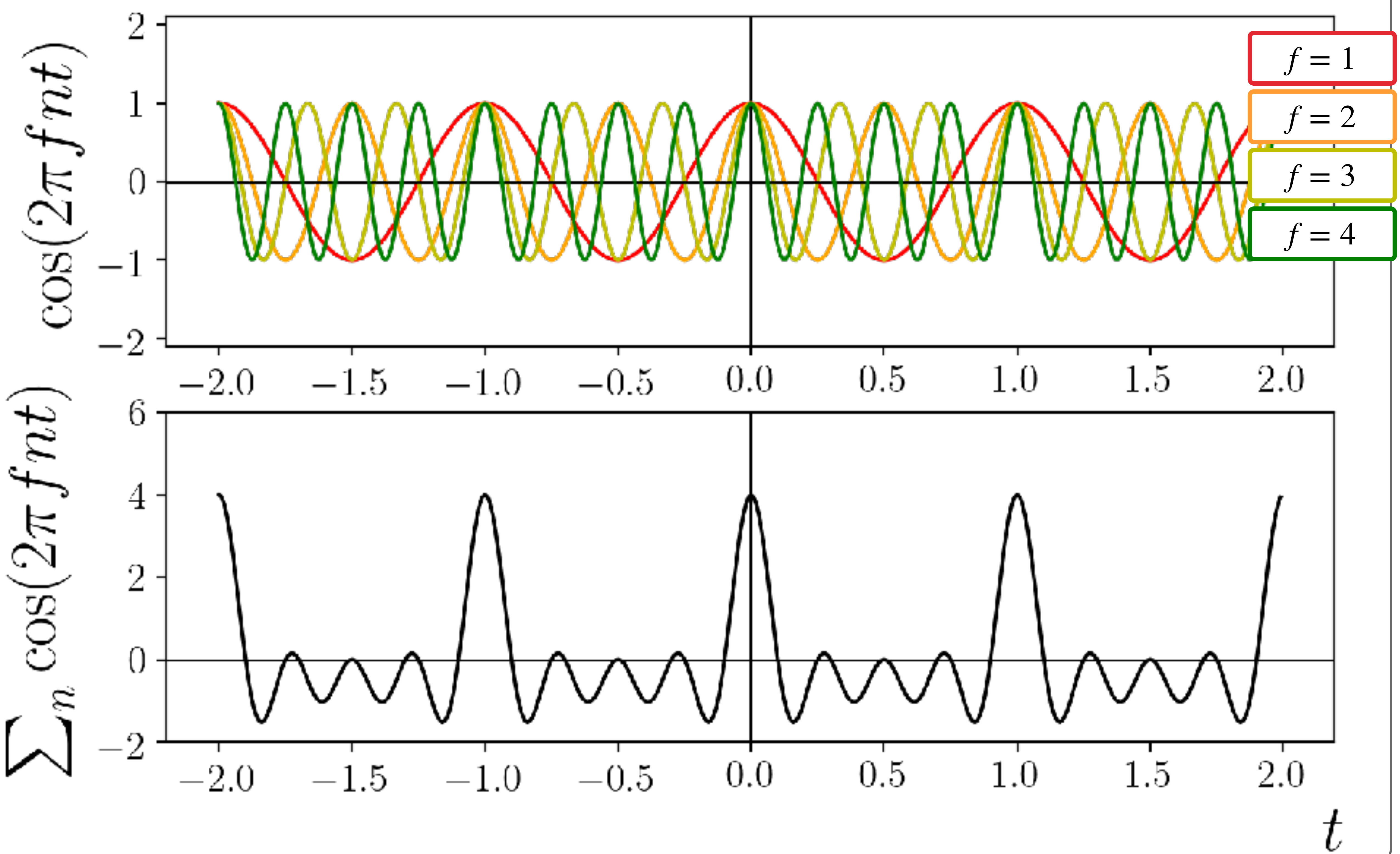
EXERCISE #2 - PLOT THE SUM OF WAVES



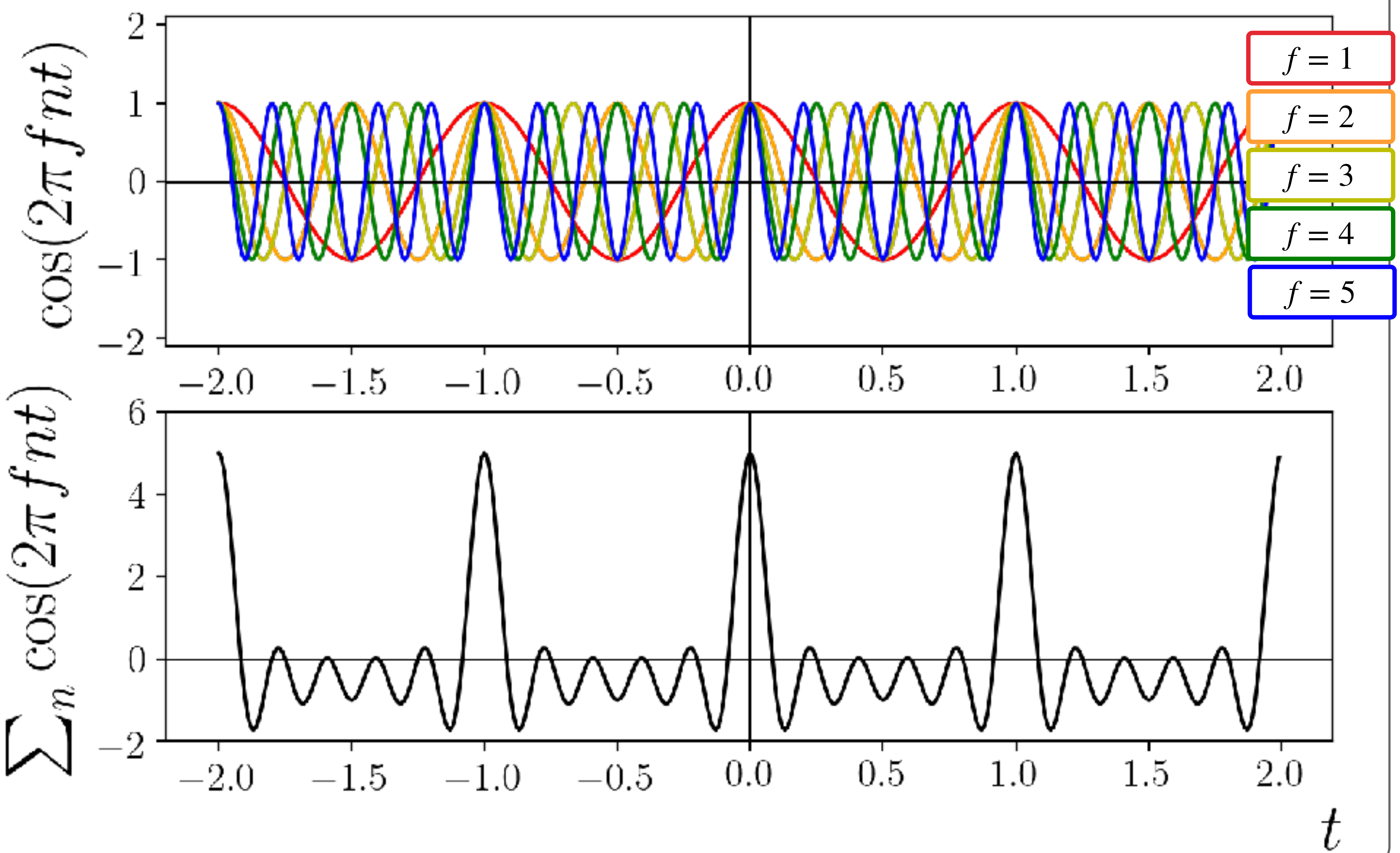
EXERCISE #2 - PLOT THE SUM OF WAVES



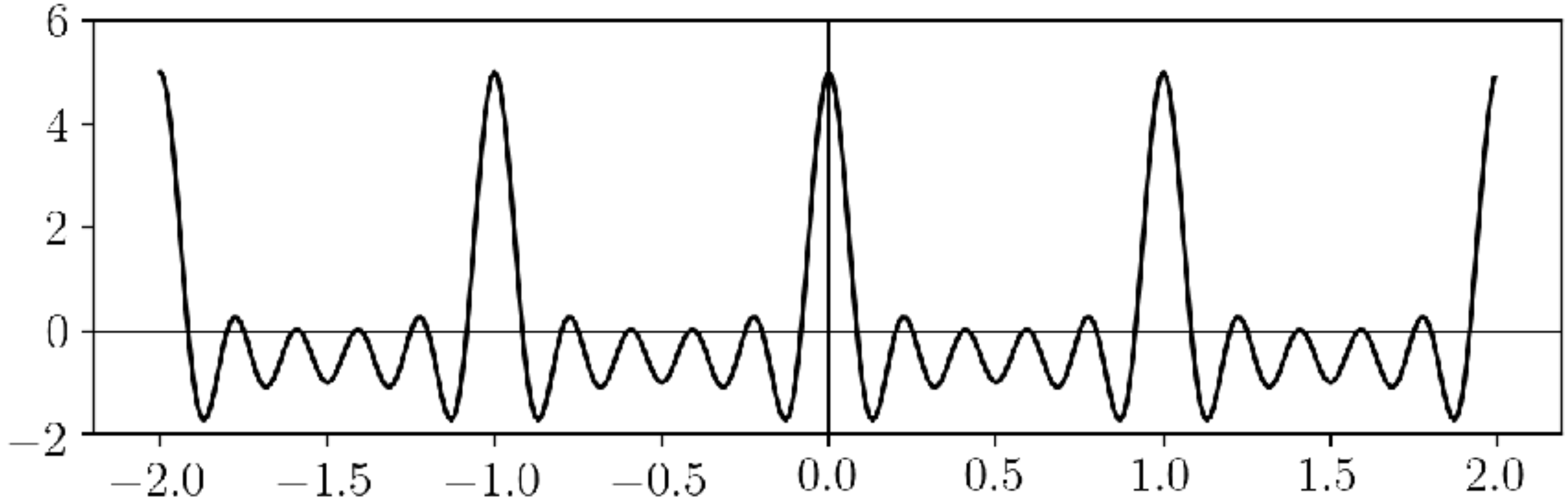
EXERCISE #2 - PLOT THE SUM OF WAVES



EXERCISE #2 - PLOT THE SUM OF WAVES



Now imagine you have some light wave packet as a function of time,



and you would like to know which frequencies of light are contributing to that light packet, and determine by which amount each wave contributes.

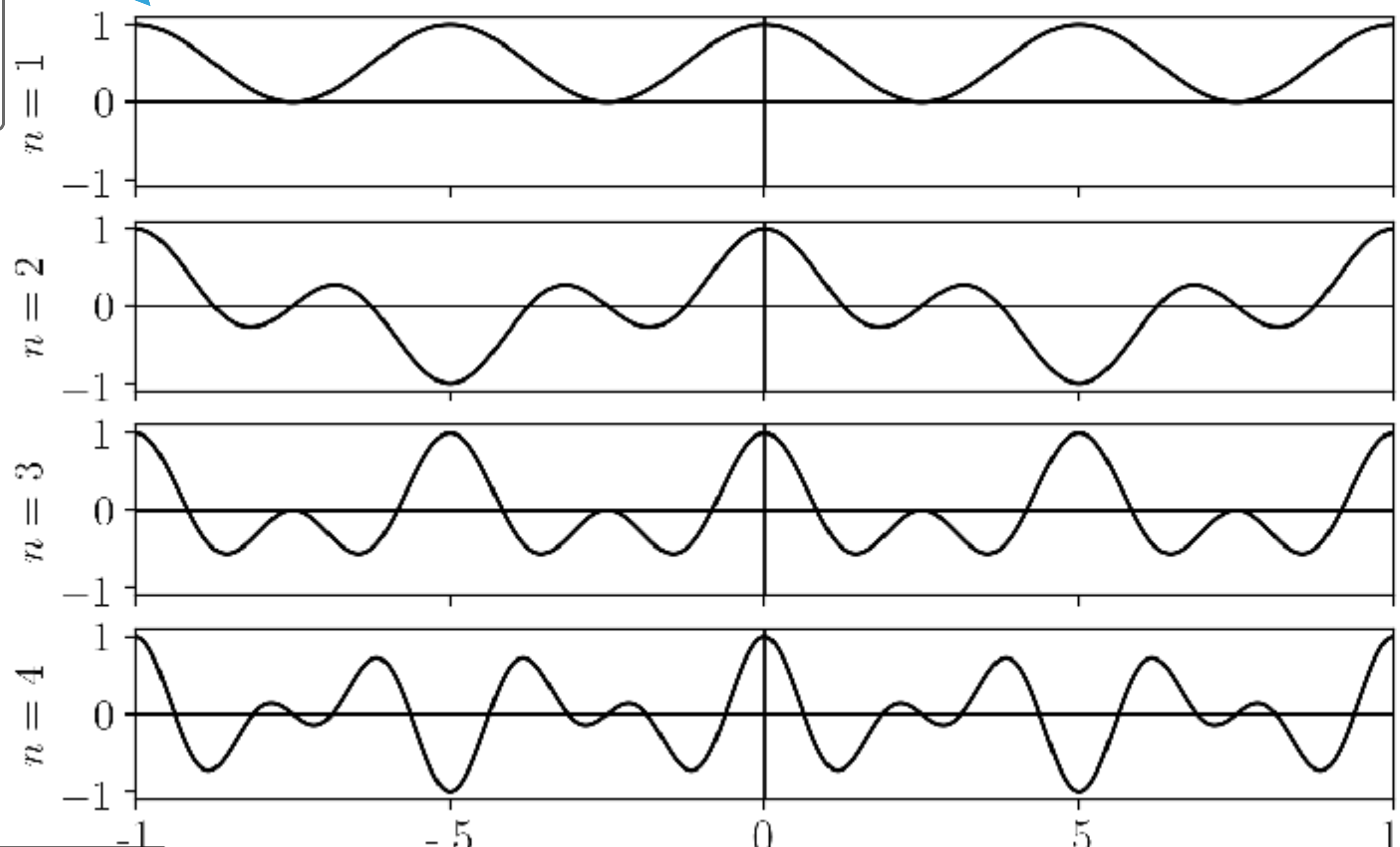
This can be achieved using **Fourier transforms**.

