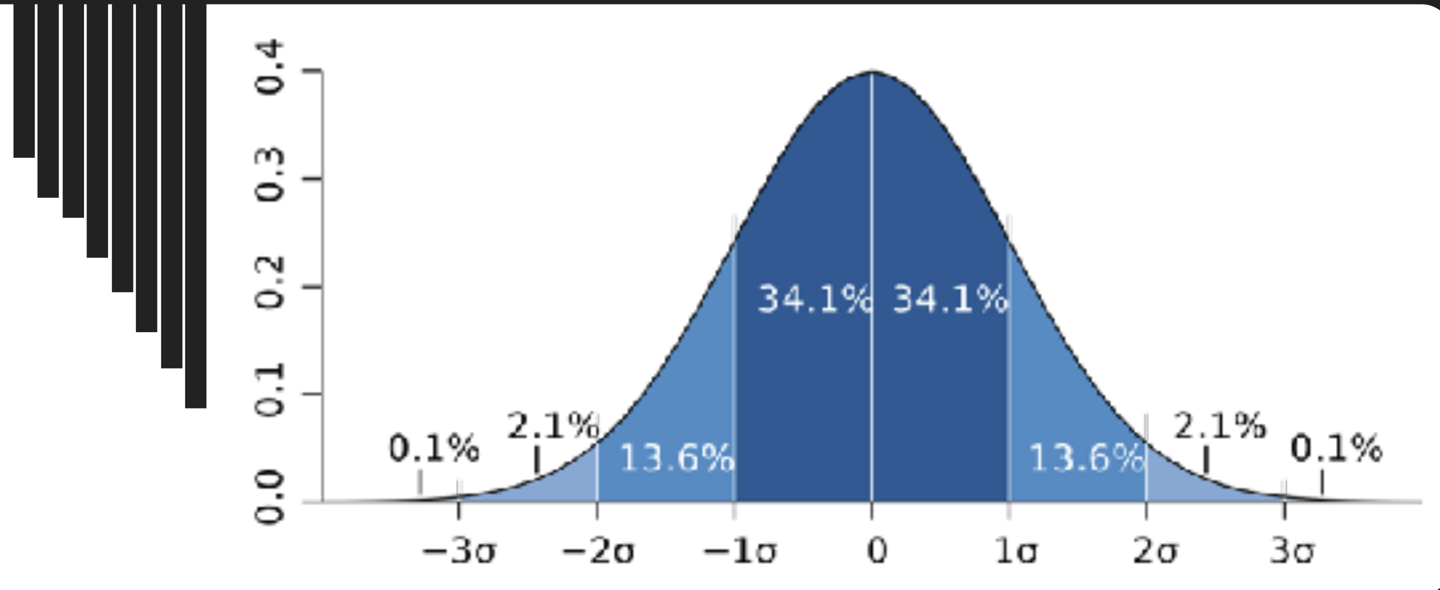

R. BRICEÑO, T. ROGERS

MODELING AND DIFFERENTIAL EQUATIONS

ADMIN STUFF

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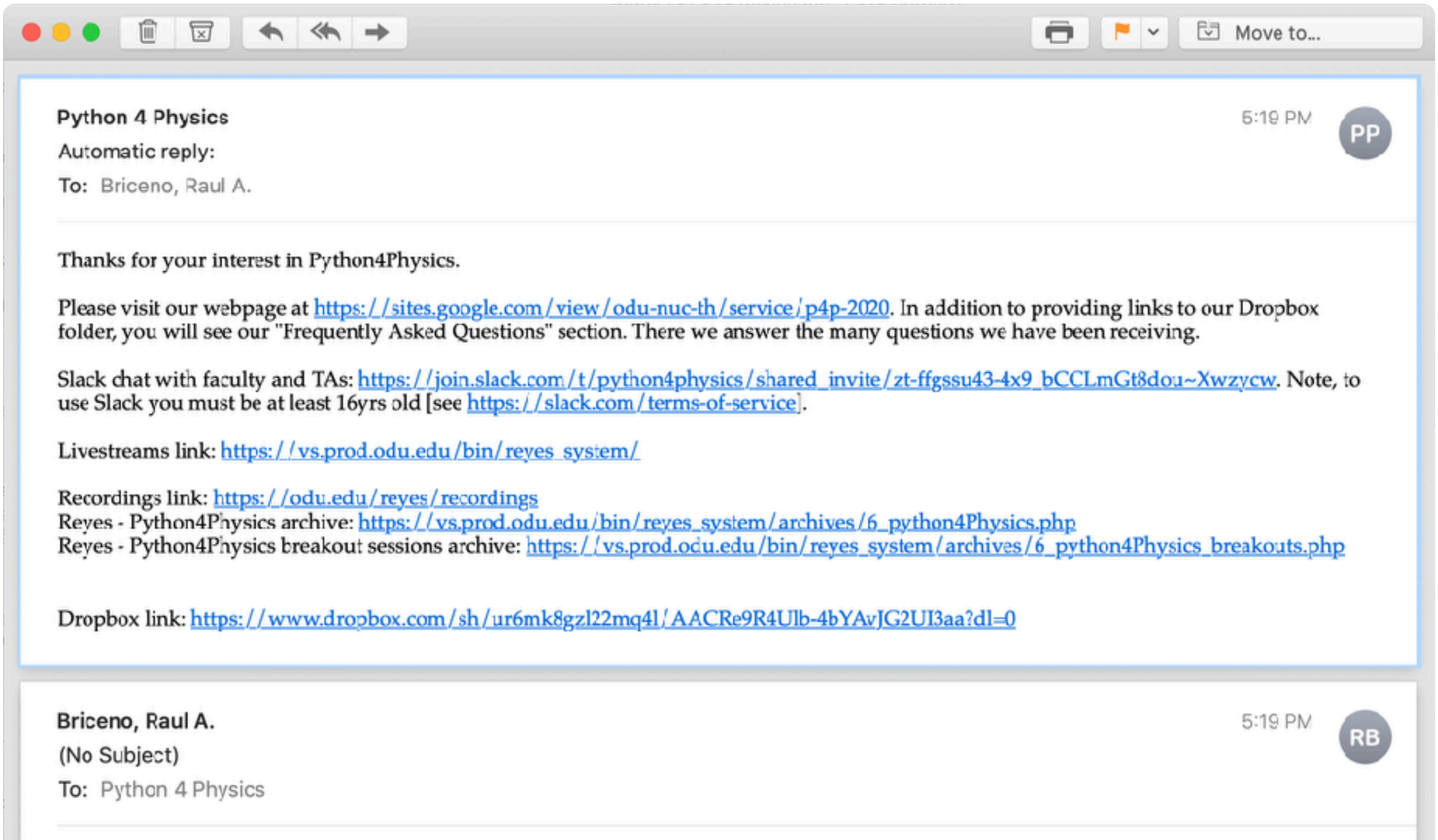

Move to...

Python 4 Physics 5:19 PM PP

Automatic reply:

To: Briceno, Raul A.

Thanks for your interest in Python4Physics.

Please visit our webpage at https://sites.google.com/view/odu-nuc-th/service/p4p-2020. In addition to providing links to our Dropbox folder, you will see our "Frequently Asked Questions" section. There we answer the many questions we have been receiving.

Slack chat with faculty and TAs: https://join.slack.com/t/python4physics/shared_invite/zt-ffgssu43-4x9_bCCLmGt8dou~Xwzycw. Note, to use Slack you must be at least 16yrs old [see https://slack.com/terms-of-service].

Livestreams link: https://vs.prod.odu.edu/bin/reyes_system/

Recordings link: https://odu.edu/reyes/recordings
Reyes - Python4Physics archive: https://vs.prod.odu.edu/bin/reyes_system/archives/6_python4Physics.php
Reyes - Python4Physics breakout sessions archive: https://vs.prod.odu.edu/bin/reyes_system/archives/6_python4Physics_breakouts.php

Dropbox link: https://www.dropbox.com/sh/ur6mk8gzl22mq41/AACRe9R4Ulb-4bYAvJG2UI3aa?dl=0

Briceno, Raul A. 5:19 PM RB

(No Subject)

To: Python 4 Physics

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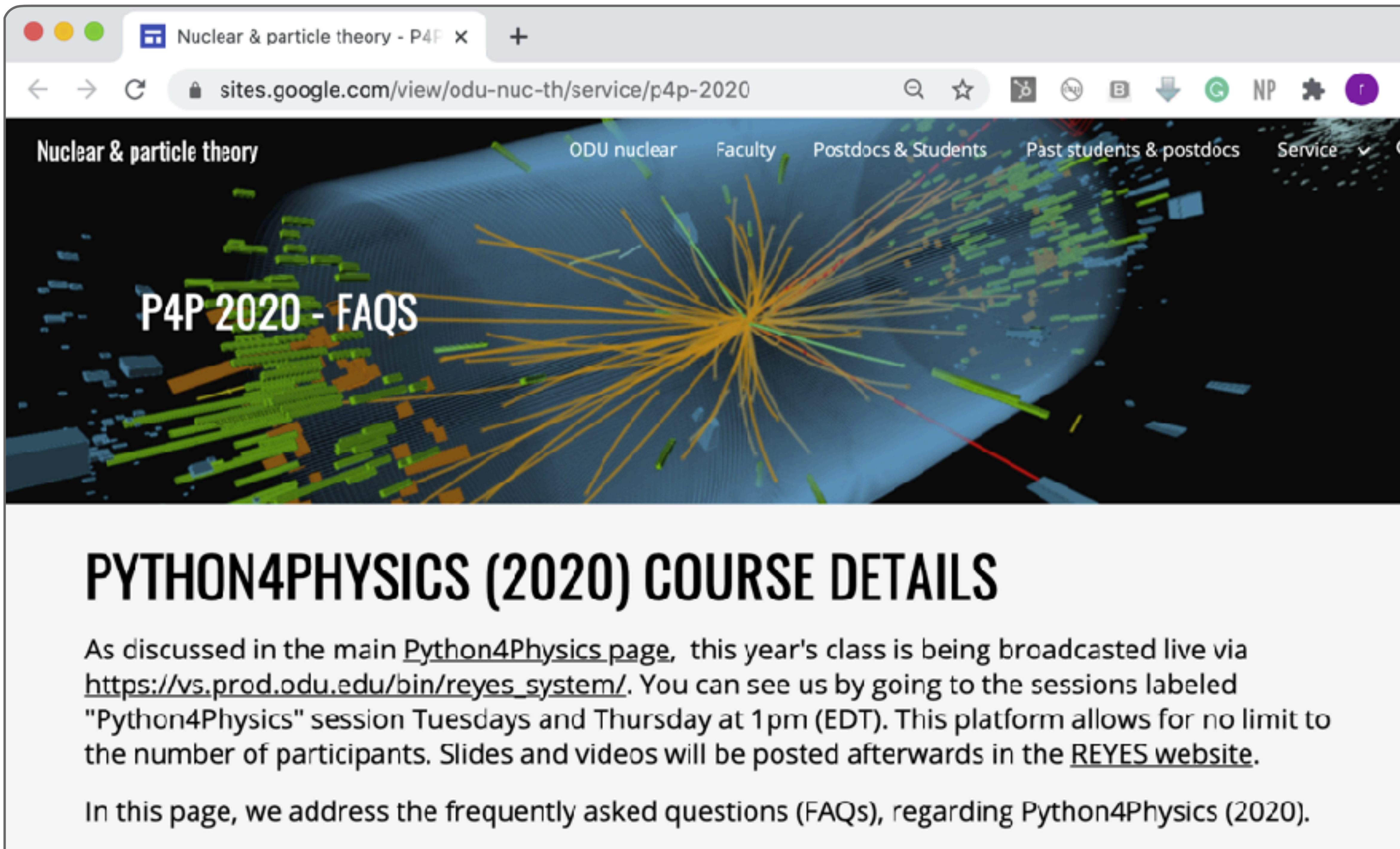

