Introduction on Confluence

International School on Rewriting 2022

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Term Rewriting

Term rewriting is a very simple and general model of computation at the heart of Computer Science which:

1. Is an essential technique in equational logic, rewriting logic and concurrency.
2. Can be directly used to write declarative programs in both:
   - functional programming, and
   - concurrent and distributed programming.
3. Can give formal semantics to convencional programming languages.
4. Is at the heart of various theorem proving and model checking verification techniques.

Deterministic vs. Concurrent Programming Languages

Programs comes in many different languages and styles.

A first useful distinction is deterministic vs. concurrent:

- **deterministic programs**, for each input either yield a single answer or loop; they are usually written in sequential programming languages and run on sequentical computers, but sometimes they can be parallelized;
- **concurrent programs** may yield different answers, or no answer at all, in the sense of being a reactive system constantly interacting with their environment; they usually run simultaneously on different processors.

Imperative vs. Declarative Programming Languages

A second useful distinction is imperative vs. declarative:

- **imperative programs** use a sequence of steps that change the state of the machine to perform a task;
- **declarative programs** give a mathematical axiomatization of a problem.

Declarative programs are based on what do you want to obtain and imperative programs on how you want obtain it.

The Declarative Advantage

For program reasoning and verification purposes, declarative programs have the important advantage of being already a piece of mathematics. Specifically:

- a declarative program P in a language based on given logic is tipically a logical theory in that logic;
- the properties that we want to verify are satisfied by P can be stated in another theory Q; and
- the satisfaction relation that needs to be verified is a semantic implication relation $P \models Q$ stating that any model of P is also a model of Q.

Equational Logic and Rewriting Logic

Term rewriting is at the core of equational and rewriting logic. We can use:

- **equational logic** to axiomatize the semantics of deterministic programs;
- **rewriting logic** to axiomatize the semantics of concurrent programs.

To axiomatize the **properties** satisfied by such programs we will use more expressive logics, such as full first-order logic (for equational logic) or temporal logic (for rewriting logic).

Initiality and Induction

- Equational and rewriting logic theories have **inital models**, which corresponds to its computational intuition.
- **Inductive reasoning principles** are sound principles to infer other properties satisfied by the standard model of a theory.
- The crucial satisfaction relations for declarative program verification, $P \models Q$, should be understood as **inductive** satisfaction relations correspondign to the inital model of P.

Equational Theories

An equational theory is a pair (Σ, E), where:

- Σ, the signature, describes the syntax of the theory.
- E is a set of **equations** between expressions in the syntax of Σ.

Algebras

- A signature Σ is just syntax: provides the symbols for a language; but what is that language talking about? what is its semantics?
- Algebras are the mathematical models in which we interpret the syntax of Σ, giving it concrete meaning.
- For Σ a signature, a Σ-algebra is a pair $\mathcal{A} = (A, \iota_{\mathcal{A}})$, where A is a set, specifying the data elements in the algebra, and $\iota_{\mathcal{A}} = \{f_{\mathcal{A}}\}_{f \in \Sigma}$ is a Σ-indexed set called the **interpretation** function that maps each constant $a \in \Sigma$ to $a_{\mathcal{A}} \in A$ and each n-ary function symbol $f \in \Sigma$ to a function $f_{\Sigma} : A^n \to A$.
- An obvious example of Σ-algebra is the **term algebra** T_{Σ}.

Variables

- We can extend our notion of signature $\Sigma(X)$ adding variables X. Variables are different from constants.
- $\Sigma(X)$ terms with variables in X are the elements of the term algebra $T_{\Sigma(X)}$.
- Given a set of variables X, a **substitution** is a function $\theta : X \to T_{\Sigma(X)}$.
- This substitution is homomorphically extended to terms $\theta : T_{\Sigma(X)} \to T_{\Sigma(X)}$.

Equations

- We can define a Σ-equation as an expression of the form:

$$t = t'$$

  where $t, t' \in T_{\Sigma(X)}$.
- We can **universally quantify** all its variables and get the sentence:

$$(\forall X)t = t'$$

Conditional Equations

- Sometimes equations are conditional.
- We may have a conjunction of **several** equations in the condition, with the general form:

$$t = t' \Leftarrow u_1 = v_1 \wedge \cdots \wedge u_n = v_n$$

  with all $t = t', u_1 = v_1, \ldots u_n = v_n$ are Σ-equations.
- Again we can **universally quantify** all its variables and get the sentence:

$$(\forall X) t = t' \Leftarrow u_1 = v_1 \wedge \cdots \wedge u_n = v_n$$

Term Rewriting

- Usually, equations are given in such a way that they can be efficiently executed by applying them from left to right.
- This process is called **term rewriting**, because we rewrite each lefthand side instance of an equation by its corresponding rigthand side instance, obtaining a special form of **equational deduction**, replacing equals by equals.

Term Rewriting

Given an equations of the form:

$$(\forall X) t = t' \Leftarrow u_1 = v_1 \wedge \cdots \wedge u_n = v_n$$

- Lefthand sides of the rules cannot be variables.
- Rewrite rules are classified according to the distribution of variables among $t, t', u_1, v_1, \ldots, u_n, v_n$ as follows:
  - type 1, if $Var(t') \cup Var(u_1) \cup \cdots Var(v_n) \subseteq Var(t)$;
  - type 2, if $Var(t') \subseteq Var(t)$;
  - type 3, if $Var(t') \subseteq Var(t) \cup Var(u_1) \cup \cdots Var(v_n)$; and
  - type 4, if no restriction is given.
- type 3 rules are called **deterministic** if $Var(u_i) \subseteq Var(t) \bigcup_{j=1}^{i-1} Var(v_j)$

Positions

- Each term can be viewed as a tree where each **position** in the tree can be denoted by a string of natural numbers, indicating the path from the root of the tree.
- Given a term t an a position π, $t|_\pi$ is the subterm of t at position π.
- For example $f(a, b, f(a, b, c(a, b)))_{3,3} = c(a, b)$.
- The root position is identified by ϵ.

