Contents

Chapter 1

The Basics

1.1 Antiderivatives

Solving an integral like $x(t) = \int g(t)\,\mathrm{d}t$ is the problem of finding the function $x(t)$ such that its derivative is equal to $g(t)$ i.e. it holds that:

$$x(t) = \int g(t)\,\mathrm{d}t \iff x' = g(t)$$

So, for solving an equation of type $x' = g(t)$ i.e. the derivative of the solution is a function only of the independent variable t, we just have to integrate the function $g(t)$ to obtain the solution.

1.1 Theorem (Fundamental Theorem of Calculus)

If $g(t)$ is a continuos function then:

$$\mathrm{D}_t \int_a^t g(s)\,\mathrm{d}s = g(t) \tag{1.1.1}$$

That means that the function

$$G(s) = \int_a^t g(s)\,\mathrm{d}s$$

is an antiderivative.

Functions defined from integrals

This definition of antiderivative let's us define some special functions as solutions to certain derivatives. Perhaps the most common example is the logarithm funtion. It can be defined as:

$$\ln(t) = \int_1^t \frac{1}{s}\,\mathrm{d}s, \quad t > 0$$

Or, equivalently, as a DE:

$$x' = \frac{1}{t}, \quad x(1) = 0$$

Defines the logarithm function.

Another useful function defined as a solution to a integral is the *error function*. It is defined as:

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t \mathrm{e}^{-s^2}\,\mathrm{d}s$$

This integral is of most use in statistics and diffusion problems.

1.2 Separable equations

I don't really get (nor find) the formal aproach to the derivation of this method without the use (or misuse) of the $\frac{\mathrm{d}x}{\mathrm{d}t}$ notation . So I will use it as a black box.

1.1 Proposition (Solving Separable equations)

If an equation can be written as:

$$x' = f(x)g(t)$$

It can be solved as

$$x' = f(x)g(t) \iff \int \frac{1}{f(x)}\,\mathrm{d}x = \int g(t)\,\mathrm{d}t$$

This gives the *implicit* solution to the DE. If we can then solve for x we'll have the *explicit* solution.

1.3 Linear equations

A DE of the form:

$$x' + p(t)x = q(t)$$

is called a *first order linear equation*. If a first order equation can not be put into this form its then called a first order *nonlinear* equation. Some terminology:

- The function $q(t)$ is called the *forcing* or *source term*
- The form with $q(t) \neq 0$, is called the *normal form*.
- When $q(t) = 0$ then in called the *homogeneous form*

This kind of equation have a nice property: the LHS of the equation can be multiplied by a function $\mu = \mu(t)$ that transforms it into a total derivative i.e.:

$$\mu(t)(x' + p(t)x) = (\mu(t)x)'$$

Such a function is called *integrating factor* and is given by:

$$\mu(t) = \exp\left(\int p(t)\,\mathrm{d}t\right) = \mathrm{e}^{P(t)}; \quad P(t) = \int p(t)\,\mathrm{d}t$$

1.2 Proposition (Solving First Order Linear Equations)

1. Multiply both sides by the integrating factor. With this we obtain a simple DE:

$$x' + p(t)x = q(t) \implies \mu(x' + px) = (\mu x)' \iff (\mathrm{e}^{P(t)}x)' = \mathrm{e}^{P(t)}q(t)$$

2. And this equation defines the integral:

$$\mathrm{e}^{P(t)}x = \int \mathrm{e}^{P(t)}q(t)\,\mathrm{d}t + C$$

3. And the solution is the obtained multiplying both sides by $\mathrm{e}^{-P(t)}$:

$$\boxed{x(t) = \mathrm{e}^{-P(t)} \int \mathrm{e}^{P(t)}q(t)\,\mathrm{d}t + C\mathrm{e}^{-P(t)}}$$

1.2 Theorem (Structure Theorem)

The general solution of the linear DE $x' + p(t)x = q(t)$ can be expressed in the form:

$$x(t) = x_h(t) + x_p(t)$$

Where $x_h(t)$ is the solution to the *homogeneous* equation, and $x_p(t)$ is the solution to one nonhomonegeous equation:

$$x_h(t) = Ce^{P(t)}, \quad x_p(t) = e^{-P(t)} \int q(t)e^{P(t)}\,dt; \quad P(t) = \int p(t)\,dt$$

Both function receive a special name in the applied math world: $x_h(t)$ is called the *transient solution*, as it envolves the initial condition. And $x_p(t)$ is the so called *steady state solution*.