To introduce Fourier transforms consider the product of $\cos(2\pi t)$ times $\cos(2\pi\, n\, t)$ for a range of n. In other words, let us plot $\cos(2\pi t)\ \cos(2\pi\, nt)$ as a function of t for different values of n.

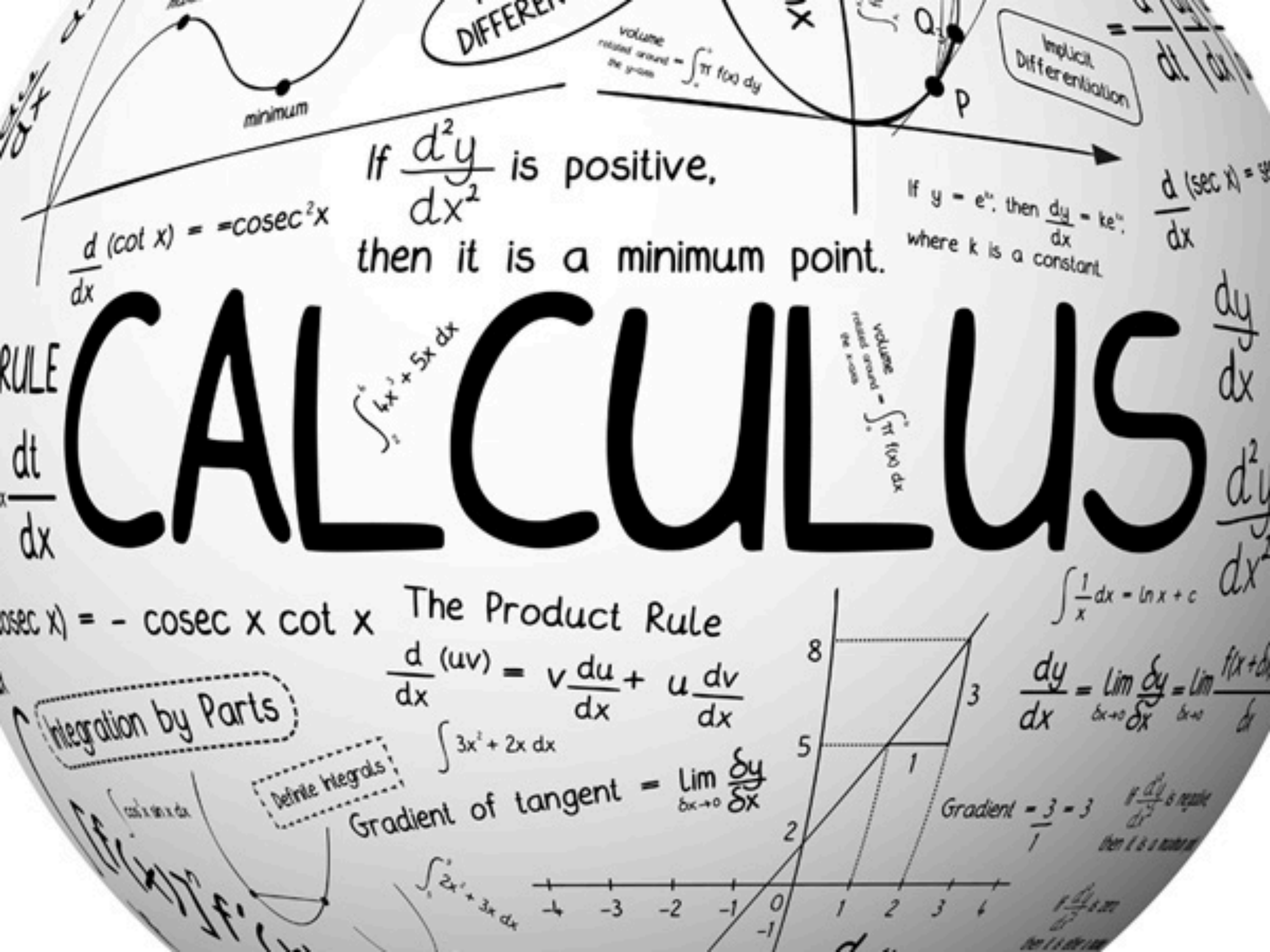
EXERCISE #3 - REPRODUCE THESE PLOTS

Here we plot $\cos(2\pi t)\ \cos(2\pi\, nt)$ as a function of t. Each panel is labeled by the choice of n.

Note, this is the only that is always positive. This makes sense, since $(\cos(2\pi t))^2 \geq 0$.



While all others have positive and negative values. In fact, it seems that if we sum the negative and positive values, we would get zero.


CALCULUS
If $\frac{d^2y}{dx^2}$ is positive, then it is a minimum point.
minimum
Implicit Differentiation
$\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$
If $y = e^{kx}$, then $\frac{dy}{dx} = ke^{kx}$, where k is a constant
RULE
The Product Rule
$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$
Integration by Parts
Definite Integrals
$\int 3x^2 + 2x\,dx$
Gradient of tangent = $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$
$\int \frac{1}{x} dx = \ln x + c$
Gradient = $\frac{3}{1}$ = 3

CALCULUS

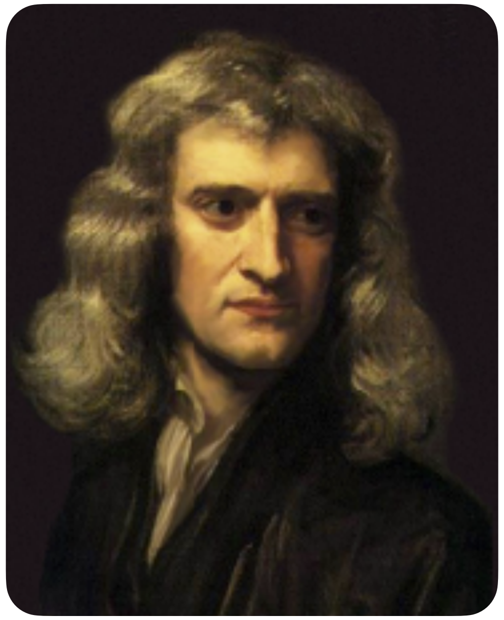
Calculus is a remarkably powerful field of mathematics. Although its history dates back to ancient Greece, the formalization of the field is largely attributed to **Newton** and **Leibniz**, who made important independent contributions.

Leibniz's interest lied primarily in setting foundational notation and mathematical rules. While **Newton's** interest was always describing the motion of physical objects, which is our interest as well.

Calculus can be divided into two fields:

differential calculus: interested in the rate of change of quantities, or equivalently the slopes of the corresponding curves

integral calculus: interested in the continuous sum of function, or equivalent the area under the curve of a function or between different functions.



Isaac Newton
England, 1643 - 1727

Gottfried Wilhelm Leibniz
France, 1646 - 1716

CALCULUS

Calculus is a remarkably powerful field of mathematics. Although its history dates back to ancient Greece, the formalization of the field is largely attributed to **Newton** and **Leibniz**, who made important independent contributions.

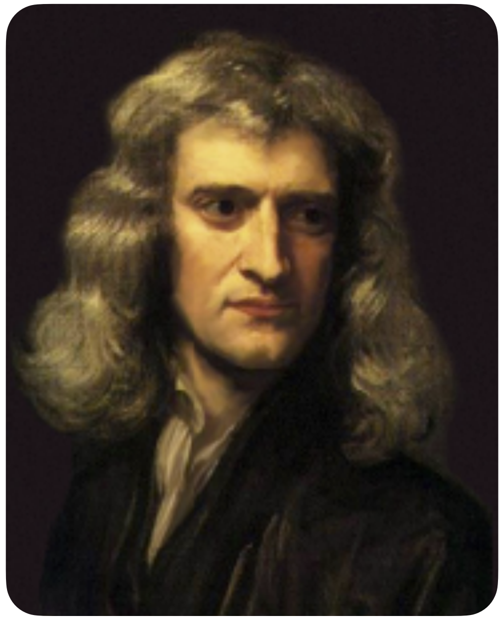
Leibniz's interest lied primarily in setting foundational notation and mathematical rules. While **Newton's** interest was always describing the motion of physical objects, which is our interest as well.

Calculus can be divided into two fields:

<u>differential calculus:</u> interested in the rate of change of quantities, or equivalently the slopes of the corresponding curves

<u>integral calculus:</u> interested in the continuous sum of function, or equivalent the area under the curve of a function or between different functions.

this was what we learnt about in the sixth lecture series



<u>Isaac Newton</u>
England, 1643 - 1727

<u>Gottfried Wilhelm Leibniz</u>
France, 1646 - 1716

Calculus is a remarkably powerful field of mathematics. Although its history dates back to ancient Greece, the formalization of the field is largely attributed to **Newton** and **Leibniz**, who made important independent contributions.

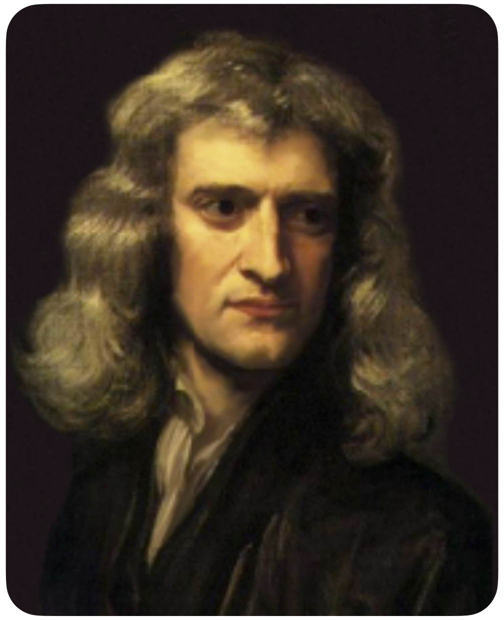
Leibniz's interest lied primarily in setting foundational notation and mathematical rules. While **Newton's** interest was always describing the motion of physical objects, which is our interest as well.

Calculus can be divided into two fields:

differential calculus: interested in the rate of change of quantities, or equivalently the slopes of the corresponding curves

integral calculus: interested in the continuous sum of function, or equivalent the area under the curve of a function or between different functions.

what we will learn about today!