Replacements

- Given a term t, the **replacement** of $t|_\pi$ by u in position π is denoted by $t[u]_\pi$.
- t and u must be Σ-terms.

Unification

- Two terms t, t' are unfiable if there is a substitution θ such that $\theta(t) = \theta(t')$.
- θ is a most general unifier if there is no other unifier θ' such that $\theta'\sigma = \theta$.

The Rewrite Relation

Let $T = (\Sigma, E)$ be a theory and E a set of equations.

- We define two binary relations on $T_{\Sigma(X)}$: $\rightarrow_E$ and $\rightarrow_E^*$
- $\rightarrow_E^*$ is the reflexive and transitive closure of $\rightarrow_E$.

The Rewrite Relation in Maude

For $t, t' \in T_{\Sigma(X)}$, we have $t \to_E t'$ iff either:

- there is an equation $u = v \in E$, a position π in t, a substitution $\theta : X \to T_{\Sigma(X)}$ such that $t|_\pi = \theta(u)$ and $t' = t[\theta(v)]_\pi$, or
- there is a conditional equation $u = v \Rightarrow u_1 = v_1 \wedge \cdots \wedge u_n = v_n \in E$, a position π in t, a substitution $\theta : X \to T_{\Sigma(X)}$ such that:
  1. $\theta(u_i) \to_E^* w_i$ and $\theta(v_i) \to_E^* w_i$, $1 \leq i \leq n$, and
  2. $t|_\pi = \theta(u)$ and $t' = t[\theta(v)]_\pi$.

The Satisfaction of the Condition

- Conditions can be interpreted as:
  - Reachability tests: $u_i \to_E^* v_i$.
  - Joinability tests: $u_i \to_E^* w_i$ and $v_i \to_E^* w_i$, also written as $u_i \downarrow v_i$.
  - Convertibility tests: $u_i \to_E^* v_i$ or $v_i \to_E^* u_i$.
- Maude applies joinability tests on conditions.

Operational Semantics and Proof Trees

$(Rf) \quad \frac{}{x \rightarrow^* x}$ $\qquad (Cg)_{f,i} \quad \frac{x_i \rightarrow y_i}{f(x_1,\ldots,x_i,\ldots,x_k) \rightarrow f(x_1,\ldots,y_i,\ldots,x_k)}$
for all $f \in \Sigma$ and $1 \leq i \leq k$

$(Tr) \quad \frac{x \rightarrow y \qquad y \rightarrow^* z}{x \rightarrow^* z}$ $\qquad (Rp)_\alpha \quad \frac{u_1 \downarrow v_1 \quad \cdots \quad u_n \downarrow v_n}{t \rightarrow t'}$
for $\alpha : t = t' \Leftarrow u_1 = v_1, \ldots, u_n = v_n \in E$

Inference rules are schematic, each inference rule $\frac{B_1 \ \cdots \ B_n}{A}$ can be used under any **instance** $\frac{\theta(B_1) \ \cdots \ \theta(B_n)}{\theta(A)}$ of the rule by a substitution θ.

Determinism

- Rule application may obtain different and unrelatable results because term rewriting can be nondeterministic.
- The minimum requirement to make term rewriting deterministic is **confluence**, also called Church-Rosser property.
- We say that the term rewriting is **ground confluence** if the property is satisfied by all the terms without variables.

Confluence on reduction relations (1/2)

1. **Confluence** is the property of reduction relations guaranteeing that whenever s has two different reducts t and t' (i.e., $s \to^* t$ and $s \to^* t'$), both t and t' are joinable, i.e., they have a common reduct u (hence $t \to^* u$ and $t' \to^* u$ holds for some u).
2. Confluence is one of the most important properties of reduction relations: for instance,
   1. it ensures that for all expressions s, at most one irreducible reduct t of s can be obtained;
   2. it ensures that two divergent computations can always join in the future.
3. Thus, the semantics and implementation of rewriting-based languages is less dependent on specific strategies to implement reductions.

1. Local confluence is a property weaker than confluence that can be used to prove confluence in the presence of termination.

$$
\begin{tikzcd}
& t \arrow[dr, dotted, "*"] & \\
s \arrow[ur] \arrow[dr] & & u \\
& t' \arrow[ur, dotted, "*"'] &
\end{tikzcd}
$$

Local confluence

$$
\begin{tikzcd}
& t \arrow[dr, dotted, "*"] & \\
s \arrow[ur, "*"] \arrow[dr, "*"'] & & u \\
& t' \arrow[ur, dotted, "*"'] &
\end{tikzcd}
$$

confluence

Local Confluence in Term Rewriting

If two rules overlap there is a critical pair.

