Tema-1-Ejercicios-RECURRENCIA-Re...

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Fundamentos de análisis de algoritmos

1º Grado en Ingeniería Informática

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1. Calcular la eficiencia del siguiente algoritmo:

```
int ejemplo1(int n)
{
   if (n <= 1)
         return 1;
   else
         return (ejemplo1(n - 1) + ejemplo1(n - 1));
}
```

T_{if} = condición + devuelve = $1+1=2$ $\longrightarrow$ cuando $n \leq 1$

T_{else} = condición + devuelve + llamada + $T(n-1)$ + resta + suma + llamada + $T(n-1)$ + resta

$T_{else} = 7 + 2T(n-1)$ $\longrightarrow$ Cuando $n>1$

$$T(n) \begin{cases} 2 & \text{si } n \leq 1 \\ 2T(n-1) + 7 & \text{si } n > 1 \end{cases}$$

$T(n) = 2T(n-1) + 7$; $T(n) - 2T(n-1) = 7$ No Homogénea

$T(n) - 2T(n-1) = 1^n \cdot 7 \cdot n^0$; $(x-2)(x-1)^{0+1} = 0 \longrightarrow$ Raices $\begin{cases} r_1 = 2 & \text{simple} \\ r_2 = 1 & \text{simple} \end{cases}$

$T(n) = C_1 \cdot r_1^n \cdot n^0 + C_2 \cdot r_2^n \cdot n^0 = C_2 \cdot 1^n + C_1 \cdot 2^n = C_2 + C_1 \cdot 2^n$

$T(1) = 2 \longrightarrow C_1 \cdot 2^n + C_2 = 2$; $C_2 = 2 - 2C_1$ Cogemos $T(1)$ por ser caso base

$T(2) = 2T(2-1) + 7 = 2T(1) + 7 = 2(2) + 7 = 4 + 7 = 11$

$T(2) = C_1 \cdot 2^2 + C_2 = C_1 \cdot 2^2 + 2 - 2C_1 = 11$; $4C_1 - 2C_1 = 11 - 2$; $2C_1 = 9$; $C_1 = 9/2$

$C_2 = 2 - 2C_1 = 2 - 2 \cdot 9/2 = 2 - 9 = -7$

$$T(n) = \frac{9}{2} \cdot 2^n - 7$$

2. Calcular la eficiencia del siguiente algoritmo:

```
int ejemplo2(int n)
{
   if (n == 1)
         return n;
   else
         return (ejemplo2(n/2) + 1);
}
```

T_{if} = condicion + devuelve = 1+1 = 2 $\longrightarrow$ cuando $n = 1$

T_{else} = condicion + devuelve + llamada + $T(n/2)$ + division + suma $\longrightarrow$ Cuando $n \neq 1$

$$T(n) \begin{cases} 2 & \text{si } n = 1 \\ T(n/2) + 5 & \text{si } n \neq 1 \end{cases}$$

$T(n) = T(n/2) + 5$; $T(n) - T(n/2) = 5$ No Homogénea

cambio base $n = 2^k$ $T(2^k) - T(2^{k-1}) = 5$ cambio base $2^k = t_k$

$t_k - t_{k-1} = 1^k \cdot 5 \cdot k^0$ cambio base $t_k = x$ $(x-1)(x-b)^{d+1} = 0$; $(x-1)(x-1) = 0$

Raiz doble = 1

$T(n) = T(2^k) = c_1 \cdot r_1^k \cdot k^0 + c_2 \cdot r_1^k \cdot k^1 = c_1 \cdot r_1^k + c_2 \cdot r_1^k \cdot k$

tenemos 2^k y queremos $n \to 2^k = n$; $\log_2 n = k$ realizamos dicho cambio de base

$T(n) = c_1 \cdot r_1^{\log_2 n} + c_2 \cdot r_1^{\log_2 n} \cdot \log_2 n$; $T(n) = c_1 + c_2 \cdot \log_2 n$; $T(1) = 2 = c_1 + c_2 \log_2 n$

$c_1 = 2 - c_2 \cdot 0$; $c_1 = 2$

$T(2) = T(2/2) + 5 = T(1) + 5 = 2 + 5 = 7$; $c_1 + c_2 \log_2 2 = 7$; $2 + c_2 \cdot 1 = 7$; $c_2 = 7 - 2 = 5$

