```
using Plots
using LinearAlgebra
using JuMP
using Gurobi
using MLDatasets
import Statistics
import Random
    solve_LR_coef(X,y,b; LIMIT)
Logistic Regression
Computes parameters b for logistic regression using the Newton-Rhapson Algorithm.
X must have 1's appended as first column. y must have 1,0 encoding.
function solve_LR_coef(X,y,b; LIMIT = 1000)
    n = size(X,1)
    # Probability function
    prob(x, \beta) = 1 / (1 + exp(-(dot(x, \beta))))
    # Gradient of Loss Function
    \nabla l(\beta, X, y, n) = sum(X[i,:] * (y[i] - prob(X[i,:], \beta)) for i = 1:n)
    \# Hl(\beta,X,n) = -sum(X[i,:] * transpose(X[i,:]) * prob(X[i,:], \beta)*(1-prob(X[i,:], \beta)) for
    global counter = 0
    tol = 1.e-6
    # Gradient descent
    while norm(\nabla l(b,X,y,n)) > tol && counter < LIMIT
        W = zeros(n,n)
        P = zeros(n)
        for i = 1:n
            W[i,i] = prob(X[i,:], b)*(1 - prob(X[i,:], b))
            P[i] = prob(X[i,:], b)
        end
        approx = W \setminus (y-P)
        z = X*b + approx
        \# z = X*b + inv(W)*(y-P) \# may not be invertible
        b = inv((X'*W*X)) * X'*W*z
        global counter += 1
    end
    println("Number of iterations: $counter.")
    if counter == LIMIT
        println("Max iterations reached.")
    end
    return b
end
```

```
kernel_ridge(X,y,\lambda,K)
Kernel Ridge Regression
Computes parameters \alpha for the kernel ridge estimator with kernel function K and
hyperparameter \lambda.
function kernel_ridge(X,y,λ,K)
    n = length(y)
    k = zeros(n,n)
    for i = 1:n
         for j = i:n
             k[i,j] = K(X[i,:],X[j,:])
             k[j,i] = k[i,j]
             # println("$(k[i,j])")
         end
    end
    # Solution \alpha to min (norm(y-k*\alpha))^2 + \lambda \alpha'*k*\alpha
    \alpha = (k + \lambda * I) \setminus y
    # \hat{f}(x^*) = sum(K(x^*,X[i,:]) * \alpha[i])
    return \alpha
end
    prox_grad_desc(X,y,β,λ; LIMIT)
Proximal Gradient Descent
Computes parameters \beta given initial guess for least squares regression with \ell-1 penalty.
Has hyperparameter \lambda.
function prox_grad_desc(X,y,β,λ; LIMIT=1000)
    # Soft-thresholding function
    S(y,1) = begin
         if abs(y) <= 1
             return 0
         elseif y > 1
             return y - 1
         else
             return y + 1
         end
    end
    # Solving min 1/2*norm(Y-X\beta,2)^2 + \lambda*norm(\beta,1)
    h = 1 / max(eigvals(X'X)...) # learning rate
    \nabla L(\beta) = -X'^*(y-X^*\beta)
    tol = 1.e-6
    counter = 0
    while norm(\nabla L(\beta)) > tol \&\& counter < LIMIT
         β = S.(β - h*∇L(β), λ*h)
         counter += 1
```

```
println("Iteration $counter, gradient norm (\nabla L(\beta))")
         # end
    end
    println("Number of iterations: $counter.")
    if counter == LIMIT
         println("Max iterations reached.")
    end
    return β
end
    elastic_net(X,y,β,λ,α; LIMIT)
Elastic Net
Computes parameters β given initial guess for least squares regression with elastic net
penalty.