Isaac Newton

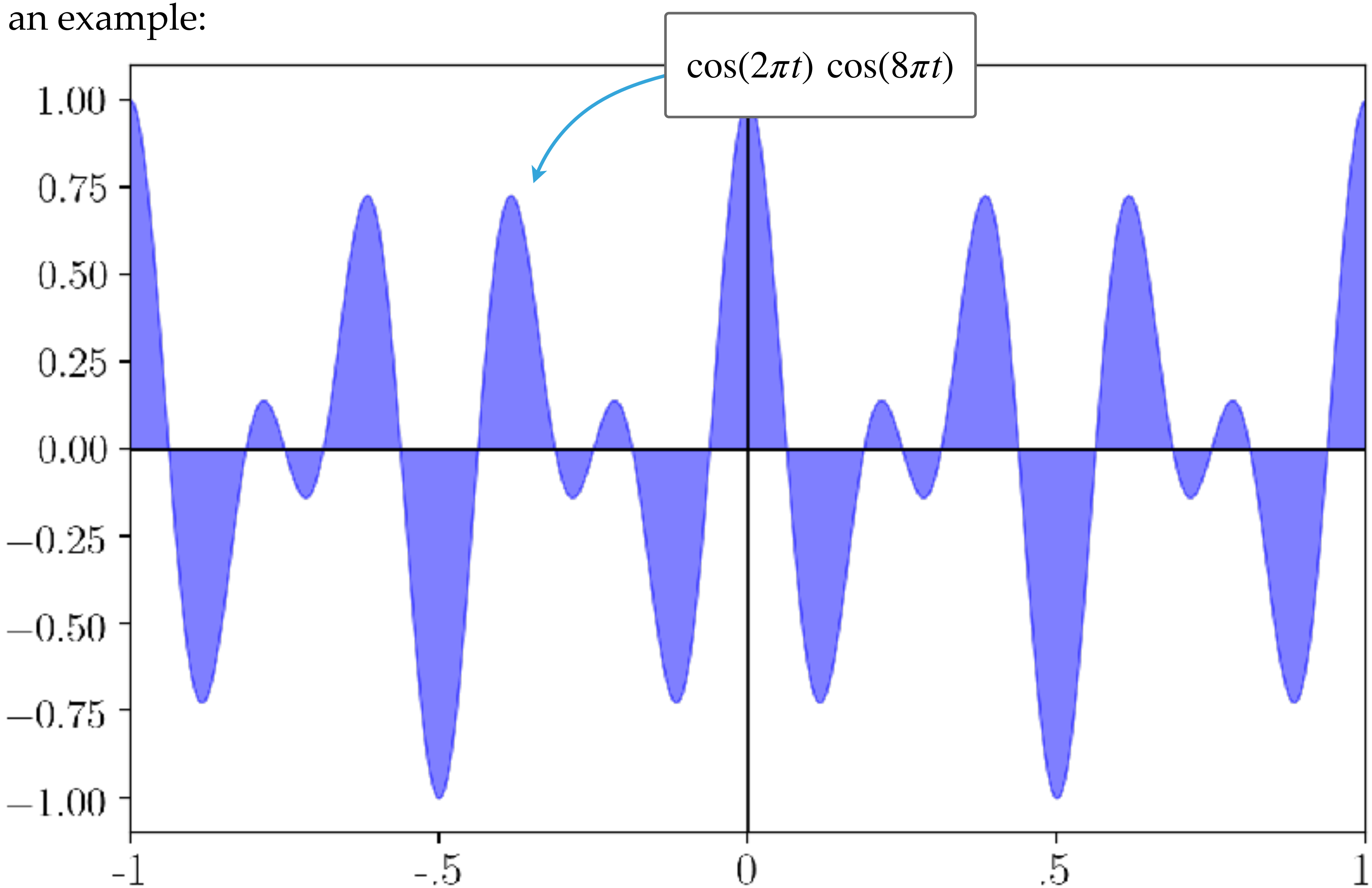
England, 1643 - 1727

Gottfried Wilhelm Leibniz

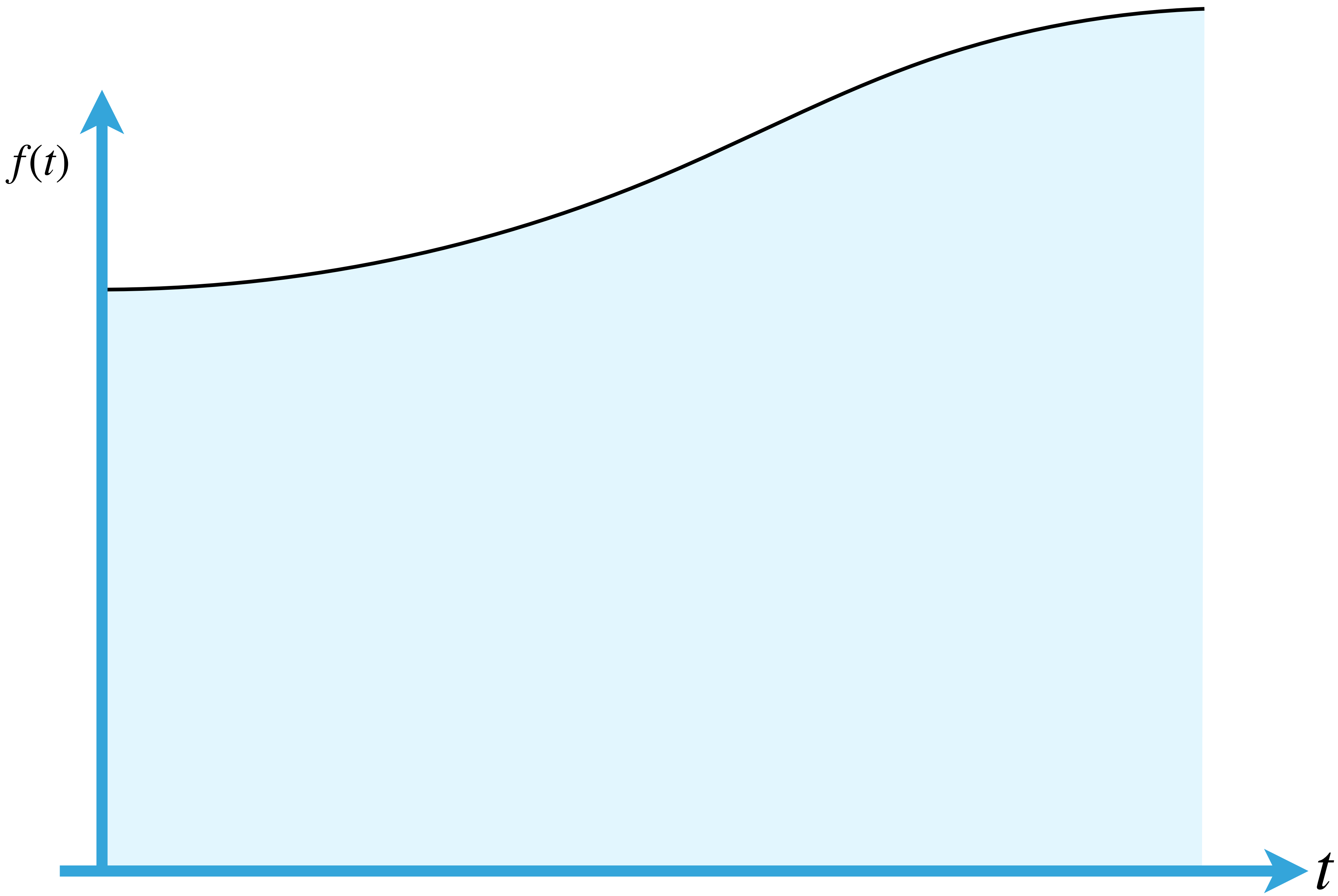
France, 1646 - 1716

One of the first things we learn in integral calculus is to evaluate *"the area under the curve"*. More specifically this means to evaluate the area between two lines: 1) the line of the function you are interested in, and 2) the line defined by the horizontal axis.

Here is an example:



Let us consider a generic function of the kind shown below. Imagine we want to calculate the area underneath the curve defined by $f(t)$ on the top.

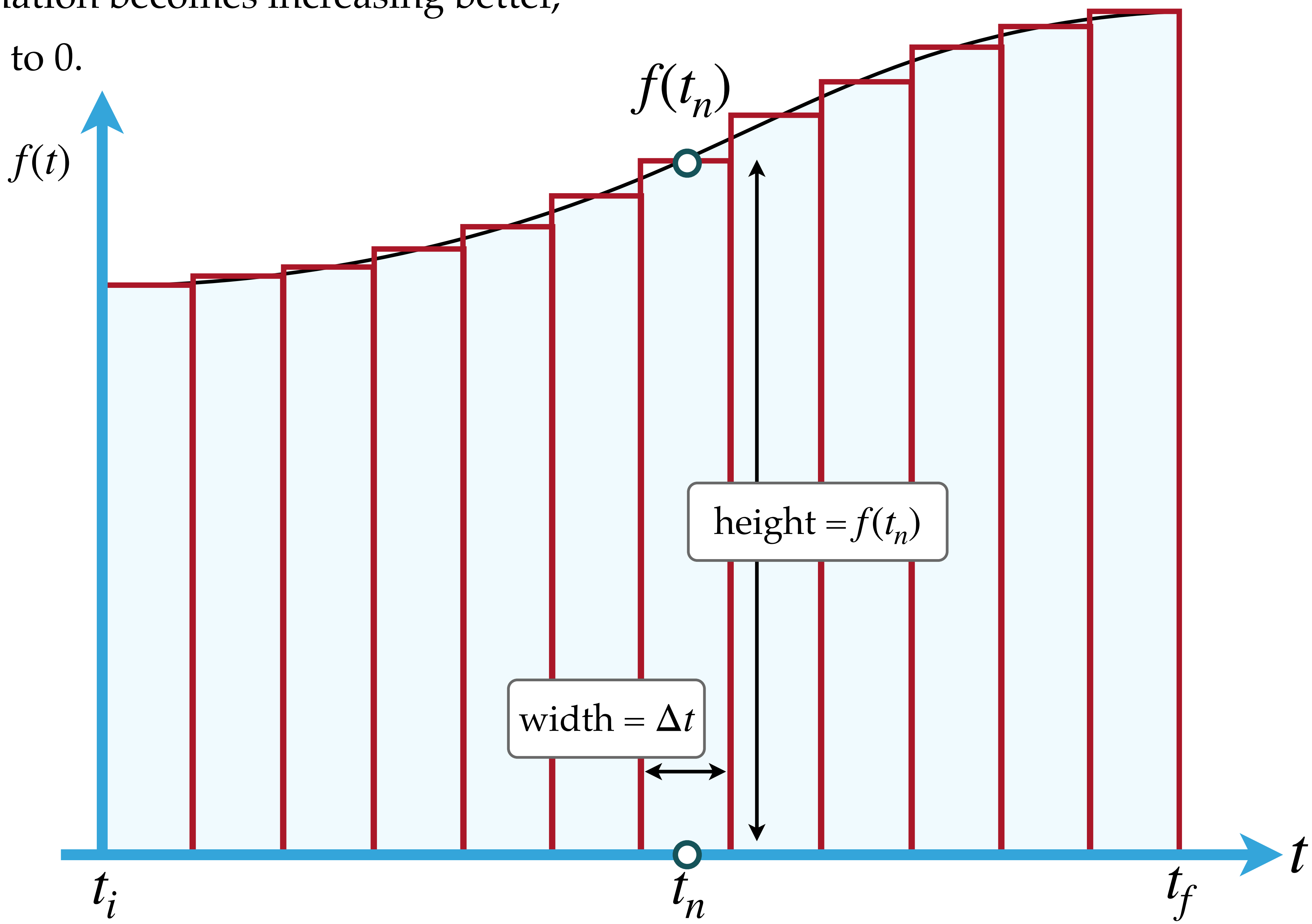


Let us consider a generic function of the kind shown below. Imagine we want to calculate the area underneath the curve defined by $f(t)$ on the top.

We can approximate this by approximating the area by introducing rectangles of width Δt and a height defined by the value of $f(t)$ at the value of t defined by the center of the rectangle.

This approximation becomes increasing better,

as we take Δt to 0.



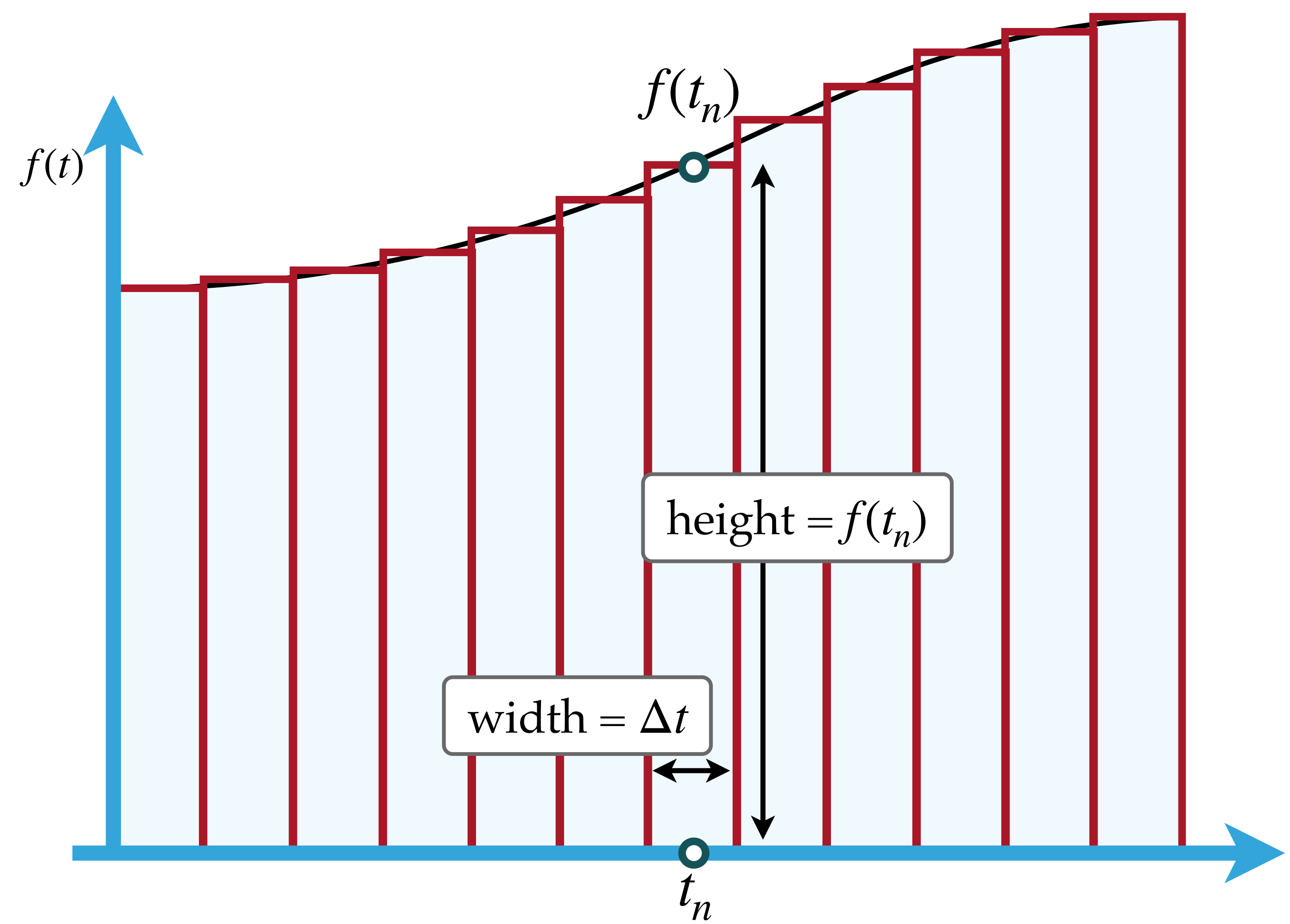
Σ is a Greek capital letter, which is equivalent to "S", symbolizing *sum*

For nonzero values of Δt, we can write a fairly simple expression for our approximation of the area

$$A_N = \sum_{n=1}^{N} f(t_n)\,\Delta t,$$

where N is the total number of points used to discretize the area. This is known as a Riemann sum.

Ultimately, we can recover the exact area in the limit that Δt goes to zero. Note in this limit, we also need to take N to infinity. In other words, we sum an infinite number of terms, each having an infinitesimal size.



RIEMANN SUMS - PART2

Σ is a Greek capital letter, which is equivalent to "S", symbolizing *sum*

For nonzero values of Δt, we can write a fairly simple expression for our approximation of the area

$$A_N = \sum_{n=1}^{N} f(t_n)\,\Delta t,$$

where N is the total number of points used to discretize the area. This is known as a Riemann sum.