$T(n) = 2 + 5 \cdot \log_2 n$

3. Calcular la eficiencia del algoritmo de las Torres de Hanoi por expansión de la recurrencia.

```
Hanoi(origen,destino,pivote,discos):
    si discos=1
            moveruno(origen,destino)
    en otro caso
            Hanoi(origen,pivote,destino,discos-1)
            moveruno(origen,destino)
            Hanoi(pivote,destino,origen,discos-1)
```

Hanoi (origen, destino, pivote, disco) ⟶ disco nos dice el tamaño, el número de discos, por tanto, disco = n, quedando de la siguiente forma

T_{si} = condición + llamada (moveruno)

T_{sino} = condición + llamada (Hanoi) + $T(n-1)$ + resta + llamada (moveruno) + + llamada (Hanoi) + $T(n-1)$ + resta

$$T(n) \begin{cases} T_{si} & \text{si } n=1 \\ T_{sino} & \text{si } n \neq 1 \end{cases} \longrightarrow T(n) \begin{cases} 2 & \text{si } n=1 \\ 2T(n-1)+6 & \text{si } n\neq 1 \end{cases}$$

$$T(n) = 2T(n-1)+6 = 2(2T(n-2)+6)+6 = 2^2 \cdot T(n-2)+6+12 = 2^2(2T(n-3)+6)+6+12 =$$
$$= 2^3 \cdot T(n-3)+6+12+6\cdot 2^2 = 2^n \cdot T(1) + 6 + 6\cdot 2^1 + 6 \cdot 2^2 + \dots 6 \cdot 2^{n-1};$$

$$T(n) = 2^n + 6\cdot 2^{n-1} - 6$$

$T(1) = 2$	$T(1) = 2$
$T(2) = 2\cdot T(2-1)+6 = 2\cdot 2+6 = 10$	$T(2) = 2^2 + 6\cdot 2^{2-1} - 6 = 10$
$T(3) = 2\cdot T(3-1)+6 = 2\cdot 10+6 = 26$	$T(3) = 2^3 + 6\cdot 2^{3-1} - 6 = 26$
$T(4) = 2\cdot T(4-1)+6 = 2\cdot 26+6 = 58$	$T(4) = 2^4 + 6\cdot 2^{4-1} - 6 = 58$

4. Resolver las siguientes ecuaciones y dar su orden de complejidad:

a) $f(n) = \begin{cases} n & si\ n = 0\ ó\ n = 1 \\ f(n-1) + f(n-2) & en\ otro\ caso \end{cases}$

b) $T(n) = \begin{cases} n & si\ n = 0,1\ ó\ 2 \\ 5T(n-1) - 8T(n-2) + 4T(n-3) & en\ otro\ caso \end{cases}$

c) $T(n) = \begin{cases} 0 & si\ n = 0 \\ 2T(n-1) + 1 & en\ otro\ caso \end{cases}$

d) $T(n) = 4T\left(\frac{n}{2}\right) + n, \quad n > 1\ y\ potencia\ de\ 2$

e) $T(n) = 4T\left(\frac{n}{2}\right) + n^2, \quad n > 1\ y\ potencia\ de\ 2$

f) $T(n) = 2T\left(\frac{n}{2}\right) + n.logn, \quad n > 1\ y\ potencia\ de\ 2$

g) $T(n) = \begin{cases} 1 & si\ n = 2 \\ 2T(\sqrt{n}) + log\ log\ n & con\ n \geq 4 \end{cases}$

h) $T(n) = T\left(\frac{n}{2}\right) + T^2\left(\frac{n}{4}\right) \quad \sin caso\ base$

Ejercicio 4a:

$f(n) = f(n-1) + f(n-2)$; $f(n) - f(n-1) - f(n-2) = 0$ Homogénea

$$x^2 - x - 1 = 0 \longrightarrow \frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a} = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} \quad r_1 = \frac{1+\sqrt{5}}{2} \quad r_2 = \frac{1-\sqrt{5}}{2}$$

$$f(n) = C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n \cdot n^0 + C_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n \cdot n^0$$

$$f(0) = 0 = C_1 \cdot 1 + C_2 \cdot 1 \ ; \ 0 = C_1 + C_2 \ ; \ C_1 = -C_2$$

$$f(1) = 1 = C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^1 + C_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^1 ; \ C_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^1 = 1 - C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^1 ;$$

$$C_2 = \frac{1 - C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^1}{\left(\frac{1-\sqrt{5}}{2}\right)^1} = -\left(\frac{1+\sqrt{5}}{2}\right)^1 + C_1 \cdot \left(\frac{3+\sqrt{5}}{2}\right)^1$$

$$C_1 = -\left(-\left(\frac{1+\sqrt{5}}{2}\right)^1 + C_1 \cdot \left(\frac{3+\sqrt{5}}{2}\right)^1\right) ; \ C_1 \cdot \left(\frac{5+\sqrt{5}}{2}\right)^1 = \left(\frac{1+\sqrt{5}}{2}\right)^1 ; \ C_1 = \frac{\sqrt{5}}{5}$$

$$-C_2 = C_1 \ ; \ C_2 = -C_1 = -\frac{\sqrt{5}}{5}$$

$$f(n) = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Ejercicio 4b:

$T(n) = 5T(n-1) - 8T(n-2) + 4T(n-3)$; $T(n) - 5T(n-1) + 8T(n-2) - 4T(n-3) = 0$ Homogenea

$x^3 - 5x^2 + 8x - 4 = 0 \rightarrow$ simplificamos mediante método de Ruffini

$$\begin{array}{c|cccc} & 1 & -5 & 8 & -4 \\ 2 & & 2 & -6 & 4 \\ \hline & 1 & -3 & 2 & 0 \\ 2 & & 2 & -2 & \\ \hline & 1 & -1 & 0 & \\ 1 & & 1 & & \\ \hline & 1 & 0 & & \end{array}$$

$(x-2)(x-2)(x-1) = 0$

Raices $= \begin{cases} r1 = 2 & \text{doble} \\ r2 = 1 & \text{simple} \end{cases}$

$T(n) = C_1 \cdot 2^n \cdot n^0 + C_2 \cdot 2^n \cdot n^1 + C_3 \cdot 1^n \cdot n^0 = C_1 \cdot 2^n + C_2 \cdot 2^n \cdot n + C_3$

$T(0) = 0 = C_1 \cdot 2^0 + C_2 \cdot 2^0 \cdot 0 + C_3 = C_1 + C_3$; $C_1 = -C_3$

$T(1) = 1 = C_1 \cdot 2^1 + C_2 \cdot 2^1 \cdot 1 + C_3 = 2C_1 + 2C_2 + C_3 = -2C_3 + 2C_2 + C_3$; $2C_2 - C_3 = 1$;

$C_3 = -1 + 2C_2$

$T(2) = 2 = C_1 \cdot 2^2 + C_2 \cdot 2^2 \cdot 2 + C_3 = 4C_1 + 8C_2 + C_3 = -4C_3 + 8C_2 + C_3$

$= -4(-1+2C_2) + 8C_2 - 1 + 2C_2 = 4 - 8C_2 + 8C_2 - 1 + 2C_2$; $2C_2 + 3 = 2$; $C_2 = -1/2$

$C_3 = -1 + 2(-1/2) = -1 - 1 = -2$

$C_1 = -(-2) = 2$

$T(n) = 2 \cdot 2^n - \frac{1}{2} \cdot 2^n \cdot n - 2$; $T(n) = 2^{n+1} - n \cdot 2^{n-1} - 2$

Ejercicio 4c:

$T(n) = 2T(n-1) + 1$; $T(n) - 2T(n-1) = 1$ No Homogénea

$f(n) = 1 = 1^n \cdot 1 \cdot n^0 \rightarrow (x-2)(x-1)^{0+1} = 0$

Raices $= \begin{cases} r1 = 2 \\ r2 = 1 \end{cases}$

$T(n) = C_1 \cdot 2^n \cdot n^0 + C_2 \cdot 1^n \cdot n^0 = C_1 \cdot 2^n \cdot C_2$

$T(0) = 0 = C_1 \cdot 2^n + C_2$; $C_2 = -C_1 \cdot 2^0$; $C_2 = -C_1$

$T(1) = 2T(1-1) + 1 = 2 \cdot 0 + 1 = 1 = C_1 \cdot 2^n + C_2$; $C_2 = 1 - 2C_1$; $C_2 = 1 + 2C_2$;

$-C_2 = 1$; $C_2 = -1$; $C_1 = -C_2 = -(-1) = 1$

$T(n) = 2^n - 1$

Ejercicio 4d:

$T(n) = 4T(n/2) + n \begin{bmatrix} \text{cambio de base} \\ n = 2^K \end{bmatrix} T(2^K) = 4T(2^K/2) + 2^K$;

$T(2^K) - 4T(2^{K-1}) = 2^K$ No Homogénea

$b^K \cdot p(a) = 2^K = 2^K \cdot K^0$; $b = 0$, $a = 2 \rightarrow (X-a)^{b+1} = (X-2)^1$

