Lecture 10 — Wed Feb/21/24

Last time

- Symmetric random walks
- Branching process

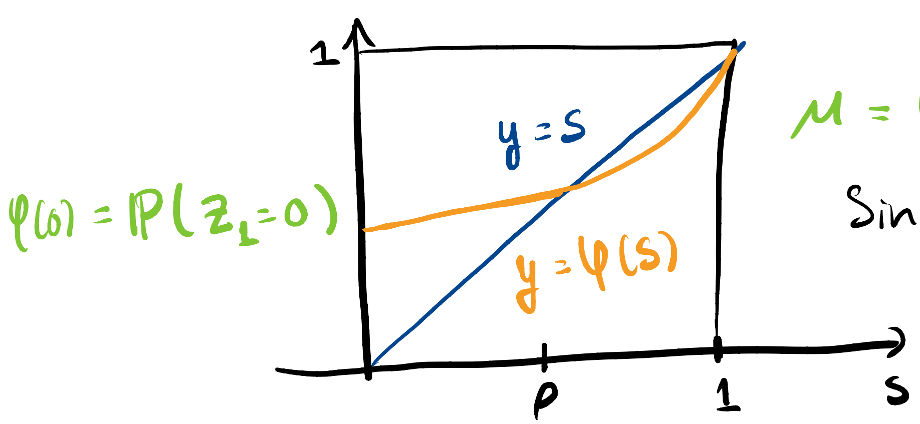
Today

- Branching process continued.
- Convergence in $\mathcal{L}^p$

What happens when $\mu > 1$? Let
$$\varphi(s) = \mathbb{E}(s^{\xi_i}) = \sum_{k \geq 0} p_k s^k \quad \text{with } p_k = \mathbb{P}(\xi_i^m = k).$$
$\uparrow$ generating function

Theorem Suppose $\mu > 1$. If $Z_0 = 1$, then $\lim_{n\to\infty} \mathbb{P}(Z_n = 0) = \rho$ where ρ is the only solution of $\varphi(\rho) = \rho$ in $[0,1)$.

Proof: Note that for $s \in [0,1)$ we have
$$\varphi'(s) = \sum_{k=1} k s^{k-1} p_k \geq 0 \quad \text{and} \quad \varphi''(s) = \sum_{k=2} k(k-1) s^{k-2} p_k > 0$$
Further, $\mu = \mathbb{E}(\xi) = \varphi'(1) = \lim_{s \uparrow 1} \varphi'(s) > 1$



$\mu = \varphi'(1) > 1 \Rightarrow \rho$ exists.

Since φ is strongly convex $\Rightarrow$ there is a unique ρ.

Notice that the generating function $\varphi_n(s) = \mathbb{E}(s^{Z_n})$ can be written as $\varphi_n(s) = \varphi_{n-1}(\varphi(s)) = \underbrace{\varphi \circ \cdots \circ \varphi}_{n}(s)$. To see this

$$\mathbb{E}(s^{Z_{n+1}} | Z_n = k) = \mathbb{E}(s^{\xi_1^{n+1} + \xi_2^{n+1} + \cdots + \xi_k^{n+1}})$$
$$= \prod \mathbb{E}\, s^{\xi_i^{n+1}}$$
$$= \varphi(s)^{Z_n}.$$

Thus, by Tower Law

$$\varphi_{n+1}(s) = \mathbb{E}(s^{Z_{n+1}}) = \mathbb{E}[\varphi(s)^{Z_n}] = \varphi_n(\varphi(s)).$$

In particular,

$$\pi_k = \mathbb{P}(Z_k = 0) = \varphi_n(0) = \varphi(\varphi_{n-1}(0)) = \varphi(\pi_{k-1}).$$

Notice that $\pi_k \leq \rho$. To see this we use induction: $\pi_0 = \varphi(0) \leq \varphi(\rho) = \rho$.

Assuming that $\pi_{k-1} \leq \rho$

$$\Rightarrow \quad \pi_k = \varphi(\pi_k) \leq \varphi(\rho) = \rho.$$

This implies that $\varphi(\pi_k) \geq \pi_k \ \forall k$.

Thus, $\pi_k \uparrow \pi_\infty \leq \rho$. Taking limits

$$\pi_\infty = \varphi(\pi_\infty) \quad \Rightarrow \quad \pi_\infty = \rho.$$

$\square$

Convergence in $\mathcal{L}^p$

When can we have a.s. convergence AND $\mathcal{L}^p$ convergence?

Today we cover the case $1<p<\infty$.

Recall that

Convergence in $\mathcal{L}^p \Rightarrow$ Convergence in $\mathcal{L}^1$.

$\mathcal{L}^1$ is the weaker notion (next class)!

