Lecture 22

Last time

- Fuzzy calculus rules.

Today

- Limiting subdifferential
- Calculus rules

Limiting subdifferential

These fuzzy calculus rules and the lack of closedness of the $\operatorname{grph} \partial(-|\cdot|)$ seem to be suggesting that the issue is that $\operatorname{grph} \partial f$ is not closed (i.e., limits might not be in the set). This motivates the following definition.

Def: (Limiting subdifferential) Let $f: E \to \mathbb{R} \cup \{+\infty\}$ be a function with $x \in \operatorname{dom} f$. Then, $v \in \partial_L f(x)$ if, and only if, there exists a sequence $x_n \in E$ and $v_n \in \partial f(x_n)$ s.t.

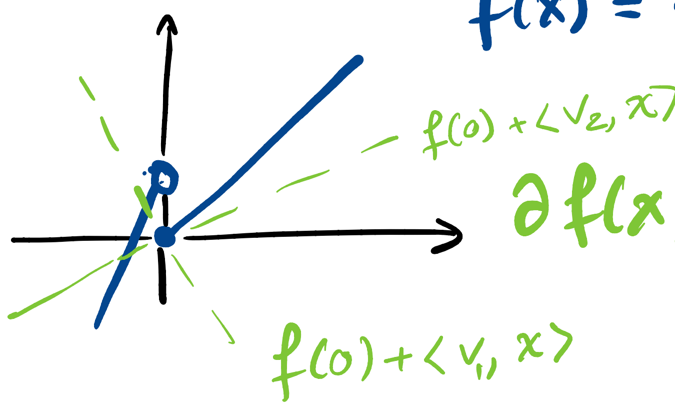
$$(x_n, f(x_n), v_n) \to (x, f(x), v).$$

$\dashv$

Examples

- We include the constraint $f(x_n) \to f(x)$ since otherwise the definition doesn't reflect the local geometry of epi f.

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ 2x+1 & \text{otherwise.} \end{cases}$$



$$\partial f(x) = \begin{cases} 2 & \text{if } x < 0, \\ (-\infty, 1] & \text{if } x = 0, \\ 1 & \text{otherwise.} \end{cases}$$

- Going back to $f(x) = |x|$, we now get

$$\partial f(x) = \begin{cases} -1 & \text{if } x < 0, \\ \{-1, 1\} & \text{if } x = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Notice that in this case

$$\begin{aligned} \partial_L (|\cdot| - |\cdot|)(0) &= \{0\} \\ &\subset [-1, 1] + \{-1, 1\} \\ &= \partial_L (|\cdot|)(0) + \partial_L (-|\cdot|)(0). \end{aligned}$$

$\dashv$

Lemma (Properties of $\partial_L f$): Consider a proper convex function $f: E \to \mathbb{R} \cup \{\infty\}$ with $x \in \operatorname{dom} f$. Then, the following hold.

1. The set $\partial_L f(x)$ is a closed set.
2. If a sequence $x_n \in E$ and $v_n \in \partial f_L(x_n)$ satisfies $(x_n, f(x_n), v_n) \to (x, f(x), v)$, then $v \in \partial_L f(x)$.
3. If f is locally Lipschitz at x, then $\partial_L f(x)$ is nonempty and compact. $\dashv$

Calculus rules

Recall that last time we established that for any $v \in \partial(h \circ c)(\bar{x})$ we have the existance of x_n, y_n and w_n such that

$$\|v - \nabla c(x_n) w_n\| \leq \frac{1}{n}$$

and $x_n \to \bar{x}$, $y_n \to c(\bar{x})$, $h(y_n) \to h(c(\bar{x}))$, and $w_n \in \partial h(y_n)$. The main problem that arises taking $n \to \infty$ is that w_n

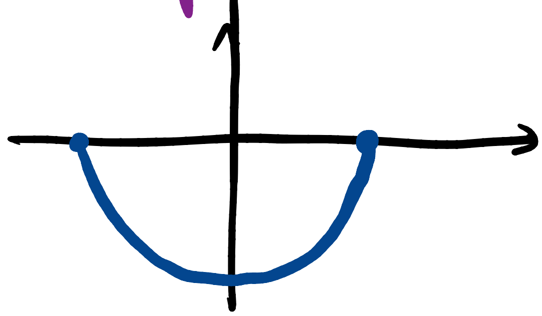
could diverge to infinity. To rule out this possibility we need to require that $\nabla C(\bar{x})^*$ is invertable along the directions in which w_n diverges.

