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DSE 210: Probability and Statistics Winter 2020

Worksheet 1 — Sets and counting

1. *(a) (0.5 points) Write down any set A of size 5.*

   A = {1, 2, 3, 4, 5}; |A| = 5

   (b) (0.5 points) What is the formal notation for all sequences of three elements from A?

   All sequences of k elements from A is denoted A^k

   For (a) we have |A| = 5 and k = 3, therefore the result for 5^3 = 125 sequences

   (c) (0.5 points) How many such sequences are there, exactly?

   125 sequences

2. *(1.5 points) How many binary sequences of length 500 are there?*

   For a binary sequence C = {0, 1}, there are $|C|^{500} = 2^{500}$ which equals to 3.273391e+150 sequences.

3. *(1.5 points total, 0.5 points each) A and B are sets with |A| = 3 and |B| = 4.*

(a) What is the largest size $A \cup B$ could possibly have?

   The largest size would be when |A| + |B| = 7

(b) What is the smallest size $A \cup B$ could possibly have?

   The smallest size would be when A is inside B, therefore, the smallest size is 4.

(c) Repeat for $A \cap B$.

   The largest size would be the size of |A|, which is 3. And the smallest size would be an empty set of size 0.

4. *(1.5 points) A donkey, an ox, a goat, and a tiger need to cross a river. They have a boat that can only hold one animal, so they need to go one at a time. How many different orderings are there?*

   Use permutations to solve this problem, which gives 4! = 24.

5. *(1.5 points) How many sequences of 5 English characters are there?*

There are 26 letters on the English alphabet, with repetition the sequences for 5 English characters are 26^5 = 11,881,376

6. *Programming Question: (10 points) Write a Python code that calculates and returns the same answer as Question 5. You don't have to list the sequences.*

Code:

```
# import libraries
import itertools
# define list for English characters
A = {'a','b','c','d','e','f','g','h','i','j','k','l','m','n','o','p','q','r','s','t','u','v','w','x','y','z'}
# define sequence length
k = 5
# find all sequences
permute_k = set(itertools.product(A, repeat = k))
```

Output:

len(permute_k) = 11,881,376

7. *(1.5 points) You have 10 good friends, and you want to choose 3 of them to accompany you on a trip. How many groups of three friends can you choose?*

We can use combinations as order does not matter. Have n = 10 choose k = 3:

$$\binom{10}{3} = \frac{10!}{(10-3)!\,3!} = \frac{10*9*8*7!}{7!\,3!} = \frac{10*9*8}{3!} = 120$$

8. *(1.5 points) You have 10 different beer bottles, and you want to line 5 of them up on your mantelpiece. How many different arrangements can you make?*

We have to use permutations as order matters in this problem, therefore, we can have 10*9*8*7*6 = 30, 240 different arrangements.

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Worksheet 2 — Probability spaces

9. *(1.5 points total, 0.3 points each) Give a possible sample space Ω for each of the following experiments.*

   (a) An election decides between two candidates A and B.

   $\Omega = \{A, B\}$

   $|\Omega| = 2$

   (b) A two-sided coin is tossed.

   $\Omega = \{H, T\}$

   $|\Omega| = 2$

   (c) A student is asked for the month and day-of-week on which her birthday falls.

   m = month

   dofw = day-of-week

   $\Omega = \{(m, \text{dofw})\text{: where } m = \{1, 2, 3, \ldots, 12\} \text{ and dofw} = \{1, 2, 3, \ldots, 7\}\}$

   $|\Omega| = 84$

   (d) A student is chosen at random from a class of ten students.

   Ω = {student 1, student 2, student 3,…, student 9, student 10}

   $|\Omega| = 10$

   (e) You choose the color of your new car's exterior (choices: red, black, silver, green) and interior (choices: black, beige).

   Ω = {(red, black), (black, black), (silver, black), (green, black), (red, beige), (black, beige), (silver, beige), (green, beige)}

   $|\Omega| = 8$

10. (1.5 points total, 0.5 points each) In each of the following situations, define the sample space Ω.