Uses soft-thresholding function update derived in Homework 2. Hyperparameters are \lambda and \alpha.
function elastic_net(X,y,\beta,\lambda,\alpha; LIMIT=1000)
    # Soft-thresholding function
    S(y,1) = begin
         if abs(y) <= 1
             return 0
         elseif y > 1
             return y - 1
         else
             return y + 1
         end
    end
    # Solving min 1/2*norm(Y-X\beta,2)^2 + \lambda*(\alpha*norm(\beta,1) + (1-\alpha)*norm(\beta,2)^2)
    η = 1 / max(eigvals(X'X)...) # learning rate
    \nabla L(\beta) = -X'^*(y-X^*\beta) + 2\lambda^*(1-\alpha)^*\beta
    tol = 1.e-6
    counter = 0
    while norm(\nabla L(\beta)) > tol \&\& counter < LIMIT
         \beta = S.(\beta - \eta*\nabla L(\beta), \lambda*\eta*\alpha)
         counter += 1
                println("Iteration $counter, gradient norm (\nabla L(\beta))")
         # end
    end
    println("Number of iterations: $counter.")
    if counter == LIMIT
         println("Max iterations reached.")
    end
```

```
return β
end
    SVM(X,y,C)
Support Vector Machines
Computes parameters \beta_0, \beta and the margins \xi for linear SVMs. Uses Gurobi to solve the
optimization problem. Has hyperparameter C. y must have -1,1 encoding.
function SVM(X,y,C)
    n,p = size(X)
    model = JuMP.Model(JuMP.optimizer_with_attributes(Gurobi.Optimizer, "MIPGap" => .01,
"TimeLimit" => 180))
    JuMP.@variable(model, β 0)
    JuMP.@variable(model, β[1:p])
    JuMP.@variable(model, \xi[1:n] >= 0)
    JuMP.@objective(model, Min, \beta'*\beta)
    for i = 1:n
        JuMP.@constraint(model, y[i]*(X[i,:]'*\beta + \beta_0) >= 1 - \xi[i])
    end
    JuMP.@constraint(model, sum(\xi[i] for i = 1:n) <= C)
    JuMP.optimize!(model)
    return value(\beta_0), value.(\beta), value.(\xi)
end
    kernel_SVM(X,y,C,K)
Kernel Support Vector Machines
Computers paramters lpha for Kernel SVMs with kernel function K and hyperparameter C. Uses
Gurobi
to solve the optimization problem. y must have -1,1 encoding.
function kernel SVM(X,y,C,K)
    n = size(X,1)
    k = zeros(n,n)
    for i = 1:n
        for j = i:n
            k[i,j] = K(X[i,:],X[j,:])
            k[j,i] = k[i,j]
            # println("$(k[i,j])")
        end
    end
    model = JuMP.Model(JuMP.optimizer_with_attributes(Gurobi.Optimizer, "NonConvex" => 2,
"MIPGap" => .01, "TimeLimit" => 180))
```

```
JuMP.@variable(model, 0 \le \alpha[1:n] \le C)
    JuMP.@objective(model,
                     sum(\alpha[i] \text{ for } i = 1:n) - 1/2 * sum(sum(\alpha[i]*\alpha[j]*y[i]*y[j]*k[i,j] \text{ for } j
= 1:n ) for i = 1:n)
    JuMP.@constraint(model, sum(\alpha[i]*y[i] for i = 1:n) == 0)
    JuMP.optimize!(model)
    return value.(\alpha)
end
    bootstrapper(X,y,B)
Bootstrapping Procedure
Returns B bootstrapped samples of input data X and y. Samples are stored in an array. The
indices used and not used in each sample are also returned for out-of-bag error calculation.
function bootstrapper(X,y,B)
    n,p = size(X)
    b samples = []
    ind used = []
    ind_not_used = []
    X_b = zeros(n,p)
        y b = zeros(n)
        ind_used_b = Set([])
        for i=1:n
            ind = rand(1:n)
            X_b[i,:] = X[ind,:]
            y_b[i] = y[ind]
            push!(ind_used_b, ind)
        end
        ind all = Set(1:n)
        ind_not_used_b = setdiff(ind_all,ind_used_b)
        push!(b_samples, (X_b,y_b))
        push!(ind used, ind used b)
        push!(ind_not_used, ind_not_used_b)
    end
    return b_samples, ind_used, ind_not_used
end
    train_neural_network(X,y,num_hidden_layers,size_hidden_layers;h,activation,problem_type,n
um_classes,num_epochs)
Initiate Neural Network
```

```
Computes weight matrices W and biases b and prints the training error. W and b are
initialized in this
function. Activation options are 'sigmoid' and 'ReLu'. Problem types are 'classification' and
'regression'.