1.4 One-Dimensional Dynamical Systems

Many aplications lead to differential equations that have no *explicit* dependence in time i.e. the change $x'(t)$ only dependends on the state itself $x = x(t)$.

Such equation have the form:

$$x' = f(x)$$

They do not have a *explicit* dependence of t, but remember that x does depend of time.

This type of equations receive the name of *autonomous*. The common way of representing the solution to a DE is to plot x vs t in the tx plane. But here we'll interpret a solution as a *state* x moving along a one-dimensional line i.e. the x axis. The x axis is a state space called the *phase line*.

1.4.1 Autonomous Equations

In animal populations, the number of specimens does not follow a exponential grow. That is because the all the population has to compete for a limited number or resources, so the increase in population decreases as the number in specimens grow i.e. x' decreases as x increases. The simplest assumption is that the relative growth rate ($\frac{x'}{x}$ drecreases linearly as the population increases, and the rate becomes zero at some maximum *carrying capacity* K. This is called the *logistic model* of population growth.

$$\frac{x'}{x} = r(1 - \frac{x}{K}) \quad \text{or} \quad x' = r(1 - \frac{x}{K})x \quad \text{or} \quad x' = rx - \frac{r}{K}x^2$$

And note that it is an autonomous equation. In the last equation the first term is the *growth term*. The second term, which is a negative quadratic in x, is the *competition term*.

We note that this equation has two constant solutions: $x(t) = 0$, $x(t) = K$. Corresponding to extinction and maximun animal capacity. These solutions are found by setting $x' = 0$ as that forces $x =$ constant. These constant solutions to autonomous equations are called *steady-state* or *equilibrium* solutions.

It makes sense (and we can see it in the equations) that if $0 < x < K$ the population tends to increase i.e. $x' > 0$. And if $x > K$ i.e. there is an over-population, then $x' < 0$; the population tends to decrease. All this information can be synthesized in the *phase line*.

1.4.2 Phase Line

We indicate this trends by arrows in the equilibrium points. If the population in greater than 0 and less than K the population tends to K. If it's greater than K then it also thends to K.

The equilibrium point that behaves like $x(t) = 0$ in this equation i.e. they are a constant solution but if we deviate just a little the solution diverges from it, are called points of *unstable equilibrium*. The points that behave like K i.e. the solution tends to convege to them, they are called *stable equilibrium* points.

1.4.3 Analytic Study

We can derive a simple analitycal criterion for equilibrium point and for the stability of an equilibrium point.

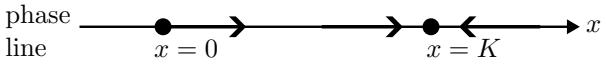

Figure 1.1: Phase line. We can see that if x is different from 0 or K the solution tends to K.

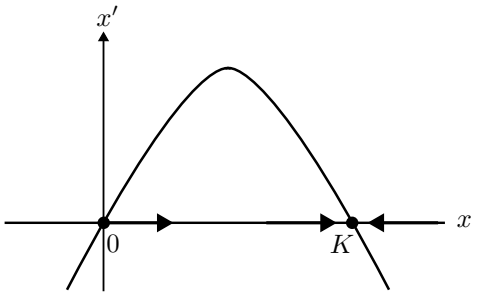

Figure 1.2: This is the essence of the phase line. Here we plot $x' = f(x)$, the derivative of the solution for every posible state of the solution. The interpretation is as follows: if the x' is positive, the value of x will increase (that's the reason of the arrow pointing towards ∞) and if x' is negative, the solution will decrease (that's the reason of the arrow pointing towards $-\infty$). So if we aproach a point from the left and the derivative is positive: the solution is aproaching that point as x increases. And if we then aproach the same point from the right and the derivative is negative: we also are aproaching the point as x increases i.e. we have a stable point, because we always aproach that point.