(a) A fair coin is tossed 200 times in a row.

$\Omega = \{H, T\}^{200}$

$|\Omega| = 2^{200}$

(b) You count the number of people who enter a department store on a particular Sunday.

Number of people 0, 1, 2, 3, …, k; where k is the max number of people.

$\Omega = \{0, 1, 2, 3, \dots, k\}$

(c) You open up Hamlet and pick a word at random.

Ω = {all words in Hamlet}

11. (2 points total) Let A, B, and C be events defined on a particular sample space Ω. Write expressions for the following combinations of events:

(a) (0.5 points) All three events occur.

We can express this with intersection of all three events as:

$(A \cap B \cap C)$

(b) (0.5 points) At least one of the events occurs.

We can express this with union of all three events as:

$(A \cup B \cup C)$

(c) (1 point) A and B occur, but not C.

We can express this with the intersection of A and B not C as:

$(A \cap B) \setminus C$

12. (2 points total) Consider a sample space $\Omega = \{a, b, c\}$ with probabilities $Pr(a) = 1/2$ and $Pr(b) = 1/3$.

(a) (0.5 points) What is Pr(c)?

Pr(c) = Pr(Ω) – Pr(a) – Pr(b) = 1 – 1/2 – 1/3 = 1/6

(b) (0.5 points) How many distinct events can be defined on this space?

A total of 8 events can be defined:

A1={a,b}; A2={a,c}; A3={b,c}; A4={a}; A5={b}; A6={c}; A7={a,b,c}; A8={0}

(c) (1 point) Find the probabilities of each of these possible events.

Pr(A1) = Pr(a) + Pr(b) = 1/2 + 1/3 = 5/6

Pr(A2) = Pr(a) + Pr(c) = 1/2 + 1/6 = 2/3

Pr(A3) = Pr(b) + Pr(c) = 1/3 + 1/6 = 1/2

Pr(A4) = Pr(a) = 1/2

Pr(A5) = Pr(b) =1/3

Pr(A6) = Pr(c) = 1/6

Pr(A7) = Pr(a) + Pr(b) + Pr(c) = 1/2 + 1/3 + 1/6 = 1

Pr(A8) = Pr(0) = 0

13. (1.5 points total, 0.5 points each) A fair coin is tossed three times in succession. Describe in words each of the following events on sample space $\{H, T\}^3$.

$|\Omega| = 2^3 = 8$

(a) E_1 = {HHH,HHT,HTH,HTT}

All the elements where the first toss is heads.

(b) E_2 ={HHH,TTT}

All the elements where all the tosses are either heads or tails.

(c) E3 = {HHT,HTH,THH}

All the elements where exactly two tosses are heads.

What are the probabilities of each of these events?

$P(E_1) = |E_1| / |\Omega| = 4/8 = 1/2$

$P(E_2) = |E_2| / |\Omega| = 2/8 = 1/4$

$P(E_3) = |E_3| / |\Omega| = 3/8$

14. *(1.5 points) Let A and B be events defined on a sample space Ω such that $Pr(A \cap B) = 1/4$, $Pr(A^C) = 1/3$, and $Pr(B) = 1/2$. Here $A^C = \Omega \setminus A$ is the event that A doesn't happen. What is $Pr(A \cup B)$?*

$Pr(A) = 1 - Pr(A^c) = 1 - 1/3 = 2/3$

$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B) = 2/3 + 1/2 - 1/4 = 11/12$

15. *(2 points) A pair of dice are rolled. What is the probability that they show the same value?*

$\Omega = \{(1, 1), (1, 2), \ldots, (1, 6), (2, 1), (2, 2), \ldots, (2, 6), (3, 1), (3, 2), \ldots, (3, 6), (4, 1), (4, 2), \ldots, (4, 6), (5, 1), (5, 2), \ldots, (5, 6), (6, 1), (6, 2), \ldots, (6, 6)\}$

