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Chapter 1

Introduction to Partial Differential Equations

1.1 Notation

We start by defining the typical function that we'll be solving for in PDE's, in this case:

$$\begin{aligned} u : \mathbb{R}^2 &\to \mathbb{R} \\ (x,t) &\mapsto u = u(x,t) \end{aligned}$$

Here we have a function that maps *2-tuples* (in this case position and time) to real numbers. The elements of the *domain*, x and t, are called *independent* variables and the element of the *codomain*, u is called the *dependent* variable. This notation can be a little confusing because we are calling the function and the element of the codomain with the same letter, but it's the more used notation and after adaptation is very clear. Note that in this example is a function from $\mathbb{R}^2$ to $\mathbb{R}$ but it can have whatever domain it needs.

Once we've defined the function we are solving for, we'll define the *partial derivative with respect to* t of this function as:

$$\begin{aligned} u_t : \mathbb{R}^2 &\to \mathbb{R} \\ (x,t) &\mapsto u_t = (\partial_t u)(x,t) \end{aligned}$$

Here the *independent* variables are still x and t, and the *dependent* variable is u_t (the same as the function name). In this case we also use the same name for the function and the element of the codomain. Note the use of the ∂_t *operator*.

- $\partial_t u$: is the partial derivate of u w.r.t. t. It is a *function* not a *number*.
- $(\partial_t u)(x,t)$: is the partial derivative of u w.r.t. t at the space-time point (x,t). It is a *number*.

It's fundamental to undestand the difference between this two statements.

For repeated differentiation we use the notation:

$$u_{tt}, u_{tx}, u_{txx}, \dots$$

and using the differential operator:

$$\partial_t^2 u, \partial_{tx} u, \partial_{txx} u \dots$$

1.1.1 Kinds of PDE's

First we define a *linear equation* as:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

Where A, B, C, D, E, F, G are all *constants* or *functions* of the independet variables (x and y).

For linear equation the are three basic types:

- *Parabolic*: usually describe heat flow and diffusion processes. They satisfy the property $B^2 - 4AC = 0$.
- *Hyperbolic*: usually describe vibrating systems and wave motion. They satisfy the property $B^2 - 4AC > 0$.
- *Elliptic*: usually describe steady-state phenomena. They satisfy the property $B^2 - 4AC < 0$.

Chapter 2

Diffusion-Type Problems

First lets start by interpreting one of the most simple PDE's, the one dimensional *heath equation.*

2.1 Interpretation of the heat equation

First let's state the problem that we want to solve. Imagine a straight bar of metal. The position in the bar is modeled by the independet variable x and time with t. The function $u(x,t)$ models the distribution of temperature in the bar at the point x and time t. We can see this function in two forms. As a surface in the xt plane such that for every time t_0 we have a function $u(x)$ that represents the distribution of temperatures in the bar. Or we can conceptually think of $u(x,t)$ as a animation of a a function $u(x)$ that changes form as the animation advances.

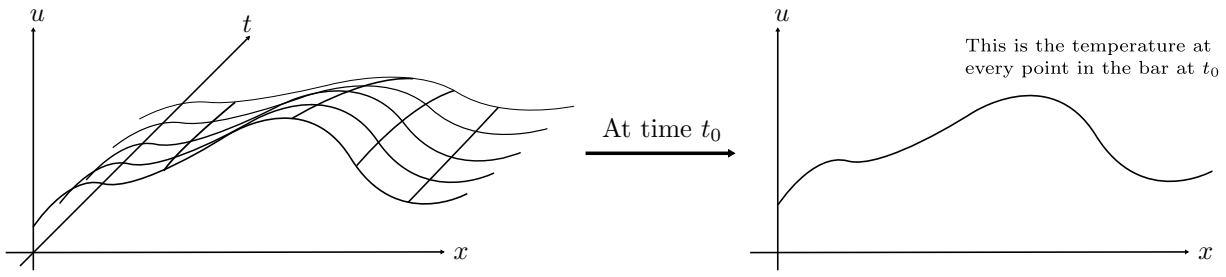

Figure 2.1: Visualization of $u(x,t)$ as a surface in the xt plane. We can think of the function evolving in time as we travel through the t line.

The one dimensional heath equation is a *parabolic* equation of the form:

$$u_t = \alpha^2 u_{xx}; \quad 0 < x < L, \quad 0 < t < \infty$$

which relates the quantites:

- u_t = the rate of change of the temperature of a point as time advances.
- u_{xx} = the *concavity* of the temperature distribution at the point (x,t).