1.3 Proposition (Equilibrium point)

Let $x' = f(x)$ be an autonomous equation. The equilibrium points of the DE are the ones that make $f(x) = 0$, i.e. x_0 is a equilibrium point if and only if $f(x_0) = 0$.

1.4 Proposition (Criterion for the stability of an equilibrium point)

If we have the autonomous equation $x' = f(x)$ a equilibrium point x_0 is:

- *Stable* $\iff f'(x) < 0$
- *Unstable* $\iff f'(x) > 0$

That's because if $f'(x)$ is negative at a equilibrium point that means that $x' > 0$ from the left and $x' < 0$ from the right. And that causes the solution to converge to that point from however side is aproaching it from. From the top: x decreases to the equilibrium. From the bottom: x increases to the equilibrium. And the reverse is true for an unstable point: x scapes the point from both sides.
(Nota para mi: creo que desde fuera de mi cabeza esto no lo entiende nadie)

1.4.4 Bifurcation

Let's say we have this autonomous equation

$$x' = x(1 - x) - h, \quad h > 0$$

Where h is a positive parameter. Now we'll study it's equilibrium points:

$$f(x) = x(1 - x) - h = 0 \implies -x^2 + x - h = 0 \iff x^* = 0.5 \pm 0.5\sqrt{1 - 4h}$$

Both of the solutions of this quadratic equations are the positions of the equilibrium points in x. As we can see this equation depends on one parameter h. So we can plot the values of x^* vs h.

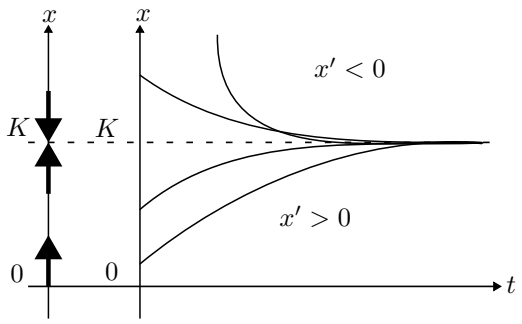

Figure 1.3: Phase line and xt plane plot. We can see from the phase line and plot that x (the population) ends converging to K. We could have just painted the phase line to give a qualitative solution to the DE without even solving it. Very useful in a lot of use cases.

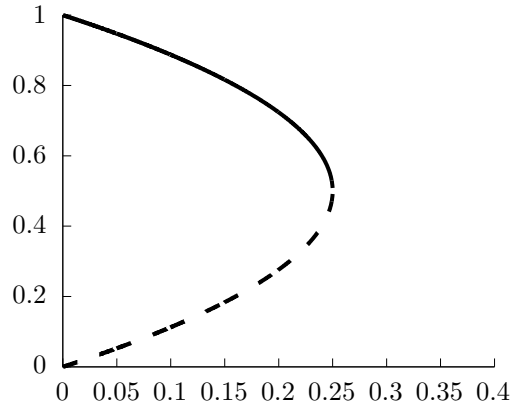

Figure 1.4: Position of the equilibrium points x^* vs the parameter h

In the plot we see the position of both of the equilibrium points with respect to the value of the parameter h. We can see that as h increases both points get closer. When $h = 0.25$ the points converges together and if $h > 0.25$ there are no equilibrium points! We say that at $h = 0.25$ ocurrs a *bifurcation* and we call the plot of the position of the equilibrium points vs the *bifurcation parameter* the *bifurcation diagram*.