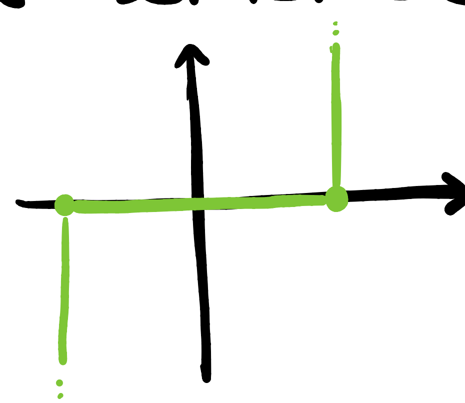
Def: Consider a proper, closed function $f: E \to \mathbb{R}$ and a point $x \in \operatorname{dom} f$. A vector $v \in E$ is a *horizontal subdifferential*, denoted $v \in \partial^\infty f(x)$ if there are sequences $x_i \in E$, $v_i \in \partial f(x_i)$, and $t_i \downarrow 0$ s.t.

$$(x_i, f(x_i), t_i v_i) \longrightarrow (x, f(x), v).$$

$\dashv$

Example: Consider a semicircle





$$f(x) = \begin{cases} -\sqrt{1-x^2} & \text{if } x \in [-1, 1], \\ +\infty & \text{otherwise} \end{cases}$$

$$\partial^\infty f(x) = \begin{cases} (-\infty, 0] & \text{if } x = -1 \\ [0, \infty) & \text{if } x = 1 \\ \{0\} & \text{if } x \in (-1, 1) \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\Rightarrow \quad f'(x) = \frac{x}{\sqrt{1-x^2}} \quad \forall x \in (-1, 1) \quad \Rightarrow$$

$\dashv$

Lemma ∞ (Properties of horizontal subdiff.) Consider a proper closed function $f: E \to \mathbb{R} \cup \{+\infty\}$ with $x \in \operatorname{dom} f$. Then,

1. The set $\partial^\infty f(x)$ is a closed cone.
2. If f is locally Lipschitz around x, then $\partial^\infty f(x) = \{0\}$.
3. If f is convex, $\partial^\infty f(x) = N_{\operatorname{dom} f}(x)$. ⊣

Corollary (Chain rule in limiting form): Consider the function $f(x) = h(c(x))$ with $h: Y \to \mathbb{R} \cup \{\infty\}$ closed and proper, and $c: E \to Y$ a C^1 map around $\bar{x} \in E$. Then,

$$\hat{\partial} f(\bar{x}) \supseteq \nabla c(\bar{x})^* \hat{\partial} h(c(\bar{x})).$$

No limits

Moreover, if we have

$$(T) \qquad \partial^\infty h(\underset{c(\bar{x})}{\bar{y}}) \cap \operatorname{Ker}(\nabla c(\bar{x})^*) = \{0\},$$

then,

$$\partial_L f(\bar{x}) \subseteq \nabla c(\bar{x})^* \partial_L h(c(\bar{x})).$$

Proof: The first inclusion follows by our previous chain rule. For the reverse inclusion, take $v \in \partial f(\bar{x})$ and $n > 0$, then $\exists\, x \in E$ and $y_n \in Y$ s.t.

$$\|x_n - \bar{x}\| \le \tfrac{1}{n}, \quad \|y_n - c(\bar{x})\| \le \tfrac{1}{n}, \quad |h(y_n) - h(c(\bar{x}))| < \tfrac{1}{n},$$

such that

$$v \in \nabla c(x)^* w_n + \tfrac{1}{n}\mathbb{B} \quad \text{for some } w_n \in \partial h(y_n).$$

We claim that the w_n are bounded. Suppose this is not the case, then $\|w_n\| \to \infty$. Then $w_n/\|w_n\|$ as at least one limit point $\bar{w}$. Suppose $\frac{w_n}{\|w_n\|} \to \bar{w}$. Then,

$$\frac{v}{\|w_n\|} \in \nabla c(x_n)^* \frac{w_n}{\|w_n\|} + \frac{1}{n\|w_n\|}\mathbb{B}$$

taking limits as $n \to \infty$ yields

$$0 = \nabla c(\bar{x})^* \bar{w} \quad \text{and} \quad \bar{w} \in \partial^\infty h(\bar{y})$$

and by (T) we conclude $\bar{w} = 0$, this is a contradiction since $\|\bar{w}\| = 1$. This concludes the proof. $\square$

Just as before, the sum rule is an immediate consequence.

Corollary (Limiting sum rule): Consider two proper, closed functions $f_1, f_2 : E \to \mathbb{R} \cup \{+\infty\}$ with $\bar{x} \in \operatorname{dom}(f_1 + f_2)$. Then,

$$\partial(f_1 + f_2)(\bar{x}) \supseteq \partial f_1(\bar{x}) + \partial f_2(\bar{x}).$$

Moreover, if

$$(T_+) \qquad \partial^\infty f_1(x) \cap -\partial^\infty f_2(x) = \{0\}$$

then,

$$\partial_L(f_1 + f_2)(\bar{x}) \subset \partial_L f_1(x) + \partial_L f_2(\bar{x}).$$

Proof: Apply previous corollary with $c(x) = (x, x)$ and $h(x_1, x_2) = f_1(x_1) + f_2(x_2)$.

$\square$

Notice that (T_+) holds trivially if either f_1 or f_2 is locally Lipschitz near $\bar{x}$ (by Lemma ∞).