$(X-4)(X-2) = 0$ Raices $\begin{cases} r_1 = 4 \\ r_2 = 2 \end{cases}$

$T(2^K) = C_1 \cdot 4^K + C_2 \cdot 2^K = C_1 \cdot (2^K)^2 + C_2 \cdot 2^K \begin{bmatrix} \text{cambio de base} \\ 2^K = n \end{bmatrix} T(n) = C_1 \cdot n^2 + C_2 \cdot n$

$T(n) - 4T(n/2) = n$; $C_1 \cdot n^2 + C_2 \cdot n - 4 \cdot \left(C_1 \cdot \left(\frac{n}{2}\right)^2 + C_2 \cdot \left(\frac{n}{2}\right)\right) = n$;

$C_1 \cdot n^2 + C_2 \cdot n - 4 \cdot C_1 \frac{n^2}{4} - 4C_2 \frac{n}{2} = n$; $C_1 \cdot n^2 + C_2 \cdot n - C_1 \cdot n^2 - 2C_2 \cdot n = n$;

$-C_2 \cdot n = n$; $-C_2 = 1$; $C_2 = -1 \longrightarrow T(n) = C_1 \cdot n^2 - n$ No podemos averiguar más sin condiciones iniciales

Ejercicio 4e:

$T(n) = 4T(n/2) + n^2 \begin{bmatrix} \text{cambio de base} \\ n = 2^K \end{bmatrix} T(2^K) = 4T(2^K/2) + (2^K)^2$;

$T(2^K) - 4T(2^{K-1}) = (2^K)^2$; No Homogénea ; $4^K = b^K \cdot p(K) = 4^K \cdot K^0$

$p(x) = (x-4)(x-4) = 0$ Raices $= r_1 = 4$ doble

$T(2^K) = C_1 \cdot 4^K + C_2 \cdot 4^K \cdot K \begin{bmatrix} \text{cambio de base} \\ 2^K = n \end{bmatrix} T(n) = C_1 \cdot n^2 + C_2 \cdot n^2 \cdot \log_2(n)$

$T(n) - 4T(n/2) = n^2$; $C_1 \cdot n^2 + C_2 \cdot n^2 \cdot \log_2(n) - 4\left(C_1 \cdot \left(\frac{n}{2}\right)^2 + C_2 \cdot \left(\frac{n}{2}\right)^2 \log_2\left(\frac{n}{2}\right)\right) = n^2$

$C_2 n^2 \log_2(n) - C_2 n^2 \log_2\left(\frac{n}{2}\right) = n^2$; $C_2 n^2 \left(\log_2(n) - \log_2(n/2)\right) = n^2$

$C_2\left(\log_2(n) - \log_2(n/2)\right) = 1$; $C_2 = 1 \longrightarrow T(n) = C_1 \cdot n^2 + n^2 \cdot \log_2(n)$

Ejercicio 4f:

$\log_2 n = \log_2 2^k = K$

$T(n) = 2T\left(\frac{n}{2}\right) + n\log_2(n) \left[\begin{matrix}\text{cambio de base} \\ n = 2^k\end{matrix}\right] \; T(2^k) - 2T(2^k/2) = 2^k K$

$T(2^k) - 2T(2^{k-1}) = 2^k K$ No Homogénea $\rightarrow b^k \cdot p(k) = 2^k \cdot K \rightarrow a = 2,\ b = 1$

$p(x) = (x-2)(x-2)^2 = 0$ Raices $= r_1 = 2$ Triple

$T(2^k) = c_1 \cdot 2^k + c_2 \cdot 2^k \cdot K + c_3 \cdot 2^k \cdot k^2 \left[\begin{matrix}\text{cambio de base} \\ 2^k = n\end{matrix}\right]$

$T(n) = c_1 n + c_2 n \cdot \log_2(n) + c_3 n \log_2^2(n)$

$c_1 n + c_2 n \cdot \log_2(n) + c_3 n \log_2^2(n) - 2\left(c_1 \frac{n}{2} + c_2 \frac{n}{2} \cdot \log_2\left(\frac{n}{2}\right) + c_3 \frac{n}{2} \log_2^2\left(\frac{n}{2}\right)\right) = n\log_2 n$

$n\log_2(n)\left(c_2 + c_3 \log_2(n)\right) - n\log_2 \frac{n}{2}\left(c_2 + c_3 \log_2\left(\frac{n}{2}\right)\right) = n\log_2(n)$;

$n\left(\log_2(n)\left(c_2 + c_3 \log_2(n)\right) - \log_2\left(\frac{n}{2}\right)\left(c_2 + c_3 \log_2\left(\frac{n}{2}\right)\right)\right) = n\log_2(n)$;