Critical pair

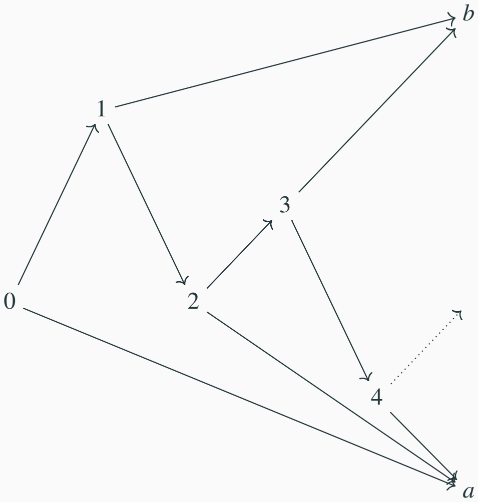
Given $\ell_1 \to r_1$, $\ell_2 \to r_2$ whose variables have been renamed such that $Vars(\ell_1 \to r_1) \cap Vars(\ell_2 \to r_2) = \varnothing$. Let $\pi \in Pos(\ell_1)$ be such that $\ell_1|_\pi$ is not a variable and θ be $mgu(\ell_1|_\pi, \ell_2)$. This determines a critical pair:

$$\{\theta(r_1), \theta(\ell_1)[\theta(r_2)]_\pi\}$$

A relation is locally confluent iff $t \leftarrow s \to t' \Rightarrow t \downarrow t'$.

Local Confluence is Weaker than Confluence

$$\begin{tikzcd} a & 0 \arrow[l] \arrow[r, bend left] & 1 \arrow[l, bend left] \arrow[r] & b \end{tikzcd}$$



Does the absence of critical pairs imply onfluence?

A system with no critical pairs (Huet80)

$$\begin{array}{rcl} c & \to & g(c) \\ f(x,x) & \to & a \\ f(x,g(x)) & \to & b \end{array}$$

Termination in proofs of confluence

A **terminating TRS** is confluence iff all its critical pairs are joinable.

A TRS is **terminating** iff there is no infinite chains of rewrites:

$$t \rightarrow_E t_1 \rightarrow_E t_2 \rightarrow_E \cdots \rightarrow_E t_n \rightarrow_E \cdots$$

Mathematical Semantics

- If the rules $\overrightarrow{E}$ associated to $T = (\Sigma, E)$ are confluent and terminating then there is a unique term called its canonical form such that $t \to_E can_E(t)$ and $can_E(t)$ cannot be rewritten.
- In this situation, there is a set $C_{\Sigma/E} = \{can_E(t) \mid t \in T_\Sigma\}$, called the **canonical term algebra**.
- In this situation, there is an agreement between the **mathematical semantics** and the **operational semantics**.
- If we can also add sufficient completeness, we can extract a constructor subsignature.

Logicality

Decidability

If E is finite and (Σ, E) is convergent, then $=_E$ is decidable.

$$E \vdash (\forall vars(t = t'))t = t' \Leftrightarrow t \downarrow_E t'$$

Avoiding Termination

- Can we obtain a proof of confluence if we don’t know if the system is terminating?
- Check the relation between rules: **Orthogonality**.
- Look for rewrite steps that do not interfere each other.
- Ways of interfere:
  - Overlaps.
  - No-linearity.

Orthogonality

Definition

A TRS is **orthogonal** if the rewrite rules are left-linear and it has no critical pairs.

Orthogonality implies confluence.

Orthogonality and left-linearity

Orthogonality is necessary

$$\begin{aligned} a &\to b \\ f(x,x) &\to a \end{aligned}$$

Of course, no critical pairs is also necessary.

Weak Orthogonality

Definition

A TRS is **weakly orthogonal** if the rewrite rules are left-linear and all critical pairs are trivial.

Weak orthogonality implies confluence.

Parallel Closed Critical Pairs

Definition

A left-linear and parallel closed TRS is confluente.

Definition

$$
\begin{aligned}
f(g(x), b) &\rightarrow f(g(x), b') \\
g(a') &\rightarrow g(a) \\
a &\rightarrow a' \\
b &\rightarrow b'
\end{aligned}
$$

Modularity on Disjoint Unions

- Confluence is a modular property.
- Uniqueness of normal forms is a modular property.

Confluence of Conditional Rewriting

Termination, Operational Termination and Confluence

(Rf) $\quad \dfrac{}{x \to^* x}$ $\qquad (Cg)_{f,i}$ $\quad \dfrac{x_i \to y_i}{f(x_1,\ldots,x_i,\ldots,x_k) \to f(x_1,\ldots,y_i,\ldots,x_k)}$
for all $f \in \Sigma$ and $1 \leq i \leq k$

(Tr) $\quad \dfrac{x \to y \qquad y \to^* z}{x \to^* z}$ $\qquad (Rp)_\alpha$ $\quad \dfrac{u_1 \to^* v_1 \quad \cdots \quad u_n \to^* v_n}{t \to t'}$
for $\alpha : t = t' \Leftarrow u_1 = v_1, \ldots, u_n = v_n \in E$

Inference rules are schematic, each inference rule $\frac{B_1 \ \cdots \ B_n}{A}$ can be used under any **instance** $\frac{\theta(B_1) \ \cdots \ \theta(B_n)}{\theta(A)}$ of the rule by a substitution θ.

Dealing with conditions

Termination and operational termination are different properties.
Operational termination implies termination but not the opposite.

For proving confluence we are interested on termination.

First-Order Theories

A first-order theory $\overline{\mathcal{R}}$ is obtained by instantiating the inference rules with our input CTRS.

Example - GLV21

$$(\forall x)\; x \to^* x$$

$$(\forall x, y, z)\; x \to y \wedge y \to^* z \Rightarrow x \to^* z$$

$$(\forall x, y, z)\; x \to y \Rightarrow f(x, z) \to f(y, z)$$

$$(\forall x, y, z)\; x \to y \Rightarrow f(z, x) \to f(z, y)$$

$$a \to b$$

$$(\forall x)\; f(x, a) \to^* f(b, b) \Rightarrow f(c, x) \to x$$

$$(\forall y)\; f(y, y) \to b$$

Joinability instead of reachability:

$$(\forall x)\; f(x, a) \to^* z \wedge f(b, b) \to^* z \Rightarrow f(c, x) \to x$$

Deducibility

Logical consequence

If $Th_E \vdash \varphi$, then φ is deducible from Th_E or φ is a logical consequence of Th_E.