$A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

$Pr(A) = |A| / |\Omega| = 6/36 = 1/6$

16. *Programming Question: (10 points) Write a Python function that simulates rolling a pair of dice in Question 15. Simulate this experiment for $n \geq 1000$ trials and return the fraction that they show the same value. Is this fraction close to the probability you calculated before? (Hint: You can use random.choice([1, 2, 3, 4, 5, 6]) function to simulate a single die roll.)*

<u>Code</u>

```
# import libraries
import numpy as np

# define simulation function
def two_dice_simulation(n=1000):
    """
    roll a pair of dice to simulate the probability they show the same value
    """
    count = 0
    for _ in range(n):
        dice1 = np.random.choice([1,2,3,4,5,6], p = [1/6]*6)
        dice2 = np.random.choice([1,2,3,4,5,6], p = [1/6]*6)
        if dice1 == dice2:
        count+=1
    print("Probability that two dice when rolled show the same value:\t",count/n)

# call function for 10k simulations
two_dice_simulation(n=10000)
```

<u>Output</u>

Probability that two dice when rolled show the same value: 0.1658

Yes, the output of the simulation for n = 10, 000 shows a very similar probability from the previous calculation of 1/6.

17. *(1.5 points) In Morse code, each letter is formed by a succession of dashes and dots. For instance, the letter S is represented by three dots and the letter O is represented by three dashes. Suppose a child types a sequence of 9 dots/dashes at random (each position is equally likely to be a dot or a dash). What is the probability that it spells out SOS?*

$\Omega = \{ \bullet, - \}^9$; $|\Omega| = 2^9 = 512$

A = {(•, •, •, -, -, -, •, •, •)}; |A| = 1

Pr(A) = 1/512

18. *(1.5 points) A die is loaded in such a way that the probability of each face turning up is proportional to the number of dots on that face (for instance, a six is three times as probable as a two). What is the probability of getting an even number in one throw?*

Ω = {1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6}; |Ω| = 21

Pr(1) = 1/21; Pr(2) = 2/21; Pr(3) = 3/21; Pr(4) = 4/21; Pr(5) = 5/21; Pr(6) = 6/21

Pr(even) = Pr(2) + Pr(4) + Pr(6) = 2/21 + 4/21 + 6/21 = 12/21 = 4/7

19. *(Bonus: 4 points) A certain lottery has the following rules: you buy a ticket, choose 3 different numbers from 1 to 100, and write them on the ticket. The lottery has a box with 100 balls numbered 1 to 100. Three (different) balls are chosen. If any of the balls has one of the numbers you have chosen, you win. What is the probability of winning?*

Ω = {1, 2, 3, 4, …, 100}; |Ω| = 100

Pr(winning) = Pr(ball_1) + Pr(ball_2) + Pr(ball_3) = 3/100 + 3/99 + 3/98 = 0.0909 -> 9.1%

20. *(1.5 points) Five people of different heights are lined up against a wall in random order. What is the probability that they just happen to be in increasing order of height (left-to-right)?*

Ω = {all possible orderings of 5 people}; |Ω| = 5! = 120

A = {possibility to line up in increasing order of height (left-to-right)}; |A| = 1

Pr(A) = |A| / |Ω| = 1/120 = 0.0083 -> 0.83%

21. (1.5 points) Five people get on an elevator that stops at five floors. Assuming that each person has an equal probability of going to any one floor, find the probability that they all get off at different floors.

Pr (each person gets off at different floors) = $5! / 5^5$ = 120 / 3125 = 24 / 625 -> 3.8%

22. (1.5 points) You are dealt five cards from a standard deck. What is the probability that the first four are aces and the fifth is a king?

The probability for the first four are aces is equal to: (4/52) * (3/51) * (2/50) * (1/49)

Then, the probability for the firth to be a king is: 4/48

Therefore, the total probability equals the product of the two:

Pr ({first four are aces and the fifth is a king}) = (4*3*2*1*4)/(52*51*50*49*48) = 1/3248700 = 3.08e-7

23. (2 points) A barrel contains 90 good apples and 10 rotten apples. If ten of the apples are chosen at random, what is the probability that they are all good?