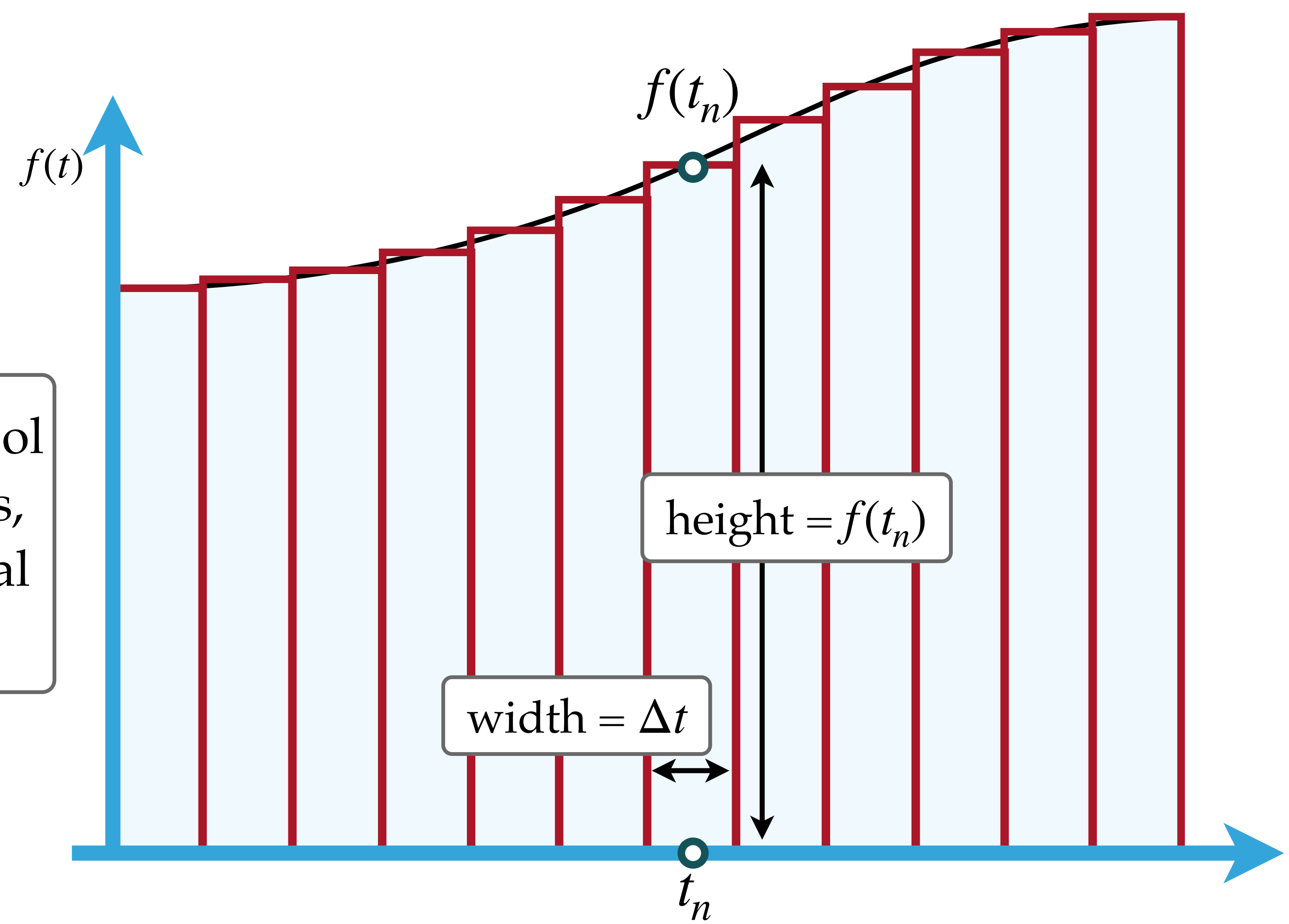
Ultimately, we can recover the exact area in the limit that Δt goes to zero. Note in this limit, we also need to take N to infinity. In other words, we sum an infinite number of terms, each having an infinitesimal size.

We give this limit a special symbol

$$A = \int_{t_i}^{t_f} f(t)\,dt = \lim_{\Delta t \to 0} \sum_{n=1}^{N} f(t_n)\,\Delta t$$

calligraphic capital "S" for *continuous sum*

note, dt is the same symbol introduce for derivatives, denoting the infinitesimal value of Δt



RIEMANN SUMS - PART2

For nonzero values of Δt, we can write a fairly simple expression for our approximation of the area

$$A_N = \sum_{n=1}^{N} f(t_n)\,\Delta t,$$

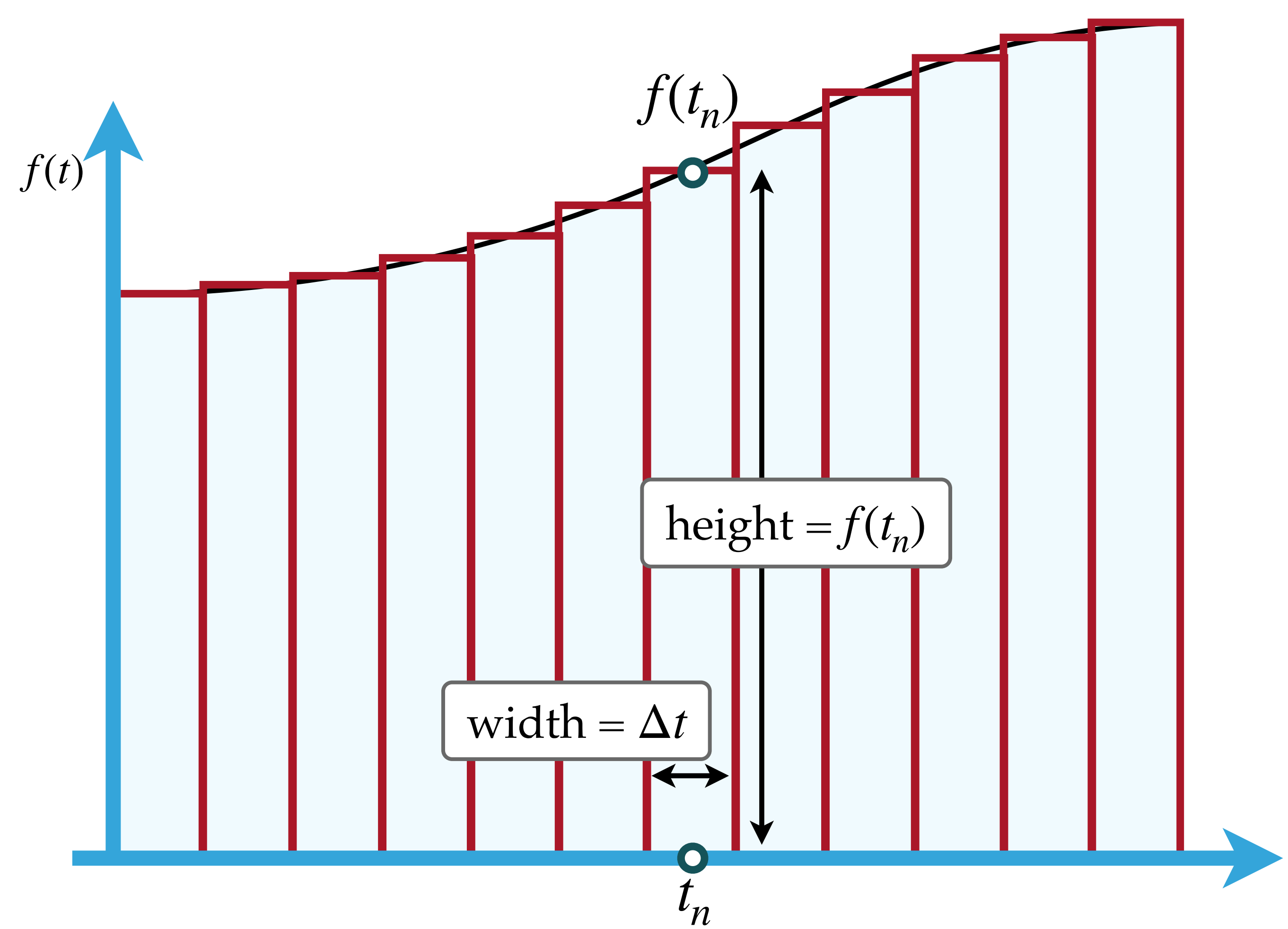
where N is the total number of points used to discretize the area. This is known as a Riemann sum.

Ultimately, we can recover the exact area in the limit that Δt goes to zero. Note in this limit, we also need to take N to infinity. In other words, we sum an infinite number of terms, each having an infinitesimal size.

We give this limit a special symbol

$$A = \int_{t_i}^{t_f} f(t)\,dt = \lim_{\Delta t \to 0} \sum_{n=1}^{N} f(t_n)\,\Delta t$$

$\int_{t_i}^{t_f} f(t)\,dt$ is called the integral of $f(t)$ in $[t_i, t_f]$.



RIEMANN SUMS - PART2

For nonzero values of Δt, we can write a fairly simple expression for our approximation of the area

$$A_N = \sum_{n=1}^{N} f(t_n)\,\Delta t,$$

where N is the total number of points used to discretize the area. This is known as a Riemann sum.

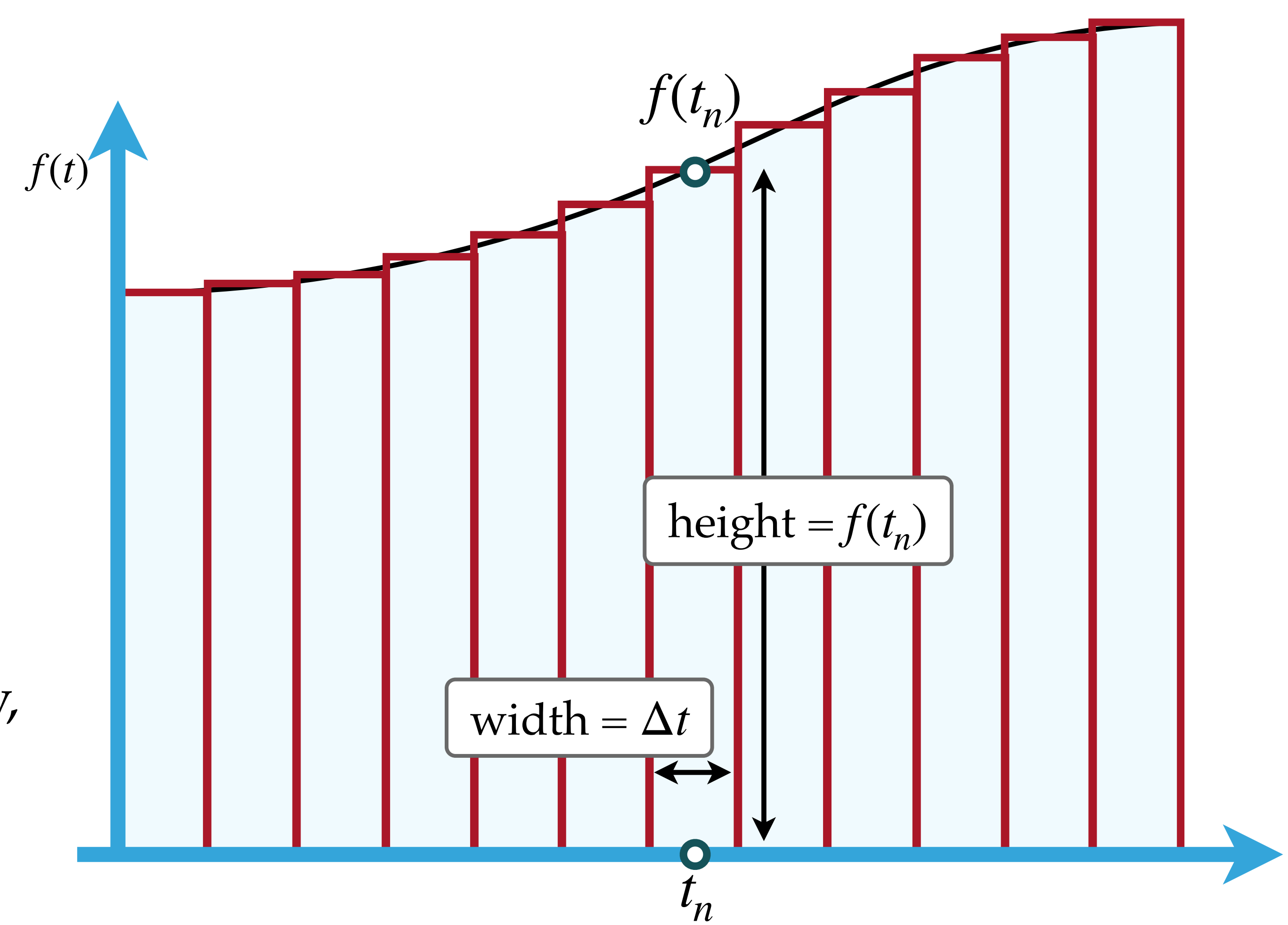
Ultimately, we can recover the exact area in the limit that Δt goes to zero. Note in this limit, we also need to take N to infinity. In other words, we sum an infinite number of terms, each having an infinitesimal size.

We give this limit a special symbol

$$A = \int_{t_i}^{t_f} f(t)\,dt = \lim_{\Delta t \to 0} \sum_{n=1}^{N} f(t_n)\,\Delta t$$

$\int_{t_i}^{t_f} f(t)\,dt$ is called the integral of $f(t)$ in $[t_i, t_f]$.

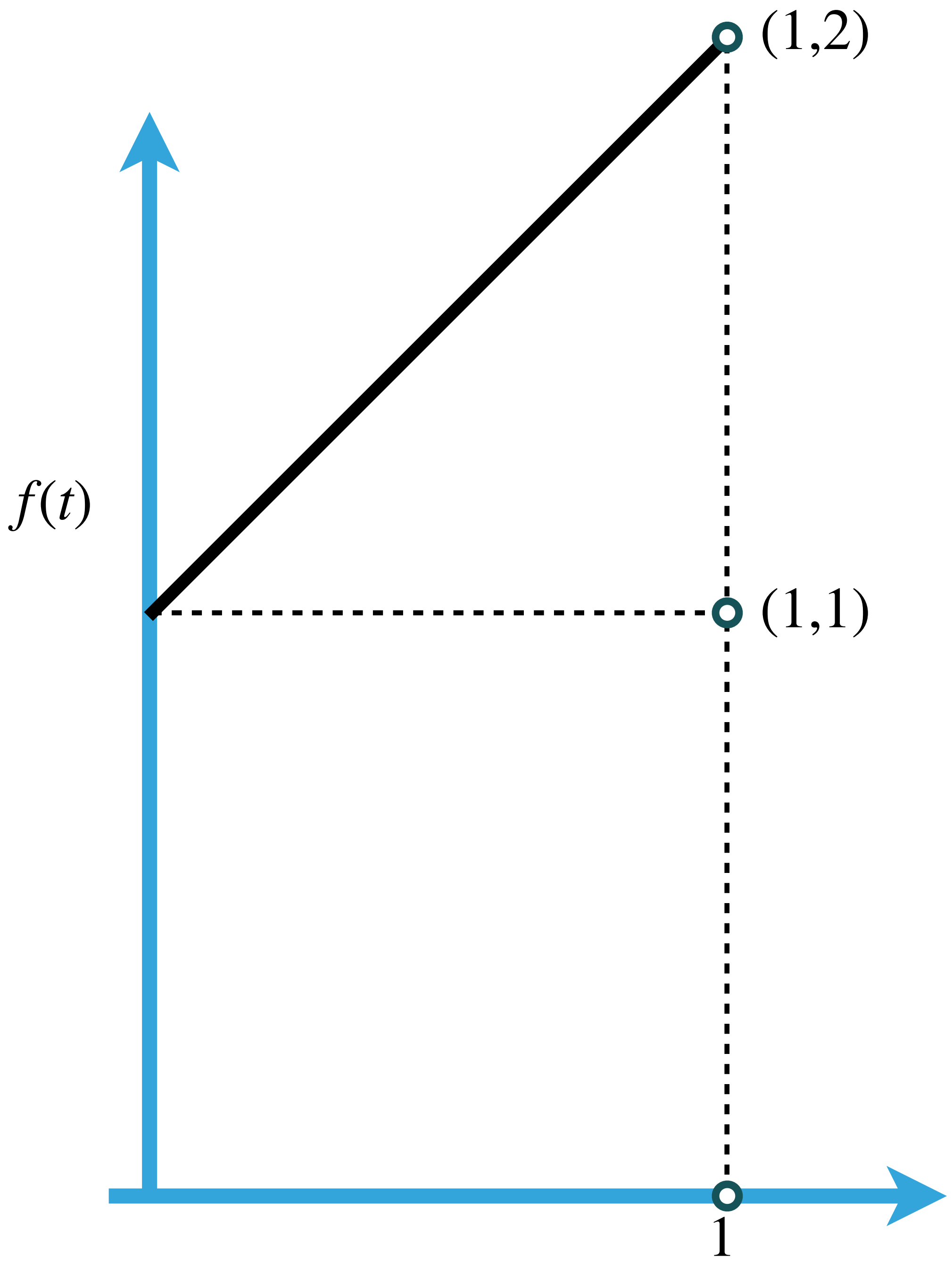
In practice, in order to evaluate this numerically, it is necessary to keep Δt fixed and test how the result converges in the limit that Δt goes to 0.