1.5 Existence of solutions

1.3 Theorem (Theorem of *local* existence)

Assume the function $f(t, x)$ and its partial derivative $(\partial_x f)(x, t) = f_x(x, t)$ are continuous in a rectangle $a < t < b$, $c < x < d$. Then, for any value t_0 and x_0 the initial value problem:

$$\begin{cases} x' = f(x, t) \\ x(t_0) = x_0 \end{cases}$$

Has a unique solution valid on some *open* interval $a < \alpha < t < \beta < b$ containing t_0

When the DE is a linear equation there is a nice propertie:

1.5 Proposition (Existance and uniquiness of a linear equation)

Consider the IVP:

$$\begin{cases} x' + p(t)x = q(t) \\ x(t_0) = x_0 \end{cases}$$

If p and q are continuous on any open interval I containing t_0 then there is a *unique* solution to the IVP on the *entire* interval I

Chapter 2

Second-Order Linear Equations

In this chapter we study DE's of the type:

$$ax'' + bx' + c = f(t)$$

This type of equation is of great use in chassical mechanics and electrical circuit, as they serve as prototypes for oscillationg systems, oscillation systems with damping and forced vibrations.

2.1 Equations with constant coeficients

As with first-order ODE we have that:

$$ax'' + bx' + cx = 0$$

is called an *homogeneous linear equation with constant coeficients*. The *homogeneous* part refers to the fact that the RHS is equal to zero.

When we have also that:

$$x(0) = x_0 \quad x'(0) = x_1$$

we are faced with an *initial value problem* (IVP).

2.1.1 Constant coeficients interpretation

Let's interpret the equation:

$$mx'' + \gamma x' + kx = 0 \iff LI'' + RI' + \frac{1}{C}I = 0$$

These are the equations for the damped spring and the LCR circuit. The constant coneficients: m/L,γ/R and $k/\frac{1}{C}$ have a physical interpretation:

- $m/L \longrightarrow$ It's the inertial term: in mechanics is the *mass* and in LCR oscillator is the *inductance*
- $\gamma/R \longrightarrow$ It's the dissipation or energy loss term: is analogous to the resistance or damping.
- $k/\frac{1}{C} \longrightarrow$ It's the energy storage term: is analogous to the (inverse) capacitance and the spring constant.

2.1.2 Solutions

2.1 Theorem (Existence-Uniqueness)

The initial value problem given by:

$$\begin{cases} ax'' + bx' + cx = 0 \\ x(0) = x_0 \\ x'(0) = x_1 \end{cases}$$

has a unique solution that exist on $-\infty < t < \infty$. That means that an *homogeneous* linear IVP has a unique solution for all reals.

But, how do we find such solutions? This type of equations have a solution of the form:

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$

Where $x_1(t)$ and $x_1(t)$ are independt (not multiple of each other) solutions.

We can spect the solution to be of the form $x(t) = \mathrm{e}^{\lambda t}$ because the terms in the equation involve the derivatives of the solution multiplied by a constant. If we substitute $\mathrm{e}^{\lambda t}$ in the general equation we get:

$$a\lambda^2 + b\lambda + c = 0$$

that equation is called the *characteristic equation*, and it's solutions *eigenvalues*. Each value of λ gives a solution $\mathrm{e}^{\lambda t}$ to the equation. As any second-order equation it can have real or complex conjugate solutions, let's see what happends with the complex solutions.

2.1.3 Complex eigenvalues

2.2 Theorem (Complex eigenvalues)

If $x(t) = g(t) + jh(t)$ is a complex-valued solution of a second-order LDE then it's real and imaginary parts ($g(t)$ and $h(t)$ respectively) are real valued solutions.

So, let's study the solution $x(t) = \mathrm{e}^{\lambda t} = \mathrm{e}^{(\alpha + j\beta)t}$, using Euler's formula:

$$\mathrm{e}^{(\alpha+j\beta)t} = \mathrm{e}^{\alpha t}\mathrm{e}^{j\beta t} = \mathrm{e}^{\alpha t}(\cos(\beta t) + j\sin(\beta t)) = \mathrm{e}^{\alpha t}\cos(\beta t) + j\mathrm{e}^{\alpha t}\sin(\beta t)$$

So, using the previous theorem we can conclude that:

$$x_1(t) = \mathrm{e}^{\alpha t}\cos(\beta t) \quad x_2(t) = \mathrm{e}^{\alpha t}\sin(\beta t)$$

are both *real-valued* solutions to the equation. The other comples root can be proven to give the same solutions (only difering in a sign, but they are the same *independent* solutions).