Pr (all ten are good) = (90/100) * (89/99) * (88/98) * (87/97) * (86/96) * (85/95) * (84/94) * (83/93) * (82/92) * (81/91) = 0.33 -> 33%

24. (2 points) Four women check their hats at a concert, but when each woman returns after the per- formance, she gets a hat chosen randomly from those remaining. What is the probability that each woman gets her own hat back?

Ω = {all possible orderings for the 4 hats}; |Ω| = 4! = 24

A = {Each woman gets her own hat back}; |A| = 1

Pr (A) = |A| / |Ω| = 1/24 -> 4.17%

25. (2 points) Assume that whenever a child is born, it is equally likely to be a girl or boy, independent of any earlier children. What is the probability that a randomly-chosen family with six children has exactly three girls and three boys?

$\Omega = \{Boy, Girl\}^6$; $|\Omega| = 2^6 = 64$

A = {ω ∈ Ω: ω has exactly three boys and three girls}; |A| = $\binom{6}{3} = \frac{6!}{(6-3)!\,3!} = 20$

Pr(A) = |A| / |Ω| = 20/64 = 5/16 -> 31.25%

26. (3 points total, 1 point each) Snow White asks three of the seven dwarfs, chosen at random, to accom- pany her on a trip.

Ω = {Sneezy, Sleepy, Happy, Doc, Grumpy, Dopey, Bashful}

A = {Dopey is in the group}

B = {Sneezy is in the group}

(a) What is the probability that Dopey is in this group?

Pr({Dopey is in the group}) = 3/7

(b) What is the probability that both Dopey and Sneezy are in the group?

$\Pr((A \cap B)) = \Pr(A) * \Pr(B|A) = (3/7) * (2/6) = 6/42 = 1/7$

(c) What is the probability that neither Dopey nor Sneezy are in the group?

$\Pr((A \cup B)) = \Pr(A) + \Pr(B) - \Pr((A \cap B)) = (3/7) + (3/7) - (1/7) = 5/7$

The probability that neither are in the group is the complement of the union between A and B, hence:

$\Pr((A \cup B)^c) = 1 - \Pr((A \cup B)) = 1 - (5/7) = 2/7$

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Worksheet 3 — Multiple events, conditioning, and independence

27. *(3 points total, 0.6 points each) A coin is tossed three times. What is the probability that there are exactly two heads, given that:*

Conditional Probability Definition -> $Pr(A \mid B) * Pr(B) = Pr(A \cap B)$

Sample Space:

$\Omega = \{(H, H, H), (T, T, T), (H, H, T), (H, T, H), (T, H, H), (T, T, H), (T, H, T), (H, T, T)\}$

$|\Omega| = 2^3 = 8$

Event of Interest:

$A = \{(H, H, T), (H, T, H), (T, H, H)\}$

$|A| = 3$

A = exactly two heads

$Pr(A) = 3/8 = 0.375$

(a) the first outcome is a head?

$B = \{(H, H, H), (H, H, T), (H, T, H), (H, T, T)\}$

$Pr(A \cap B) = 2/8 = 0.25$; $Pr(B) = 4/8 = 0.5$

$Pr(A \mid B) = 0.25/0.5 = 0.5$ -> 50%

(b) the first outcome is a tail?

$B = \{(T, T, T), (T, H, H), (T, T, H), (T, H, T)\}$

$Pr(A \cap B) = 1/8 = 0.125$; $Pr(B) = 4/8 = 0.5$

$Pr(A \mid B) = 0.125/0.5 = 0.25$ -> 25%

(c) the first two outcomes are both heads?

B = {(H, H, H), (H, H, T)}

Pr (A $\cap$ B) = 1/8 = 0.125; Pr (B) = 2/8 = 0.25

Pr (A | B) = 0.125/0.25 = 0.5 -> 50%

(d) the first two outcomes are both tails?