EXERCISE #3 - CALCULATE A SIMPLE INTEGRAL

To put these ideas to a test, write code to evaluate the integral of $f(t) = 1 + t$ in $[0,1]$.

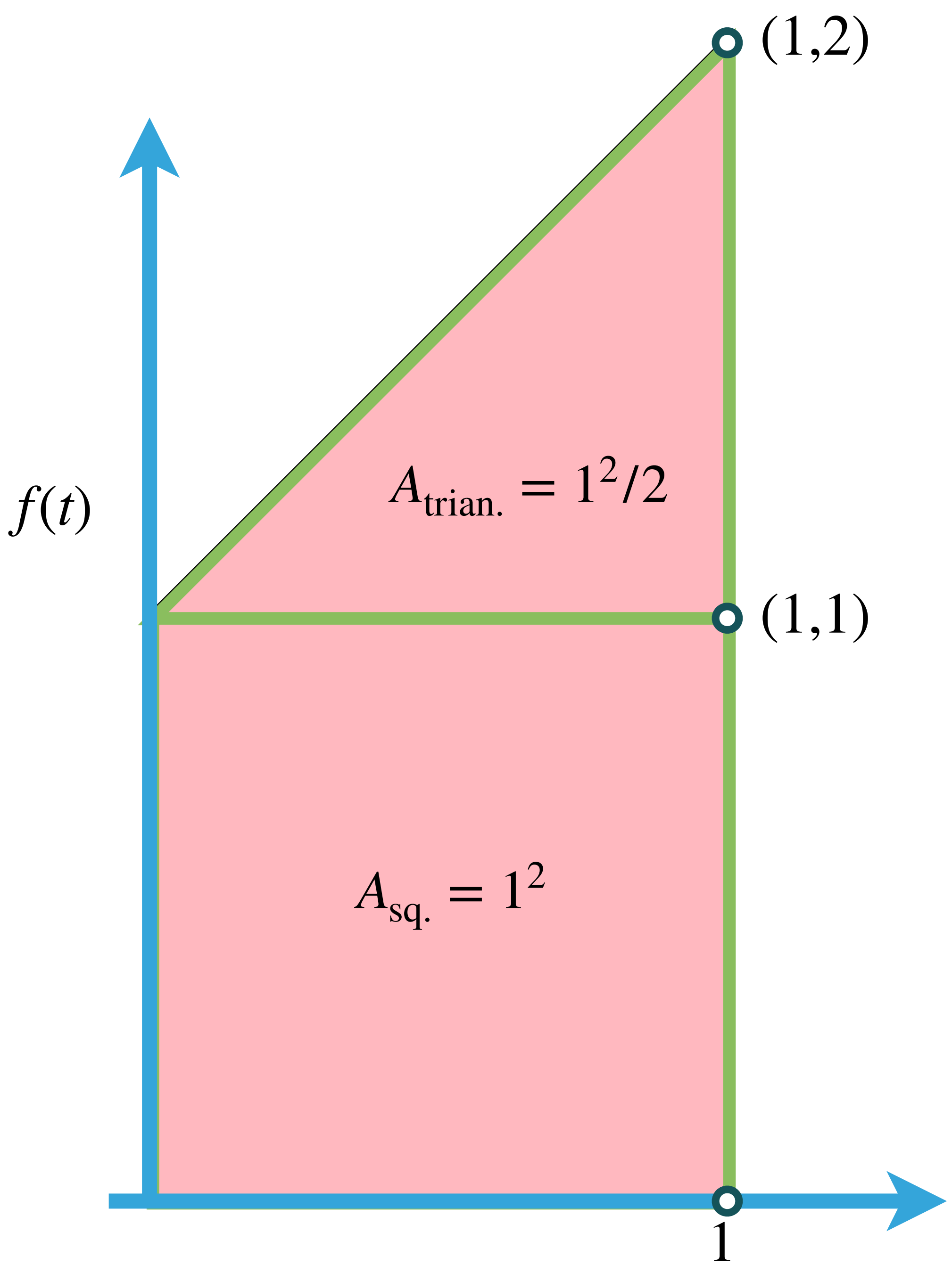
With a bit of thought…it's easy to see that the answer has to be $A = 1 + \dfrac{1}{2} = \dfrac{3}{2}$.



EXERCISE #3 - CALCULATE A SIMPLE INTEGRAL

To put these ideas to a test, write code to evaluate the integral of $f(t) = 1 + t$ in [0,1].

With a bit of thought…it's easy to see that the answer has to be $A = 1 + \dfrac{1}{2} = \dfrac{3}{2}$.

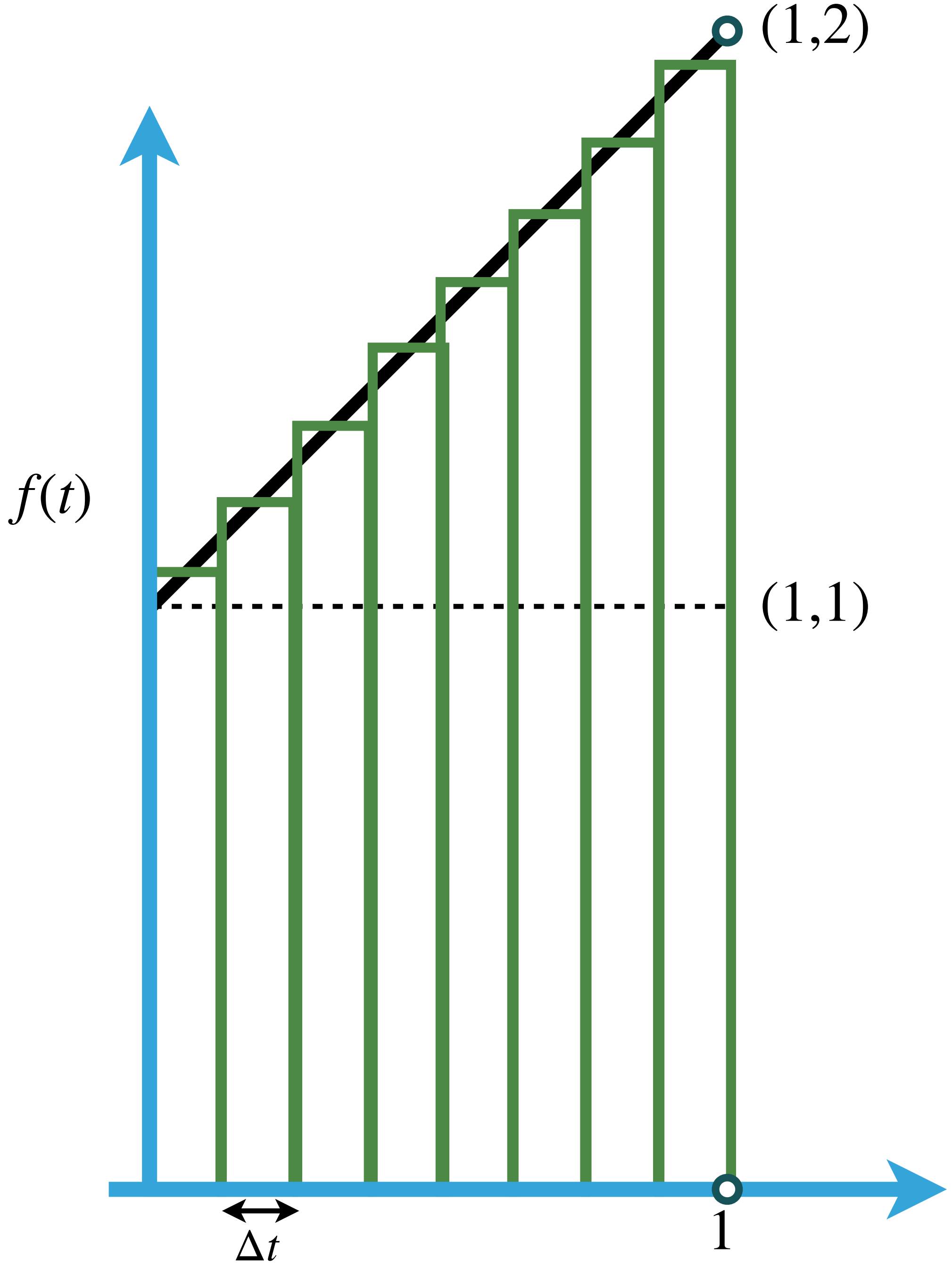


Note, the area has to be the sum of the area of a square and a triangle with $A_{\text{sq.}} = 1^2$ and $A_{\text{trian.}} = 1^2/2$

EXERCISE #3 - CALCULATE A SIMPLE INTEGRAL

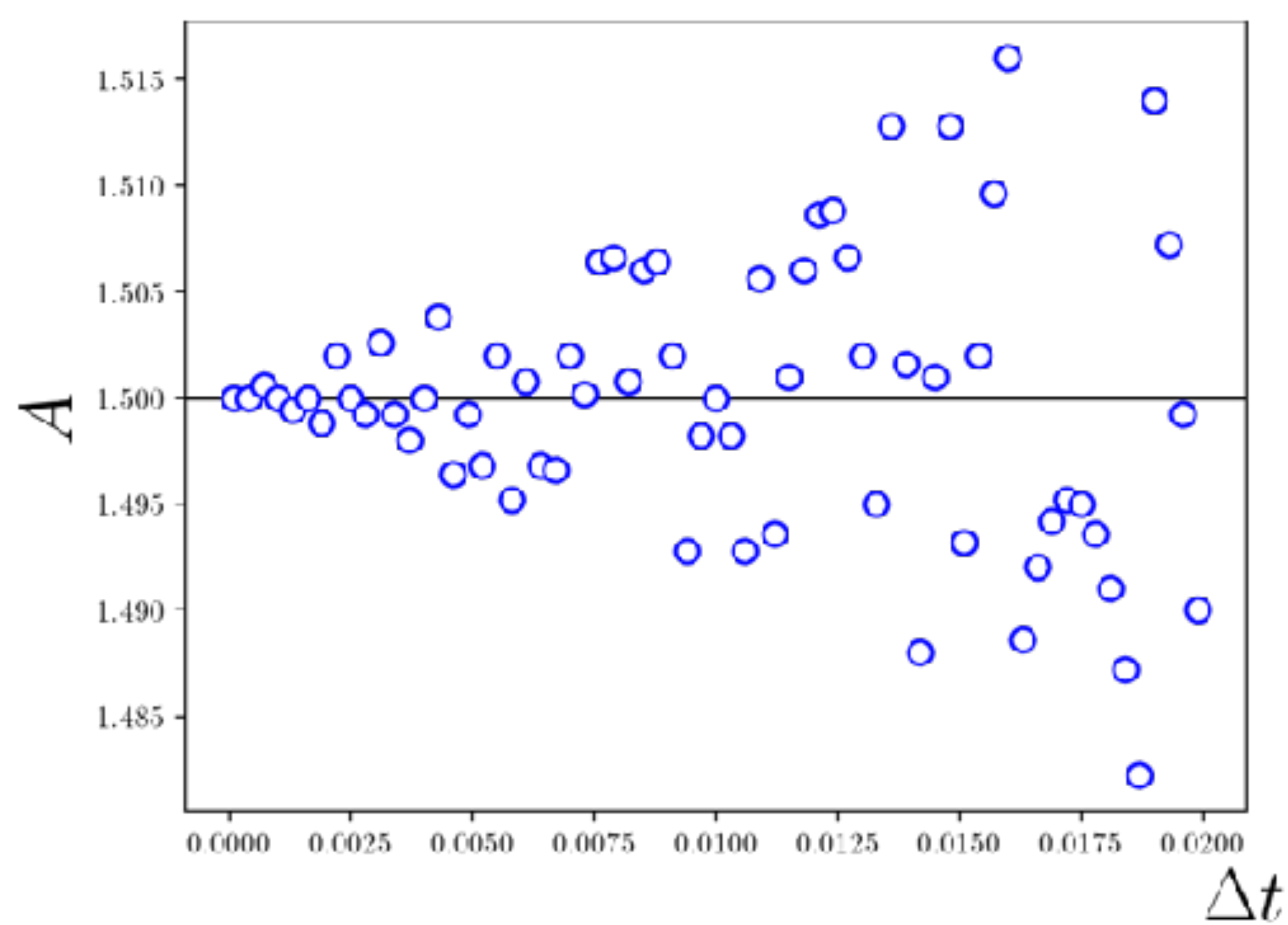
To put these ideas to a test, write code to evaluate the integral of $f(t) = 1 + t$ in [0,1].

With a bit of thought…it's easy to see that the answer has to be $A = 1 + \dfrac{1}{2} = 1.5$.



