Resource Economics (Spring Term 2025)

Refreshing Optimization

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Learning Goals

- **Refreshing Optimization** (self study as home work)
  - Repetition on minima and maxima of higher polynomials in 2 and 3-dimensional functions.
  - Recap of constrained optimization problems (Lagrangian)
  - Understand the limits of pen and paper for optimization tasks.
  - *I assume you are familiar with the content in this section or are able to catch up with some self-studying. If you still struggle after reading through this handout, please come and see me / email me, so we can close the gaps.*

Refresher in Optimization

When we talk about optimizations in this course, we often mean an optimization function that states a certain outcome given one or several input variables. A result is deemed "optimal" when this outcome variable is maximized or minimized given the available range of input variables. We will now repeat how to optimize different types of functions by hand.

To find maxima and minima in any sort of function, we need derivatives. Here is a short recap of how we can use the different types of derivatives for a hypothetical function $f(x)$:

- "first order" derivatives (also written as $\frac{df}{dx}$, or $f'(x)$) represent the slope of $f(x)$ around a certain value of x. It can also be understood as the "rate of change". We often set this equation to zero $f'(x) = 0$ because a slope of zero means that the initial function $f(x)$ is flat at a certain (number) of x's, meaning it could be a peak, valley or plateau.

- "second order" derivatives (also written as $\frac{d^2f}{d^2x}$, or $f''(x)$), represent the curvature, the change in slope, or the "rate of change of the rate of change". We use this derivative to find out if the x's we found with $f'(x)$ are actually a peak, valley or plateau. $\frac{d^2f}{d^2x} > 0$ means we found a minimum, $\frac{d^2f}{d^2x} < 0$ a maximum and $\frac{d^2f}{d^2x} = 0$ a plateau.

- "third and higher order" derivatives can be calculated, but will not be needed in this course.

- "partial" derivatives (also written as $\frac{\partial f}{\partial x}$, or $f'_x(x, y)$): Sometimes an optimization function, an outcome, depends on more than just one input variable. In this case, we use partial derivatives to examine how the outcome changes if we only change one variable (here x) a little bit while leaving the other variables constant.

1. **Convex & Concave**

   Exercises in economics often feature "convex" or "concave" functions. These terms describe the slope of the function (and the first and second derivative) in such a way, that we often save the trouble of deriving the second derivative. A detailed explanation of concave and convex functions is given in this summary slide deck. In our course, we consider two type of functions: quadratic / two times differentiable functions and exponential functions. For both cases, we can state:

$$
\begin{aligned}
f''(x) > 0 \quad & \text{for all } x \in \mathbb{R} \rightarrow \text{convex} \\
f''(x) < 0 \quad & \text{for all } x \in \mathbb{R} \rightarrow \text{concave}
\end{aligned}
$$

   See these two examples for an illustration and the plot of the base function (green) and the second order derivative (red) in Fig 1.

quadratic function	exponential function
$f(x) = -2x + x^2$	$g(x) = ln(x)$
$f'(x) = 2x$	$g'(x) = \frac{1}{x}$
$f''(x) = 2 > 0$	$g''(x) = -\frac{1}{x^{-2}} < 0$
$\rightarrow$ *convex*	$\rightarrow$ *concave*

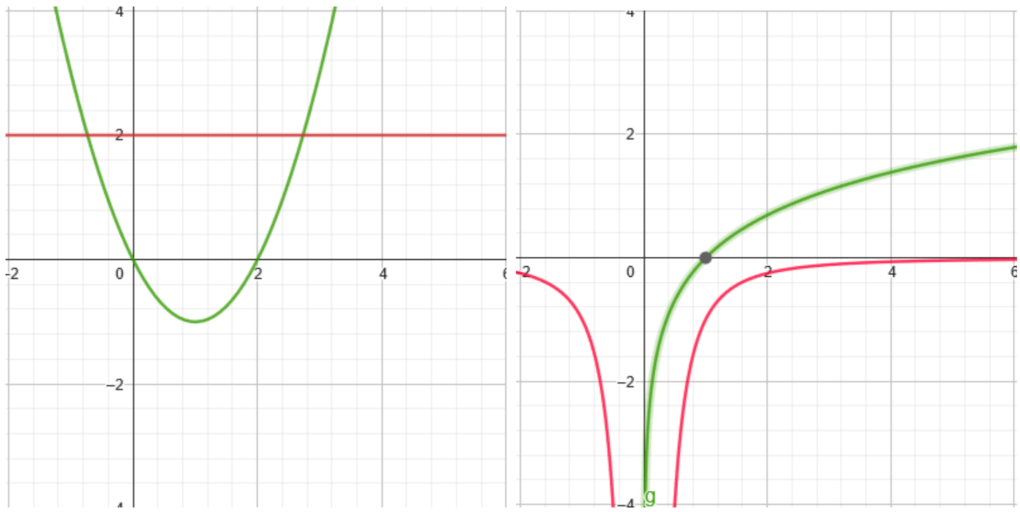

Figure 1:

2. **Higher Polynomials (Local and Global Minima/Maxima)**
   It can be easy to determine if a critical point is a minimum or a maximum in quadratic functions. But can you quickly determine the critical points in a higher polynomial?

$$
\begin{aligned}
f(x) &= (x+2)(x-1.5)^2(x+0.75)^3 \\
&\approx x^6 + 1.25x^5 - 4.3125x^4 - 5.20313x^3 + 3.375x^2 + 6.01172x + 1.89844
\end{aligned}
$$

$$
\frac{df}{dx} = 6x^5 + 6.25x^4 - 17.25x^3 - 15.61x^2 + 6.75x - 6.01 \stackrel{!}{=} 0
$$
$$
f'(x) = 0 \rightarrow \quad x_1 = -1.73, \quad x_2 = -0.75, \quad x_3 = 0.69 \quad x_4 = 1.5
$$

$$
\frac{d^2f}{d^2x} = 30x^4 + 25x^3 + 51.75x^2 - 31.22x - 6.75
$$
$$
f''(x) \rightarrow \quad f''(x_1) = 44.82 \quad f''(x_2) = 0 \quad f''(x_3) = -24.3, \quad f''(x_4) = 79.73
$$

In this example, $f(x_2)$ is clearly "only" a plateau, $f(x_3)$ is the global maxima and $f(x_1)$ and $f(x_4)$ are minima. But which minima is the global minima? Plug the x's back into the base function to find out.

$$
f(x_1) = -2.65 < f(x_4) = 0 \qquad \rightarrow f(x_1) \text{is the global minima}
$$

Use GeoGebra to convince yourself of that fact and check the other x's to see if there are truly minia or maxima.

3. **3-Dimensional Functions**
   Here is an example of how to find maxima and minima in 3-dimensional functions. First, calculate both partial first-order derivatives for x and y. To determine which of the critical points x^*, y^* is a minimum or a maximum, we derive the discriminant $D = f_{xx}(x^*, y^*)f_{yy}(x^*, y^*) - (f_{xy}(x^*, y^*))^2$, also called "second derivative test". The following charismatics determine the critical points:

   - If $D > 0$ and $f_{xx}(x^*, y^*) > 0$ then f has a local minimum at (x^*, y^*).
   - If $D > 0$ and $f_{xx}(x^*, y^*) < 0$ then f has a local maximum at (x^*, y^*).
   - If $D < 0$ then f has a saddle point at (x^*, y^*).

- If $D = 0$ then the test is inconclusive.

$$
\begin{aligned}
& f(x,y) = 3x^3 + 2xy - 6x - 4y^2 \\
& FOC: \\
& \qquad f_x = 3x^2 + 2y - 6 = 0 \\
& \qquad f_y = 2x - 8y = 0 \qquad \to x = 4y \\
& \qquad\quad 3(4y)^2 + 2y - 6 = 0 \\
& \qquad\quad 48y^2 + 2y - 6 = 0 \\
& \qquad\quad 2(8y+3)(3y-1) = 0 \\
& \qquad\quad \to y^* = -\frac{3}{8}, \frac{1}{3}; x^* = \frac{3}{2}, \frac{4}{3}
\end{aligned}
$$

$$
\begin{aligned}
D &= f_{xx}(x^*,y^*) f_{yy}(x^*,y^*) - (f_{xy}(x^*,y^*))^2 \\
&= \underbrace{f_{xx}}_{6x} \underbrace{f_{yy}}_{-8} - (\underbrace{f_{xy}}_{2})^2
\end{aligned}
$$

$$D\left(-\frac{3}{2}, -\frac{3}{8}\right) > 0 \quad f_{xx}\left(-\frac{3}{2}, -\frac{3}{8}\right) < 0 \to \text{Local max at } \left(-\frac{3}{2}, -\frac{3}{8}\right)$$

$$D\left(\frac{4}{3}, \frac{1}{3}\right) < 0 \to \text{Saddle point at } \left(\frac{4}{3}, \frac{1}{3}\right)$$

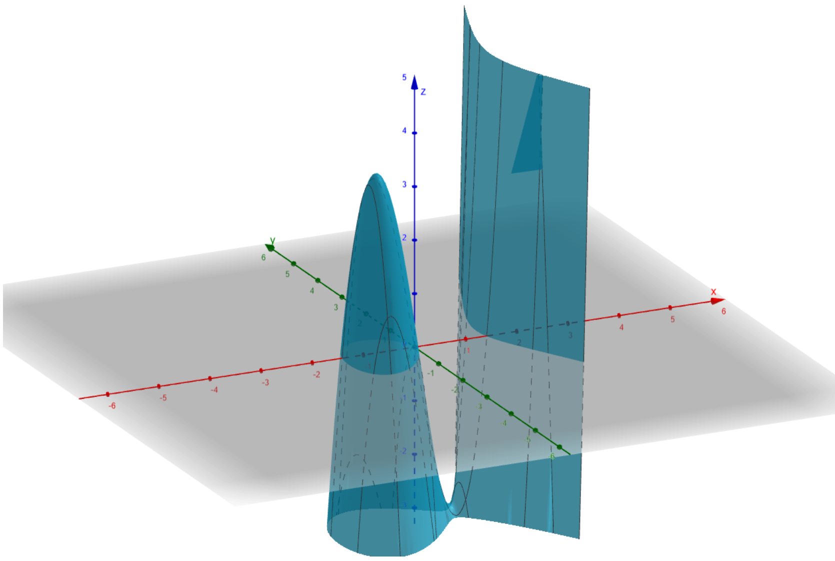

Figure 2:

4. **Lagrangian**[1]

   After considering regular optimizing problems, we now want to look at a particular type of problem: A constrained, two-variable optimization problem. The Lagrangian is a well-known and widely used method in economics to solve these types of questions. Take the following example:

   A major gas extraction company operates two sites that require different extraction technologies. A single extraction pump (e.g. $x = 1$) produces different gas quantities $q_a(x)$ and $q_b(x)$, depending on the site it is located. The quantity of extracted quantity gas from both sites $q_a + q_b$ is sold at an external, market-given price $P = 1$ USD. The firm has only a limited budget of 70 extraction pumps. Assume the gas stock is endless in consideration of human life. Help the firm to find an optimal allocation of pumps between the sites.

$$q_a(x) = 120 - 4x_a^2 \qquad q_b(x) = 80 - x_b^2$$
$$x_a + x_b = X = 70 \qquad P(q) = 1q$$

[1] If you are interested in a visualization of a Lagrangian, see GeoGebra and visualize the two functions $g(x, y) = x^2 + y^2$ and $g(x, y) = y^2 - x = 5$

This is a step-by-step instruction for solving optimization problems with a given set of constraints.

(a) Define the optimization problem. State which are the variables that can be changed and what is their constraint.

(b) Build the Lagrangian formula.

(c) Derive the first-order conditions for the variables and for λ

(d) Set the first-order conditions to zero and calculate all critical points.

(e) Plug the values for the correct critical points back into the base function and derive the result.

$$\max_{x_a, x_b} L(x_a, x_b) = \underbrace{(120 - 4x_a^2)}_{\text{max output in } a} + \underbrace{(80 - x_b^2)}_{\text{max output in } b} - \lambda \underbrace{(x_a + x_b - 70)}_{\text{budget constraint}}$$

$$\begin{aligned}
& FOC: \\
I \quad & \frac{\partial L}{\partial x_a} = -8x_a - \lambda = 0 \\
II \quad & \frac{\partial L}{\partial x_b} = -2x_b - \lambda = 0 \\
III \quad & \frac{\partial L}{\partial \lambda} = x_a + x_b - 70 = 0
\end{aligned}$$

$$\begin{aligned}
I \to II: \quad & 8x_a = \lambda = 2x_b \qquad \to x_b = 4x_a \\
(I + II) \to III: \quad & 70 = x_a + x_b = x_a + (4x_a) = 70 \qquad \to x_a = \frac{70}{5} = 14 \\
\to (I + II): \quad & x_b = 4x_a = 4(14) = 56 \\
\to II: \quad & \lambda = -2x_b = -2(56) = -112 \\
\to \quad & \mathbf{x_a^* = 14} \quad \mathbf{x_b^* = 56} \quad \lambda = \mathbf{-112}
\end{aligned}$$

$$\begin{aligned}
\text{Max output: } & q_a(x_a^*) + q_b(x_b^*) = (120 - 4(14)^2) + (80 - (56)^2) = 94 \\
\text{Budget constraint: } & q_a(x_a^*) + q_b(x_b^*) \stackrel{!}{=} 70 \to 14 + 56 = 70
\end{aligned}$$

5. **Limitations of Pen and Paper**

   Analytical solutions (derived as a mathematical formula) are often used in economics classes to solve exercises. But what if problems become too intense to be derived by hand with pen and paper?

   To start with an illustrative example, take a simple bond calculation from a basic finance class, nominal value n CHF 20'000, annual coupons c of 3%, an interest rate r of 2% and 10 years to maturity. What is its current market price?

$$\begin{aligned} PV &= \sum_{t=0}^{T-1} \frac{c}{(1+r)^t} + \frac{c+N}{(1+r)^T} \\ &= \sum_{t=0}^{9} \frac{600}{(1.02)^t} + \frac{600+20000}{(1.02)^{10}} = 21'796.5 \end{aligned}$$

   This type of question can be formulated easily. Now we could also answer other types of questions. Assuming the current market price is exactly CHF 22'000 and the interest rate remains 5%, what is the corresponding coupon rate for the bond to have this price? (Answer: 6.0295%)

   But try solving a more complicated question: "The bond has $n = 20'000$, $c = 3.25\%$, is currently priced at $PV = 18305.1$. Try to derive the current interest with the information given."

   This is impossible (for us) to solve because we would need to address the higher polynomial in the function. Even if we could do it with pen and paper, the entire process would be highly difficult and prone to errors. But luckily our computers can help. We will revisit this task and use software to solve it.