B = {(T, T, T), (T, T, H)}

Pr (A $\cap$ B) = 0/8 = 0; Pr (B) = 2/8 = 0.25

Pr (A | B) = 0/0.25 = 0 -> 0%

(e) the first outcome is a head and the third outcome is a tail?

B = {(H, H, T), (H, T, T)}

Pr (A $\cap$ B) = 1/8 = 0.125; Pr (B) = 2/8 = 0.25

Pr (A | B) = 0.125/0.25 = 0.5 -> 50%

28. (Bonus: 3 points total, 1.5 points each) A student must choose exactly two of the following three elec- tives: art, French, or mathematics. The probability that he chooses art is 5/8, the probability he chooses French is 5/8, and the probability that he chooses both art and French is 1/4.

(a) What is the probability that he chooses mathematics?

Define the variables as Art: A; French: F; Mathematics: M

Pr (A $\cap$ F) + Pr (A $\cap$ M) = 5/8 (probability of choosing art)

Pr (A $\cap$ F) + Pr (F $\cap$ M) = 5/8 (probability of choosing french)

Pr (A $\cap$ F) = 2/8 (probability of choosing both art and French)

Using the above variables to solve for the probability of choosing mathematics, gives:

Pr (A $\cap$ M) + Pr (F $\cap$ M) = 3/8 + 3/8 = 6/8 = 3/4

(b) What is the probability that he chooses either art or French?

Since the student has to pick 2 out of 3, the probability of picking either one will be 1.

29. *(2 points) For a bill to come before the president of the United States, it must be passed by both the House of Representatives and the Senate. Assume that, of the bills presented to the two bodies, 60% pass the House, 80% pass the Senate, and 90% pass at least one of the two. Calculate the probability that the next bill presented to the two groups will come before the president.*

Need to find the probability intersecting both groups, this equals:

Pr (house ∩ senate) = Pr (house) + Pr (senate) – Pr (house ∪ senate)

= 60% + 80% - 90% = 50%

30. *(Bonus: 4 points) In a fierce battle, not less than 70% of the soldiers lost one eye, not less than 75% lost one ear, not less than 80% lost one hand, and not less than 85% lost one leg. What is the minimal possible percentage of those who simultaneously lost one ear, one eye, one hand, and one leg?*

Need to find the intersection of all four events. Assume there are 100 soldiers, therefore:

$|Eye| = 70$; $|Ear| = 75$; $|Hand| = 80$; $|Leg| = 85$

$|Eye^c| = 30$; $|Ear^c| = 25$; $|Hand^c| = 20$; $|Leg^c| = 15$

$|Eye \cap Ear \cap Hand \cap Leg| >= 100 - |Eye^c| - |Ear^c| - |Hand^c| - |Leg^c|$

>= 10 or 10%

31. *(2 points) A card is drawn at random from a standard deck. What is the probability that:*

(a) it is a heart, given that it is red?

Pr (heart | red) = Pr (heart, red) / Pr (red) = (13/52) / (26/52) = 1/2 = 0.5 -> 50%

(b) it is higher than a ten, given that it is a heart (interpret J,Q,K,A as having numeric value 11, 12, 13, 14)?

Pr (>10 | heart) = Pr (>10, heart) / Pr (heart) = (4/52) / (13/52) = 4/13 = 0.307 -> 30.7%

(c) it is a jack, given that it is higher than a 10?

Pr (jack | >10) = Pr (jack, >10) / Pr (>10) = (4/52) / (16/52) = 4/16 = 1/4 -> 25%

32. (2 points) If $Pr(B^C) = 1/4$ and $Pr(A|B) = 1/2$, what is $Pr(A \cap B)$?

$Pr(B) = 1 - Pr(B^c) = 1 - 1/4 = 3/4$

$Pr(A \cap B) = Pr(A|B) * Pr(B) = (1/2) * (3/4) = 3/8$

33. (4 points total, 1 point each) A die is rolled twice. What is the probability that the sum of the two rolls is > 7, given that:

(a) the first roll is a 4?