Note, the area has to be the sum of the area of a square and a triangle with $A_{\text{sq.}} = 1^2$ and $A_{\text{trian.}} = 1^2/2$

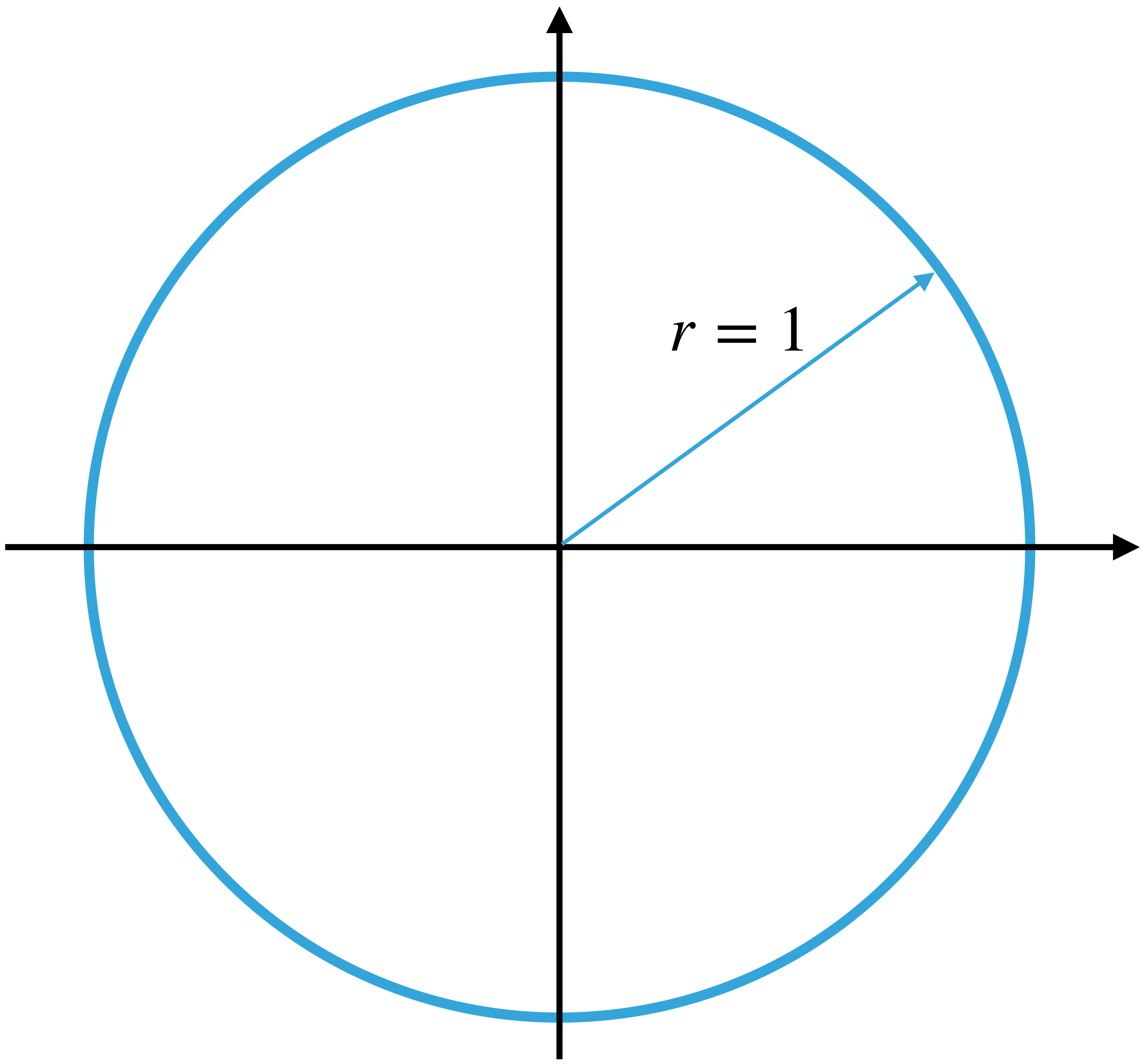
Now, write code to evaluate this using rectangles.



EXERCISE #4 - CALCULATE Π…AGAIN!

Now, let's calculate π again. This time, we will do it by calculating the area of a circle with a radius of $r = 1$ using the concepts of numerical integration.

We know the answer has to be $A = \pi r^2 = \pi$.



EXERCISE #4 - CALCULATE Π…AGAIN!

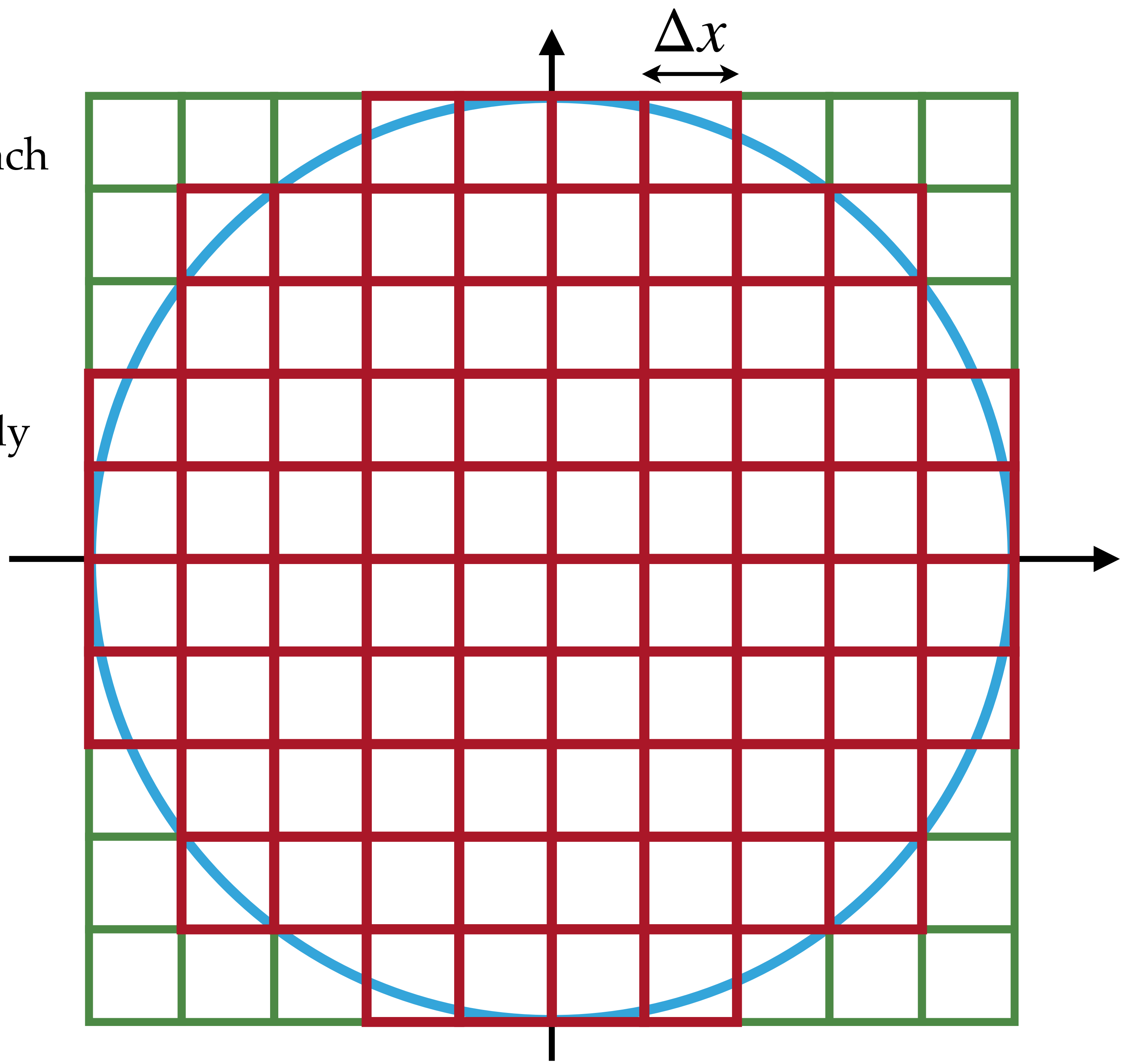
Now, let's calculate π again. This time, we will do it by calculating the area of a circle with a radius of $r = 1$ using the concepts of numerical integration.

We know the answer has to be $A = \pi r^2 = \pi$.

To do this, we breakup the area into little square of length Δx, making the area of each square $(\Delta x)^2$. If the center of the square is inside the circle we will label it as red.

Then, the area of the circle is approximately the sum of the area of all the red squares.

Notice, this crude approximation because arbitrarily precise as we decrease Δx.

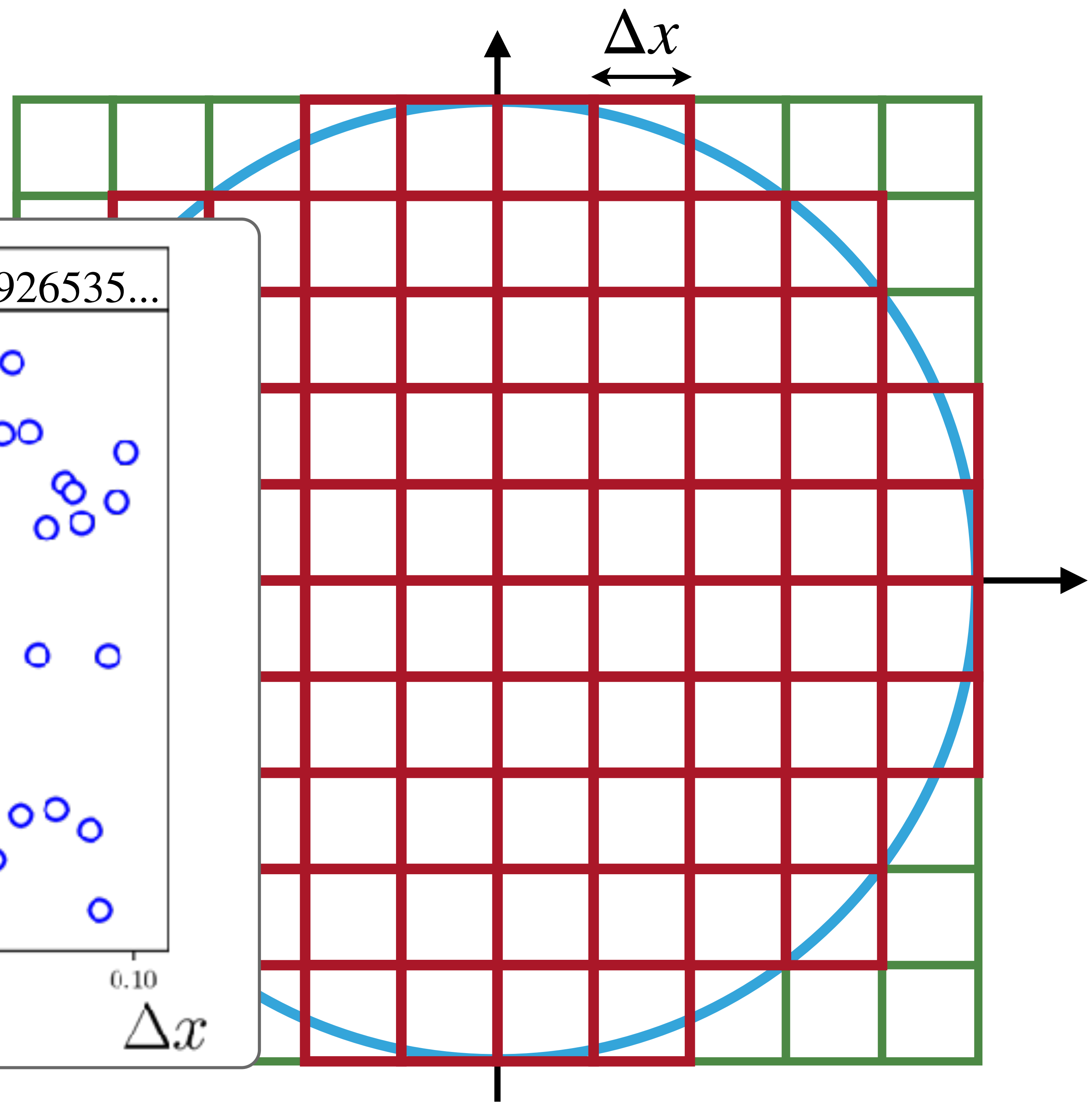


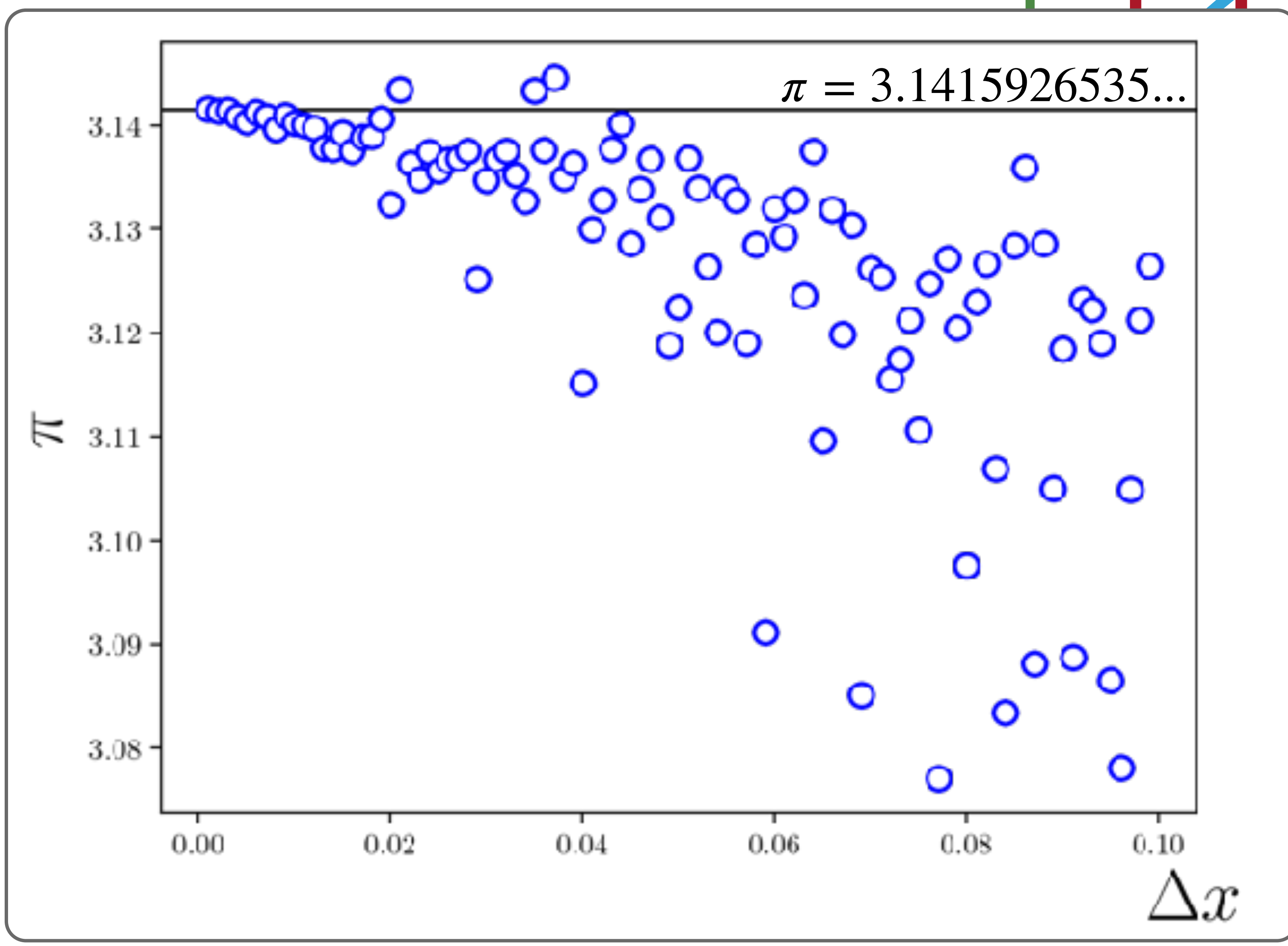
EXERCISE #4 - CALCULATE Π…AGAIN!