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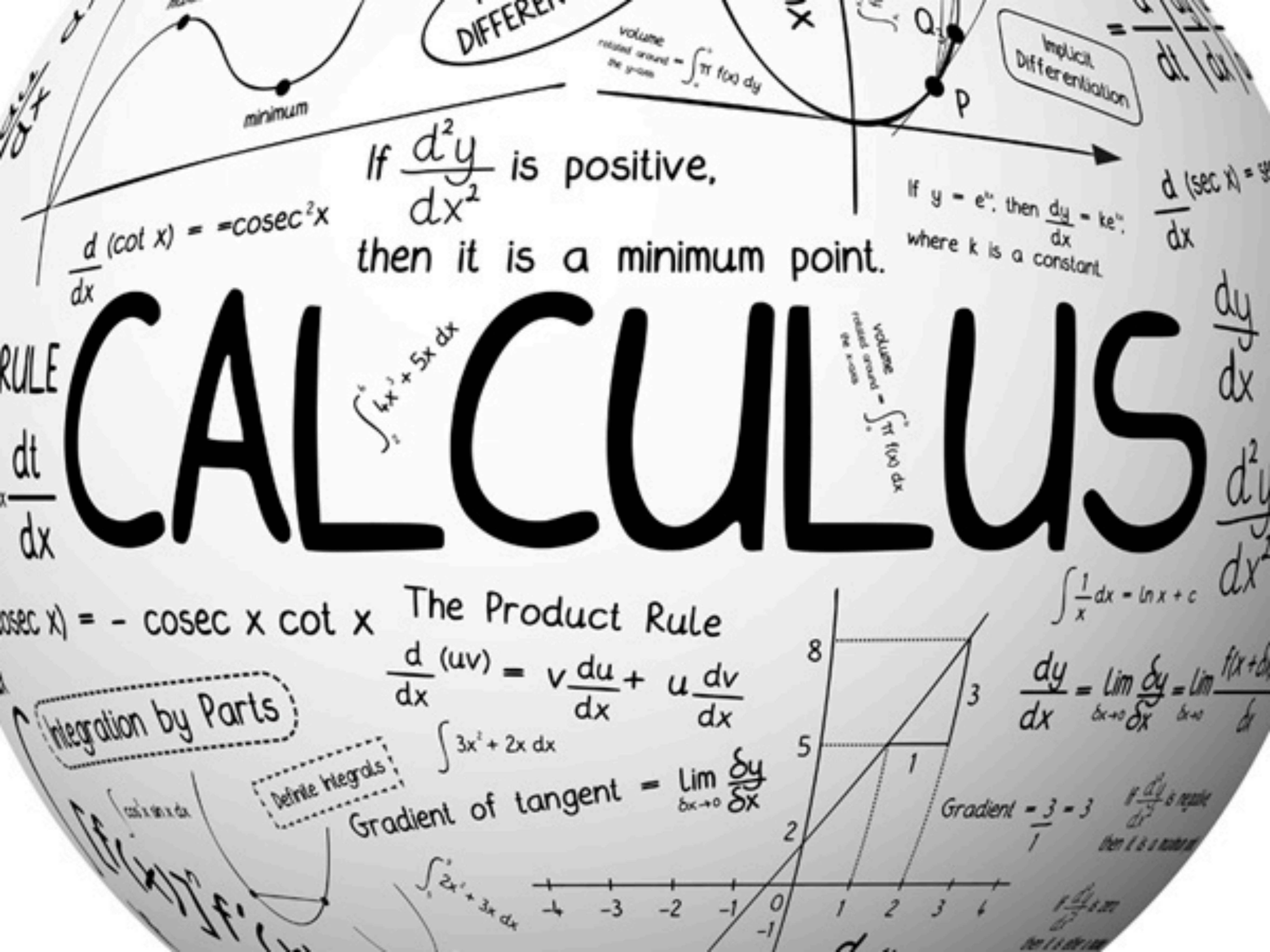
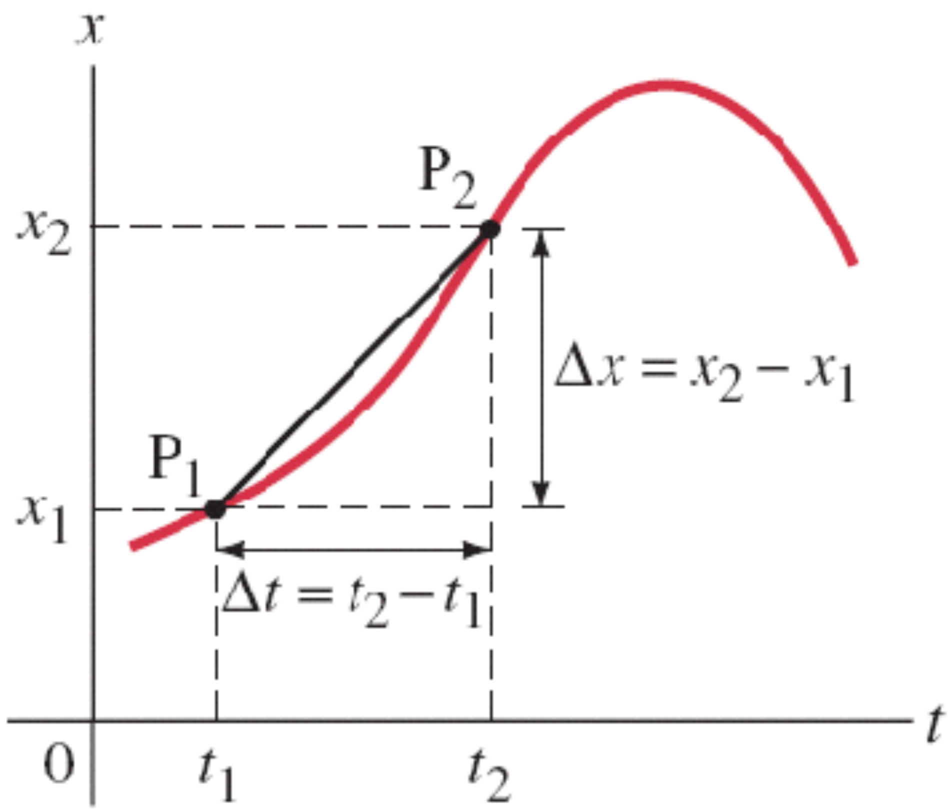
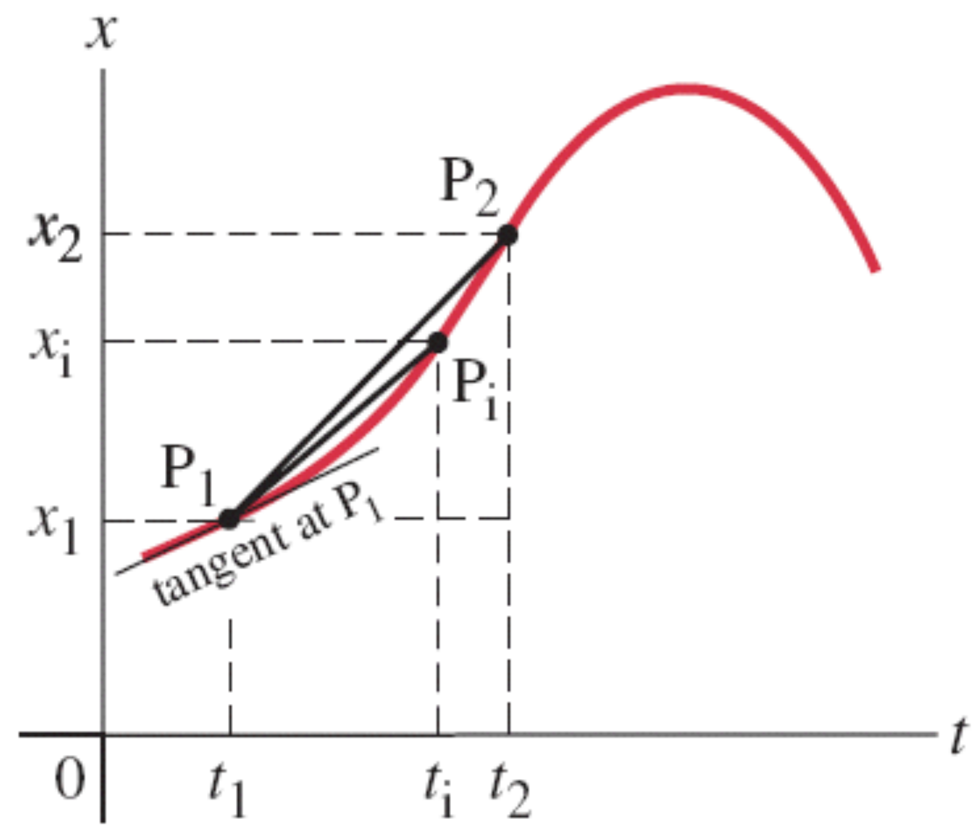
Dropbox for students, email us! Subject: “Dropbox access”

REVIEW


CALCULUS
If $\frac{d^2y}{dx^2}$ is positive, then it is a minimum point.
minimum
Implicit Differentiation
$\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$
If $y = e^{kx}$, then $\frac{dy}{dx} = ke^{kx}$, where k is a constant
RULE
The Product Rule
$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$
Integration by Parts
Definite Integrals
$\int 3x^2 + 2x\,dx$
Gradient of tangent = $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$
$\int \frac{1}{x} dx = \ln x + c$
$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$
Gradient = $\frac{3}{1}$ = 3
volume rotated around the x-axis

Instantaneous velocity is a vector defined as “how fast” a particle is moving at a given instant.





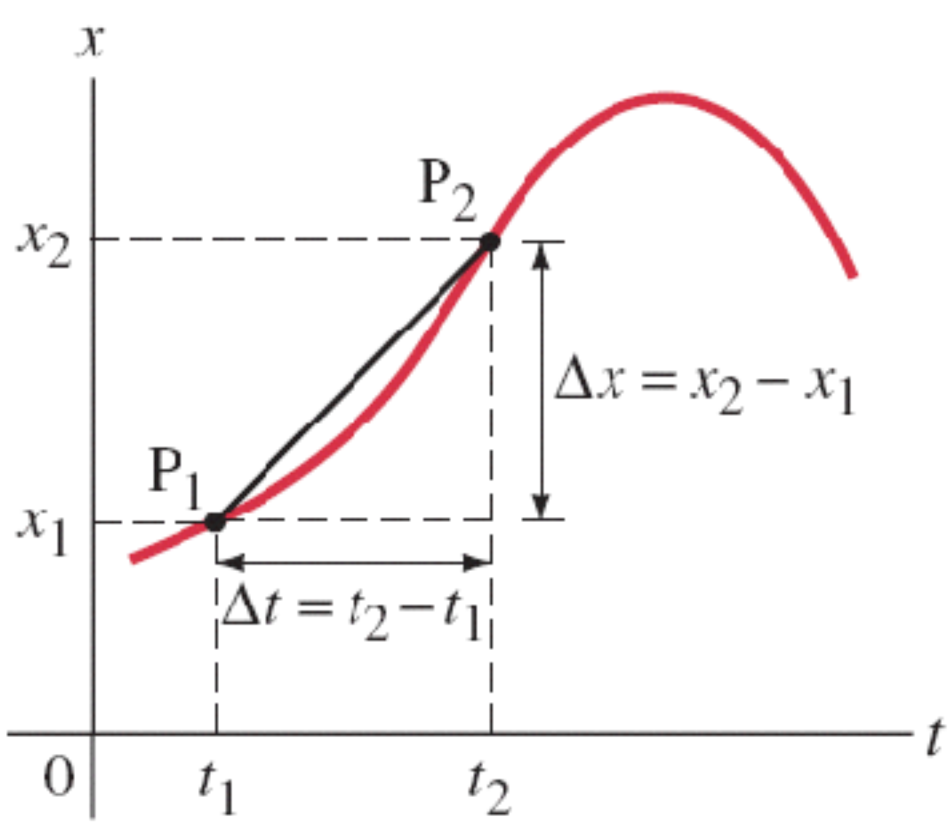
$$\bar{\nu} = \frac{x_2 - x_1}{t_2 - t_1}$$

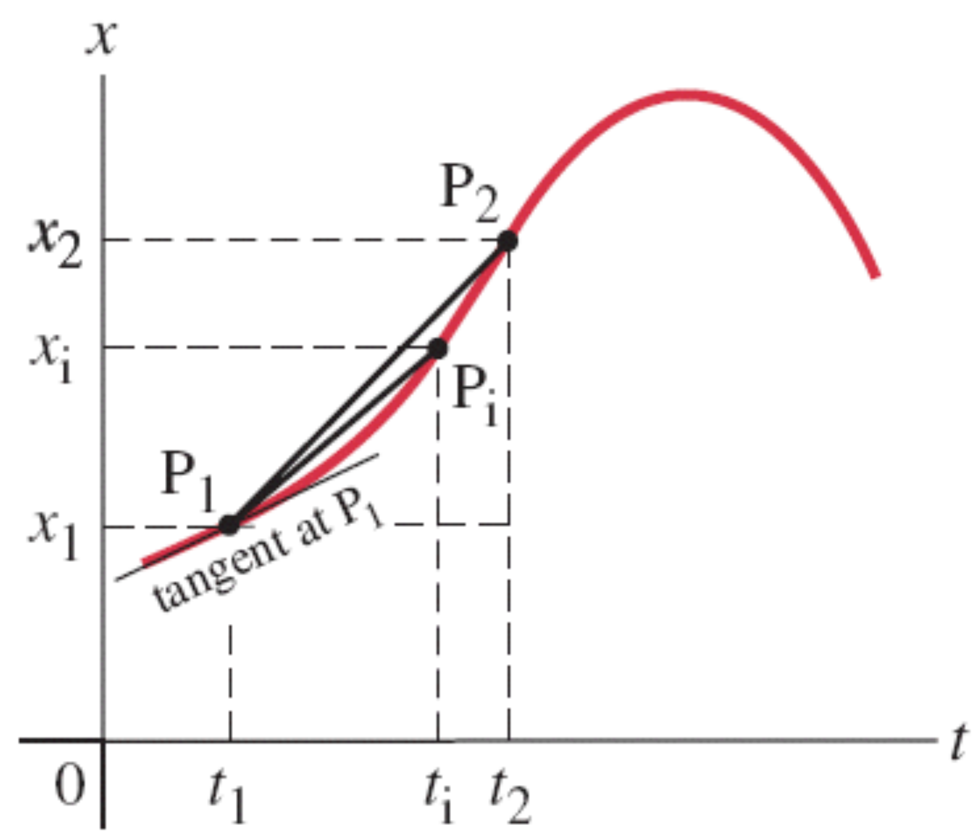
$$\boxed{\nu = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}}$$

Instantaneous velocity is the tangent to the curve at any point.