2.1.4 Resonance

The phenomenon of *resonance* is a key element of vibrating systems. It occurst when the *frequency* of a forcing term has the same frequency as the natural oscillations in the system. It gives rise to large amplitude oscillations. In the book they only give examples, boring to write in LaTeX.

Chapter 3

Linear Systems

Let's start stating the any second-order linear differential equation is equivalent to a two unknow function first-order linear system, i.e.:

$$ax'' + bx' + cx = 0 \iff \begin{cases} y = x' \\ ay' + by + cx = 0 \end{cases}$$

The general form of a linear system is:

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$$

3.1 Insight from the equations

As for the first-order linear equations, we can get some visual insight by inspection/ploting without even solving them. Similarly to the slope fields, we can plot the so called *vector field* of the equation just by ploting the function:

$$\vec{F}(x, y) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

This plot gives the tangent vector of the solution in each point of the xy plane. There are two special lines where the tangent vector is either vertial or horizontal, they are called the *nullclines*. The *x-nullcline* is the line $ax + by = 0$ i.e. where $x' = 0$ so the vector field is vertical. IDEM with the *y-nullcline*. Note: *null-cline* comes from *in-cline* but *null* because there zero (*null*) incline in one component? Maybe.

Now let's assume that we know the solution to the linear system i.e. we have the equations $x(t)$ and $y(t)$. We can plot the solution in two ways: as a time series or as a parametric plot. The former is the simplest and can give information about periocity, they tell us how both the states vary with time. But the latter is the most insteresting because we can think of the solutions as a parametric curve in the xy plane i.e. we plot $\mathbf{x}(t) = (x(t), y(t))$ and we get an *orbit*. The evolution in time of the orbit can be illustrated by also plotting the vector field for that solution.

In some ocasions we could alse plot the solution as a *one function* i.e. if we can eliminate the time parameter we can plot the obtained equation as a normal function $y(x)$.

3.2 The eigenvalue problem

We can see that a linear system can be expressed in matrix form as:

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \iff \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \iff A\mathbf{x} = \mathbf{x}'$$

As the derivative of the solution involves multiples of the solution function, we can attemp to find one of the form:

$$x(t) = \mathbf{v}\mathrm{e}^{\lambda t}$$

Where $\mathbf{v}$ and λ are constant. Substituting this and its derivative $x'(t) = \lambda\mathbf{v}\mathrm{e}^{\lambda t}$ in the equation in matrix form gives:

$$A\mathbf{x} = \mathbf{x}' \implies A\mathbf{v}\mathrm{e}^{\lambda t} = \lambda\mathbf{v}\mathrm{e}^{\lambda t} \iff A\mathbf{v} = \lambda\mathbf{v}$$

And this is a classic *eigenvalue problem*. And if we can find the *eigenpair* $(\mathbf{v}, \lambda)$ we have the solution to the linear system as $\mathbf{x} = \mathbf{v}\mathrm{e}^{\lambda t}$. We have transformed the problem of solving a differential equation to one of solving an algebraic problem.

Chapter 4

Numerical methods

4.1 Euler Method

Here I will show three derivations of the *Euler Method*: geometric, Taylor series, Fundamental Theorem of calculus.

In every derivation we'll be trying to solve numericaly the following initial value problem:

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

And assume that it has a solution in a interval $[t_0, T]$. We then partition such interval in N parts of length h to give $N + 1$ time instants such that: $t_{n+1} = t_n + h$ or $t_n = t_0 + nh$.

4.1.1 Geometric derivation

With the geometrical derivation we start by noting that for t_0 we can compute the value of the derivative $x'(t_0)$ as $x'(t_0) = f(t_0, x_0)$. This gives us the slope of the solution at that point i.e. the first order approximation of the solution (by a straight line). If we then assume that for a little deviation in t, say $t_0 + h$, the solution stays in the line with the slope given by the derivative, we can compute the value of the solution there as:

$$x(t_0 + h) = x(t_0) + hx'(t_0) \iff x(t_0 + h) = x(t_0) + f(t_0, x(t_0))h$$

Because with these assumptions $hx'(t_0)$ gives how much does the solution increase from t_0 to $t_0 + h$. If we then repeat this process for every $t_n \in [t_0, T]$ we have numerically computed the solution to the DE with the *Euler method*.

