Homework 1

DSE 210

Guidelines for Homework Submission

1. Each HW is released on the day of the lecture. Students are expected to submit their solutions for the respective HW until 11:59 PM, the day before the next lecture.

2. Each HW is divided into multiple worksheets based on the concepts discussed in the class. Students are expected to attempt all the worksheets in a given HW.

3. HW has to be done individually. NO group work is allowed.

4. The solutions for the HW can be handwritten or typed. (Note: If the handwriting is illegible or if the pages are not scanned properly then the student shall receive 0 points for the respective question)

5. The solutions have to be uploaded to Gradescope in PDF format only. (Note: Do not forget to map question numbers and the pages containing the respective answers while uploading on Gradescope)

6. Some questions in the HW have the tag "***Programming Question***". Students are expected to implement the solution for the respective question in Python. PDF version of the source code has to be uploaded to Gradescope. (Note: Students should combine the theoretical solutions and Python source codes into one PDF file and then upload it to Gradescope)

DSE 210: Probability and Statistics **Winter 2020**

Worksheet 1 — Sets and counting

1. (a) (0.5 points) Write down any set A of size 5.
   (b) (0.5 points) What is the formal notation for all sequences of three elements from A?
   (c) (0.5 points) How many such sequences are there, exactly?
2. (1.5 points) How many binary sequences of length 500 are there?
3. (1.5 points total, 0.5 points each) A and B are sets with $|A| = 3$ and $|B| = 4$.
   (a) What is the largest size $A \cup B$ could possibly have?
   (b) What is the smallest size $A \cup B$ could possibly have?
   (c) Repeat for $A \cap B$.
4. (1.5 points) A donkey, an ox, a goat, and a tiger need to cross a river. They have a boat that can only hold one animal, so they need to go one at a time. How many different orderings are there?
5. (1.5 points) How many sequences of 5 English characters are there?
6. ***Programming Question:*** (10 points) Write a Python code that calculates and returns the same answer as Question 5. You don't have to list the sequences.
7. (1.5 points) You have 10 good friends, and you want to choose 3 of them to accompany you on a trip. How many groups of three friends can you choose?
8. (1.5 points) You have 10 different beer bottles, and you want to line 5 of them up on your mantelpiece. How many different arrangements can you make?

DSE 210: Probability and Statistics **Winter 2020**

Worksheet 2 — Probability spaces

9. (1.5 points total, 0.3 points each) Give a possible sample space Ω for each of the following experiments.
   (a) An election decides between two candidates A and B.
   (b) A two-sided coin is tossed.
   (c) A student is asked for the month and day-of-week on which her birthday falls.
   (d) A student is chosen at random from a class of ten students.
   (e) You choose the color of your new car's exterior (choices: red, black, silver, green) and interior (choices: black, beige).

10. (1.5 points total, 0.5 points each) In each of the following situations, define the sample space Ω.
   (a) A fair coin is tossed 200 times in a row.
   (b) You count the number of people who enter a department store on a particular Sunday.
   (c) You open up *Hamlet* and pick a word at random.

11. (2 points total) Let A, B, and C be events defined on a particular sample space Ω. Write expressions for the following combinations of events:
   (a) (0.5 points) All three events occur.
   (b) (0.5 points) At least one of the events occurs.
   (c) (1 point) A and B occur, but not C.

12. (2 points total) Consider a sample space $\Omega = \{a, b, c\}$ with probabilities $\Pr(a) = 1/2$ and $\Pr(b) = 1/3$.
   (a) (0.5 points) What is $\Pr(c)$?
   (b) (0.5 points) How many distinct events can be defined on this space?
   (c) (1 point) Find the probabilities of each of these possible events.

13. (1.5 points total, 0.5 points each) A fair coin is tossed three times in succession. Describe in words each of the following events on sample space $\{H, T\}^3$.
   (a) $E_1 = \{HHH, HHT, HTH, HTT\}$
   (b) $E_2 = \{HHH, TTT\}$
   (c) $E_3 = \{HHT, HTH, THH\}$

   What are the probabilities of each of these events?

14. (1.5 points) Let A and B be events defined on a sample space Ω such that $\Pr(A \cap B) = 1/4$, $\Pr(A^c) = 1/3$, and $\Pr(B) = 1/2$. Here $A^c = \Omega \setminus A$ is the event that A *doesn't* happen. What is $\Pr(A \cup B)$?