Now, let's calculate π again. This time, we will do it by calculating the area of a circle with a radius of $r = 1$ using the concepts of numerical integration.

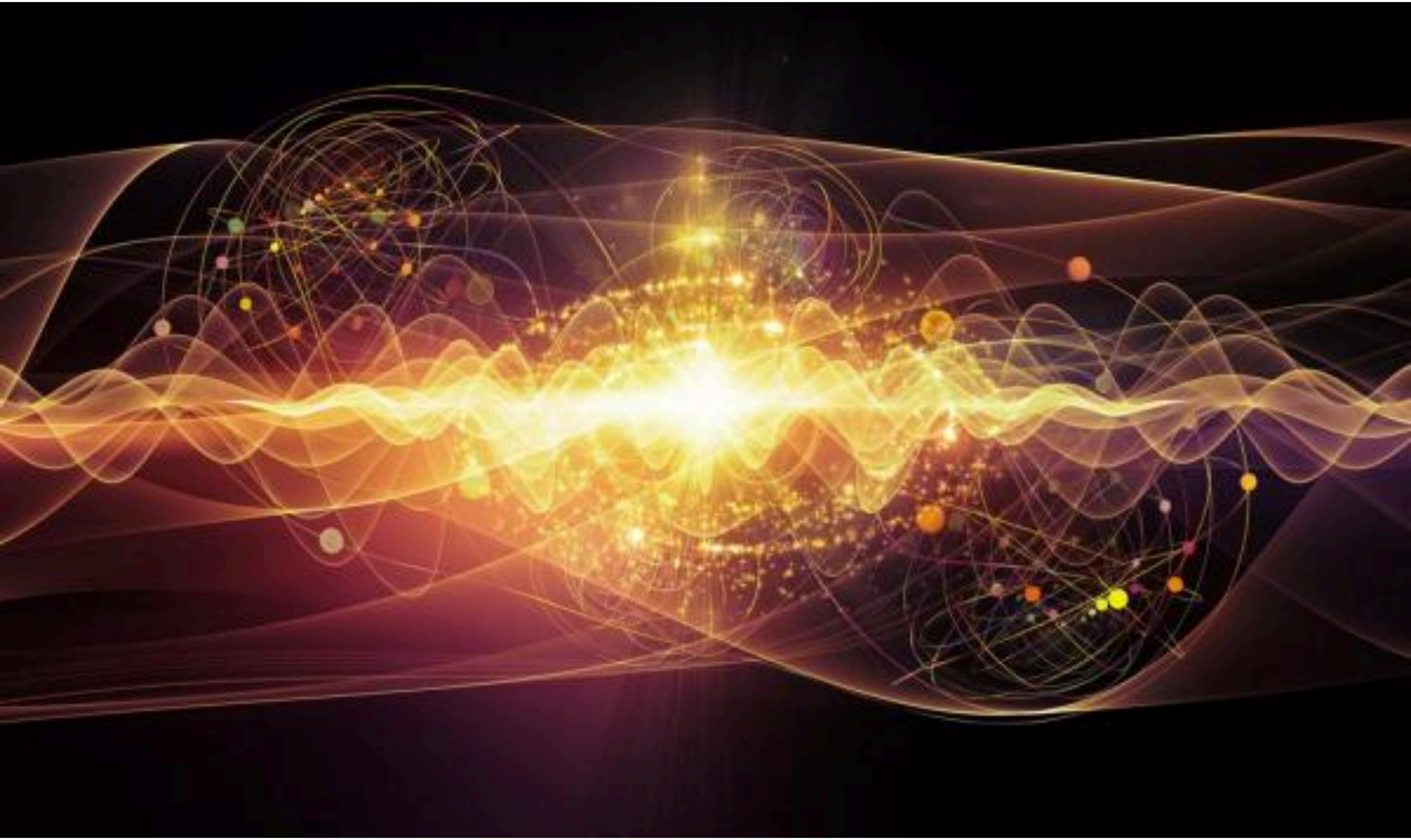
We know the answer has to be $A = \pi r^2 = \pi$.

To do this, we breakup the area into little square of length Δx, making the area of each

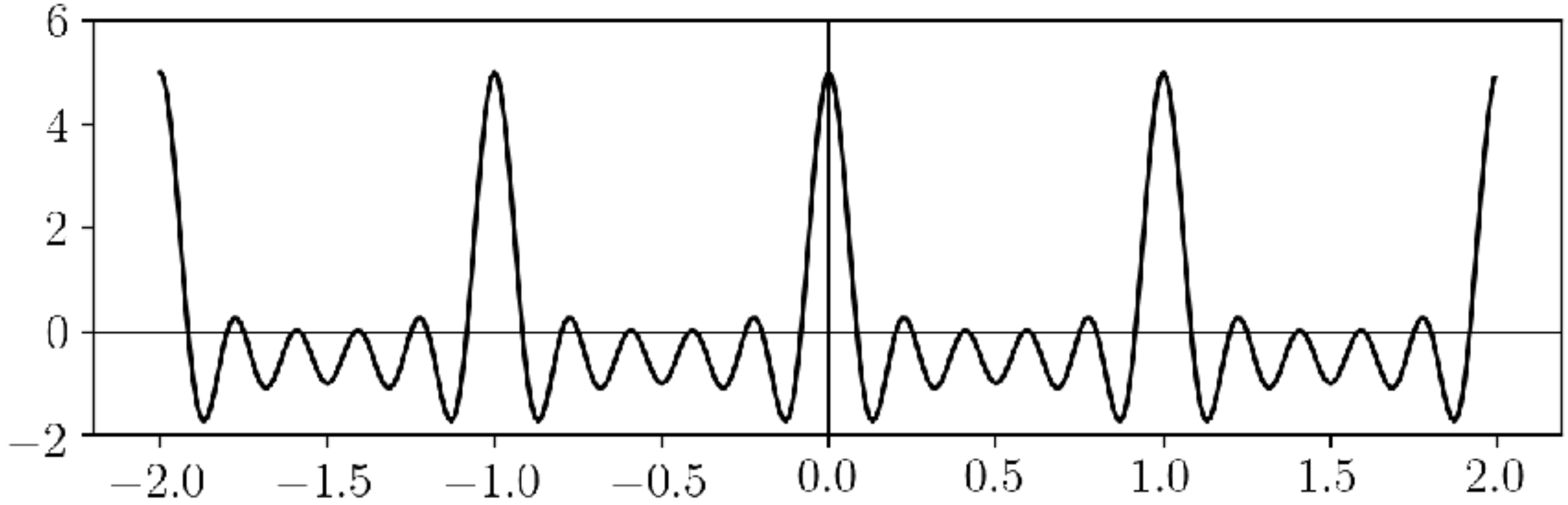




BACK TO WAVES!



Imagine you have some light wave packet as a function of time,



and you would like to know which frequencies of light are contributing to that light packet, and determine by which amount each wave contributes.

This can be achieved using [Fourier transforms](). Fourier transforms is a way to decompose a function of time to its constituent frequencies. In other words, it maps a function of time to a function of frequencies.

Fourier transforms play an essential role in modern physics.

Check this [illustrative video]() for a nice pedagogical introduction into Fourier transforms.

For the case of interest, let us call the signal of time $f(t)$, and it has been determined in a range of $[t_i, t_f]$.

Its Fourier transform will be labeled $\hat{f}(k)$ where k is a frequency, and $\hat{f}(k)$ is defined as

$$\hat{f}(k) = \lim_{\Delta t \to 0} \sum_{n=1}^{N} e^{-i2\pi kt} f(t_n)\, \Delta t = \int_{t_i}^{t_f} e^{-i2\pi kt} f(t)\, dt.$$

Where $i = \sqrt{-1}$, and $e^{-i2\pi kt}$ is a complex number. If you are not familiar with this, you can use Euler's formula, to write this exponential as

$$e^{-i2\pi kt} = \cos(2\pi kt) - i\sin(2\pi kt).$$

For the cases we are interested in, $f(t)$ is real. This means that $\hat{f}(k)$ is a complex number, with

$$\mathrm{Re}\,\hat{\mathrm{f}}(\mathrm{k}) = \int_{\mathrm{t_i}}^{\mathrm{t_f}} \cos(2\pi \mathrm{kt})\, \mathrm{f(t)}\, \mathrm{dt},$$

$$\mathrm{Im}\,\hat{\mathrm{f}}(\mathrm{k}) = -\int_{\mathrm{t_i}}^{\mathrm{t_f}} \sin(2\pi \mathrm{kt})\, \mathrm{f(t)}\, \mathrm{dt}.$$

If this is too confusing, don't worry! We will only consider functions that are symmetric over the time range, which makes the imaginary parts vanish.

COMPLEX NUMBERS - BRIEF INTRODUCTION

In physics, complex numbers and functions play a key role. In particle physics, the functions that describe subatomic particles are in general complex. This is manifest in the Schrödinger Eq. which we saw last time,

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[\frac{-\hbar^2}{2m}\nabla^2 + V(x,t)\right]\Psi(x,t).$$

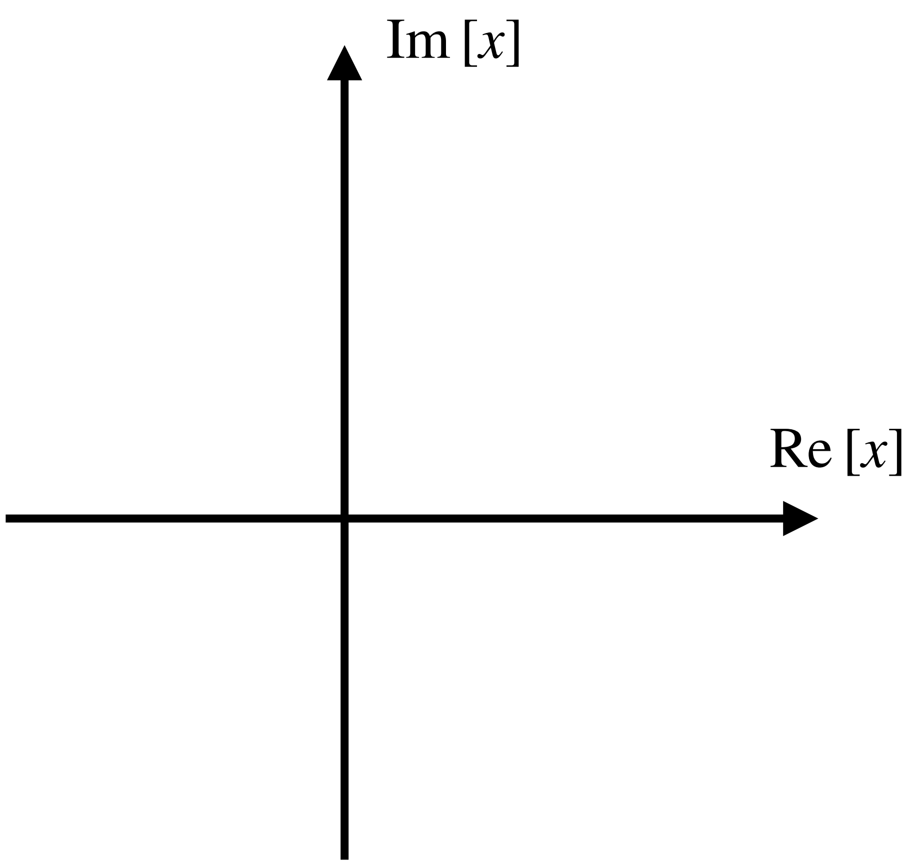
note, and explicit factor of $i = sqrt(-1)$

COMPLEX NUMBERS - BRIEF INTRODUCTION

In physics, complex numbers and functions play a key role. In particle physics, the functions that describe subatomic particles are in general complex. This is manifest in the Schrödinger Eq. which we saw last time,

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[\frac{-\hbar^2}{2m}\nabla^2 + V(x,t)\right]\Psi(x,t).$$

Complex numbers can be understood in the "complex plane". This is a plane where the horizontal axis represents the real value of the number, and the vertical axis represents the imaginary component of the number.



In physics, complex numbers and functions play a key role. In particle physics, the functions that describe subatomic particles are in general complex. This is manifest in the Schrödinger Eq. which we saw last time,

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[\frac{-\hbar^2}{2m}\nabla^2 + V(x,t)\right]\Psi(x,t).$$

Complex numbers can be understood in the "complex plane". This is a plane where the horizontal axis represents the real value of the number, and the vertical axis represents the imaginary component of the number.

Examples: $e^{i\pi/4}$ and $e^{-i\pi/4}$.