Theorem ($\mathcal{L}^p$ convergence) If X_n is a martingale with $\sup \mathbb{E}|X_n|^p < \infty$ where $p>1$. Then, $X_n \to X_\infty$ a.s. and in $\mathcal{L}^p$. $\dashv$

This is a consecuence of

Theorem ($\mathcal{L}^p$ maximum inequality) If X_n is a submartingale, and $\bar{X}_n = \max_{k \le n} X_k^+$ ← positive part. Then,

$$\mathbb{E}[\bar{X}_n^p] \le \left(\frac{p}{p-1}\right)^p \mathbb{E}[X_n^+]^p. \quad \dashv$$

Proof of $\mathcal{L}^p$ convergence:

Since $(\mathbb{E}|X_n|)^p \le \mathbb{E}|X_n|^p$, we get from Doob's convergence Thm. that

$X_n \to X_\infty$ a.s. Notice that

$$|X_n - X|^p \le (2 \sup_m |X_m|)^p \quad \forall n$$

submartingale

by the $\mathcal{L}^p$ maximum inequality $\sup_m \underbrace{|X_m|}^p$ is integrable, so by DCT $\mathbb{E}|X_n - X|^p \to 0$

$\square$

The beauty of $\mathcal{L}^p$ convergence is that it implies $\mathcal{L}^1$ convergence so

$$|\mathbb{E}X_n - \mathbb{E}X| \le \mathbb{E}|X_n - X_0| \to 0.$$

← convergence in expected value!

Proof of $\mathcal{L}^p$ maximum inequality

We start by stating some facts

1) If N is a stopping time with $\mathbb{P}(N \le K) = 1$ for some $K > 0$, then

$$\mathbb{E}X_0 \le \mathbb{E}X_N \le \mathbb{E}X_K$$

2) For any $\lambda > 0$ and $A = \{\bar{X}_n \ge \lambda\}$,

$$\lambda \mathbb{P}(A) \le \mathbb{E}[X_n^+ \mathbb{1}_A]$$

Proof: Let $N = \inf\{m \mid X_m \ge \lambda \text{ or } m = n\}$.

$$\Rightarrow \lambda \mathbb{P}(A) = \lambda \mathbb{P}(X_N \mathbb{1}_A \ge \lambda)$$

$$\le \mathbb{E}(X_N \mathbb{1}_A) \le \mathbb{E}(X_n \mathbb{1}_A) = \mathbb{E}[X_n^+ \mathbb{1}_A]$$

where the last inequality follows from 1) and the fact that $X_N = X_n$ on A^c. $\square$

3) If X is a positive r.v. then

$$\mathbb{E} X^p = \int_0^\infty \mathbb{P}(X^p \geq t)\, dt$$

$\lambda^p = t$

$p\lambda^{p-1} d\lambda = dt$

$$= \int_0^\infty p \lambda^{p-1} \mathbb{P}(X \geq \lambda)\, d\lambda$$

Now we are ready to prove the result. We truncate with M (will take $M \uparrow \infty$ later):

$$\mathbb{E}\left[(\bar{X}_n \wedge M)^p\right] \overset{3)}{=} \int_0^\infty p\lambda^{p-1} \mathbb{P}(\bar{X}_n \wedge M \geq \lambda)\, d\lambda$$

2)
$$\leq \int_0^\infty p \lambda^{p-2} \int X_n^+ \mathbb{1}_{\{\bar{X}_n \wedge M \geq \lambda\}}\, d\mathbb{P}\, d\lambda$$

Fubini
$$= p \int X_n^+ \int_0^{\bar{X}_n \wedge M} \lambda^{p-2}\, d\lambda\, d\mathbb{P}$$

$$= \frac{p}{p-1} \int X_n^+ (\bar{X}_n \wedge M)^{p-1}\, d\mathbb{P}$$

Holder with $q = \frac{p}{p-1}$

$$\leq \frac{p}{p-1} \left(\mathbb{E}|X_n^+|^p\right)^{1/p} \left(\mathbb{E}|\bar{X}_n \wedge M|^p\right)^{1/q}$$

Noticing that $\frac{\mathbb{E}(\bar{X}_n \wedge M)^p}{(\mathbb{E}(\bar{X}_n \wedge M)^p)^{1/q}} = (\mathbb{E}|X_n \wedge M|^p)^{1/p}$,

we have

$$\|\bar{X}_n \wedge M\|_p \leq \frac{p}{p-1} \|X_n^+\|_p .$$

Then, an application of MCT gives the result.

$\square$

This concludes the proof of the $\mathcal{L}^p$ maximum inequality.

Arguably $p=2$ is the most useful case.

The next result is useful in this case.

Theorem Let X_n be a martingale s.t. $X_n \in \mathcal{L}^2$. Then, $\sup |X_n|^2 < \infty$ iff

$$\sum E(X_n - X_{n-1})^2 < \infty.$$

$\dashv$

Proof: The key observation is

Lemma For any $l < s \le m < n$

$$\langle X_n - X_m, X_m - X_s \rangle = 0$$

Proof: Assume $Y \in \mathcal{F}_m$, by Cauchy-Schwarz $\mathbb{E}\,|(X_n - X_m) Y| < \infty$. Then using the def of a martingale

$$E((X_n - X_m)Y) = \mathbb{E}[(X_n - X_m)Y \mid \mathcal{F}_m] = (\mathbb{E}[X_n \mid \mathcal{F}_m] - X_m)Y$$
$$= 0.$$

$\square$

Then it is easy to see that

$$X_n = X_0 + \sum_{k=1}^{n} X_k - X_{k-1}$$

and so

$$\mathbb{E}X_n^2 = \mathbb{E}X_0^2 + \sum^{n} (X_k - X_{k-1})^2.$$

The conclusion of the Theorem follows $\square$