Other parameters settings are number of hidden layers, number of neurons per hidden layer,
number of epochs,
number of classes (1 if regression), and learning rate h.
function train_neural_network(X,y,num_hidden_layers,size_hidden_layers;h=0.1,
activation="sigmoid",problem type="classification",num classes=1,num epochs=1)
    # Create activation functions
    if activation == "sigmoid"
        \sigma = (z) \rightarrow 1 / (1+exp(-z))
        D\sigma = (z) \rightarrow \sigma(z)^2 * \exp(-z)
                                         # gradient with respect to z
        println("Sigmoid used.")
    elseif activation == "ReLU"
        \sigma = (z) \rightarrow \max(z,0)
        D\sigma = (z) \rightarrow begin
            if z >= 0.0
                 return 1.0
            else
                 return 0.0
            end
        end
        println("ReLU used.")
    end
    # Create appropriate loss function
    if problem_type == "classification"
        y indicator = []
        classes = vcat(collect(1:9), 0)
        for i = 1:length(y)
            push!(y_indicator, map(k -> y[i] == k, classes))
        end
        L = (y_indicator,a_L_prob) -> -sum(y_indicator[k] * log(a_L_prob[k]) for
k=1:num classes)
        DL = (y_indicator,a_L_prob) -> [-y_indicator[k] / a_L_prob[k] for k=1:num_classes]
        # These are gonna return vectors
        f = (z) \rightarrow [exp(z[j]) / sum(exp(z[k]) for k=1:num classes) for j=1:num classes]
        Df = (z) \rightarrow [(sum(exp(z[k]) for k=1:num_classes) - exp(z[j]))*exp(z[j]) /
(sum(exp(z[k]) for k=1:num classes))^2 for j=1:num classes]
    elseif problem_type == "regression"
        L = (y,a_L) \rightarrow norm(y - a_L)^2
        DL = (y,a_L) -> 2*(a_L - y)
        f = (z) \rightarrow z
        Df = (z) \rightarrow 1
    end
    # Total number of layers
```

```
LL = num_hidden_layers + 2
   # Stepsize
   \eta = h
   # To store weighted inputs
   z = map(_ -> zeros(size_hidden_layers), 1:num_hidden_layers)
   pushfirst!(z, zeros(size(X,2)))
   push!(z, zeros(num classes))
   # To store activations
   a = map(_ -> zeros(size_hidden_layers), 1:num_hidden_layers)
   pushfirst!(a, zeros(size(X,2)))
   push!(a, zeros(num classes))
   # To store errors
   δ = map(_ -> zeros(size_hidden_layers), 1:num_hidden_layers)
   pushfirst!(\delta, zeros(size(X,2)))
   push!(δ, zeros(num_classes))
   # Create biases
   b = map(_ -> rand(size_hidden_layers), 1:num_hidden_layers)
   pushfirst!(b, rand(size(X,2)))
   push!(b, rand(num classes))
   # Create weight parameters
   W = [Matrix{Float64}(I(size(X,2)))]  # store identity for sake
   push!(W, rand(size_hidden_layers, size(X,2)) ) # W[2] = weights from layer 1 to layer 2
   for = 1:(num hidden layers-1)
       push!(W, rand(size_hidden_layers, size_hidden_layers))
   push!(W, rand(num_classes, size_hidden_layers))
   for = 1:num epochs
        for i = 1:size(X,1)
            # Begin forward pass
            z[1] = X[i,:] # identical transformation
            a[1] = Vector{Float64}(\sigma.(z[1]))
            for 1 = 2:(LL-1) \# LL \text{ and } LL-1
                z[1] = W[1]*a[1-1] + b[1]
                a[1] = Vector{Float64}(\sigma.(z[1]))
            end
            z[LL] = W[LL]*a[LL-1] + b[LL]
            \# a[LL] = f(z[LL]) \# Regression requires a broadcasting with a . while
classification does not
            # Begin backward pass
            if problem type == "classification"
                # \delta[LL] = DL(y_indicator[i], a[LL]) .* Do.(z[LL])