VELOCITY - DERIVATIVE OF POSITION

Instantaneous velocity is a vector defined as “how fast” a particle is moving at a given instant.





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$$\boxed{\nu = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}}$$

Instantaneous velocity is the tangent to the curve at any point.

Differential calculus focuses on making sense and defining this limit:

$$\boxed{\frac{d}{dx}f(x) \equiv \lim_{a\to 0}\frac{\Delta_a f(x)}{2a} = \lim_{a\to 0}\frac{f(x+a)-f(x-a)}{2a}}$$

derivative of $f(x)$

This is of course the instantaneous rate of change of the $f(x)$.

Determining this for different functions analytically is not always easy…

…numerically it is much easier 🤓

Differential equations are equations that relate one or more functions and their derivations.

Here is a simple example of a differential equation: $\frac{d}{dx}f(x) = g(x)$.

In general, differential equations can take a more complicated form.

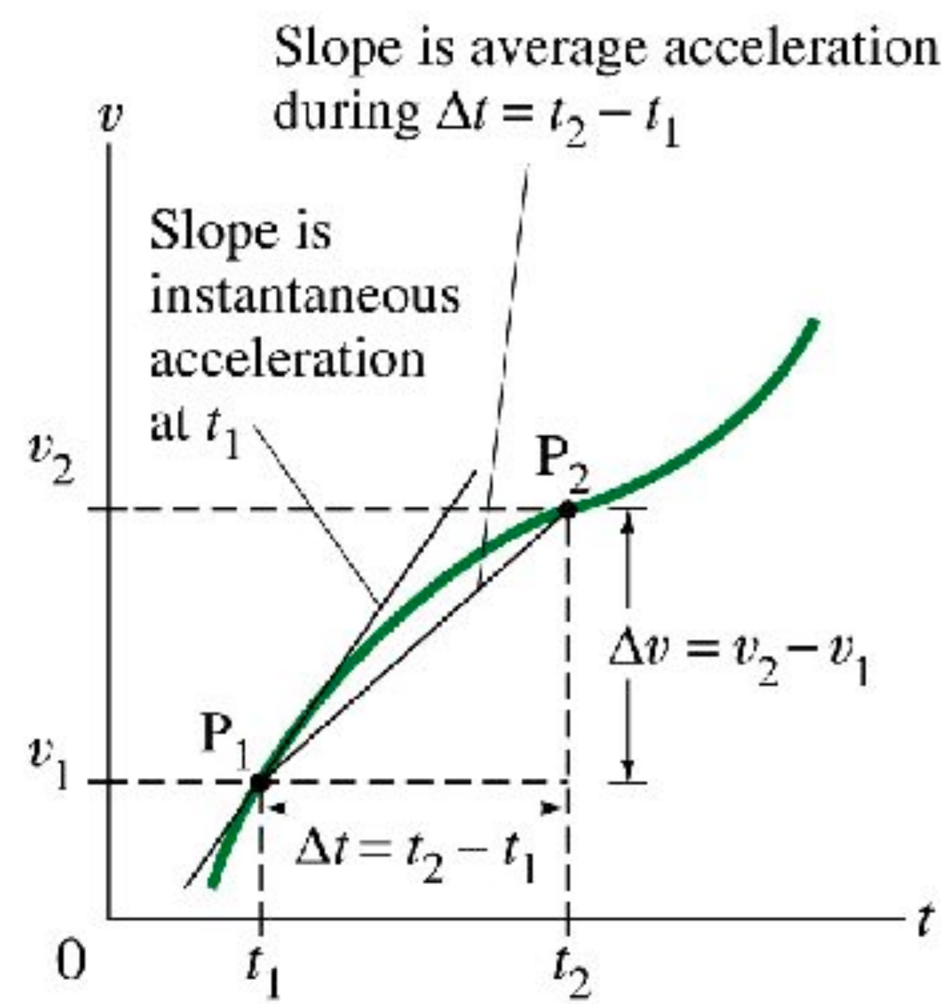
A significant fraction of the most important equations in Physics are differential equations or can be written as such. Examples include:

- $F = ma$, which can be written as $F = m\frac{d^2}{dt^2}x(t)$
- Wave equations
- Schrödinger Equation
- Maxwells Equations

Instantaneous acceleration is the average acceleration in the limit as the time interval becomes infinitesimally short.

$$\boxed{a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2x}{dt^2}}$$

Instantaneous acceleration is the slope of the tangent to the v vs. t curve at that time.



SUMMARY - CONCEPTS COVERED

Physics concepts/equations

Position

Velocity : $v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$

Acceleration : $a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$

Calculus

Derivatives: $\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$

Numerical solution to differential equations

Equation: $\frac{d}{dx} f(x) = g(x)$ Solution: $f(x + 2a) = f(x) + g(x + a)2a$

Equation: $\frac{d^2}{dt^2} f(t) = h(t)$

Solution: $g(t + 2a) = g(t) + h(t + a)2a,\ f(t + 2a) = f(t) + g(t + a)2a.$

Python

Functions of function: def $f(g, x)$: return $g(x)$

EXERCISE #1 - DEFINE DERIVATIVE

Write code for a "*derivative*" of any function, f, at x for a non-zero value of a.

Note, in Python, you can define functions which take function as inputs. Here is an example of this.

```python
def fin_deriv(f, x, a):
    """
    this defines the rate of change of the function f
    as a function of x

    the final point is at xf = x + a
    the initial point is at xi = x - a
    the difference between these two is xf - xi = 2*a
    """

    "the numerator"
    num = f(x+a) - f(x-a)
    "the denominator"
    denum = 2.0* a

    return num / denum
```

look inside my code 😎

EXERCISE #2 - TEST CONVERGENCE

In what follows we will vary $a = [10^{-5}, .1)$ and fix x = 1.

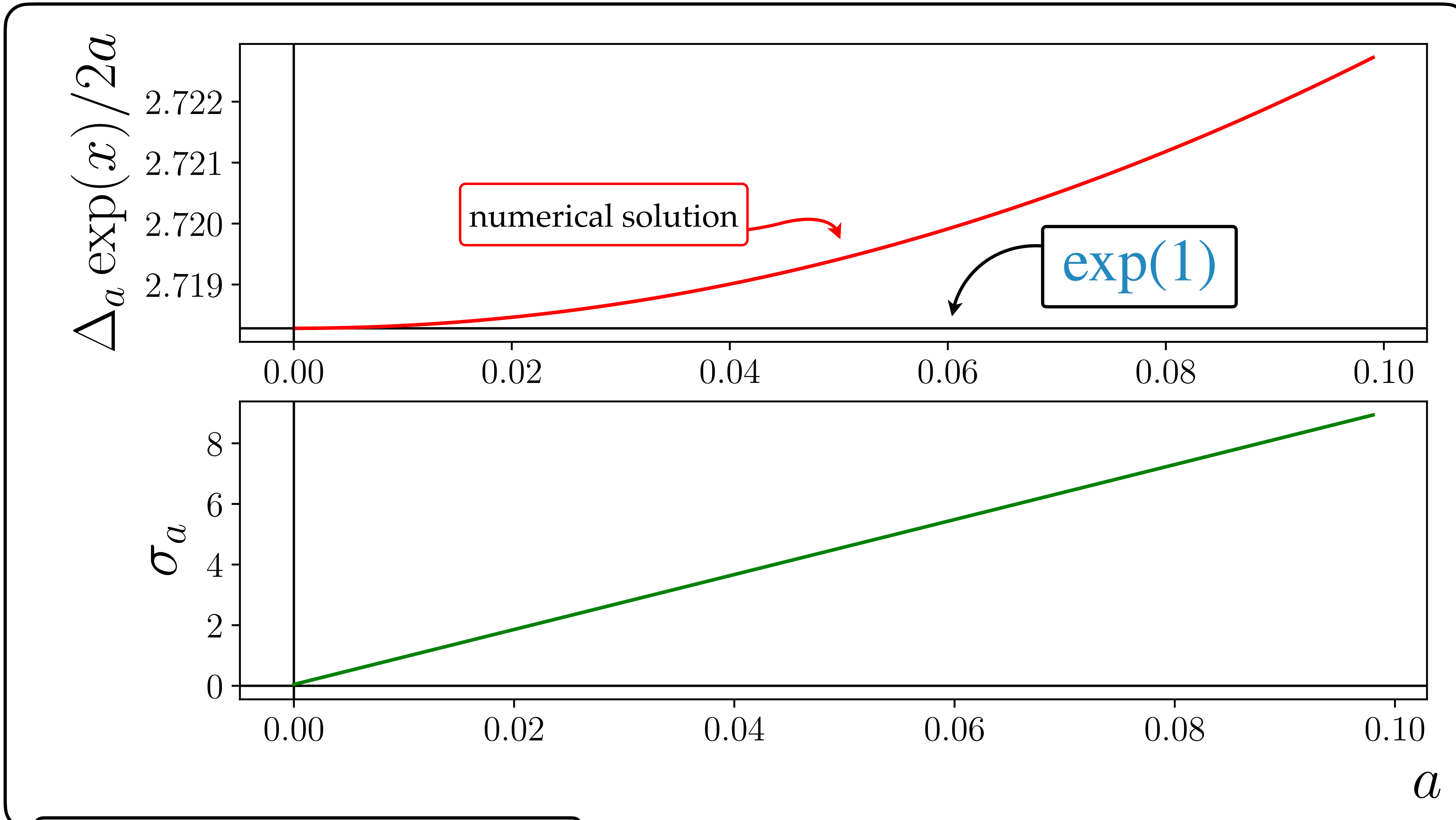
```
x = 1
a=np.arange(pow(10,-5),.1,pow(10,-3))
```

Note, I chose to vary a in steps of 10^{-3}, but you are welcomed to try a different step size.

Let us test your code by taking the numerical derivative of the following functions:

$$\exp(x),\ \log(x),\ \sin(x),\ \cos(x),\ \frac{x^3}{6} + \frac{x^4}{8}$$

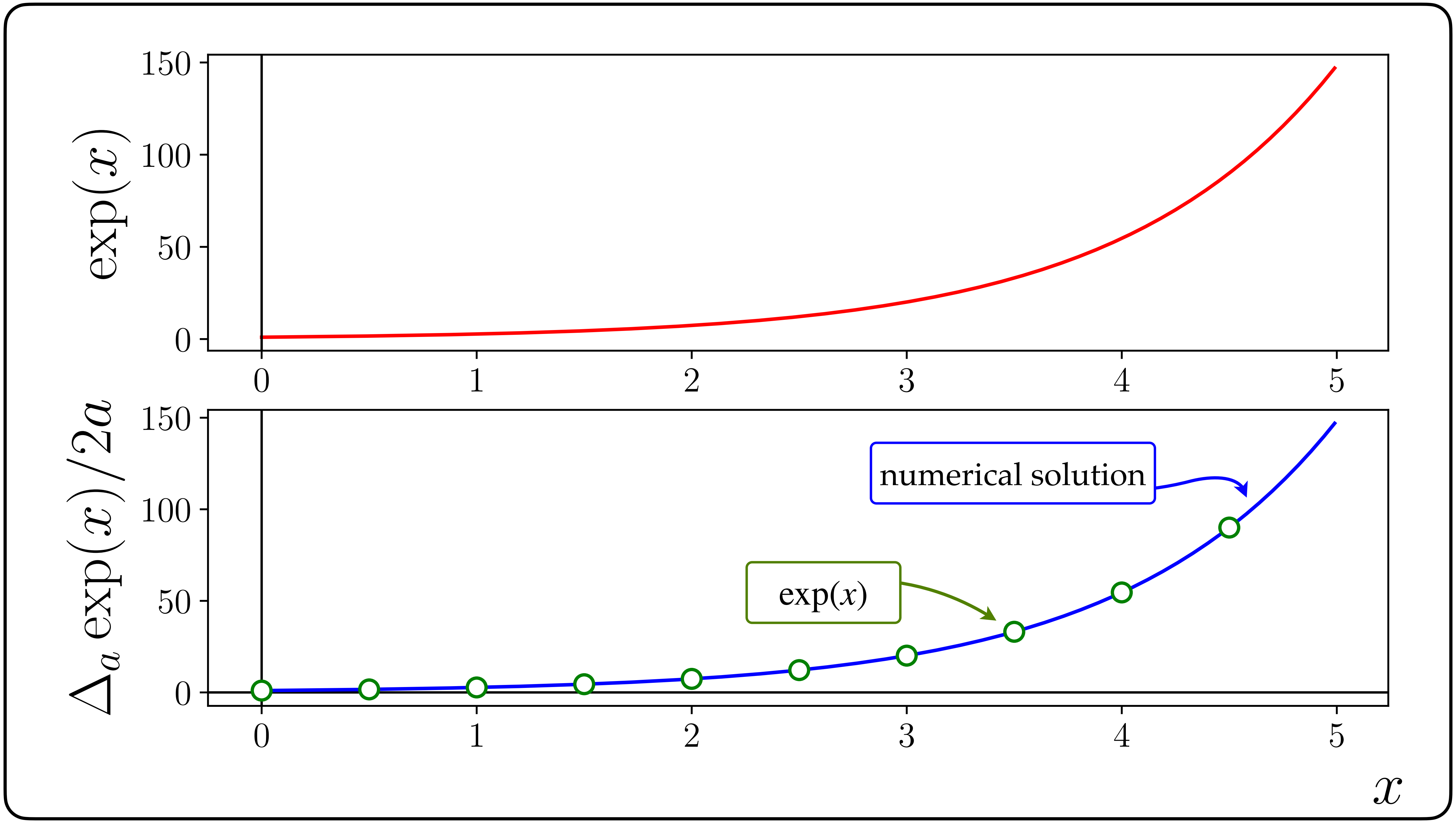
EXERCISE #2 - TEST CONVERGENCE: $\exp(x)$



$$\sigma_a = \frac{100}{\delta} \times \left| \frac{\Delta_a f(x)}{2a} - \frac{\Delta_{a+\delta} f(x)}{2(a+\delta)} \right|$$