Confluence and local confluence as first-order formualae

$$(\forall x, y, z, u)\ x \rightarrow^* y \wedge x \rightarrow^* z \Rightarrow y \rightarrow^* u \wedge z \rightarrow^* u$$

$$(\forall x, y, z, u)\ x \rightarrow y \wedge x \rightarrow z \Rightarrow y \rightarrow^* u \wedge z \rightarrow^* u$$

Confluence is not a Logical Consequence of $\overline{\mathcal{R}}$

We can find a model $\mathcal{A}$ of $\overline{\mathcal{R}}$ which is not a model of φ_{WCR}.

Locally confluent but not confluent example

$$(\forall x)\ x \to^* x$$

$$(\forall x, y, z)\ x \to y \wedge y \to^* z \Rightarrow x \to^* z$$

$$b \to a \qquad b \to c$$

$$c \to b \qquad c \to d$$

Locally confluent but not confluent example

$$a^{\mathcal{A}} = 0 \quad b^{\mathcal{A}} = 0 \quad c^{\mathcal{A}} = 0 \quad d^{\mathcal{A}} = 0$$

$$\to^{\mathcal{A}} = \{(1,0),(1,2),(2,1),(2,3),(4,0),(4,3)\}$$

$$(\to^*)^{\mathcal{A}} = \{(1,0),(1,2),(2,1),(2,3),(4,0),(4,3)\}$$

$$\cup\{(0,0),(1,1),(2,2),(3,3),(4,4)\} \cup \{(2,0),(1,3)\}$$

Canonical Model for Conditional Rewriting

- We can consider variables as constants: $C_X = \{c_X \mid x \in X\}$ and $t^{\downarrow}$ is obtained replacing each occurrence of x by c_X.
- Each k-ary symbol is interpreted as $f^{\mathcal{M}_{\mathcal{R}}}(t_1, \ldots, t_k) = f(t_1, \ldots, t_k)$ for all $t_1, \ldots, t_k \in T_{\Sigma(X)}$.
- $\rightarrow^{\mathcal{M}_{\mathcal{R}}} = \{(s^{\downarrow}, t^{\downarrow}) \mid s, t \in T_{\Sigma(X)}, s \rightarrow_{\mathcal{R}} t\}$.
- $(\rightarrow^*)^{\mathcal{M}_{\mathcal{R}}} = \{(s^{\downarrow}, t^{\downarrow}) \mid s, t \in T_{\Sigma(X)}, s \rightarrow^*_{\mathcal{R}} t\}$.

Confluence as satisfiability

Let $\mathcal{R}$ be a CTRS. If $\mathcal{M}_{\mathcal{R}} \vdash \varphi_{CR}$ (resp. $\mathcal{M}_{\mathcal{R}} \vdash \varphi_{WCR}$) holds, then $\mathcal{R}$ is (locally) confluent.

Conditional Critical Pairs

Definition

Let $\mathcal{R}$ be a CTRS. Let $\alpha : \ell \to r \Leftarrow C$ and $\alpha' : \ell' \to r' \Leftarrow C'$ be rules of $\mathcal{R}$ sharing no variable (rename if necessary). Let p be a nonvariable position of ℓ such that $\ell|p$ and ℓ' unify with mgu σ. Then, we call the expression

$$\langle \sigma(\ell[r']_p), \sigma(r') \rangle \Leftarrow \sigma(C), \sigma(C')$$

a conditional critical pair (CCP) of $\mathcal{R}$.

Joinability

We say that $\langle s, t \rangle \Leftarrow C$ is joinable if $\theta(s) \downarrow_{\mathcal{R}} \theta(t)$ for all substitutions θ such that $\theta(C)$ holds.

Logical Characterization of Joinability of CCPs

Definition

Let $\mathcal{R}$ be a CTRS, $\langle s, t \rangle \Leftarrow C$ is joinable if and only if $\mathcal{M}_{\mathcal{R}} \models (\forall \overrightarrow{x})(\exists z) C \Rightarrow s \rightarrow^* z \wedge t \rightarrow^* z$ holds.

Joinability

We say that $\langle s, t \rangle \Leftarrow C$ is joinable if $\theta(s) \downarrow_{\mathcal{R}} \theta(t)$ for all substitutions θ such that $\theta(C)$ holds.

Feasibility

Definition

Let $\bowtie \in \{\rightarrow, \rightarrow^*, \downarrow, \ldots\}$. A condition $s \bowtie t$ is feasible if $T_{\bowtie} \vdash \sigma(s) \bowtie \sigma(t)$ holds; otherwise, it is infeasible. A sequence F is feasible if there exists σ that makes all the conditions feasible.

Proving Conditional Joinability

Joinability

We say that $\langle s, t \rangle \Leftarrow C$ is joinable if $\theta(s) \downarrow_{\mathcal{R}} \theta(t)$ for all substitutions θ such that $\theta(C)$ holds.

Provability

Let $\mathcal{R}$ be a CTRS and $\pi : \langle s, t \rangle \Leftarrow C$ be a critical pair. If $\overline{\mathcal{R}} \vdash (\forall \vec{x})(\exists z) C \Rightarrow s \to^* z \wedge t \to^* z$ holds, then π is joinable.

Provability without considering C

Let $\mathcal{R}$ be a CTRS and $\pi : \langle s, t \rangle \Leftarrow C$ be a critical pair. If $s^{\downarrow} \to^* z, t^{\downarrow} \to^* z$ is $\overline{\mathcal{R}}$-feasible, then π is joinable.