                a[LL] = f(z[LL])
                \delta[LL] = DL(y_indicator[i], a[LL]) .* Df(z[LL])
            elseif problem_type == "regression"
                # \delta[LL] = DL(y[i], a[LL][1]) .* D\sigma.(z[LL])
                a[LL] = f.(z[LL])
```

```
\delta[LL] = DL(y[i], a[LL][1]) .* Df.(z[LL])
        end
        # Compute weighted errors
        for 1 = (LL-1):-1:2
             \delta[1] = W[1+1]' * \delta[1+1] .* Vector{Float64}(D\sigma.(z[1]))
        end
        # Update weights and biases
        for 1 = 2:LL
            W[1] = W[1] - \eta * \delta[1] * a[1-1]'
            b[1] = b[1] - \eta * \delta[1]
        end
    end
end
# for 1=2:LL
      println("W $1 is:\n$(W[1])")
      println("b_$1 is:\n$(b[1])")
# end
function f_predictor(x)
    z[1] = x  # identical transformation
    a[1] = Vector{Float64}(\sigma.(z[1]))
    for 1 = 2:(LL-1)
        z[1] = W[1]*a[1-1] + b[1]
        a[1] = Vector{Float64}(\sigma.(z[1]))
    end
    z[LL] = W[LL]*a[LL-1] + b[LL]
    #println("$(a[L])")
    if problem_type == "classification"
        a[LL] = f(z[LL])
        y_pred = argmax(a[LL])
        if y_pred == 10
            y_pred = 0
        end
        return y_pred, a[LL]
    elseif problem_type == "regression"
        a[LL] = f.(z[LL])
        return a[LL][1] # if regression, final layer has 1 output
    end
end
if problem_type == "classification"
    # loss = [L(y_indicator[i], f_predictor(X[i,:])) for i = 1:size(X,1)]
    y_pred_vec = zeros(size(X,1))
    a_L_vec = map(k -> zeros(num_classes), 1:size(X,1))
    for i = 1:size(X,1)
```

```
y_pred_vec[i], a_L_vec[i] = f_predictor(X[i,:])
        end
        total loss = sum([L(y indicator[i], a L vec[i]) for i = 1:size(X,1)])
        println("Total loss is $total loss.")
        ave_loss = Statistics.mean([L(y_indicator[i], a_L_vec[i]) for i = 1:size(X,1)])
        println("Average loss $ave_loss out of $(size(X,1)) training images.")
        # Compute misclassication
        misclass = 0
        for i = 1:size(X,1)
            if y_pred_vec[i] != y[i]
                 misclass += 1
            end
        println("$misclass of $(size(X,1)) training images misclassified.")
        return W,b
    elseif problem type == "regression"
        ave loss = Statistics.mean([L(y[i], f predictor(X[i,:])) f or i = 1:size(X,1)])
        println("Average loss $ave_loss out of $(size(X,1)) training images.")
        return W,b
    end
end
.. .. ..
    update_neural_network(X,y,W,b,num_hidden_layers,size_hidden_layers;h,activation,problem_t
ype,num classes,num epochs)
Update Neural Network
Computes weight matrices W and biases b and prints the training error. W and b are passed as
arguments in this
function. The network settings should be the same as those passed when creating W and b.
function
update_neural_network(X,y,W,b,num_hidden_layers,size_hidden_layers;h=0.1,activation="sigmoid"
,problem type="classification",num classes=1,num epochs=1)
    # Create activation functions
    if activation == "sigmoid"
        \sigma = (z) \rightarrow 1 / (1+exp(-z))
        D\sigma = (z) \rightarrow \sigma(z)^2 * \exp(-z)
                                         # gradient with respect to z
        println("Sigmoid used.")
    elseif activation == "ReLU"
        \sigma = (z) \rightarrow \max(z,0)
        D\sigma = (z) \rightarrow begin
            if z >= 0.0
                 return 1.0
            else
                 return 0.0
            end
        end
        println("ReLU used.")