EXERCISE #3 - PLOT DERIVATIVES AS A FUNCTION OF X

Having tested the convergence of your code, you are now in a place to plot the derivative of the same functions as a function of x. We will fix $a = 10^{-3}$.



EXERCISE #4 - WRITE CODE THAT SOLVES DIFF. EQ. FOR ANY $g(x)$

Write code for obtaining $f(x)$ for a range of values of x given any function of $g(x)$ and some initial value of $f(0)$.

To do this, you can write a “for loop” that reiterated the equation we obtained in the previous slide:

$$f(x + 2a) = f(x) + g(x + a)2a.$$

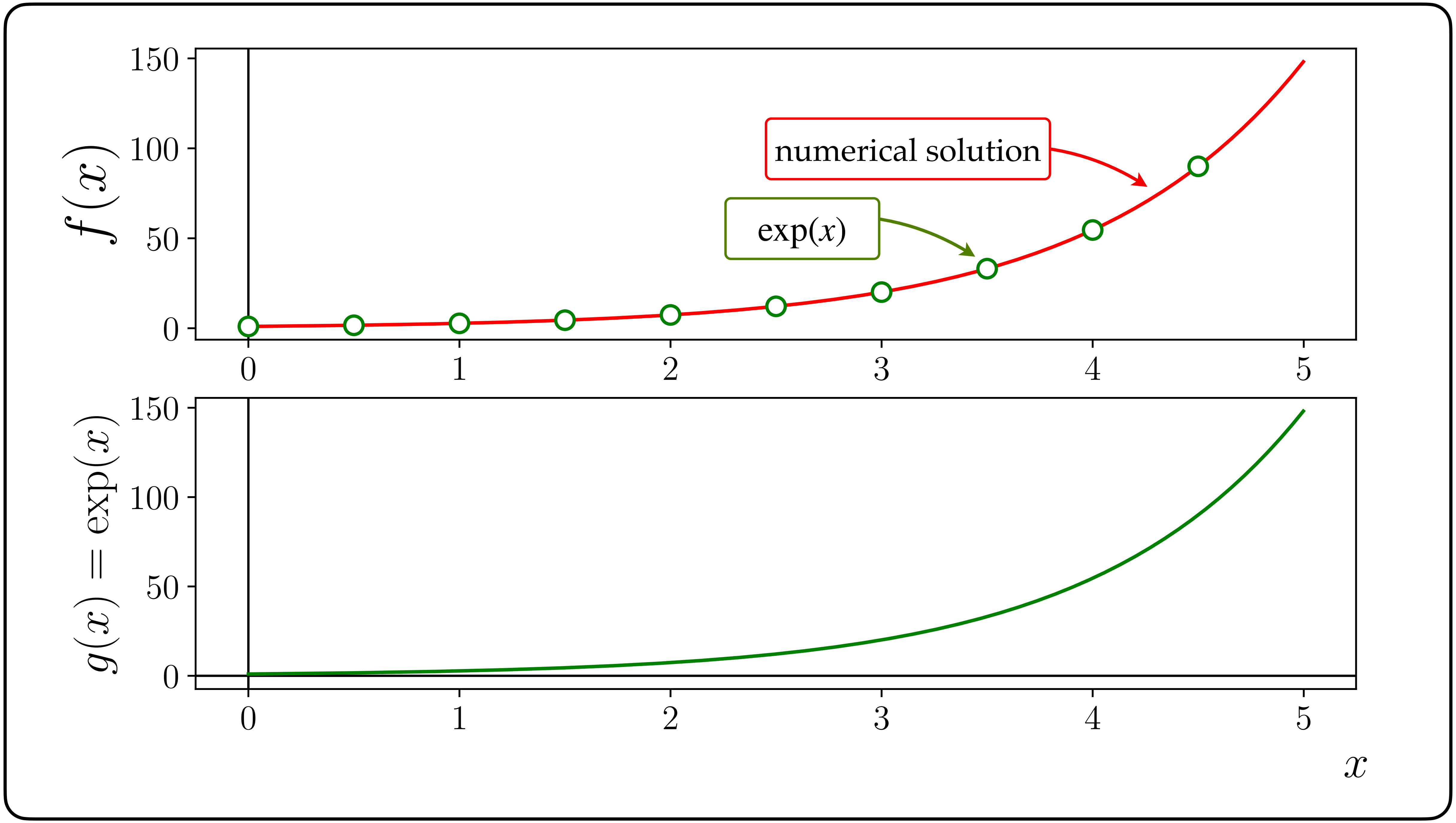
look inside my code 😎

Don’t be scared! The code is very short, I just wrote lots of comment to explain its logic 🤓

```python
def diff_eq(g, f0, xs, a):

    """
    this solves the diff. eq. df/dx = g(x) by discretizing it
    (f(x+a) - f(x-a))/2a = g(x)
    f(x+a) = f(x-a) + 2*a*g(x)

    or equivalently

    f(x+2 a) = f(x) + 2*a*g(x+a)

    we put the list of points into fs

    we initiate this with
    the initial value of f, which is f0 = f(xs[0])

    """

    fs=[f0]

    "note in this loop, we skip over the first element"
    for i0 in range(1,len(xs)):

        "we grab the previous term in the list"
        f0 = fs[-1]

        """
        we need the derivative at x+a,
        since xs[i0] =  x+2*a
        we can use the fact that x+a = xs[i0]-a
        """
        df = g(xs[i0]-a)*2*a

        "we add these together"
        fs.append(f0+ df)

    return fs
```

EXERCISE #5 - PUT YOUR CODE TO A TEST BY SOLVING FOR $f(x)$

Solve for $f(x)$ in this differential equation $\dfrac{df(x)}{dx} = g(x)$, by fixing $a = 10^{-3}$ and $x = [0,5)$, when $f(0) = 1$ and $g(x) = \exp(x)$.



EXERCISE #6 - WRITE CODE FOR $x(t)$

First, write code for the expected behavior of $x(t)$ and $v(t)$ as a function of time.

Position as a function of time	$x(t) = x_0 + v_0^x t + \frac{a_x}{2} t^2$ $y(t) = y_0 + v_0^y t + \frac{a_y}{2} t^2$
Velocity as a function of time	$v_x(t) = x'(t) = v_0^x + a_x t$ $v_y(t) = y'(t) = v_0^y + a_y t$

```
def x_func(t, x0, v0, a0):
    x1 = v0 * t
    x2 = a0 * pow(t,2)/2.0
    return x0 + x1 + x2

def v_func(t, v0, a0):
    return v0 + a0 * t
```

look inside my code 😎

EXERCISE #7 - WRITE CODE FOR $f(t)$, $g(t)$ GIVEN $h(t)$

Use these iterative equations

$$g(t+2a) = g(t) + h(t+a)2a,$$
$$f(t+2a) = f(t) + g(t+a)2a.$$

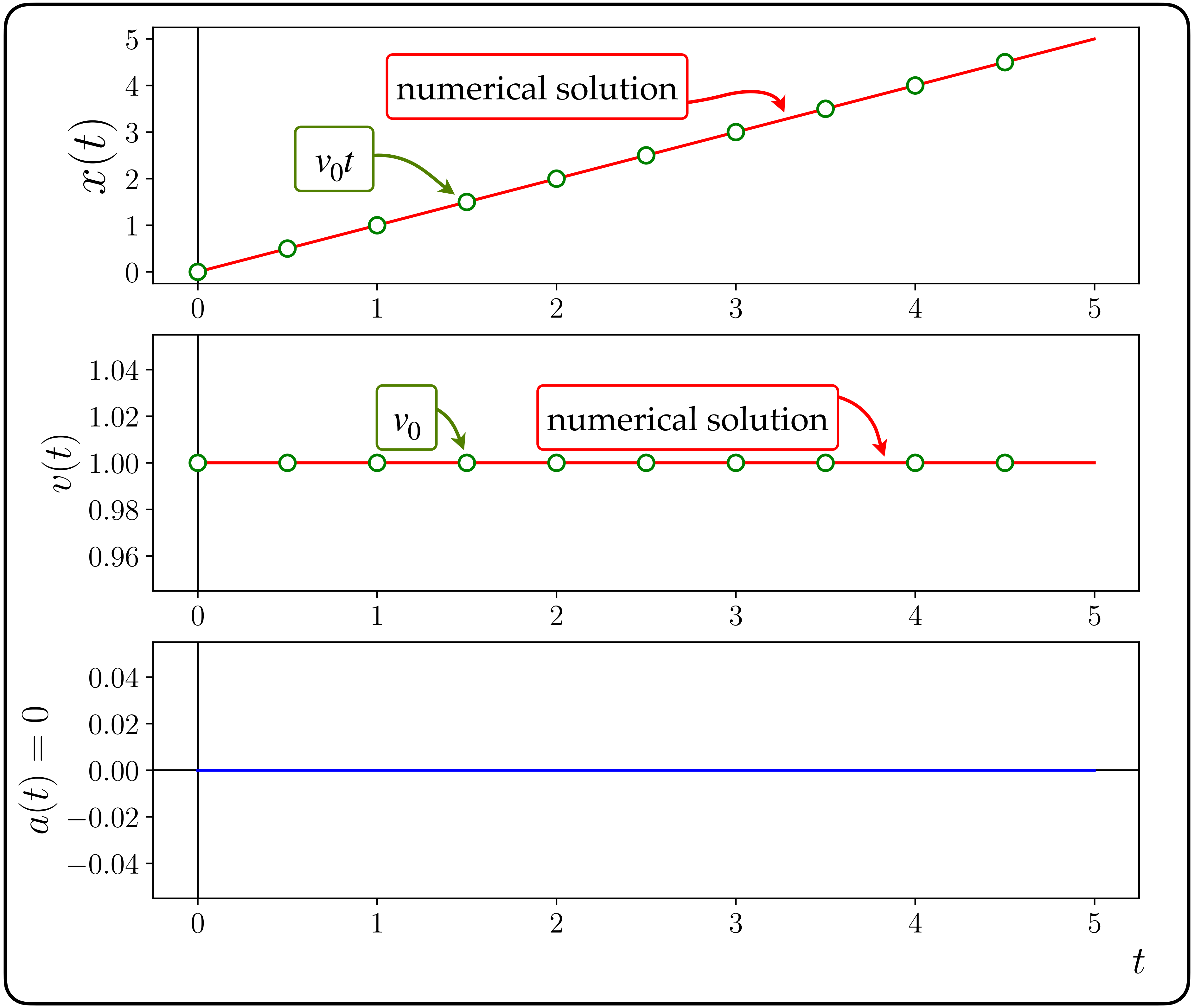
```python
def second_diff_eq(h, f0, g0, ts, a):

    """
    this solves the diff. eq. d^2f/dx^2 = h(x) by discretizing
    re-writting it as a coupled linear equations

    dg/dt = h(t)
    df/dt = g(t)

    g(t + 2*a) = g(t) + 2 * a * h(t + a)
    f(t + 2*a) = f(t) + 2 * a * g(t + a)

    we put the list of points into fs, gs

    we initiate this with
    the initial value of f, which is f0 = f(ts[0])
    the initial value of g, which is g0 = g(ts[0])

    note, we need to calculate g(t + a), but we have g(t + 2*a) and g(t)
    we will estimate this with the average of these two, which we will call

    gbar = ( g(t + 2*a) + g(t) ) / 2
    """

    fs=[f0]
    gs=[g0]

    "note in this loop, we skip over the first element"
    for i0 in range(1,len(ts)):
        "we start with g(t): we grab the previous term in the list"
        g0 = gs[-1]
        "the shift in g"
        dg = h(ts[i0] + a) * 2*a

        gs.append(  g0 + dg )

        "next we calculate f(t): we grab the previous term in the list"
        f0 = fs[-1]
        "we estimate g(t+a) with the average of the last two point in gs"
        gbar = (gs[-1] + gs[-2]) /2.0
        df = gbar * 2 * a

        "we add these together"
        fs.append(f0 + df)

    return fs, gs
```

look inside my code 😎

Don’t be scared! The code is very short, I just wrote lots of comment to explain its logic 🤓