Pr (sum two rolls > 7 | first is 4) = Pr (sum two rolls > 7, first is 4) / Pr (first is 4)

= (3/36) / (1/6) = 0.5 -> 50%

(b) the first roll is a 1?

Pr (sum two rolls > 7 | first is 1) = Pr (sum two rolls > 7, first is 1) / Pr (first is 1)

= (0/36) / (1/6) = 0%

(c) the first roll is > 3?

Pr (sum two rolls > 7 | first is > 3) = Pr (sum two rolls > 7, first is > 3) / Pr (first is > 3)

= (12/36) / (3/6) = 2/3 = 66.67%

(d) the first roll is < 5?

Pr (sum two rolls > 7 | first is < 5) = Pr (sum two rolls > 7, first is < 5) / Pr (first is < 5)

= (6/36) / (4/6) = 1/4 = 25%

34. (4 points total, 2 points each) Two cards are drawn successively from a deck of 52 cards.

(a) Find the probability that the second card is equal in rank to the first card. (Rank is defined according to the following ordering: 2, 3, . . . , 10, J, Q, K, A. The suit is irrelevant.)

Pr (second card is equal in rank to first card) = 3/51 = 1/17 *(b) Find the probability that the second card is higher in rank than the first card.*

Pr (two cards have different value) = 1 - Pr (second card is equal in rank to first card)

= 1 – 1/17 = 16/17

Therefore, the probability of having the second card higher in rank is half of the probability of two cards having different value. This equals to 8/17

35. (4 points total, 2 each) A particular car manufacturer has three factories F_1, F_2, F_3 making 25%, 35%, and 40%, respectively, of its cars. Of their output, 5%, 4%, and 2%, respectively, are defective. A car is chosen at random from the manufacturer's supply.

(a) What is the probability that the car is defective?

Need to use the summation rule as these events are disjoint.

Pr (defective) = Pr (defective, F1) + Pr (defective, F2) + Pr (defective, F3)

= Pr (defective | F1) * Pr (F1) + Pr (defective | F1) * Pr (F1) + Pr (defective | F1) * Pr (F1)

= (0.05)*(0.25) + (0.04)*(0.35) + (0.02)*(0.40) = 0.0345 -> 3.45%

(b) Given that it is defective, what is the probability that it came from factory F_1?

Pr (F1 | defective) = Pr (F1, defective) / Pr (defective)

= (0.25)*(0.05) / 0.0345 = 0.3623 -> 36.23%

36. (3 points) Suppose that there are equal numbers of men and women in the world, and that 5% of men are colorblind whereas only 1% of women are colorblind. A person is chosen at random and found to be colorblind. What is the probability that the person is male?

Use summation rule again.

Pr (colorblind) = Pr (colorblind, male) + Pr (colorblind, female)

= Pr (colorblind | male) * Pr (male) + Pr (colorblind | female) * Pr (female)

= (0.05)*(0.5) + (0.01)*(0.5) = 0.03

Pr (male | colorblind) = Pr (male, colorblind) / Pr (colorblind)

= (0.5)*(0.05) / 0.03 = 0.833 -> 83.3%

37. *Programming Question: (10 points) Write a Python function that simulates the experiment above. Simulate this experiment for n ≥ 10000 people and return the experimental probability P(Male | Colorblind). Check if the simulated answer matches the calculated probability. (Hint: For each person in the sim- ulation, first simulate the gender then simulate the color blindness based on the gender.)*

Code

```
# import libraries
import numpy as np

# define simulation function
def colorblind_simulation_male(n=10000):
    """
    Suppose that there are equal numbers of men and women in the world, and that 5%
    of men are colorblind whereas only 1% of women are colorblind.
    """
    # set counters
    count_m, blind_m, count_f, blind_f = 0, 0, 0, 0
    # iterate over every simulation
    for _ in range(n):
        # find gender first
        gender = np.random.choice(['m','f'], p = [0.5,0.5])
        # find if the gender is color blind
        if gender == 'm':
            count_m+=1
            if np.random.choice([True,False], p = [0.05, 0.95]):
                blind_m+=1
        elif gender == 'f':
            count_f+=1
            if np.random.choice([True,False], p = [0.01, 0.99]):
                blind_f+=1
    print("The probability of a colorblind person being male is
    {}".format(blind_m/(blind_m+blind_f)))

# call function for 1M simulations
colorblind_simulation_male(n=1000000)
```