```

```
end
         # Create appropriate loss function
         if problem_type == "classification"
                  y indicator = []
                  classes = vcat(collect(1:9), 0)
                  for i = 1:length(y)
                            push!(y indicator, map(k -> y[i] == k, classes))
                   end
                   L = (y_indicator,a_L_prob) -> -sum(y_indicator[k] * log(a_L_prob[k]) for
k=1:num classes)
                  DL = (y indicator, a L prob) -> [-y indicator[k] / a L prob[k] for k=1:num classes]
                  # These are gonna return vectors
                  f = (z) \rightarrow [exp(z[j]) / sum(exp(z[k]) for k=1:num_classes) for j=1:num_classes]
                   Df = (z) \rightarrow [(sum(exp(z[k]) for k=1:num_classes) - exp(z[j]))*exp(z[j]) / (constant) + (constan
(sum(exp(z[k]) for k=1:num_classes))^2 for j=1:num_classes]
         elseif problem type == "regression"
                  L = (y,a_L) -> norm(y - a_L)^2
                  DL = (y,a_L) -> 2*(a_L - y)
                  f = (z) \rightarrow z
                  Df = (z) \rightarrow 1
         end
         # Total number of layers
         LL = num_hidden_layers + 2
         # Stepsize
         \eta = h
         # To store weighted inputs
         z = map(_ -> zeros(size_hidden_layers), 1:num_hidden_layers)
         pushfirst!(z, zeros(size(X,2)))
         push!(z, zeros(num_classes))
         a = map( -> zeros(size hidden layers), 1:num hidden layers)
         pushfirst!(a, zeros(size(X,2)))
         push!(a, zeros(num classes))
         # To store errors
         δ = map(_ -> zeros(size_hidden_layers), 1:num_hidden_layers)
         pushfirst!(\delta, zeros(size(X,2)))
         push!(δ, zeros(num_classes))
         for _ = 1:num_epochs
                  for i = 1:size(X,1)
                            # Begin forward pass
                            z[1] = X[i,:] # identical transformation
                            a[1] = Vector{Float64}(\sigma.(z[1]))
                            for 1 = 2:(LL-1) # LL and LL-1
                                      z[1] = W[1]*a[1-1] + b[1]
                                      a[1] = Vector{Float64}(\sigma.(z[1]))
```

```
end
        z[LL] = W[LL]*a[LL-1] + b[LL]
        \# a[LL] = f(z[LL]) \# Regression requires a broadcasting with a . while
        # Begin backward pass
        if problem_type == "classification"
             # \delta[LL] = DL(y_indicator[i], a[LL]) .* D\sigma.(z[LL])
             a[LL] = f(z[LL])
             \delta[LL] = DL(y_indicator[i], a[LL]) .* Df(z[LL])
        elseif problem_type == "regression"
             # \delta[LL] = DL(y[i], a[LL][1]) .* D\sigma.(z[LL])
             a[LL] = f.(z[LL])
             \delta[LL] = DL(y[i], a[LL][1]) .* Df.(z[LL])
        end
        # Compute weighted errors
        for 1 = (LL-1):-1:2
             \delta[1] = W[1+1]' * \delta[1+1] .* Vector{Float64}(D\sigma.(z[1]))
        end
        # Update weights and biases
        for 1 = 2:LL
            W[1] = W[1] - \eta * \delta[1] * a[1-1]'
             b[1] = b[1] - \eta * \delta[1]
        end
    end
end
      println("W_$1 is:\n$(W[1])")
      println("b_$1 is:\n$(b[1])")
function f_predictor(x)
    z[1] = x  # identical transformation
    a[1] = Vector{Float64}(\sigma.(z[1]))
    for 1 = 2:(LL-1)
        z[1] = W[1]*a[1-1] + b[1]
        a[1] = Vector{Float64}(\sigma.(z[1]))
    end
    z[LL] = W[LL]*a[LL-1] + b[LL]
    #println("$(a[L])")
    if problem_type == "classification"
        a[LL] = f(z[LL])
        y_pred = argmax(a[LL])
        if y_pred == 10
            y_pred = 0
```

```
return y_pred, a[LL]
        elseif problem type == "regression"
            a[LL] = f.(z[LL])
            return a[LL][1] # if regression, final layer has 1 output
        end
    end
    if problem_type == "classification"
        # loss = [L(y_indicator[i], f_predictor(X[i,:])) for i = 1:size(X,1)]
        y_pred_vec = zeros(size(X,1))
        a_L_vec = map(k -> zeros(num_classes), 1:size(X,1))
        for i = 1:size(X,1)
            y_pred_vec[i], a_L_vec[i] = f_predictor(X[i,:])
        end
        total loss = sum([L(y_indicator[i], a_L_vec[i]) for i = 1:size(X,1)])
        println("Total loss is $total loss.")
        ave loss = Statistics.mean([L(y indicator[i], a L vec[i]) for i = 1:size(X,1)])
        println("Average loss $ave_loss out of $(size(X,1)) training images.")