Proving Conditional Joinability

Example

$$a \to b$$
$$f(c, x) \to a \Leftarrow f(x, a) \to^* f(b, b)$$
$$f(y, y) \to b$$

Critical pair

$$\langle a, b \rangle \Leftarrow f(c, a) \to^* f(b, b)$$

Disproving Conditional Joinability

Non-Joinability

Let $\langle s, t\rangle \Leftarrow C$ be a CCP such that $C^{\downarrow}$ is $\overline{\mathcal{R}}$-feasible. If $s^{\downarrow} \to^* z, t^{\downarrow} \to^* z$ is $\overline{\mathcal{R}}$-infeasible, then π is not joinable.

Example

$$f(x,x) \to x \Leftarrow f(x,x) \to^* b$$
$$f(y,y) \to b$$

Critical pair

$$\langle x, b\rangle \Leftarrow f(x,x) \to^* b$$

We can prove that $f(c_x, c_x) \to^* b$ is $\overline{\mathcal{R}}$-feasible, but $c_x \to^* z, b \to^* z$ is infeasible.

Disproving Joinability

Non-Joinability

A critical pair $\langle s, t\rangle$ is joinable if and only if $s^{\downarrow} \to^{*} z, t^{\downarrow} \to^{*} z$ is $\overline{\mathcal{R}}$-feasible.

Example (check)

$$\begin{array}{cc} a(b(x)) \to b(c(x)) & c(b(x)) \to b(c(x)) \\ c(b(x)) \to c(c(x)) & b(b(x)) \to a(c(x)) \\ a(b(x)) \to a(b(x)) & c(c(x)) \to c(b(x)) \end{array}$$
$$a(c(x)) \to c(a(x))$$

We have the following critical pair:

$$\langle a(a(c(x))), b(c(b(x)))\rangle$$

We can prove that $a(a(c(c_x))) \to^{*} z, b(c(b(c_x))) \to^{*} z$ is $\overline{\mathcal{R}}$-infeasible.

Confluence of CTRSs

Non-Joinability

Let $\mathcal{R}$ be a CTRS. If $CCP(\mathcal{R})$ contains a non-joinable CCP, then $\mathcal{R}$ is not (locally) confluent.

Example (check)

$$\begin{array}{cc} a(b(x)) \to b(c(x)) & c(b(x)) \to b(c(x)) \\ c(b(x)) \to c(c(x)) & b(b(x)) \to a(c(x)) \\ a(b(x)) \to a(b(x)) & c(c(x)) \to c(b(x)) \end{array}$$

$$a(c(x)) \to c(a(x))$$

We have the following critical pair:

$$\langle a(a(c(x))), b(c(b(x))) \rangle$$

We can prove that $a(a(c(c_x))) \to^* z, b(c(b(c_x))) \to^* z$ is $\overline{\mathcal{R}}$-infeasible.

Confluence of CTRSs

Non-Joinability

Let $\mathcal{R}$ be a CTRS. If $CCP(\mathcal{R})$ contains a non-joinable CCP, then $\mathcal{R}$ is not (locally) confluent.

Example (check)

$$a \to b$$
$$f(c,x) \to x \Leftarrow f(x,a) \downarrow f(b,b)$$
$$f(y,y) \to b$$

We have the following critical pair:

$$\langle c, b\rangle \Leftarrow f(c,a) \downarrow f(b,b)$$

We can prove that $f(c,a) \downarrow f(b,b)$ is $\overline{\mathcal{R}}$-feasible, but $c \downarrow b$ is $\overline{\mathcal{R}}$-infeasible.

Confluence of CTRSs

Termination and CCPs

A terminating (noetherian) join CTRSs is confluent if all its critical pairs are joinable overlays, where a (conditional) critical pair is an overlay if the critical position is the top position.

Example (check)

$$a \to b$$
$$f(c,x) \to a \Leftarrow f(x,a) \downarrow f(b,b)$$
$$f(y,y) \to b$$

The underlying TRS is terminating, we have the CCP:

$$\langle a, b\rangle \Leftarrow f(c,a) \downarrow f(b,b)$$

$f(c,a) \downarrow f(b,b)$ is $\overline{\mathcal{R}}$-feasible, but $c \downarrow b$ is $\overline{\mathcal{R}}$-infeasible.

Confluence of CTRSs

This doesn't work for oriented CTRSs:

Example

$$a \to b$$
$$f(x) \to c \Leftarrow x \to^* a$$

The underlying TRS is terminating and we have no CCPs, but $f(b) \leftarrow f(a) \to c$.

Normal CTRSs

Definition

Normal CTRSs are CTRSs where terms t in conditions $s \rightarrow^* t$ of the conditional part of rules are ground, irreducible terms.

Confluence

A terminating normal CTRS is confluent if all its critical pairs are joinable overlays.

Normal CTRSs and Termination

Example

$$
\begin{aligned}
c &\to b \\
d &\to b \\
f(a, x) &\to c \Leftarrow x \to^* a \\
f(x, x) &\to d \Leftarrow x \to^* a \\
g(x) &\to d \Leftarrow g(x) \to^* b \\
g(x) &\to f(a, a)
\end{aligned}
$$

We have the following CCPs:

$$
\begin{aligned}
\langle c, d\rangle &\Leftarrow a \to^* a \\
\langle d, f(a, a)\rangle &\Leftarrow g(x) \to^* b
\end{aligned}
$$

We can prove that both CCPs are joinable.