Output

The probability of a colorblind person being male is 0.8318142505488657

38. *(Bonus: 2 points) A doctor assumes that his patients have one of the three diseases d_1, d_2, or d_3, each with probability 1/3. He carries out a test that will be positive with probability 0.8 if the patient has d_1, with probability 0.6 if the patient has d_2, and with probability 0.4 if the patient has d_3.*

    (a) *What is the probability that the test will be positive?*

    Pr (positive) = Pr (positive, d1) + Pr (positive, d2) + Pr (positive, d3)

    = Pr (positive | d1)*Pr (d1) + Pr (positive | d2)*Pr (d2) + Pr (positive | d3)*Pr (d3)

    = (1/3)*(0.8) + (1/3)*(0.6) + (1/3)*(0.4) = 0.6 -> 60%

    (b) *Suppose that the outcome of the test is positive. What probabilities should the doctor now assign to the three possible diseases?*

    Use Bayes' rule to solve this problem:

    Pr (d1 | positive) = Pr (positive | d1)*Pr (d1) / Pr (positive)

    = (1/3)*(0.8) / 0.6 = 0.444 -> 44.4%

    Pr (d2 | positive) = Pr (positive | d2)*Pr (d2) / Pr (positive)

    = (1/3)*(0.6) / 0.6 = 0.333 -> 33.3%

    Pr (d3 | positive) = Pr (positive | d3)*Pr (d3) / Pr (positive)

    = (1/3)*(0.4) / 0.6 = 0.222 -> 22.2%

39. *(1.5 points) One coin in a collection of 65 coins has two heads; the rest of the coins are fair. If a coin, chosen at random from the lot and then tossed, turns up heads six times in a row, what is the probability that it is the two-headed coin?*

    Pr (6H | 2H coin) = 1

    Pr (2H coin) = 1/65

    Pr (6H) = (1)*(1/65) + (1/2^6)*(64/65) = 2/65

    Then, using Bayes' rule this gives:

    Pr (2H coin | 6H) = Pr (6H | 2H coin) * Pr (2H coin) / Pr (6H)

    = (1) * (1/65) / (2/65) = 0.5

40. *(1.5 points) A scientist discovers a fossil fragment that he believes is either some kind of tiger (with probability 1/3) or mammoth (with probability 2/3). To shed further light on this question, he conducts a test which has the property that for tigers, it will come out positive with probability 5/6 whereas for mammoths it will come out positive with probability just 1/3. Suppose the test comes out negative. What is the probability, given the outcome of the test, that the fossil comes from a tiger?*

Define the variables as M for mammoth and T for tiger, and the probabilities as:

Pr (T) = 1/3; Pr (M) = 2/3; Pr (T test is pos) = 5/6; Pr (M test is pos) = 1/3

Pr (T test is neg) = 1 – Pr (T test is pos) = 1 – (5/6) = 1/6

Pr (M test is neg) = 1 – Pr (M test is pos) = 1 – (1/3) = 2/3

Pr (test is neg) = Pr (test is neg | T test is neg) + Pr (test is neg | M test is neg)

= Pr (T) * Pr (T test is neg) + Pr (M) * Pr (M test is neg)

= (1/3) * (1/6) + (2/3) * (2/3) = 1/2

Use Bayes' rule to solve the problem:

Pr (T | test is neg) = Pr (T test is neg | T) * Pr (T) / Pr (test is neg)