        # Compute misclassication
        misclass = 0
        for i = 1:size(X,1)
            if y_pred_vec[i] != y[i]
                misclass += 1
            end
        println("$misclass of $(size(X,1)) training images misclassified.")
        return W,b
    elseif problem type == "regression"
        ave_loss = Statistics.mean([L(y[i], f_predictor(X[i,:])) for i = 1:size(X,1)])
        println("Average loss $ave_loss out of $(size(X,1)) training images.")
        return W,b
    end
end
    predict_neural_network(X,W,b,num_hidden_layers,size_hidden_layers;activation,problem_type
,num classes)
Predictions via Trained Neural Network
Computes predictions to input data X given weights W and biases b. Network settings should be
the same
as those which were used to learn W and b.
function
predict neural network(X,W,b,num hidden layers,size hidden layers;activation="sigmoid",proble
m_type="classification",num_classes=1)
   # Create activation functions
```

```
if activation == "sigmoid"
        \sigma = (z) \rightarrow 1 / (1+exp(-z))
        D\sigma = (z) \rightarrow \sigma(z)^2 * \exp(-z)
                                          # gradient with respect to z
        println("Sigmoid used.")
    elseif activation == "ReLU"
        \sigma = (z) \rightarrow \max(z,0)
        D\sigma = (z) \rightarrow begin
            if z >= 0.0
                 return 1.0
             else
                 return 0.0
             end
        end
        println("ReLU used.")
    end
    # Create appropriate loss function
    if problem type == "classification"
        y indicator = []
        classes = vcat(collect(1:9), 0)
        for i = 1:length(y)
             push!(y_indicator, map(k -> y[i] == k, classes))
        end
        L = (y_indicator,a_L_prob) -> -sum(y_indicator[k] * log(a_L_prob[k]) for
k=1:num classes)
        DL = (y_indicator,a_L_prob) -> [-y_indicator[k] / a_L_prob[k] for k=1:num_classes]
        # These are gonna return vectors
        f = (z) \rightarrow [exp(z[j]) / sum(exp(z[k]) for k=1:num_classes) for j=1:num_classes]
        Df = (z) \rightarrow [(sum(exp(z[k]) for k=1:num classes) - exp(z[j]))*exp(z[j]) /
(sum(exp(z[k]) for k=1:num_classes))^2 for j=1:num_classes]
    elseif problem_type == "regression"
        L = (y,a_L) \rightarrow norm(y - a_L)^2
        DL = (y,a_L) -> 2*(a_L - y)
        f = (z) \rightarrow z
        Df = (z) \rightarrow 1
    end
    # Total number of layers
    LL = num hidden layers + 2
    # To store weighted inputs
    z = map(_ -> zeros(size_hidden_layers), 1:num_hidden_layers)
    pushfirst!(z, zeros(size(X,2)))
    push!(z, zeros(num_classes))
    # To store activations
    a = map(_ -> zeros(size_hidden_layers), 1:num_hidden_layers)
    pushfirst!(a, zeros(size(X,2)))
    push!(a, zeros(num_classes))
```

```
println("W_$1 is:\n$(W[1])")
          println("b_$1 is:\n$(b[1])")
    # end
    function f_predictor(x)
        z[1] = x  # identical transformation
        a[1] = Vector{Float64}(\sigma.(z[1]))
        for 1 = 2:(LL-1)
            z[1] = W[1]*a[1-1] + b[1]
            a[1] = Vector{Float64}(\sigma.(z[1]))
        end
        z[LL] = W[LL]*a[LL-1] + b[LL]
        #println("$(a[L])")
        if problem_type == "classification"
            a[LL] = f(z[LL])
            y_pred = argmax(a[LL])
            if y_pred == 10
                y_pred = 0
            end
            return y_pred
        elseif problem_type == "regression"
            a[LL] = f.(z[LL])
            return a[LL][1] # if regression, final layer has 1 output
        end
    end
    predictions = zeros(size(X,1))
    for i = 1:size(X,1)
        predictions[i] = f_predictor(X[i,:])
    end
    return predictions
end
```