15. (2 points) A pair of dice are rolled. What is the probability that they show the same value?

16. ***Programming Question:*** (10 points) Write a Python function that simulates rolling a pair of dice in Question 15. Simulate this experiment for $n \geq 1000$ trials and return the fraction that they show the same value. Is this fraction close to the probability you calculated before? (Hint: You can use random.choice([1, 2, 3, 4, 5, 6]) function to simulate a single die roll.)

17. (1.5 points) In Morse code, each letter is formed by a succession of dashes and dots. For instance, the letter S is represented by three dots and the letter O is represented by three dashes. Suppose a child types a sequence of 9 dots/dashes at random (each position is equally likely to be a dot or a dash). What is the probability that it spells out SOS?

18. (1.5 points) A die is loaded in such a way that the probability of each face turning up is proportional to the number of dots on that face (for instance, a six is three times as probable as a two). What is the probability of getting an even number in one throw?

19. (Bonus: 4 points) A certain lottery has the following rules: you buy a ticket, choose 3 different numbers from 1 to 100, and write them on the ticket. The lottery has a box with 100 balls numbered 1 to 100. Three (different) balls are chosen. If any of the balls has one of the numbers you have chosen, you win. What is the probability of winning?

20. (1.5 points) Five people of different heights are lined up against a wall in random order. What is the probability that they just happen to be in increasing order of height (left-to-right)?

21. (1.5 points) Five people get on an elevator that stops at five floors. Assuming that each person has an equal probability of going to any one floor, find the probability that they all get off at different floors.

22. (1.5 points) You are dealt five cards from a standard deck. What is the probability that the first four are aces and the fifth is a king?

23. (2 points) A barrel contains 90 good apples and 10 rotten apples. If ten of the apples are chosen at random, what is the probability that they are all good?

24. (2 points) Four women check their hats at a concert, but when each woman returns after the performance, she gets a hat chosen randomly from those remaining. What is the probability that each woman gets her own hat back?

25. (2 points) Assume that whenever a child is born, it is equally likely to be a girl or boy, independent of any earlier children. What is the probability that a randomly-chosen family with six children has exactly three girls and three boys?

26. (3 points total, 1 point each) Snow White asks three of the seven dwarfs, chosen at random, to accompany her on a trip.

    (a) What is the probability that Dopey is in this group?

    (b) What is the probability that both Dopey and Sneezy are in the group?

    (c) What is the probability that neither Dopey nor Sneezy are in the group?

DSE 210: Probability and Statistics **Winter 2020**

Worksheet 3 — Multiple events, conditioning, and independence

27. (3 points total, 0.6 points each) A coin is tossed three times. What is the probability that there are exactly two heads, given that:
    (a) the first outcome is a head?
    (b) the first outcome is a tail?
    (c) the first two outcomes are both heads?
    (d) the first two outcomes are both tails?
    (e) the first outcome is a head and the third outcome is a tail?

28. (Bonus: 3 points total, 1.5 points each) A student must choose exactly two of the following three electives: art, French, or mathematics. The probability that he chooses art is 5/8, the probability he chooses French is 5/8, and the probability that he chooses both art and French is 1/4.
    (a) What is the probability that he chooses mathematics?
    (b) What is the probability that he chooses either art or French?

29. (2 points) For a bill to come before the president of the United States, it must be passed by both the House of Representatives and the Senate. Assume that, of the bills presented to the two bodies, 60% pass the House, 80% pass the Senate, and 90% pass at least one of the two. Calculate the probability that the next bill presented to the two groups will come before the president.

30. (Bonus: 4 points) In a fierce battle, not less than 70% of the soldiers lost one eye, not less than 75% lost one ear, not less than 80% lost one hand, and not less than 85% lost one leg. What is the minimal possible percentage of those who simultaneously lost one ear, one eye, one hand, and one leg?

31. (2 points) A card is drawn at random from a standard deck. What is the probability that:
    (a) it is a heart, given that it is red?
    (b) it is higher than a ten, given that it is a heart (interpret J, Q, K, A as having numeric value $11, 12, 13, 14$)?
    (c) it is a jack, given that it is higher than a 10?

32. (2 points) If $\Pr(B^c) = 1/4$ and $\Pr(A|B) = 1/2$, what is $\Pr(A \cap B)$?