= (1/6) * (1/3) / (1/2) = 1/9 = 0.111 -> 11.1%

41. *(Bonus: 2 points) Sherlock Holmes finds paw prints at the scene of a murder, and thinks that they are either from a dog, with probability 3/4, or from a small bear, with probability 1/4. He then discovers some unusual scratches on a nearby tree. The probability that a dog would produce these scratches is 1/10, while the probability that a bear would is 3/5. What is the probability, given the presence of scratches, that the animal is a bear?*

Define probabilities from the problem:

Pr (dog) = 3/4; Pr (bear) = 1/4; Pr (scratches | dog) = 1/10; Pr (scratches | bear) = 3/5

Pr (scratches) = Pr (dog) * Pr (scratches | dog) + Pr (bear) * Pr (scratches | bear)

= (3/4)*(1/10) + (1/4)*(3/5) = 9/40

Use Bayes' to solve the problem:

Pr (bear | scratches) = Pr (scratches | bear) * Pr (bear) / Pr (scratches)

= (3/5) * (1/4) / (9/40) = 0.666 -> 66.6%

42. *(2 points total, 1 point each) A coin is tossed three times. Consider the following five events:*

- *A: Heads on the first toss*
- *B: Tails on the second toss*
- *C: Heads on the third toss*
- *D: All three outcomes the same*
- *E: Exactly one head*

(a) Which of the following pairs of events are independent? Both (1) and (2)

(1) A,B **Independent**: two tosses are done separately

(2) A,D **Independent**: one toss first and three tosses after, no relation with one another

(3) A,E **Dependent**: toss A depends on Toss E condition of exactly one head

(4) D,E **Dependent**: toss E is limited to one outcome and toss D has three

(b) Which of the following triples of events are independent? Only (1)

(1) A,B,C **Independent**: three tosses are done separately

(2) A,B,D **Dependent**: toss D forces A and B to be the same

(3) C,D,E **Dependent**: toss E limits the length of outcomes to one

43. *(Bonus: 4 points total, 1 point each) You randomly shuffle a standard deck and deal two cards. Which of the following pairs of events are independent?*

Check if condition of independence is meet: Pr $(A \cap B)$ = Pr (A) Pr (B)

(1) A = {first card is a heart}, B = {second card is a heart}

Pr $(A \cap B) = (13/52) * (13/51) > (13/52) * (13/52)$, therefore they are **Dependent**

(2) A = {first card is a heart},B = {first card is a 10}

Pr $(A \cap B) = (13/52) * (4/52)$ = Pr (A) Pr (B), therefore they are **Independent**

(3) A = {first card is a 10}, B = {second card is a 9}

Pr $(A \cap B) = (4/52) * (4/51) > (4/52) * (4/52)$, therefore they are **Dependent**

(4) A = {first card is a heart}, B = {second card is a 10}

Pr (A ∩ B) = (12/52)*(4/51) + (1/52)*(3/51) = 1/52 = (13/52) * (1/13)

= Pr (A) Pr (B), therefore they are **Independent**

44. (2 points total, 1 point each) A student applies to UCLA and UCSD. He estimates that he has a probability of 0.5 of being accepted at UCLA and a probability of 0.3 of being accepted at UCSD. He further estimates that the probability that he will be accepted by both is 0.2.

Define the probabilities:

Pr (UCLA) = 0.5; Pr (UCSD) = 0.3; Pr (UCLA ∩ UCSD) = 0.2

Pr (UCLA ∪ UCSD) = Pr (UCLA) + Pr (UCSD) - Pr (UCLA ∩ UCSD)

= 0.5 + 0.3 – 0.2 = 0.6

(a) What is the probability that he is accepted at UCSD if he is accepted at UCLA?

Pr (UCSD | UCLA) = Pr (UCLA ∩ UCSD) / Pr (UCLA) = 0.2 / 0.5 = 0.4 -> 40%

(b) Is the event "accepted at UCLA" independent of the event "accepted at UCSD"?

To test for independence, the following must hold true:

Pr (UCLA ∩ UCSD) = Pr (UCLA) * Pr (UCSD)

0.2 is not equal to (0.5)*(0.3) = 0/15, therefore, they are **not independent**