Transfomations

U transformations

Let $\mathcal{R}$ be a deterministic 3-CTRS. For each conditional rule as $\ell \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$ we introduce $n+1$ unconditional rules

$$\begin{array}{rcl} \ell & \to & U_1(s_1, \overrightarrow{x}_1) \\ U_{i-1}(t_{i-1}, \overrightarrow{x}_{i-1}) & \to & U_i(s_i, \overrightarrow{x}_i) \\ U_n(t_n, \overrightarrow{x}_n) & \to & r \end{array}$$

where the U_i are fresh new symbols added to Σ and the $\overrightarrow{x}_i$ are vectors of variables occurring in $\mathcal{V}ar(\ell) \cup Var(t_1) \cup \cdots \cup Var(t_{i-1})$ for all $1 \leq i \leq n$.

U-Transfomation

Example

$$a \to b$$
$$c \to k(f(a))$$
$$c \to k(g(b))$$
$$f(x) \to g(x) \Leftarrow h(f(x)) \to^* k(g(b))$$
$$h(f(a)) \to c$$
$$h(x) \to k(x)$$

We have the following transformation:

$$f(x) \to U_1(h(f(x)), x)$$
$$U_1(k(g(b)), x) \to g(x)$$

U-Transfomation

Example

$$a \to b$$
$$c \to k(f(a))$$
$$c \to k(g(b))$$
$$f(x) \to U_1(h(f(x)), x)$$
$$U_1(k(g(b)), x) \to g(x) \quad h(f(a)) \to c$$
$$h(x) \to k(x)$$

We have the following CCPs:

$$\langle k(f(a)), k(g(b)) \rangle$$
$$\langle h(U_1(f(a)), a), c \rangle$$
$$\langle h(f(b)), c \rangle$$
$$\langle c, k(f(a)) \rangle$$

We can try to prove that the CCPs are joinable.

Strongly Deterministic

Strongly deterministic CTRSs

Let $\mathcal{R}$ be a deterministic 3-CTRS, $\mathcal{R}$ is called strongly deterministic if, for every rule $\ell \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n \in \mathcal{R}$, every term t_i is strongly irreducible. A term t is called strongly irreducible w.r.t. $\mathcal{R}$ if $\sigma(t)$ is a normal form for every normalized substitution σ. $\mathcal{R}$ is called syntactically deterministic if, for every rule, every term t_i, is a constructor term or a ground $\mathcal{R}_u$ normal form.

Theorem

Every quasi-decreasing strongly-deterministic 3-CTRS with joinable critical pairs is confluent.

More Properties

Right-stable CTRSs

A CTRS is called right stable if every rewrite rule $\ell \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n \in \mathcal{R}$ satisfies the following conditions for all $i \in \{1, \ldots, k\}$:

$$(Var(\ell) \cup \bigcup_{j=1}^{i-1} Var(s_j = t_j) \cup Var(s_i)) \cap Var(t_i) = \varnothing$$

and t_i is either a linear constructor term or a ground $\mathcal{R}_u$ normal form. Every variable $y \in \bigcup_{i=1}^{k} Var(ti)$ is an extra variable ,i.e., y does not occur in ℓ.

Almost normal

A CTRS is called almost normal if it is normal or right stable.

More Properties

Level confluence

A CTRS $\mathcal{R}$ is called level-confluent if, for every $n \in \mathbb{N}$, the TRS $\mathcal{R}_n$ (meaning that every rule, independently and without conditions) is confluent.

Normal 2-CTRS

Every almost orthogonal almost normal 2-CTRS is level-confluent.

Properly oriented and right-stable

Every orthogonal properly oriented right-stable 3-CTRS is level-confluent.

Level-confluence implies confluence. However, not viceversa.

Confluence of Context-Sensitive Rewriting

Context-Sensitive Rewriting

Orthogonal and confluent

$$
\begin{array}{rl}
p(s(x)) \to x & \mathit{if}(\mathit{true}, x, y) \to x \\
0 + x \to x & \mathit{if}(\mathit{false}, x, y) \to y \\
s(x) + y \to s(x + y) & \mathit{zero}(0) \to \mathit{true} \\
0 \times y \to 0 & \mathit{zero}(s(x)) \to \mathit{false} \\
s(x) \times y \to y + (x \times y) & \mathit{fact}(x) \to \mathit{if}(\mathit{zero}(x), s(0), \mathit{fact}(p(x)) \times x)
\end{array}
$$

where $\mu(\mathit{if}) = \{1\}$, $\mu(f) = \{1, \ldots, \mathit{arity}(f)\}$ for the rest

Confluence does not imply confluence of CSR.

Context-Sensitive Rewriting

Orthogonal and confluent

$$
\begin{aligned}
f(x) &\to g(x,x) \\
g(0,x) &\to 0 \\
g(s(x),y) &\to s(x) \\
h(0) &\to 0 \\
h(s(x)) &\to 0
\end{aligned}
$$

where $\mu(g) = \{1\}$, $\mu(f) = \{1,\ldots,arity(f)\}$ for the rest

Peak:

$$f(g(x,x)) \leftarrow f(f(x)) \to g(f(x),f(x))$$

μ-Critical Pairs

Definiton

Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. A critical pair $\langle \theta(\ell)[\theta(r')]_p, \theta(r) \rangle \in CP(\mathcal{R})$ is a μ-critical pair of $\mathcal{R}$ if p is an active position. The set of μ-critical pairs of $\mathcal{R}$ is $CP(\mathcal{R}, \mu)$. A critical pair $\langle s, t \rangle$ is μ-joinable if $s \downarrow_\mu t$ holds.

Example

$$
\begin{aligned}
f(x) &\to g(h(x), x) \\
f(x) &\to x \\
g(x, x) &\to x \\
h(x) &\to x
\end{aligned}
$$

where $\mu(f) = \mu(h) = \varnothing$, $\mu(f) = \{1, \ldots, arity(f)\}$ for the rest

We obtain the following μ-joinable μ-critical pair: $\langle g(h(x), x), x \rangle$

Having no CPs does not mean μ-local confluence

Example

$$g(x,a) \to c(x)$$
$$a \to b$$

where $\mu(g) = \{1\}$, $\mu(c) = \varnothing$, $\mu(f) = \{1, \ldots, arity(f)\}$ for the rest

We obtain no CPs.

