The Matrix Game Administrator's Problem

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We consider a bilevel optimization problem that admits a finite-horizon zero-sum game with two players at its lower-level. We first describe the game and, then, present the main problem.

At each round of the game, either player selects an option (a play) from a set of available options, which determine the outcome of the round, i.e., a player wins or it is a draw. The winning of one player equals the loss of the other, as if money is exchanged directly. The winning/loss for each pair of plays is predetermined and known by the players. Therefore, the game can be summarized in a matrix, whose rows and columns identify the plays available to the players, and its entries specify the (monetary) winnings/losses, hence a *matrix game*. It is customary to present the matrix by the earnings of the "row player"; a negative value indicates a loss, equivalently a winning for the "column player".

Given a matrix $A_{m\times n}$ representing the game, a risk-averse row player seeks a *strategy* to curb her loss throughout the game. Let $x \in \mathbb{R}^m$ denote the row player's strategy; she selects play $i \in \{1, \ldots, m\}$ with the probability x_i at each round. Similarly, let $y \in \mathbb{R}^n$ be the column player's strategy, unknown to the row player. Then, the expected payoff to the row player at each round is $x^\top Ay = \sum_{i=1}^m \sum_{j=1}^n a_{ij}x_iy_j$. Patently, with a fixed strategy $\hat{x}$, the row player's expected earning at each round will not be less than $\min_y \hat{x}^\top Ay$. It is easy to show that $\min_y \hat{x}^\top Ay = \min_j \sum_{i=1}^m a_{ij}\hat{x}_i$ [1]. Therefore, an optimal strategy for the row player can be obtained by maximizing such a lower bound (on her earning) over all possible strategies, i.e.,

$$\begin{aligned} \max_x \quad & v \\ \text{s.t.} \quad & v - \sum_{i=1}^m a_{ij}x_i \le 0,\ \forall j \in \{1,\ldots,n\}, \\ & \sum_{i=1}^m x_i = 1, \\ & x_i \ge 0,\ \forall i \in \{1,\ldots,m\}. \end{aligned} \tag{1}$$

The optimal value of linear program (1) is called the *value* of the game. By duality, it can be shown that an optimal strategy for the column player that minimizes her maximum expected loss leads to the same game value. A *fair* game is a game with the value of zero.

Now, consider the following setting: Before the game starts, an administrator specifies the plays available to the players. That is, given an original matrix A, a submatrix of A is chosen first, and the game is played over the submatrix. This translates to determining the rules/scope of the game by the administrator. The administrator receives a commission from one of the players based on the value of the game. Also, there might be a cost associated with eliminating the plays as well as some constraints regarding available plays to the players. The administrator's problem is to determine the scope of the game, i.e., a submatrix of the original matrix game subject to the constraints, such that her earning, i.e., commission minus cost, is maximized. We refer to this problem as the matrix game administrator's (MGA) problem.

MGA admits a bilevel optimization formulation. Without loss of generality, we assume the administrator is receiving commission from the row player. Let $r \in \{0,1\}^m$ and $s \in \{0,1\}^n$ indicate the administrator's (upper-level) decision. That is, $r_i = 1$ ($s_j = 1$) indicates play $i \in \{1,\dots,m\}$ ($j \in \{1,\dots,n\}$) is available to the row (column) player, and $r_i = 0$ ($s_j = 0$) implies it has been eliminated. Let γ be a (fixed) commission rate, and $c_r \in \mathbb{R}^m$ and $c_s \in \mathbb{R}^n$ be the cost vectors associated with eliminating rows and columns of the original matrix, respectively. Then, MGA can be formulated as the following bilevel mixed-integer program:

$$\begin{aligned}
\max_{r,s} \quad & \gamma z - c_r^\top(\mathbf{1} - r) - c_s^\top(\mathbf{1} - s) \\
\text{s.t.} \quad & (r,s) \in \mathcal{R} \times \mathcal{S} \ \cap \ \{0,1\}^{m+n}, \\
& z = \max_x \quad v \\
& \qquad \text{s.t.} \quad v - \sum_{i=1}^m a_{ij} x_i \le M(1 - s_j), \ \forall j \in \{1,\dots,n\}, \\
& \qquad\qquad\quad \sum_{i=1}^m x_i = 1, \\
& \qquad\qquad\quad 0 \le x_i \le r_i, \ \forall i \in \{1,\dots,m\},
\end{aligned} \tag{2}$$

where $\mathbf{1}$ is the vector of all ones, M is a big-enough positive scalar, and $\mathcal{R} \times \mathcal{S}$ is a (simple) polytope characterizing the set of feasible upper-level decisions, e.g., bounds on the cardinality of eliminated rows and/or columns. Note that the cost of eliminating a row or column appears in the objective function only if the corresponding variable is zero; the big-M parameter ensures that the constraint associated with an eliminated column is relaxed, as the column player will never use that play, and the bounds on x variables guarantee that the row player will not use an eliminated play.

The structure of (2) suggests it can be transformed to an ordinary (single-level) mixed-integer program. This result is formally presented through the following proposition.

Proposition 1. *Given an $A_{m\times n} = [a_{ij}]$ matrix game, the corresponding administrator's problem can be formulated as follows:*

$$\begin{aligned}
\max_{r,s} \quad & \gamma z - c_r^\top(\mathbf{1} - r) - c_s^\top(\mathbf{1} - s) \\
\textit{s.t.} \quad & z - \sum_{i=1}^m a_{ij} x_i \le M(1 - s_j), \ \forall j \in \{1,\dots,n\}, \\
& \sum_{i=1}^m x_i = 1, \\
& 0 \le x_i \le r_i, \ \forall i \in \{1,\dots,m\}, \\
& (r,s) \in \mathcal{R} \times \mathcal{S} \ \cap \ \{0,1\}^{m+n}.
\end{aligned} \tag{MGA}$$

Proof. The proof is almost immediate. Let (r^*, s^*, x^*, z^*) be an optimal solution to (MGA), and note that (MGA) is a relaxation of (2); the requirement that (x,z) is an optimal strategy-value pair to the lower-level problem has been relaxed. Therefore, we just need to show that

(x^*, z^*) is an optimal strategy-value pair to the matrix game characterized by (r^*, s^*). Let $I = \{i \mid r_i^* = 1\}$ and $J = \{j \mid s_j^* = 1\}$; by restricting (r, s) to (r^*, s^*), (MGA) reduces to

$$\begin{aligned} \max_{r,s} \quad & \gamma z - \text{const.} \\ \text{s.t.} \quad & z - \sum_{i \in I} a_{ij} x_i \leq 0, \ \forall j \in J, \\ & \sum_{i \in I} x_i = 1, \\ & 0 \leq x_i \leq 1, \ \forall i \in I, \end{aligned}$$

which is equivalent to (1) defined on the submatrix of $A_{m \times n}$ induced by I and J. □

References

[1] V. Chvatal. *Linear programming.* W.H. Freeman, 1983.