# A8-MATH411

March 22, 2023

**Raul Escobar**
raulescobar_g@tamu.edu

## Problem 11 (textbook)

### a)

### b)

### c)

## Problem 1

$$Z = X - \frac{\mu}{\sigma}$$

$$\begin{aligned} P(\mu - \sigma \leq X \leq \mu + \sigma) &= P(\frac{(\mu - \sigma) - \mu}{\sigma} \leq \frac{X - \mu}{\sigma} \leq \frac{(\mu + \sigma) - \mu}{\sigma} \\ &= P\left(-\frac{\sigma}{\sigma} \leq \frac{X - \mu}{\sigma} \leq \frac{\sigma}{\sigma}\right) \\ &= P\left(-1 \leq \frac{X - \mu}{\sigma} \leq 1\right) \\ &= P(-1 \leq Z \leq 1) \\ &= P(Z \leq 1) - P(Z \leq -1) \end{aligned}$$

$$\boxed{= 0.68...}$$

Both intervals below follow exact same method as above:

$$P(-2 \leq Z \leq 2) = P(Z \leq 2) - P(Z \leq -2)$$

$$\boxed{= 0.95...}$$

$$P(-3 \leq Z \leq 3) = P(Z \leq 3) - P(Z \leq -3)$$

$$\boxed{= 0.997...}$$

## Problem 2

**a)**

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} f_X(x) \,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,\mathrm{d}x
\end{aligned}$$

$$u = \frac{x-\mu}{\sigma} \qquad \mathrm{d}u = \frac{1}{\sigma} \,\mathrm{d}x$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} \,\mathrm{d}u$$

We can assume that:

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \,\mathrm{d}x = \sqrt{2\pi}$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}}$$

$$\boxed{1 = 1}$$

**b)**

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x f_X(x) \,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,\mathrm{d}x
\end{aligned}$$

$$u = \left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2 \qquad \mathrm{d}u = \frac{x-\mu}{\sigma^2} \,\mathrm{d}x$$

Now we have to modify our main function in order to be able to do u-sub:

$$
\begin{aligned}
&= \frac{\sigma^2}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \left( \frac{x}{\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} - \frac{\mu}{\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} + \frac{\mu}{\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right) \mathrm{d}x \\
&= \frac{\sigma^2}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \left( \frac{x-\mu}{\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} + \frac{\mu}{\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right) \mathrm{d}x \\
&= \frac{\sigma^2}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \frac{x-\mu}{\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,\mathrm{d}x + \frac{\sigma^2}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \frac{\mu}{\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,\mathrm{d}x \\
&= \frac{\sigma^2}{\sqrt{2\pi}\sigma} \int_{\infty}^{\infty} e^{-u} \,\mathrm{d}u + \frac{\sigma^2}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \frac{\mu}{\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,\mathrm{d}x
\end{aligned}
$$

We have the same integral bounds for the first integral. This makes it zero so we can remove it:

$$
\begin{aligned}
&= \frac{\sigma^2}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \frac{\mu}{\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,\mathrm{d}x \\
&= \mu \left( \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,\mathrm{d}x \right) \\
&\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,\mathrm{d}x = 1
\end{aligned}
$$

$$
\boxed{\mathbb{E}[X] = \mu}
$$

Now to get the variance we can follow a similar pattern:

$$
\begin{aligned}
\mathrm{Var}[X] &= \int_{-\infty}^{\infty} x^2 f_X(x) \,\mathrm{d}x - \mu^2 \\
&= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,\mathrm{d}x - \mu^2 \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} \,\mathrm{d}x - \mu^2
\end{aligned}
$$

Some u-sub:

$$
u = \frac{x-\mu}{\sigma} \qquad \mathrm{d}u = \frac{1}{\sigma} \,\mathrm{d}x
$$

$$
x^2 = (\sigma u + \mu)^2
$$

$$
x^2 = \sigma^2 u^2 + 2\sigma\mu u + \mu^2
$$

Substituting $x$ for $u$:

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (\sigma^2 u^2 + 2\sigma\mu u + \mu^2) e^{-\frac{u^2}{2}} \,\mathrm{d}u - \mu^2 \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \sigma^2 u^2 e^{-\frac{u^2}{2}} \,\mathrm{d}u + \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} 2\sigma\mu u e^{-\frac{u^2}{2}} \,\mathrm{d}u + \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \mu^2 e^{-\frac{u^2}{2}} \,\mathrm{d}u - \mu^2 \\
&= \frac{\sigma^2}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} u^2 e^{-\frac{u^2}{2}} \,\mathrm{d}u + \frac{2\sigma\mu}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} u e^{-\frac{u^2}{2}} \,\mathrm{d}u + \frac{\mu^2}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} \,\mathrm{d}u - \mu^2
\end{aligned}
$$

We can assume that we already know:

$$
\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \,\mathrm{d}x = \sqrt{2\pi}
$$

$$
\int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} \,\mathrm{d}x = 0
$$

$$
\int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} \,\mathrm{d}x = \sqrt{2\pi}
$$

$$
\begin{aligned}
&= \frac{\sigma^2}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} u^2 e^{-\frac{u^2}{2}} \,\mathrm{d}u + \frac{2\sigma\mu}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} u e^{-\frac{u^2}{2}} \,\mathrm{d}u + \frac{\mu^2}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} \,\mathrm{d}u - \mu^2 \\
&= \frac{\sigma^2}{\sqrt{2\pi}\sigma} \sqrt{2\pi} + \frac{\mu^2}{\sqrt{2\pi}\sigma} \sqrt{2\pi} - \mu^2 \\
\mathrm{Var}[X] &= \frac{\sigma^2}{\sigma} + \frac{\mu^2}{\sigma} - \mu^2
\end{aligned}
$$

# Problem 3

## a)

$$Z = \frac{X - \mu}{\sigma}$$

$$P(15 - k \leq X \leq 15 + k) = 0.90$$

$$P\left(\frac{15 - k - \mu}{\sigma} \leq \frac{X - \mu}{\sigma} \leq \frac{15 + k - \mu}{\sigma}\right) = 0.90$$

$$P\left(\frac{15 - k - 15}{4} \leq \frac{X - \mu}{\sigma} \leq \frac{15 + k - 15}{4}\right) = 0.90$$

$$P\left(-\frac{k}{4} \leq Z \leq \frac{k}{4}\right) = 0.90$$

$$P\left(Z \leq \frac{k}{4}\right) - P\left(Z \leq -\frac{k}{4}\right) = 0.90$$

$$P\left(Z \leq \frac{k}{4}\right) - \left(1 - P\left(Z \leq \frac{k}{4}\right)\right) = 0.90$$

$$2P\left(Z \leq \frac{k}{4}\right) - 1 = 0.90$$

$$2P\left(Z \leq \frac{k}{4}\right) = 1.90$$

$$P\left(Z \leq \frac{k}{4}\right) = 0.95$$

$$\frac{k}{4} = 1.65$$

$$\boxed{k = 6.6}$$

**b)**

$$2P\left(Z \leq \frac{k}{4}\right) - 1 = 0.95$$

$$2P\left(Z \leq \frac{k}{4}\right) = 1.95$$

$$P\left(Z \leq \frac{k}{4}\right) = 0.975$$

$$\frac{k}{4} = 1.96$$

$$\boxed{k = 7.84}$$

## Problem 4

## Problem 5