We have, generally:

$$x(t_{n+1}) = x(t_n) + f(t_n, x(t_n)h) + O(h^2)$$

The second order error term will be explained in next section.

4.1.2 Taylor series derivation

We can arrive to the same method by the Taylor series expansion of the solution centered at t_n truncated at the second order term. This is the same as approximating the solution as a straight line that passes though t_n and has a slope aqual to the derivative of the solution at that point i.e. $x'(t_n)$. This gives us:

$$x(t) = x(t_n) + x'(t_n)(t - t_n) + O(h^2)$$

With this approximation we can compute the value of $x(t_{n+1})$ as:

$$x(t_{n+1}) = x(t_n) + x'(t_n)h + O(h^2)$$

And solving for $x'(t_n)$ (this is the deffinition of the *forward difference*):

$$x'(t_n) = \frac{x(t_{n+1}) - x(t_n)}{h} + O(h^2)$$

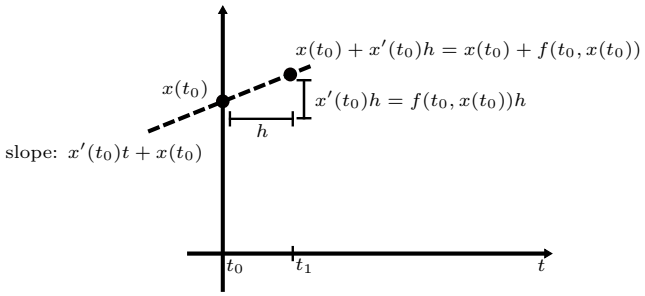

Figure 4.1: Graphical representation of the geometric derivation. We can see the approximation made with a straight line and how we obtain the next value in the solution

If we see the origina IVP we can se that $x'(t_n) = f(t_n, x(t_n))$, then:

$$x'(t_n) = f(t_n, x(t_n)) \iff \frac{x(t_{n+1}) - x(t_n)}{h} + O(h^2) = f(t_n, x(t_n)) \iff x(t_{n+1}) = x(t_n) + hf(t_n, x(t_n)) + O(h^2)$$

And we have arrive at the same equation for the *Euler method*. Note that as we have truncated the series expansion in the second term we have a $O(h^2)$ term representing the *local error* i.e. the error of each interation. This error is of *order* 2 for $h < 1$. That means that if we reduce the step size twofold the error reduces by a factor of 4, and if we decrease h tenfold, the error reduces by $10^2 = 100$.

4.1.3 Fundamental Theorem of Calculus derivation

This is a derivation that enable us to use more exact methods to compute the solution. We start by taking the integral of the DE over an interval $[t_n, t_{n+1}]$:

$$\int_{t_n}^{t_{n+1}} x'(t)\,\mathrm{d}t = \int_{t_n}^{t_{n+1}} f(t, x(t))\,\mathrm{d}t$$

The LHS can be solved using the Fundamental Theorem of Calculus giving us:

$$x(t_{n+1}) - x(t_n) = \int_{t_n}^{t_{n+1}} f(t, x(t))\,\mathrm{d}t$$

To arrive to the same equation as the *Euler method* we need to use the rectangular aproximation to compute the integral. This gives us:

$$x(t_{n+1}) = x(t_n) + hf(t_n, x(t_n))$$

As this is still the original *Euler method* it has an error of $O(h^2)$. But this derivation enables us to use wathever method we want to compute the integral. And as this integral is the only source of error, if we use a method with lesser error we will obtain smaller error in the solution of the DE.

4.1.4 Local an Global truncating error

We have shown that the *local truncating error* is of the order of h^2 i.e. $O(h^2)$ for small $h < 1$. We can see that the number of iterations N is equall to $N = \frac{t_o - T}{h}$ that is of order h. That means that (informally but correctly) after N iterations i.e. in the last point of the interval $[t_0, T]$, the *global truncating error* is of ordern h i.e. $O(h)$.