$n(c_3 - c_2) + 2c_3 n\log_2(n) = n\log_2(n)$; $2c_3 = 1$; $c_3 = \frac{1}{2}$; $c_2 = \frac{1}{2}$

$T(n) = c_1 n + \frac{1}{2} n\log_2(n) + \frac{1}{2} n\log_2^2(n)$

Ejercicio 4g:

$T(n) = 2T(\sqrt{n}) + \log\log(n) \left[\begin{matrix}\text{cambio de base} \\ n = k^2\end{matrix}\right] \; T(k^2) - 2T(\sqrt{k^2}) = \log\log(k^2)$

$T(k^2) - 2T(k^{2/2}) = \log\log(k^2)$ No Homogénea

$p(x) = (x-2)(x-2) = 0$ Raices $= r_1 = 2$ doble

$T(k^2) = c_1 \cdot 2^k \cdot k^0 + c_2 \cdot 2^k \cdot k^1 \left[\begin{matrix}\text{cambio de base} \\ 2^k = n\end{matrix}\right] \; T(n) = c_1 \cdot n + c_2 \cdot n \cdot \log(n)$

$T(2) = 1 = c_1 n + c_2 n\log(n) = c_1 \cdot 2 + c_2 \cdot 2$; $c_1 = \frac{1 - 2c_2}{2}$

$T(4) = 2T(\sqrt{n}) + \log\log(n) = 2T(\sqrt{4}) + \log\log(4) = 2T(2) + 1 = 2 \cdot 1 + 1 = 3$

$T(4) = c_1 n + c_2 n\log(n) = 4c_1 + 4c_2 \cdot 2 = 4c_1 + 8c_2 = 3$;

$4\left(\frac{1 - 2c_2}{2}\right) + 8c_2 = 3$; $2 - 4c_2 + 8c_2 = 3$; $4c_2 = 1$; $c_2 = \frac{1}{4}$

$c_1 = \frac{1 - 2c_2}{2} = \frac{1 - 2\frac{1}{4}}{2} = \frac{1}{4} \longrightarrow T(n) = \frac{1}{4} n + \frac{1}{4} n\log(n)$

Ejercicio 4h:

$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right)^2 \begin{bmatrix} \text{cambio de base} \\ n = 2^k \end{bmatrix} \; T(2^k) = T(2^{k-1}) + T(2^{k-2})^2$

Vamos a hacer uso de logaritmos ya que no es lineal:

$\text{Log}(T(2^k)) - \text{Log}(T(2^{k-1})) - 2\,\text{Log}(T(2^{k-2})) = 0$

$u_k - u_{k-1} - 2u_{k-2} = 0$ Homogenea

$p(x) = x^2 - x - 2 = 0 \longrightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 \cdot 4 \cdot 1 \cdot (-2)}}{2} = \frac{1 \pm 3}{2} = \begin{cases} r_1 = 2 \\ r_2 = -1 \end{cases}$

$p(x) = (x-2)(x+1) = 0$

$u_k = c_1 \cdot 2^k + c_2 (-1)^k = 2^k c_1 - c_2 \begin{bmatrix} \text{cambio de base} \\ t_k = 2^{u_k} \end{bmatrix} t_k = 2^{2^k c_1 - c_2}$

$\begin{bmatrix} \text{cambio de base} \\ k = \text{Log}(n) \end{bmatrix} \begin{bmatrix} \text{cambio de base} \\ n = 2^k \end{bmatrix} \; T(n) = 2^{nc_1 - c_2} = \frac{2^{nc_1}}{2^{c_2}}$

$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right)^2 \; ; \; \left(\frac{2^{nc_1}}{2^{c_2}}\right) = \left(\frac{2^{\frac{n}{2}c_1}}{2^{c_2}}\right) + 2\left(\frac{2^{\frac{n}{4}c_1}}{2^{c_2}}\right)^2 \; ;$

$\frac{2^{\frac{n}{2}c_1}}{2^{c_2}} = \frac{2^{\frac{n}{2}c_1}}{(2^{c_2})^2} \; ; \; 2^{\frac{n}{2}c_1} = \frac{2^{\frac{n}{2}c_1}}{2^{c_2}} \; ; \; 2^{c_2} = 1 \; ; \; c_2 = \text{Log}(1) \; ; \; c_2 = 0$

$T(n) = \frac{2^{nc_1}}{2^{c_2}} = \frac{2^{nc_1}}{2^0} \; ; \; T(n) = 2^{nc_1}$