EXERCISE #8 - REPRODUCE THESE PLOTS

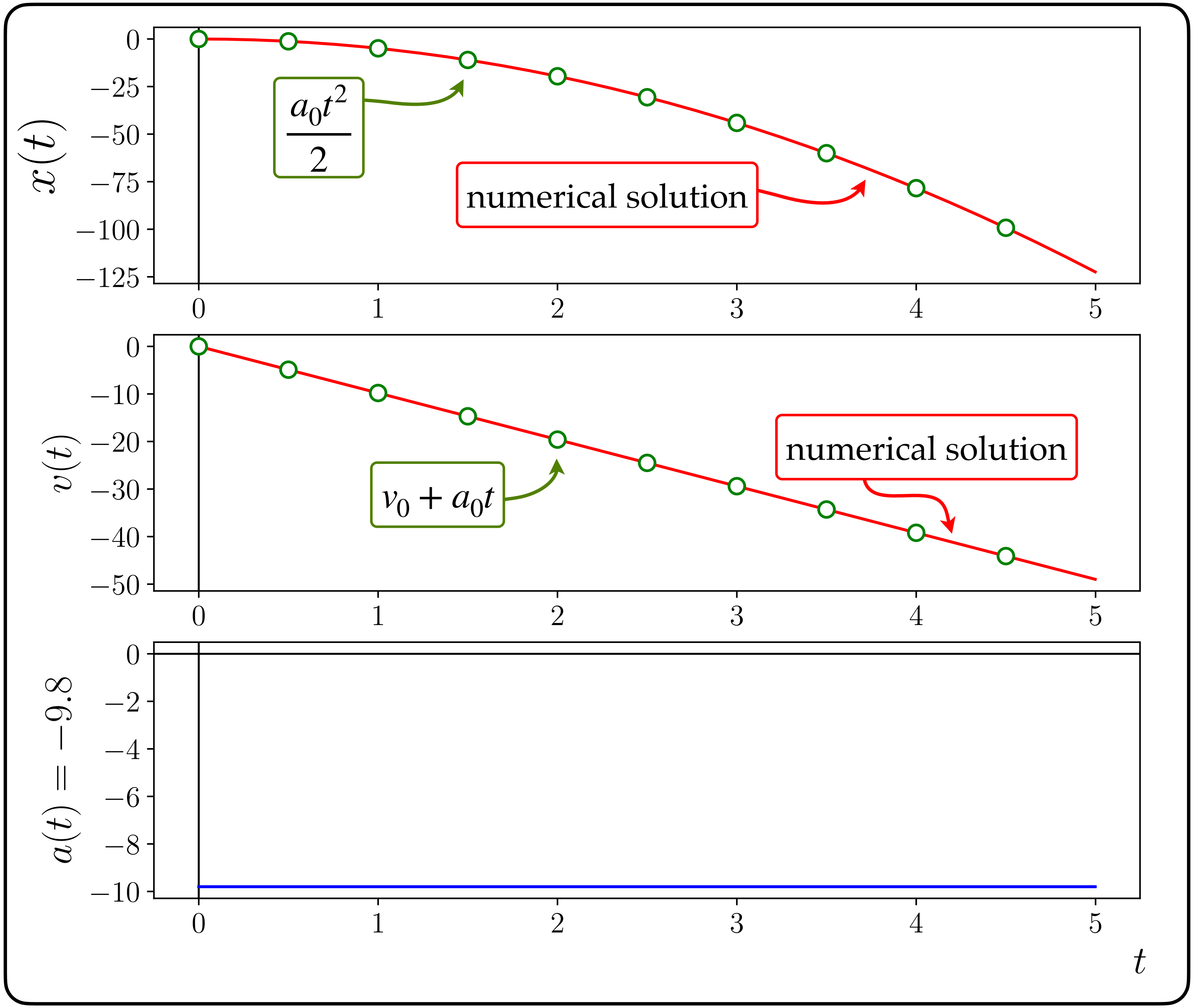
Test your code and test the kinematic equations obtained assuming a constant acceleration. For $(x_0, v_0) = (0{,}1)$ and $a(t) = 0$.



$$x(t) = x_0 + v_0^x t + \frac{a_x}{2} t^2$$

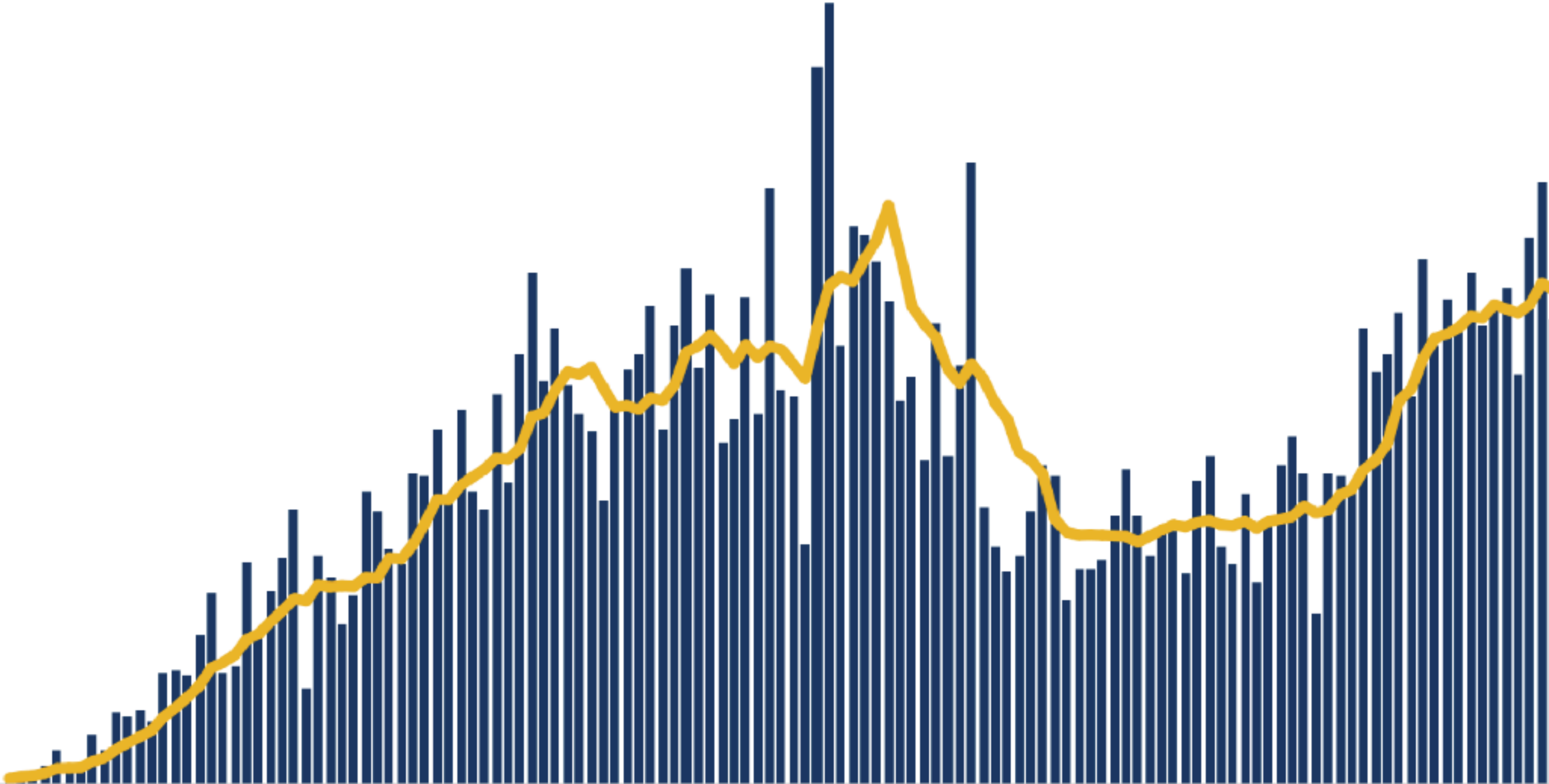
EXERCISE #8 - REPRODUCE THESE PLOTS

Test your code and test the kinematic equations obtained assuming a constant acceleration. For $(x_0, v_0) = (0{,}0)$ and $a(t) = -9.8$.