What this equation tries to say is that the temperature $u(x,t)$ will increase or decrease accoding to whether u_{xx} is positive or negative.

2.1.1 Interpretation of u_{xx}

To interpret u_{xx} lets steer to the numerical methods. Imagine that we have a discreet rod of metal (or a discreet number of thermocouples measuring the temperature of a continuos rod) with a temperature distribution

$u(x_n, t)$. We can model second derivatives as a (central) *finite diference*:

$$u_{xx}(x_n, t) = \frac{1}{h^2}[u(x_{n+1}, t) - 2u(x_n, t) + u(x_{n-1}, t)] \tag{2.1.1}$$

We can rearrange that to give:

$$u_{xx}(x_n, t) = \frac{2}{h^2}\left[\frac{u(x_{n+1}, t) + u(x_{n-1}, t)}{2} - u(x_n, t)\right]$$

This lets us study the sign of u_{xx} as the difference between the *average* of the temperature arround the current point and the temperature of the current point.

- If the *average is larger* than the temperature of the current point, then u_{xx} will be positive. So the temperture of the current point will increase with time (as $u_{xx} > 0 \implies u_t > 0$ with the heath equation).
- If the *average is lower* than the temperature of the point then the reverse happens: $u_{xx} < 0$ and the temperature of the point tends to decrease.

If we keep rearranging (2.1.1) we arrive to:

$$\begin{aligned} u_{xx} &= \frac{1}{h^2}\left[u(x_{n+1}, t) - u(x_n, y) + u(x_{n-1}, t) - u(x_n, t)\right] \\ &= \frac{1}{h^2}\left[\Delta u - \nabla u\right] \end{aligned}$$

Where Δ and ∇ are the forward and backward difference operators, respectivly.

We can interpret this as the rate of change of the rate of change i.e. Δu tells us how the temperature changes relative to the next point, and ∇u tells us how the temperature changes relative to the previous point. So the difference between this rates of change will tell us how the rate of change (u_x) changes as we move in the bar from left to right.
If this double rate of change is positive then the function at that point is concave up (like $\cup$) and so the temperature will tend to rise to equalice with it's vecinity. The reverse is also true, if the difference is negative it means that the function is concave down (like $\cap$) so the temperature will tend to decrease, givin it's energy to it's neighbors. The *sign* of u_{xx} tells the direction of the flux of energy i.e. if the point is gaining energy: $u_{xx} > 0$; or if it's losing it: $u_{xx} < 0$.

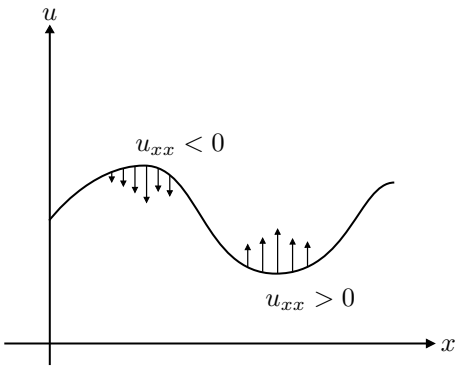

Figure 2.2: This figure illustrates the tendency of the temperature at the points where the function is concave up and down.

2.2 Boundary and Initial conditions

Following with the example of the metal rod, we need to state what happens at the boundary of such rod. Does it lose temperature from its ends? Does it maintain a fixed temperature? This statements are called the *boundary conditions*, they say how our model behaves at its domain baoundary.
All systems must start from some value of time (usually $t = 0$, we can think of it as the time at which we begin to measure) and so we need to declare the state of the system at that point, this is called the *initial condition*.

If we have a boundary condition and a initial condition we have what is called an *initial-boundary-value problem (IBVP)*. In the rod example we can have, let say, a fixed temperature at the ends of T_1 and T_2 respectively and a initial temperature of T_0 in all of rod. So the full problem would be:

$$\text{PDE} \quad u_t = \alpha^2 u_{xx} \quad 0 < x < L \quad 0 < t < \infty$$

$$\text{BCs} \quad \begin{cases} u(0,t) = T_1 \\ u(L,t) = T_2 \end{cases} \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = T_0 \quad 0 \leq x \leq L$$

2.3 Boundary conditions to diffusion type problems

There are two main types of boundary conditions for diffusion type problems: Temperature of the boundaries or flux specified.

2.3.1 Temperature of the boundaries

It's what we did in the rod example, we mantained a constant temperature in the ends of the rod (the boundary). In general the temperature at the boundary can be a function of time and/or space. It can even be the case that we want to find the boundary temperatures that will force the temperature of the object to act in a suitable maner.

2.3.2 Flux specified