33. (4 points total, 1 point each) A die is rolled twice. What is the probability that the sum of the two rolls is > 7, given that:
    (a) the first roll is a 4?
    (b) the first roll is a 1?
    (c) the first roll is > 3?
    (d) the first roll is < 5?

34. (4 points total, 2 points each) Two cards are drawn successively from a deck of 52 cards.

    (a) Find the probability that the second card is equal in rank to the first card. (Rank is defined according to the following ordering: $2, 3, \ldots, 10, J, Q, K, A$. The suit is irrelevant.)

    (b) Find the probability that the second card is higher in rank than the first card.

35. (4 points total, 2 each) A particular car manufacturer has three factories F_1, F_2, F_3 making 25%, 35%, and 40%, respectively, of its cars. Of their output, 5%, 4%, and 2%, respectively, are defective. A car is chosen at random from the manufacturer's supply.

    (a) What is the probability that the car is defective?

    (b) Given that it is defective, what is the probability that it came from factory F_1?

36. (3 points) Suppose that there are equal numbers of men and women in the world, and that 5% of men are colorblind whereas only 1% of women are colorblind. A person is chosen at random and found to be colorblind. What is the probability that the person is male?

37. ***Programming Question:*** (10 points) Write a Python function that simulates the experiment above. Simulate this experiment for $n \geq 10000$ people and return the experimental probability P(Male | Colorblind). Check if the simulated answer matches the calculated probability. (Hint: For each person in the simulation, first simulate the gender then simulate the color blindness based on the gender.)

38. (Bonus: 2 points) A doctor assumes that his patients have one of the three diseases d_1, d_2, or d_3, each with probability 1/3. He carries out a test that will be positive with probability 0.8 if the patient has d_1, with probability 0.6 if the patient has d_2, and with probability 0.4 if the patient has d_3.

    (a) What is the probability that the test will be positive?

    (b) Suppose that the outcome of the test is positive. What probabilities should the doctor now assign to the three possible diseases?

39. (1.5 points) One coin in a collection of 65 coins has two heads; the rest of the coins are fair. If a coin, chosen at random from the lot and then tossed, turns up heads six times in a row, what is the probability that it is the two-headed coin?

40. (1.5 points) A scientist discovers a fossil fragment that he believes is either some kind of tiger (with probability 1/3) or mammoth (with probability 2/3). To shed further light on this question, he conducts a test which has the property that for tigers, it will come out positive with probability 5/6 whereas for mammoths it will come out positive with probability just 1/3. Suppose the test comes out negative. What is the probability, given the outcome of the test, that the fossil comes from a tiger?

41. (Bonus: 2 points) Sherlock Holmes finds paw prints at the scene of a murder, and thinks that they are either from a dog, with probability 3/4, or from a small bear, with probability 1/4. He then discovers some unusual scratches on a nearby tree. The probability that a dog would produce these scratches is 1/10, while the probability that a bear would is 3/5. What is the probability, given the presence of scratches, that the animal is a bear?

42. (2 points total, 1 point each) A coin is tossed three times. Consider the following five events:

    - A: Heads on the first toss
    - B: Tails on the second toss
    - C: Heads on the third toss

- D: All three outcomes the same
- E: Exactly one head

(a) Which of the following pairs of events are independent?

(1) A, B
(2) A, D
(3) A, E
(4) D, E

(b) Which of the following triples of events are independent?

(1) A, B, C
(2) A, B, D
(3) C, D, E

43. (Bonus: 4 points total, 1 point each) You randomly shuffle a standard deck and deal two cards. Which of the following pairs of events are independent?

(1) $A = \{\text{first card is a heart}\}, B = \{\text{second card is a heart}\}$
(2) $A = \{\text{first card is a heart}\}, B = \{\text{first card is a 10}\}$
(3) $A = \{\text{first card is a 10}\}, B = \{\text{second card is a 9}\}$
(4) $A = \{\text{first card is a heart}\}, B = \{\text{second card is a 10}\}$

44. (2 points total, 1 point each) A student applies to UCLA and UCSD. He estimates that he has a probability of 0.5 of being accepted at UCLA and a probability of 0.3 of being accepted at UCSD. He further estimates that the probability that he will be accepted by both is 0.2.

(a) What is the probability that he is accepted at UCSD if he is accepted at UCLA?

(b) Is the event "accepted at UCLA" independent of the event "accepted at UCSD"?