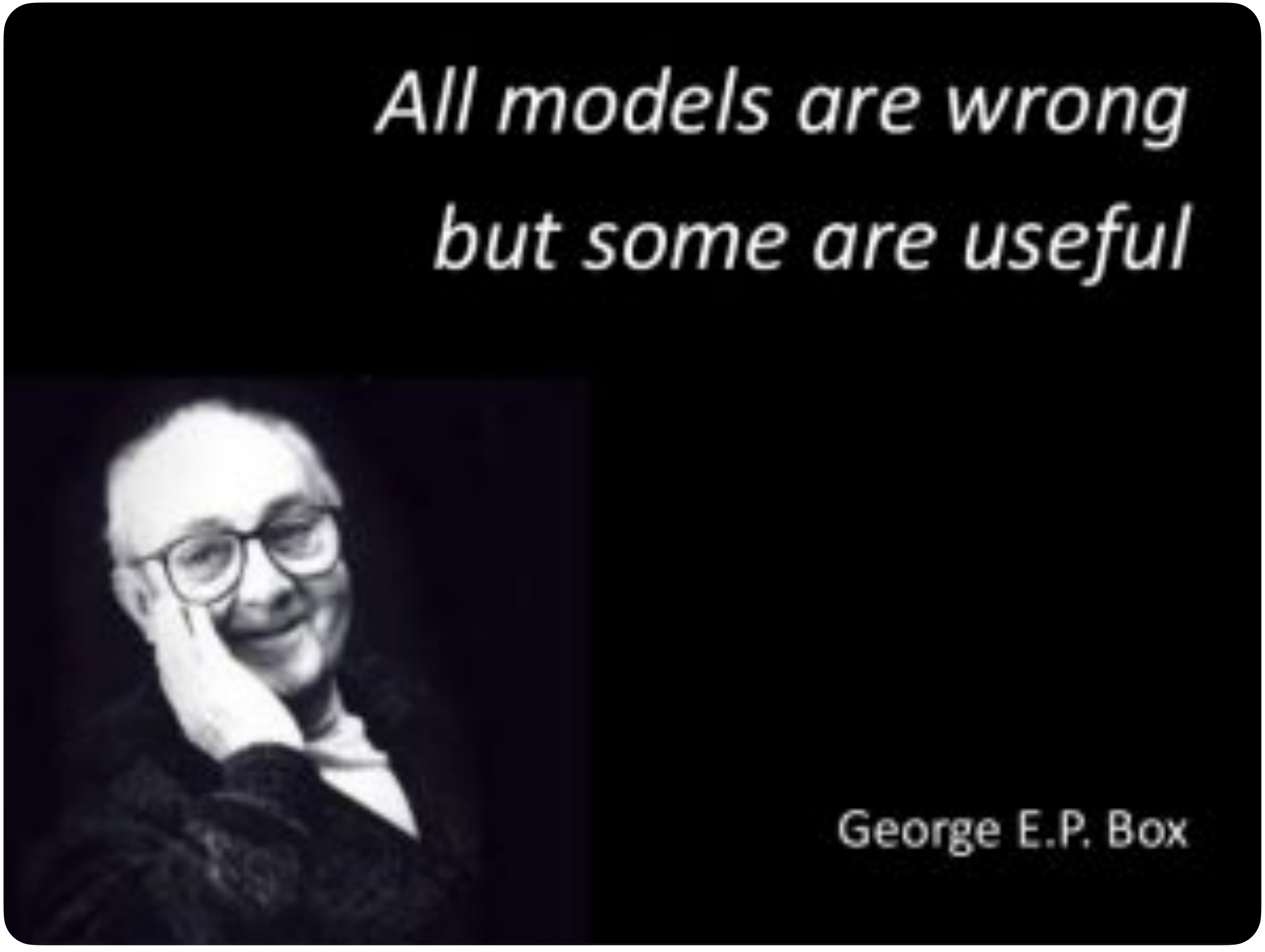


$$x(t) = x_0 + v_0^x t + \frac{a_x}{2} t^2$$

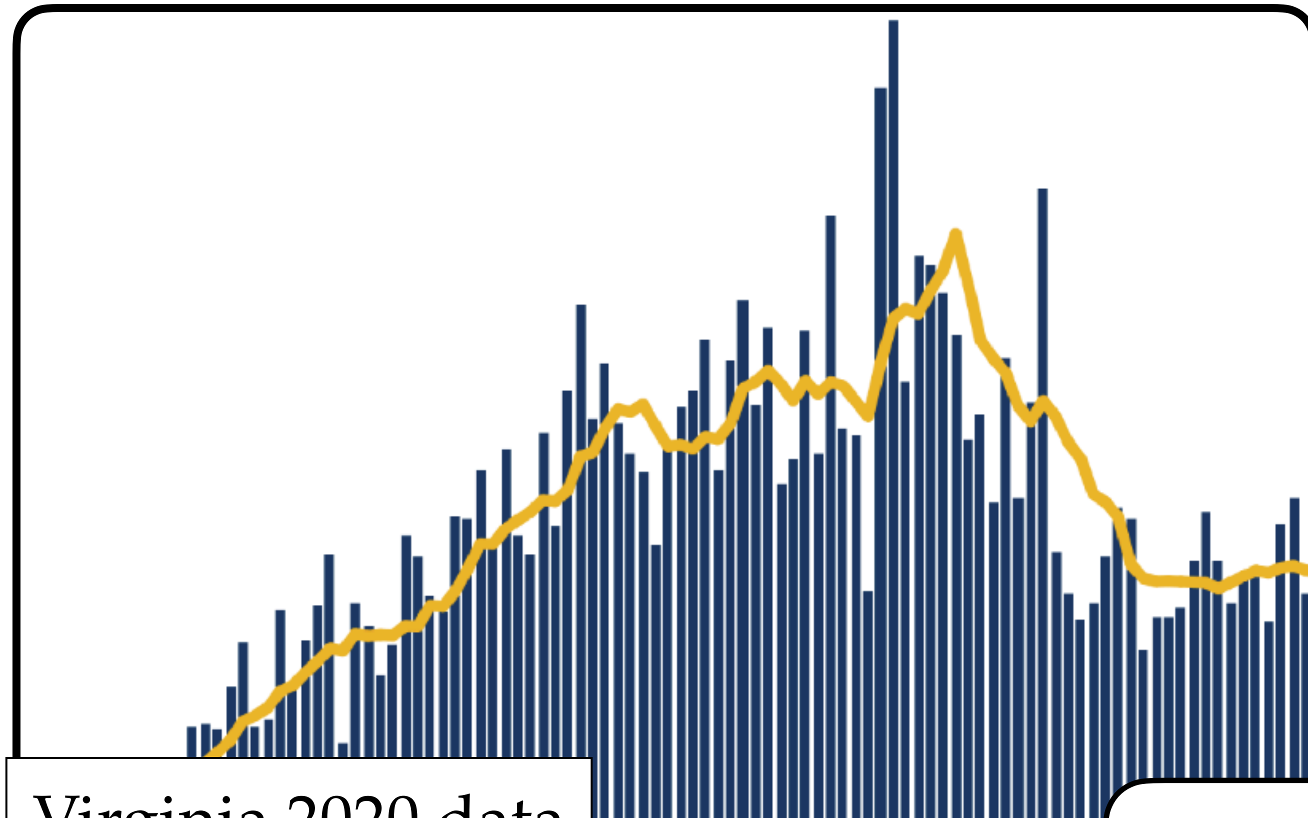
FIRST ORDER DIFFERENTIAL EQUATION



MODELS ARE USEFUL 🤓

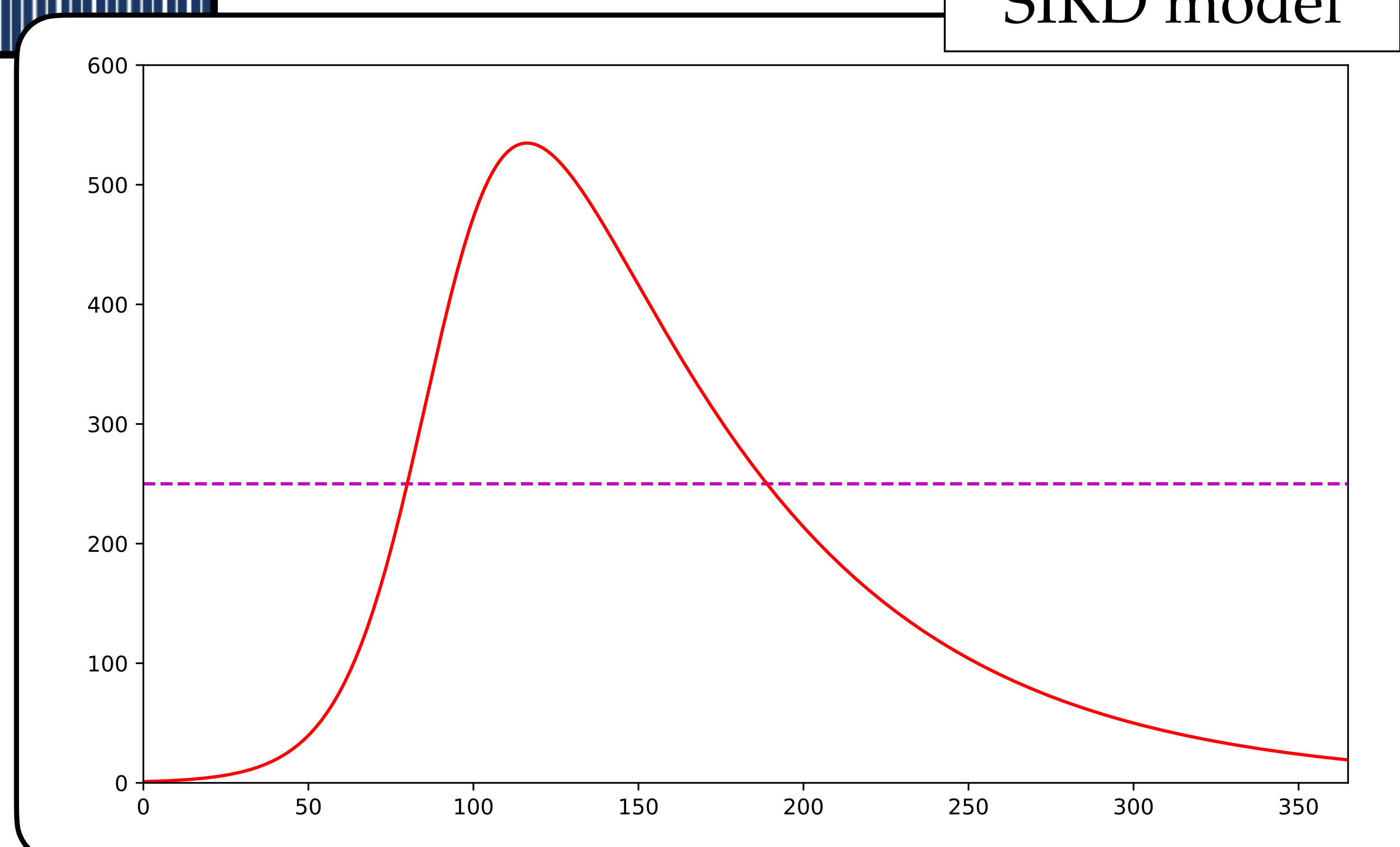


DATA VS. MODELS



Virginia 2020 data

SIRD model



WARNING: parameters have not been fix to fit the data


CAUTION: MATH

SOME DEFINITIONS - PART ONE

Epidemiology: the study and analysis of the distribution, patterns and determinants of health and disease conditions in defined populations.

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😅 / 😵 **<u>The Recovered/Removed</u>**: An individual that was previously infected but is no longer. In reality this could mean that they have recovered or are deceased, which will be labeled as $R(t)$.

<u>The total population</u>: We will label the total population as N, which at all times has to be equal to $N = S(t) + I(t) + R(t)$.

SOME DEFINITIONS - PART TWO

<u>Infection rate, β:</u> If a person is Infectious, there is some probability (P_i) that it will infect a Susceptible person if they mean. If the average infectious person sees N_0 people a day, then the expected number of people it will infect per day is $\beta = P_i N_0$.

$$\beta = \text{(probability of infecting)} \times \frac{\text{people encountered}}{\text{days}}$$

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Reproduction number, R_0: A infectious person will be sick and be able to infect others a total of D days. Since β is the number of people that an infected person can get sick per day, then the total number of people that this infectious individual can infect is equal to $R_0 = \beta D$.

$$R_0 = \text{(totals days sick)} \times \text{(probability of infecting)} \times \frac{\text{people encountered}}{\text{days}}$$

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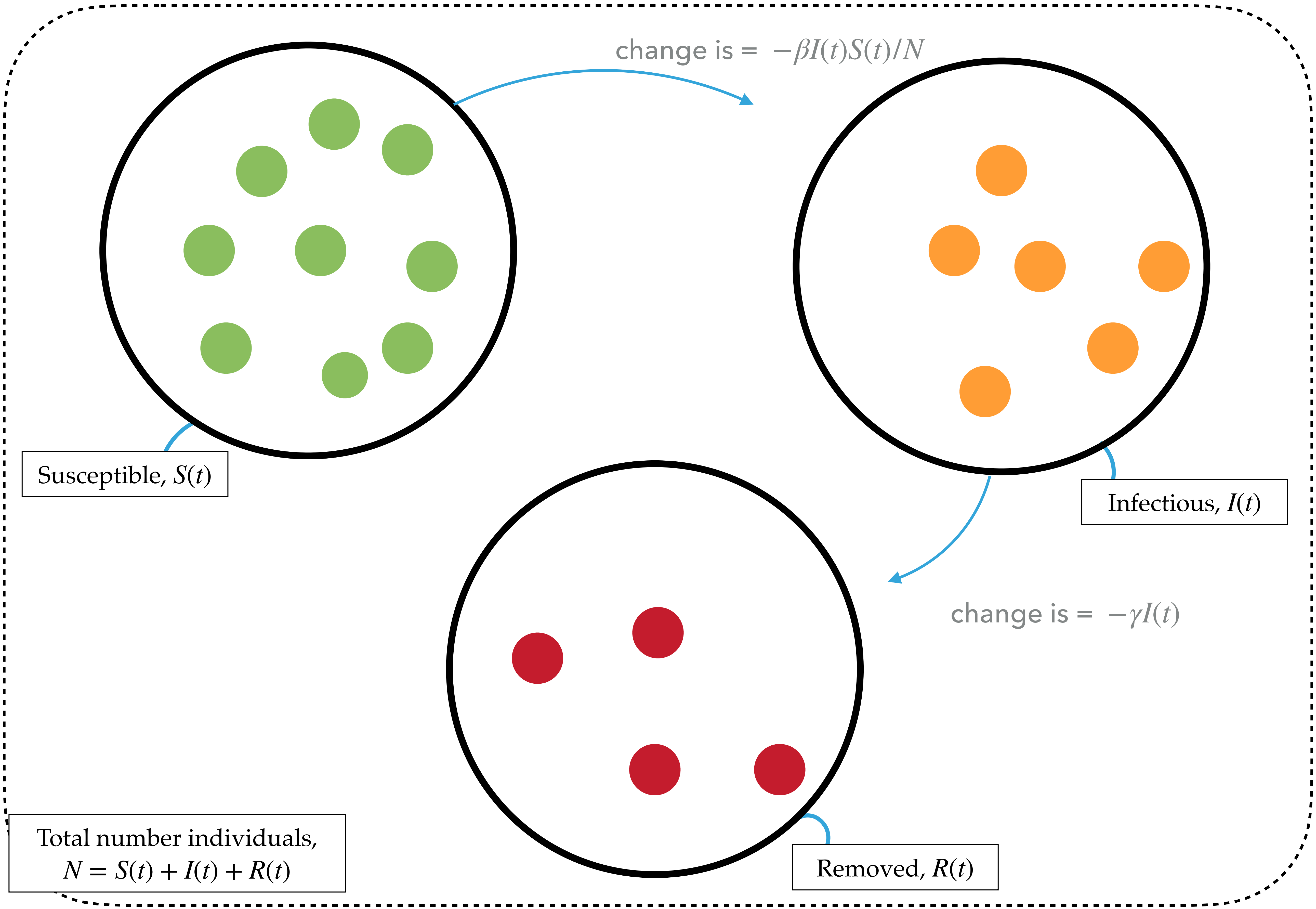
$$R_0 = \text{(totals days sick)} \times \text{(probability of infecting)} \times \frac{\text{people encountered}}{\text{days}}$$

<u>Rate of recovery, γ:</u> In what follows, it is convenient to use not D, but rather $1/D$. We will label this as $\gamma = 1/D$.

- If γ is small, this means that you are taking many days to recover. In other words, you are "*slow to recovery*".
- If γ is large, this means that you are taking few days to recover. In other words, you are quick to recovery.

As a result, it is convenient to think of γ as a **rate of recovery**.