Peak:

$$g(b,a) \hookleftarrow g(a,a) \hookrightarrow c(a)$$

LHRV-condition

Definiton

Let μ be a replacement map. A rule $\ell \to r$ has left-homogeneous μ-replacing variables (written $LHRV(\ell \to r, \mu)$), if active variables in the left-hand side ℓ have no frozen occurrence neither in ℓ nor in r. A TRS $\mathcal{R}$ has left-homogeneous μ-replacing variables if $LHRV(\ell \to r, \mu)$ holds for all rules $\ell \to r \in \mathcal{R}$.

Proving and Disproving Confluence of CSR

- If there is a non-μ-joinable μ-critical pair $\pi \in CP(\mathcal{R}, \mu)$, then $\mathcal{R}$ is not (locally) μ-confluent.
- If $\mathcal{R}$ is left-linear, $LHRV(\mathcal{R}, \mu)$ holds and $CP(\mathcal{R}, \mu)$ is empty, then $\mathcal{R}$ is μ-confluent.
- If $\mathcal{R}$ is μ-terminating, $LHRV(\mathcal{R}, \mu)$ holds, and all μ-critical pairs $\pi \in CP(\mathcal{R}, \mu)$ are μ-joinable, then $\mathcal{R}$ is μ-confluent.

Variable Peaks

$$
\begin{tikzcd}
 & \sigma(\ell) \arrow[dl] \arrow[dr] & \\
\sigma(\ell)[C[\sigma(r')]_{p'}]_p \arrow[d, "*"'] & & \sigma(r) \arrow[ddl, "*"] \\
\sigma'(\ell) \arrow[dr] & & \\
 & \sigma'(r) &
\end{tikzcd}
$$

LH_μ-Variable Peak

Definiton

Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. Let $\ell \to r \in \mathcal{R}$ non-LHRV and x, in position p, is an active variable in ℓ that is frozen in ℓ or in r. Then,

$$\sigma(\ell)[C[\sigma(r')]_{p'}]_p \hookleftarrow \sigma(\ell) \hookrightarrow \sigma(r)$$

is a LH_μ-variable peak, where p' is an active position in a context C.

LH_μ-Critical Pairs

Definiton

Let $\mathcal{R}$ be a TRS and $\mu \in M_\mathcal{R}$. Let $\ell \to r \in \mathcal{R}$ be non-LHRV and p a variable position in ℓ such that $\ell|_p = x$ also appears on a frozen position in ℓ or r. Let y be a fresh variable, not occurring in ℓ or r. Then,

$$\langle \ell[y]_p, r \rangle \Leftarrow x \hookrightarrow y$$

Example

Can you obtain the LH_μ-critical pairs from the previous examples?

μ-Joinable LH_μ-Critical Pairs

Definiton

Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. Let $\pi : \langle s, t \rangle \Leftarrow x \hookrightarrow y$ be an LH_μ-critical pair. We say that π is μ-joinable if, for all substitutions σ such that $\sigma(x) \hookrightarrow_{\mathcal{R},\mu} \sigma(y)$ implies $\sigma(s) \downarrow_\mu \sigma(t)$.

Example

$\langle f(y), g(x,x) \rangle \Leftarrow x \hookrightarrow y$ is not μ-joinable.

Local μ-Confluence

Extended μ-critical pairs

Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. The set

$$ECP(\mathcal{R}, \mu) = CP(\mathcal{R}, \mu) \cup LHCP(\mathcal{R}, \mu)$$

Theorem

Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. Then, $\mathcal{R}$ is locally μ-confluent if and only if all pairs in $ECP(\mathcal{R}, \mu)$ are μ-joinable.

Local μ-confluence and termination

Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. If $\mathcal{R}$ is μ-terminating, then $\mathcal{R}$ is μ-confluence if and only if $\mathcal{R}$ is locally μ-confluent.

Proving and Disproving Confluence of CSR

- If there is a non-μ-joinable μ-critical pair $\pi \in ECP(\mathcal{R}, \mu)$, then $\mathcal{R}$ is not (locally) μ-confluent.
- If $\mathcal{R}$ is left-linear, ~~$LHRV(\mathcal{R}, \mu)$ holds and~~ $ECP(\mathcal{R}, \mu)$ is empty, then $\mathcal{R}$ is μ-confluent.
- If $\mathcal{R}$ is μ-terminating, ~~$LHRV(\mathcal{R}, \mu)$ holds, and~~ all μ-critical pairs $\pi \in ECP(\mathcal{R}, \mu)$ are μ-joinable, then $\mathcal{R}$ is μ-confluent.

μ-Orthogonality

Definition

Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. If $ECP(\mathcal{R}, \mu)$ is empty, then $\mathcal{R}$ is called μ-orthogonal.

Corollary

μ-orthogonal TRSs are μ-confluence.

First-Order Theories in CSR

A first-order theory $\overline{\mathcal{R}_\mu}$ is obtained by instantiating the inference rules with our input CS-TRS.

Example

$$(\forall x)\ x \to^* x$$

$$(\forall x, y, z)\ x \to y \wedge y \to^* z \Rightarrow x \to^* z$$

$$(\forall x, y)\ x \to y \Rightarrow f(x) \to f(y)$$

$$a \to b$$

$$(\forall x)\ f(x) \to g(x)$$

$$(\forall x)\ g(x) \to x$$

Feasibility

Feasibility on $\overline{\mathcal{R}_\mu}$

Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. A feasibility sequence is $\mathcal{R}$-feasible if and only if there is a substitution σ such that, for all conditions $s \to^* t$ in the sequence, $\overline{\mathcal{R}_\mu} \vdash \sigma(s) \to^* \sigma(t)$ holds. Otherwise, is called $\mathcal{R}_\mu$-infeasible.