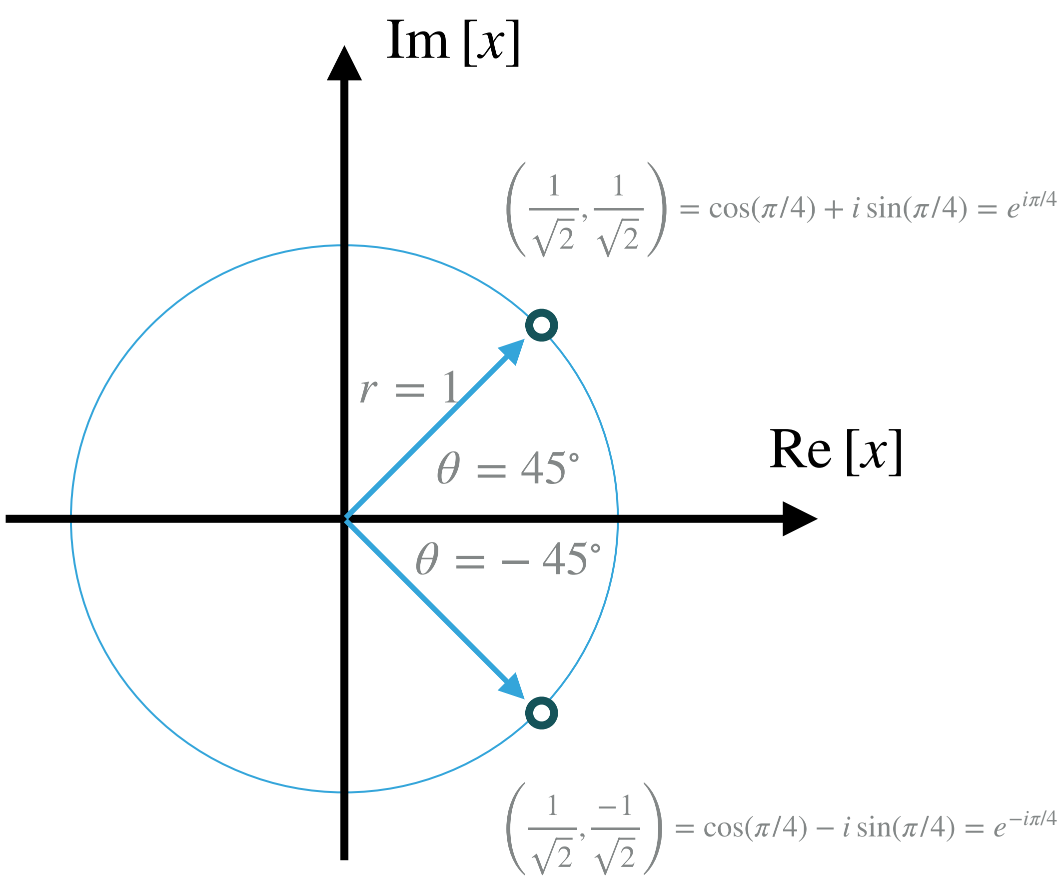
Using the Euler formula, these can be written as

$$e^{i\pi/4} = \cos(\pi/4) + i\sin(\pi/4) = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}},$$

$$e^{-i\pi/4} = \cos(\pi/4) - i\sin(\pi/4) = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}.$$

In a coordinate basis, we would denote these as

$$e^{i\pi/4} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ and } e^{-i\pi/4} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$



COMPLEX NUMBERS - IN PYTHON

Using complex numbers in python is relatively straight forward. The only thing you need to keep in mind is that the symbol for the imaginary number i is in fact $1j$.

Otherwise, you can do all the same arithmetic operations you are familiar with for real numbers.

Here is an example of addition of complex numbers using ipython.

```
In [1]: x = 1 + 1j

In [2]: y = 2 + 1.5j

In [3]: z= x + y

In [4]: x
Out[4]: (1+1j)

In [5]: y
Out[5]: (2+1.5j)

In [6]: z
Out[6]: (3+2.5j)

In [7]: z.real
Out[7]: 3.0

In [8]: z.imag
Out[8]: 2.5

In [9]:
```

note, you can return just the real or the imaginary part of the number of your interest.

COMPLEX NUMBERS - IN PYTHON

Using complex numbers in python is relatively straight forward. The only thing you need to keep in mind is that the symbol for the imaginary number i is in fact $1j$.

Otherwise, you can do all the same arithmetic operations you are familiar with for real numbers.

Here is an example of addition of complex numbers using ipython.

```
In [1]: x = 1 + 1j

In [2]: y = 2 + 1.5j

In [3]: z= x + y

In [4]: x
Out[4]: (1+1j)

In [5]: y
Out[5]: (2+1.5j)

In [6]: z
Out[6]: (3+2.5j)

In [7]: z.real
Out[7]: 3.0

In [8]: z.imag
Out[8]: 2.5

In [9]:
```

Matplotlib can not make sense of complex number for plotting. If you fid a complex number, it will just plot the real part. For more control, you can write code to specify if you want to plot the real or imaginary piece, or its absolute value and/or phase.

COMPLEX NUMBERS - IN PYTHON

Using complex numbers in python is relatively straight forward. The only thing you need to keep in mind is that the symbol for the imaginary number i is in fact $1j$.

Otherwise, you can do all the same arithmetic operations you are familiar with for real numbers.

Here is an example of addition of complex numbers using ipython.

```
In [1]: x = 1 + 1j

In [2]: y = 2 + 1.5j

In [3]: z= x + y

In [4]: x
Out[4]: (1+1j)

In [5]: y
Out[5]: (2+1.5j)

In [6]: z
Out[6]: (3+2.5j)

In [7]: z.real
Out[7]: 3.0

In [8]: z.imag
Out[8]: 2.5

In [9]:
```

Matplotlib can not make sense of complex number for plotting. If you fid a complex number, it will just plot the real part. For more control, you can write code to specify if you want to plot the real or imaginary piece, or its absolute value and/or phase.

Note, with numpy you can, not only multiply complex numbers, but also evaluate functions with complex arguments

```
In [13]: z*z
Out[13]: (2.75+15j)

In [14]: exp(z)
Out[14]: (-16.091399670844+12.020634347899307j)
```

COMPLEX NUMBERS - IN PYTHON

Using complex numbers in python is relatively straight forward. The only thing you need to keep in mind is that the symbol for the imaginary number i is in fact $1j$.

Otherwise, you can do all the same arithmetic operations you are familiar with for real numbers.

Here is an example of addition of complex numbers using ipython.

```
In [1]: x = 1 + 1j

In [2]: y = 2 + 1.5j

In [3]: z= x + y

In [4]: x
Out[4]: (1+1j)

In [5]: y
Out[5]: (2+1.5j)

In [6]: z
Out[6]: (3+2.5j)

In [7]: z.real
Out[7]: 3.0

In [8]: z.imag
Out[8]: 2.5

In [9]:
```

Matplotlib can not make sense of complex number for plotting. If you fid a complex number, it will just plot the real part. For more control, you can write code to specify if you want to plot the real or imaginary piece, or its absolute value and/or phase.

Note, with numpy you can, not only multiply complex numbers, but also evaluate functions with complex arguments

```
In [13]: z*z
Out[13]: (2.75+15j)

In [14]: exp(z)
Out[14]: (-16.091399670844+12.020634347899307j)
```

That being said, numpy has limited versatility for complex numbers. Another model, cmath, introduces more functions.

```
In [18]: import cmath

In [19]: cmath.phase(z)
Out[19]: 0.6947382761967031
```

EXERCISE #5 - EVALUATE THE FOURIER TRANSFORM

Let $f(t) = \sum_{n=1}^{4} \cos(2\pi n t)$ and assume have gotten a clean measurement of this signal in a time range of $[-10,10]$ in steps of $\Delta t = .01$.

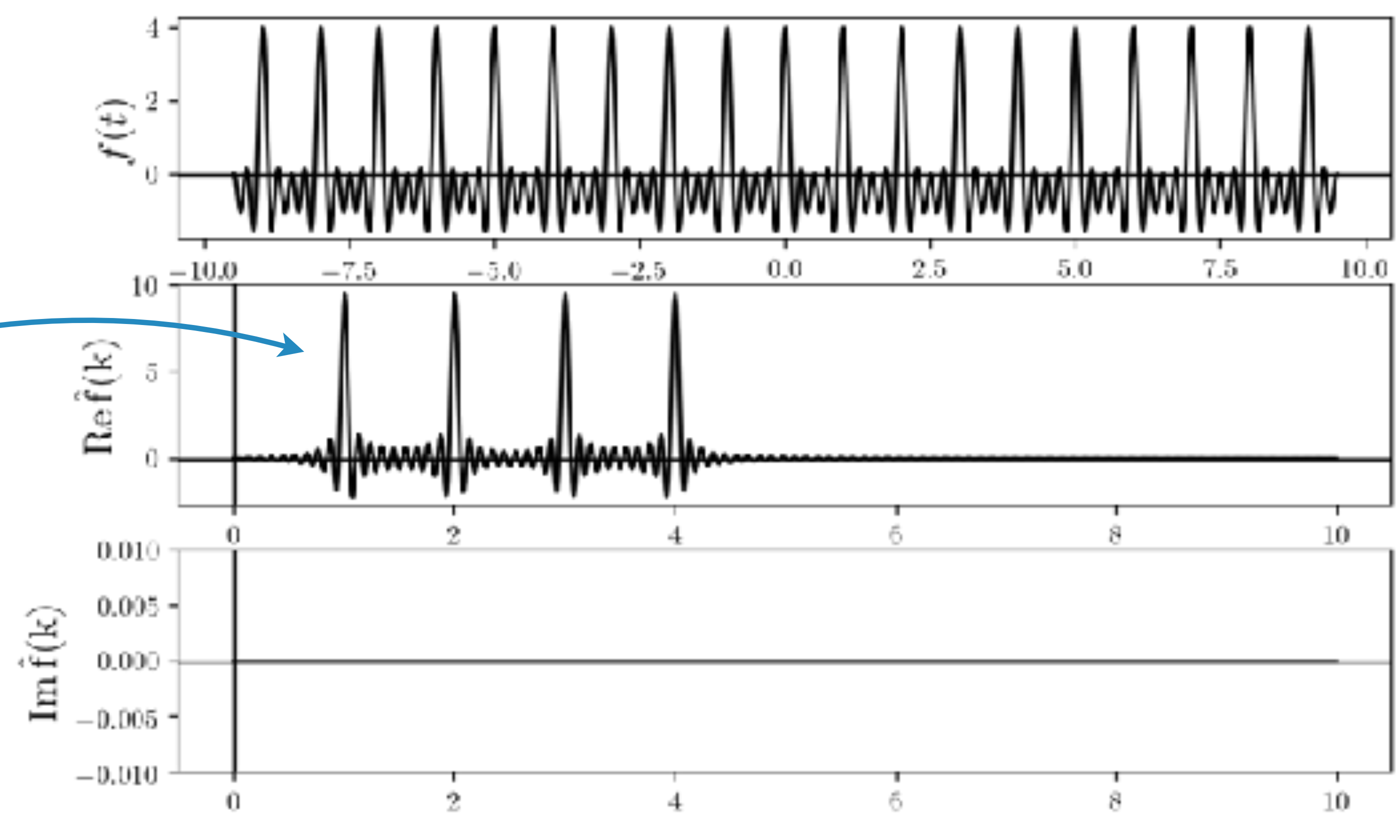
1) Plot the signal as a function of time.
2) Calculate its Fourier transform for the real and imaginary parts using,

$$\mathrm{Re}\,\hat{f}(k) = \sum_{n=1}^{N} \cos(2\pi k t)\, f(t_n)\, \Delta t,$$

$$\mathrm{Im}\,\hat{f}(k) = -\sum_{n=1}^{N} \sin(2\pi k t)\, f(t_n)\, \Delta t.$$

3) Plot the results for $k = 0 - 10$.

note, the peaks of the Fourier transform are at the frequencies of the frequencies we put in [1,2,3,4]



EXERCISE #6 - DETERMINE THE FREQUENCIES OF THE FAKE DATA

Load the data in **Dropbox:/P4P_2020/project# 8/FT_mystery.txt**

Plot the data, and use its Fourier transform to figure which frequencies were used to generate it.

t = independent variable

$f(t)$

-4.995000000000000107e+00	4.783895373538199853e+00
-4.985000000000000320e+00	3.349646056512814773e+00
-4.975000000000000533e+00	1.690645842602883198e+00
-4.965000000000000746e+00	9.255541128781336058e-01
-4.955000000000000959e+00	9.551582233017932300e-01
-4.945000000000001172e+00	7.680613576142201193e-01
-4.935000000000001386e+00	-2.141621669603087086e-01
-4.925000000000001599e+00	-1.337049598033808628e+00
-4.915000000000001812e+00	-1.405815848642620436e+00
-4.905000000000002025e+00	-8.600345050820309978e-02
-4.895000000000002238e+00	1.605238846176940903e+00
-4.885000000000002451e+00	2.307899344533481223e+00
-4.875000000000002665e+00	1.707105781186829014e+00
-4.865000000000002878e+00	7.465915875813549680e-01
-4.855000000000003091e+00	4.560279447752586535e-01
-4.845000000000003304e+00	7.583890496114416013e-01
-4.835000000000003517e+00	6.281261924006120712e-01
-4.825000000000003730e+00	-5.984951178972290897e-01
-4.815000000000003944e+00	-2.296592359774313863e+00
-4.805000000000004157e+00	-3.111119186398800807e+00
-4.795000000000004370e+00	-2.394423131709089603e+00
-4.785000000000004583e+00	-8.971045812798497110e-01
-4.775000000000004796e+00	7.151095178221766879e-02
-4.765000000000005009e+00	5.651201461217453259e-02
-4.755000000000005222e+00	-1.565855600103588374e-01
-4.745000000000005436e+00	4.050960312630382809e-01
-4.735000000000005649e+00	1.652091707358699146e+00
-4.725000000000005862e+00	2.448635069557680539e+00
-4.715000000000006075e+00	1.891435163965823258e+00
-4.705000000000006288e+00	3.833260468427656109e-01
-4.695000000000006501e+00	-7.444872422338661977e-01
-4.685000000000006715e+00	-6.511443910579606253e-01
-4.675000000000006928e+00	1.560315337323680580e-01
-4.665000000000007141e+00	4.848897259269585591e-01
-4.655000000000007354e+00	-1.354282362225517899e-01