SIR - CARTOON FLOW



SIR MODEL - SUSCEPTIBLE EQ.

Having defined the β as the number of people that an infectious person can get sick, assuming all people they encounter are susceptible. But only a fraction of $S(t)/N$ are actually susceptible.

This means in a single day, a single infectious person can actually get $\beta S(t)/N$ people infected.

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To calculate the number of people that the whole population of infectious people get sick, we just have to multiply this by $I(t)$.

$$\beta \frac{S(t)}{N} I(t) = \text{(probability of infecting)} \times \frac{\text{people encountered}}{\text{days}} \times \frac{\text{susceptible pop.}}{\text{total pop.}} \times \text{infectious pop.}$$

Notice, this is the amount by which $S(t)$ decreases per day:

$$\boxed{\frac{\Delta S(t)}{\Delta t} = -\beta \frac{S(t)}{N} I(t)}$$

that easy 😎

the sign is telling us that the number of susceptibles can only decrease!

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$$\boxed{\frac{\Delta S(t)}{\Delta t} = -\beta \frac{S(t)}{N} I(t)}$$

Differential form:

$$\boxed{\frac{dS(t)}{dt} = \lim_{\Delta t \to 0} \frac{\Delta S(t)}{\Delta t} = -\beta \frac{S(t)}{N} I(t)}$$

SIR MODEL - INFECTIOUS EQ.

The infectious population can increase when the susceptible population gets sick, but it can also decease whenever people recover. Meaning people can

The amount that it increase by the same amount that the susceptible population decreases by, $\beta \frac{S(t)}{N} I(t)$.

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To understand the people that leave infectious category to go to the recovered population, we remember we defined the **recovery rate, γ,** $\gamma = 1/D$ where D is the total number of days it take for an individual to recover.

This means that in D days all of the people in the $I(t)$ would have recovered. This means that the rate at which people are leaving the this group is $-I(t)/D = -\gamma I(t)$.

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ok, what required a bit more work 😅

Putting everything together we get

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differential form

$$\boxed{\frac{dI(t)}{dt} = \lim_{\Delta t \to 0} \frac{\Delta I(t)}{\Delta t}}$$

SIR MODEL - RECOVERED/REMOVED EQ.

Having done all of the hard work, it is relatively straightforward to determine the rate at which the number of recovered individuals are increasing. This increase has to be equal in magnitude to the rate at which the infectious group is decreasing, in other words

$$\boxed{\frac{\Delta R(t)}{\Delta t} = \gamma I(t)}$$

that easy 😎

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[NOTE: in the SIR model, people that are recovered are assumed to build immunities and stay immune. This might not be true for COVID, but it is a good approximation to start with]

SIR MODEL - SUMMARY

In summary, the SIR model describes the relationship between three population using three coupled equations,

$$\boxed{\frac{\Delta S(t)}{\Delta t} = -\beta \frac{S(t)}{N} I(t)} \quad \boxed{\frac{\Delta I(t)}{\Delta t} = \beta \frac{S(t)}{N} I(t) - \gamma I(t)} \quad \boxed{\frac{\Delta R(t)}{\Delta t} = \gamma I(t)}$$

just a bunch of first order differential equations, which we know how to solve 💪🏽

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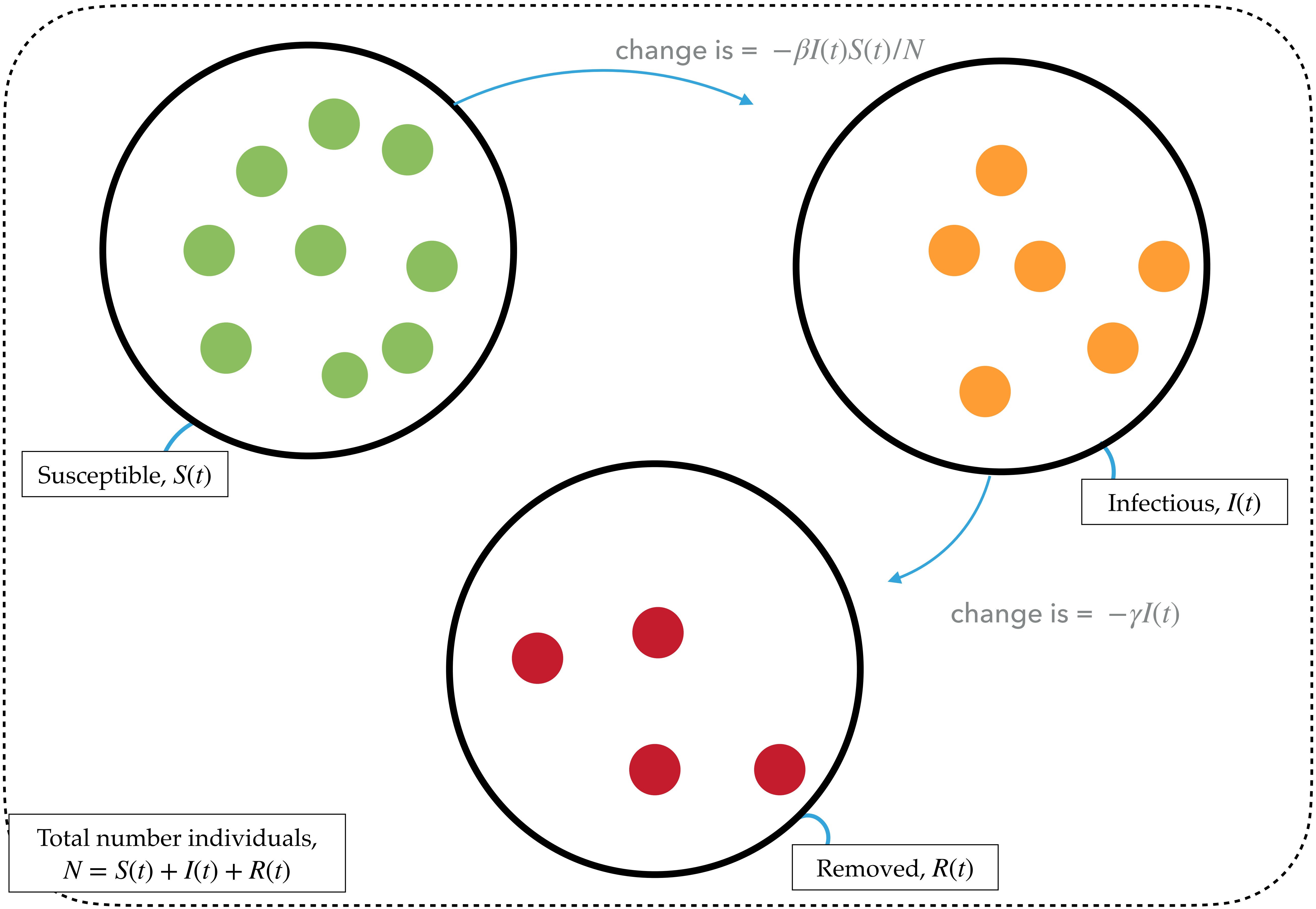
In their differential form, they take the form

$$\boxed{\frac{dS(t)}{dt} = -\beta\frac{S(t)}{N}I(t)} \quad \boxed{\frac{dI(t)}{dt} = \beta\frac{S(t)}{N}I(t) - \gamma I(t)} \quad \boxed{\frac{dR(t)}{dt} = \gamma I(t)}$$

[In order to solve numerically, we will use the discrete form]

SIR - CARTOON FLOW



Review from the last lecture!

Although finding an analytical solution to differential equations may seem like a daunting task, obtaining a numerical solution is more straightforward. In fact, all we need is skills we have developed and a bit of algebra.

To see this, let us consider the simple example previously mentioned: $\dfrac{d}{dx}f(x) = g(x)$.

Numerically we know we need to replace the derivative with a finite difference by introducing a finite value of a

$$\frac{d}{dx}f(x) = \lim_{a\to 0}\frac{\Delta_a f(x)}{2a} = \lim_{a\to 0}\frac{f(x+a) - f(x-a)}{2a}$$

Just as before, we will keep a fixed and check that our results as independent of it. By replacing the derivative with this expression, we are able to rewrite this equations as an algebraic one

$$\frac{f(x+a) - f(x-a)}{2a} = g(x).$$

We can rewrite this by solving for $f(x+a)$

$$f(x+a) = f(x-a) + g(x)2a.$$

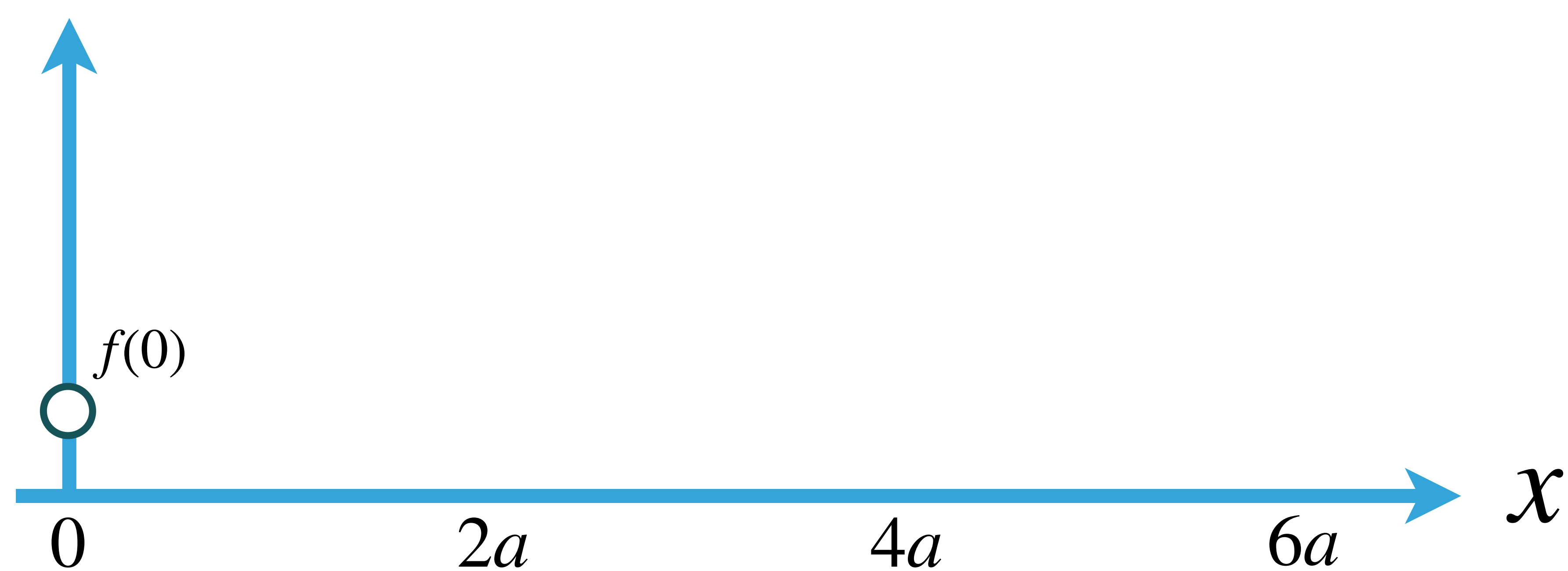
We can rewrite this by replacing $x = x_0 + a$: $f(x_0 + 2a) = f(x_0) + g(x_0 + a)2a$.

Since x_0 is the name of a generic variable, we can just leave it as x

$$f(x + 2a) = f(x) + g(x + a)2a.$$

Notice, if we know the first value of our function, we can determine all the other points using this equation.

Let the first point be a $x = 0$ with value $f(0)$.