μ-joinability

Let $\mathcal{R} = (\Sigma, R)$ be a TRS, $\mu \in M_{\mathcal{R}}$, and $s, t \in T_{\Sigma(X)}$. The s and t are μ-joinable if and only if $s^{\downarrow}$ and $t^{\downarrow}$ are μ-joinable.

$\overline{\mathcal{R}_\mu}$-feasibility

Let $\mathcal{R} = (\Sigma, R)$ be a TRS, $\mu \in M_{\mathcal{R}}$, $s, t \in T_{\Sigma(X)}$, and $z \in X$. The $s \downarrow_\mu t$ if and only if $s^{\downarrow} \hookrightarrow^* z$ and $t^{\downarrow} \hookrightarrow z$ is $\overline{\mathcal{R}_\mu}$-feasible.

μ-Critical Pairs

μ-joinability of μ-critical pairs

Let $\mathcal{R}$ be a TRS, $\mu \in M_{\mathcal{R}}$, and $\pi : \langle s, t \rangle \in CP(\mathcal{R}, \mu)$. The π is μ-joinable if and only if $s^{\downarrow}$ and $t^{\downarrow}$ are μ-joinable.

μ-joinability of LH_μ-critical pairs

Let $\mathcal{R}$ be a TRS, $\mu \in M_{\mathcal{R}}$, and $\pi : \langle \ell[y]_p, r \rangle x \hookrightarrow y$ in $LHCP(\mathcal{R}, \mu)$. If $\overline{\mathcal{R}_\mu} \vdash (\forall x)(\forall y)(\exists z) x \to y \Rightarrow \ell^{\downarrow\{\bar{x}\}}[y]_p \to^* z \wedge r^{\downarrow\{\bar{x}\}} \to^* z$ holds, the π is μ-joinable.

Example (check)

$$(\forall x)\ x \to^* x$$
$$(\forall x, y, z)\ x \to y \wedge y \to^* z \Rightarrow x \to^* z$$
$$(\forall x, y)\ x \to y \Rightarrow f(x) \to f(y)$$
$$a \to b$$
$$(\forall x)\ f(x) \to g(x)$$
$$(\forall x)\ g(x) \to x$$

We have the following ECP:

$$\overline{\mathcal{R}_\mu} \vdash (\forall x)(\forall y)(\exists z) x \to y \Rightarrow f(y) \to^* z \wedge g(x) \to^* z$$

which is μ-joinable.

Conditions cannot be grounded

Non-μ-joinability of μ-critical pairs

Let $\mathcal{R}$ be a TRS, $\mu \in M_{\mathcal{R}}$, and $\pi : \langle s, t\rangle \Leftarrow x \hookrightarrow^* y$ be a LH_μ-critical pair. If

$$x \hookrightarrow^* y, s^{\downarrow\{\overline{x,y}\}} \hookrightarrow^* z, t^{\downarrow\{\overline{x,y}\}} \hookrightarrow^* z$$

is $\overline{\mathcal{R}_\mu}$-infeasible, then π is not μ-joinable.

Orthogonal and confluent (check)

$$f(x) \to g(x,x)$$
$$g(0,x) \to 0$$
$$g(s(x),y) \to s(x)$$
$$h(0) \to 0$$
$$h(s(x)) \to 0$$

where $\mu(g) = \{1\}$, $\mu(f) = \{1,\ldots,arity(f)\}$ for the rest

The LH_μ-critical pair:

$$\langle f(y), g(x,x)\rangle \Leftarrow x \hookrightarrow y$$

is $\overline{\mathcal{R}_\mu}$-infeasible.

We can force μ-rewriting steps

Non-μ-joinability of μ-critical pairs

Let $\mathcal{R}$ be a TRS, $\mu \in M_{\mathcal{R}}$, and $\pi : \langle s, t \rangle \Leftarrow x \hookrightarrow^* y$ be a LH_μ-critical pair. Let $\ell \to r \in \mathcal{R}$, $\sigma(x) = \ell^{\downarrow}$, $\sigma(y) = r^{\downarrow}$ and $\sigma(z) = c_z$ if $z \notin \{x, y\}$. If $\sigma(s)$ and $\sigma(t)$ are not μ-joinable then π is not μ-joinable.

Orthogonal and confluent (check)

$$
\begin{array}{rl}
p(s(x)) \to x & \mathit{if}(\mathit{true}, x, y) \to x \\
0 + x \to x & \mathit{if}(\mathit{false}, x, y) \to y \\
s(x) + y \to s(x + y) & \mathit{zero}(0) \to \mathit{true} \\
0 \times y \to 0 & \mathit{zero}(s(x)) \to \mathit{false} \\
s(x) \times y \to y + (x \times y) & \mathit{fact}(x) \to \mathit{if}(\mathit{zero}(x), s(0), \mathit{fact}(p(x)) \times x)
\end{array}
$$

where $\mu(\mathit{if}) = \{1\}$, $\mu(f) = \{1, \ldots, \mathit{arity}(f)\}$ for the rest

The LH_μ-critical pair:

$$\langle \mathit{fact}(y), \mathit{if}(\mathit{zero}(x), s(0), \mathit{fact}(p(x))) \rangle \Leftarrow x \hookrightarrow y$$

can be instantiated to make it $\overline{\mathcal{R}_\mu}$-infeasible.