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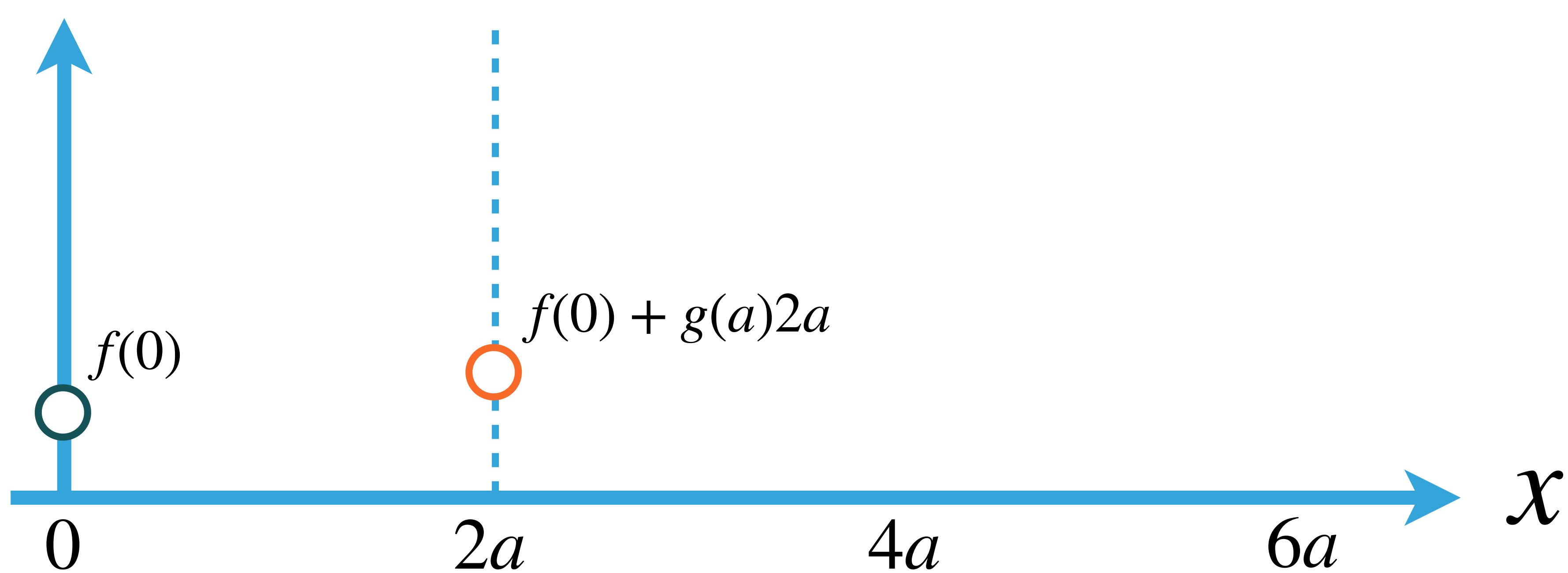
Then, the next point will be at $x = 2a$ with value $f(2a) = f(0) + g(a)2a$.



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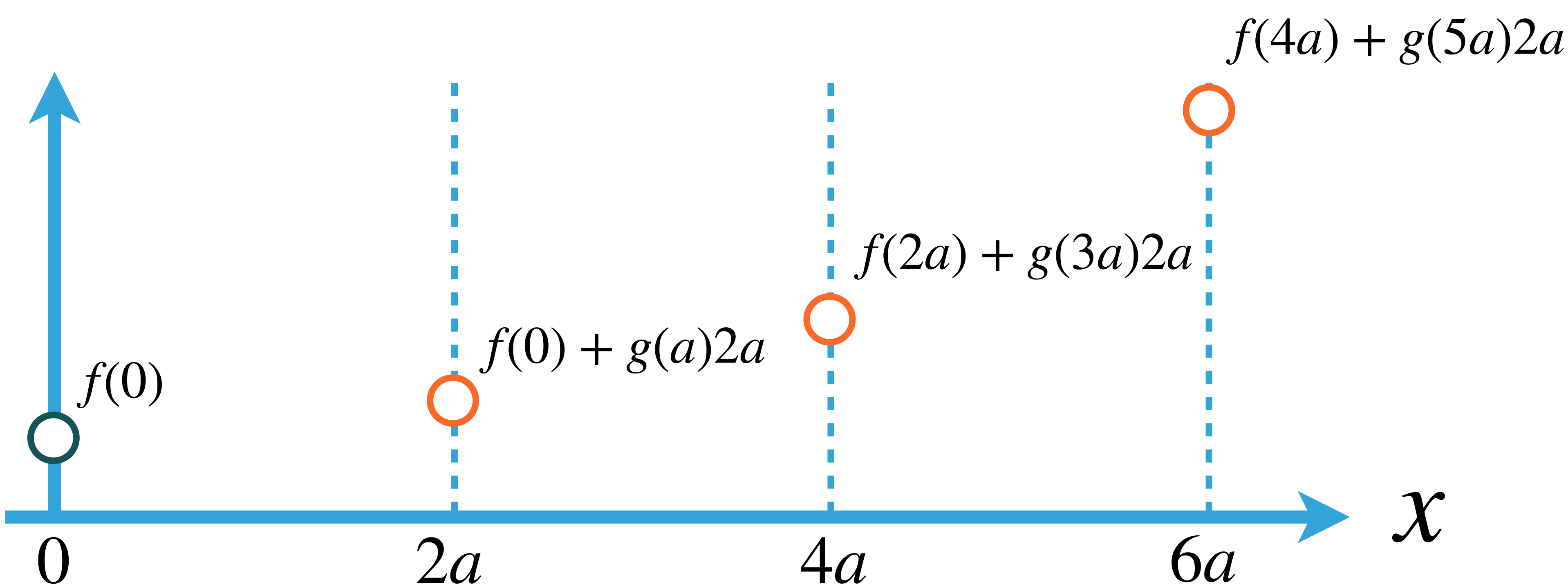
Follow by $f(4a) = f(2a) + g(3a)2a = f(0) + g(3a)2a + g(a)2a$
and $f(6a) = f(4a) + g(5a)2a = f(0) + g(5a)2a + g(3a)2a + g(a)2a$.



SIR MODEL - SOLUTION

This means that the solution to our equations:

$$\boxed{\frac{\Delta S(t)}{\Delta t} = -\beta\frac{S(t)}{N}I(t)} \quad \boxed{\frac{\Delta I(t)}{\Delta t} = \beta\frac{S(t)}{N}I(t) - \gamma I(t)} \quad \boxed{\frac{\Delta R(t)}{\Delta t} = \gamma I(t)}$$

Can be written as

$$S(t+dt) = S(t) - dt\,\beta\frac{S(t)I(t)}{N}$$

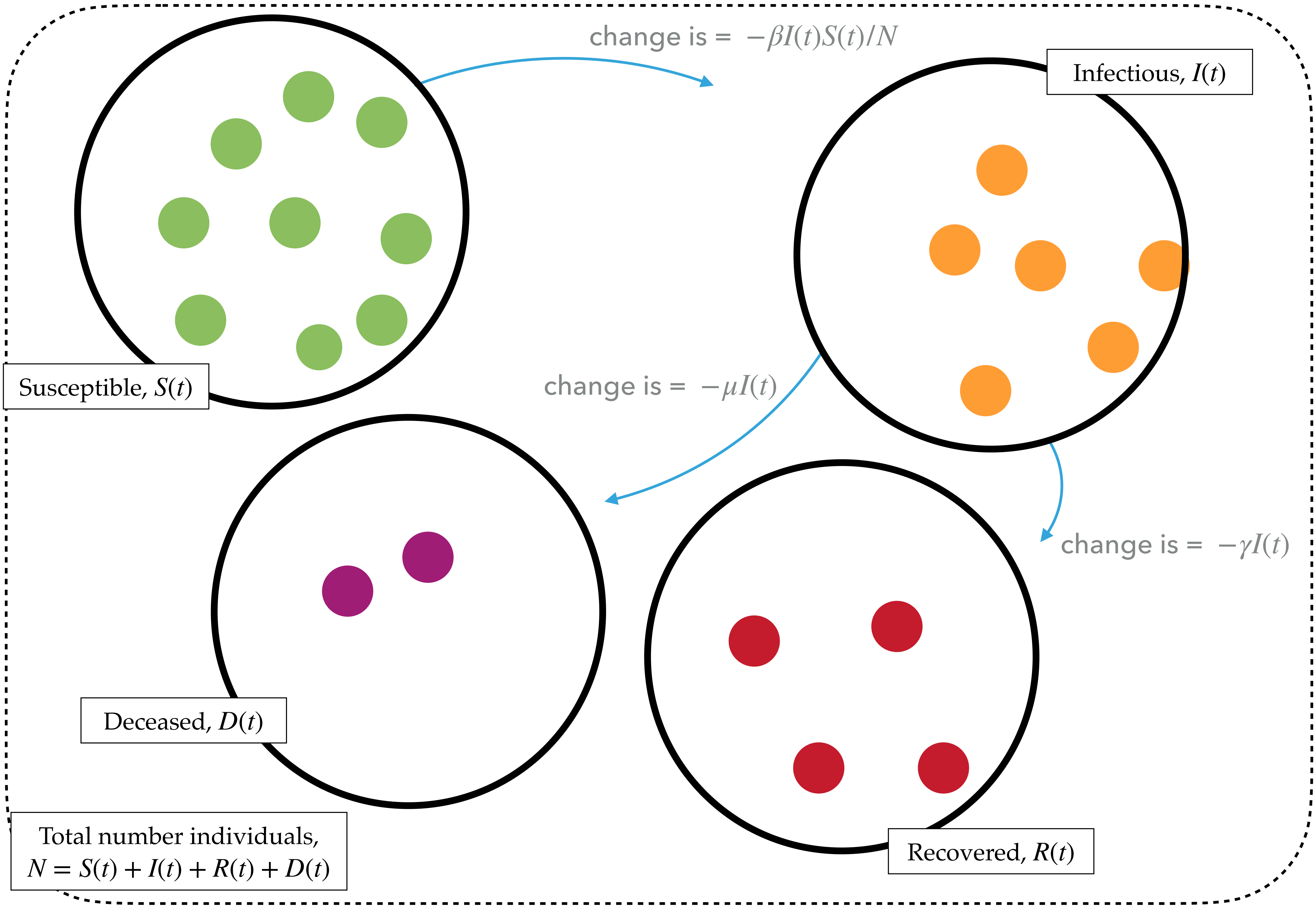
$$I(t+dt) = I(t) + dt\,\left(\beta\frac{S(t)I(t)}{N} - \gamma I(t)\right)$$

$$R(t+dt) = R(t) + dt\gamma I(t)$$

ripe for a "for loop"

Note:
we used $dt = 2a$,
and we have a made an $\mathcal{O}(a^2)$ error.

SIRD - CARTOON FLOW



SIRD MODEL - SOLUTION

For this case, we have have additional parameter (μ) for describing the probability of an infectious individual to become deceased ($D(t)$).

$$\boxed{\frac{\Delta S(t)}{\Delta t} = -\beta \frac{S(t)}{N} I(t)} \quad \boxed{\frac{\Delta I(t)}{\Delta t} = \beta \frac{S(t)}{N} I(t) - \gamma I(t) - \mu I(t)}$$

$$\boxed{\frac{\Delta R(t)}{\Delta t} = \gamma I(t)} \quad \boxed{\frac{\Delta D(t)}{\Delta t} = \mu I(t)}$$

New equation for describing the change in the deceased population.

μ: the rate of mortality

SIRD MODEL - SOLUTION

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$$\boxed{\frac{\Delta S(t)}{\Delta t} = -\beta \frac{S(t)}{N} I(t)} \quad \boxed{\frac{\Delta I(t)}{\Delta t} = \beta \frac{S(t)}{N} I(t) - \gamma I(t) - \mu I(t)}$$

$$\boxed{\frac{\Delta R(t)}{\Delta t} = \gamma I(t)} \quad \boxed{\frac{\Delta D(t)}{\Delta t} = \mu I(t)}$$

Solutions satisfy

$$S(t+dt) = S(t) - dt\,\beta \frac{S(t)I(t)}{N}$$

$$I(t+dt) = I(t) + dt\,\left(\beta \frac{S(t)I(t)}{N} - (\gamma + \mu)\,I(t)\right)$$

$$R(t+dt) = R(t) + dt\gamma I(t)$$

$$D(t+dt) = D(t) + dt\mu I(t)$$

Total number individuals,
$N = S(t) + I(t) + R(t) + D(t)$

EXERCISE #1 - WRITE CODE FOR SOLVING THE DIFFERENT EQS.

```python
"this is the change in functions describing the SIRD model, up to the dt scale"
def SIRD(s,i,r,d, beta, gamma, mu):
    "population size"
    N = s+i+r+d

    "the shift in the subsceptibles"
    s_dot = -s*i*beta / N

    "the shift in the subsceptibles"
    i_dot = (s*i*beta/N) - gamma*i - mu*i

    "the shift in the subsceptibles"
    r_dot = gamma*i

    "the shift in the subsceptibles"
    d_dot = mu*i

    return s_dot, i_dot, r_dot, d_dot


"""
this code returns the solutions for the different equation.

We added an output option command, this is to just return either the

number of infected people or the deceased people
"""
def diff_eq_sird(model,s,i,r,d,beta,gamma,mu,steps,dt, output):
    for t in steps:
        if t < len(steps)-1:

            "note: here we call the model function"
            s_dot, i_dot, r_dot, d_dot = model(s[t], i[t], r[t], d[t], beta,gamma,mu)
            s[t+1] = s[t]+dt*s_dot
            i[t+1] = i[t]+dt*i_dot
            r[t+1] = r[t]+dt*r_dot
            d[t+1] = d[t]+dt*d_dot

    "only return number of infections"
    if output =='i':
        return i

    "only return number of deceased"
    if output =='d':
        return d
```

look inside our code 😎

EXERCISE #2 - MAKE PLOTS!

Fix $\beta, \gamma, \mu = .09,\ .01,\ 0.005$, set the initial conditions of $S(0) = 998;\ \ I(0) = 1;\ \ R(0) = 0;\ \ D(0) = 0$, and calculate them as a function of time. Plot $I(t), D(t)$ vs. t.

If hospitals in your area only have capacity for 250 patients, will you have enough beds to support your sick? Which one of the variables above would you hope to reduce to assure that there are enough beds available?