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Midgets of superior Mandelbrot set

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Abstract

Considering the complex-valued polynomial $Q_c(z) = z^n + c$, $n \geqslant 2$, we discuss midgets and related properties of superior Mandelbrot sets. A comparison of the usual and superior Mandelbrot sets for $Q_c(z)$ on the basis of midgets suggests that the two types of Mandelbrot sets are effectively different in many cases.

1. Introduction

Perhaps the Mandelbrot set is the most popular object in fractal theory. It is believed that it is not only the most beautiful object, which has been made visible but the most complex as well. This object was given by Benoît B. Mandelbrot in 1979 and has been the subject of intense research right from its advent. Mandelbrot set and its various extensions and variants have been extensively studied by using the Picard iterations (see, for instance, [2–6,8,9]).

Recently Rani and Kumar [13,14] have introduced superior iterations (essentially investigated by Mann [10] in nonlinear analysis) in the study of chaos and fractals. The purpose of this paper is to study the midgets and antennas of superior Mandelbrot sets for complex-valued quadratics and polynomials of higher degree. Our study shows that, in many cases, the SM sets are effectively different from the Mandelbrot sets.

2. Preliminaries

Let z_0 be an arbitrary element of the set of complex numbers C. Construct a sequence $\{z_n\}$ of points of C in the following manner:

$$z_n = sf(z_{n-1}) + (1-s)z_{n-1}, \quad n = 1, 2, \ldots,$$

where the parameter s lies in the closed interval $[0, 1]$.

The sequence $\{z_n\}$ constructed above, denoted by $SO(f, z_0, s)$, has been called the superior orbit for the complex-valued function with initial choice z_0 and parameter s. This kind of iterations were essentially introduced in 1953, by Mann [10], in nonlinear analysis. It has various applications in several areas of applicable mathematics (see, for

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instance, [1,7,15]). Notice that the superior iteration with $s = 1$ is the function iteration (also called Picard iteration). However, Rani [12] introduced it in fractal analysis and discrete dynamics for the first time in 2002, and called it superior orbit. Subsequently, the fractals generated using superior iterations instead of the function iterations were called superior Julia sets, superior Mandelbrot sets etc.

The superior Mandelbrot (SM) set for the complex-valued polynomial

$$Q_c(z) = z^n + c, \quad n \geqslant 2$$

is the collection of all $c \in C$ for which the superior orbit with $z_0 = 0$ is bounded, that is the SM set for $Q_c(z) = \{c \in C : \{Q_c^k(0) : k = 0, 1, 2, \ldots\}$ is bounded$\}$.

This definition is due to Rani [12] (see also [14]). We remark that the usual Mandelbrot set is a special case of the SM set, i.e., the SM set with $s = 1$ is the usual Mandelbrot set.

Escape criterions play a crucial role in generation and analysis of the SM set for a polynomial. The general escape criterion for the polynomial $Q_c(z) = z^n + c$ is

$$\max\left\{|c|, \left(\frac{2}{s}\right)^{\frac{1}{(n-1)}}\right\}, \quad n \geqslant 2.$$

For details, refer to Rani and Kumar [13,14].

3. Midgets of superior Mandelbrot set

The midgets of the SM set are the small mini Mandelbrot set like images found in the scattered surroundings of the SM set. The study of midgets in the Mandelbrot set is given by Philip [11] and Romera [16].

We have generated numerous SM sets for $Q_c(z)$ for various values of n. We find fascinating new fractals having several effectively different geometric shapes. However, we have selected a few figures (see Figs. 4–13) to study midgets in SM sets with $n = 2$, 4, 6, 8, 17 and 52, wherein $s = 0.1$ or 0.5. The Mandelbrot set consists of many small decorations or bulbs attached to the main body. The main body is called "main cardioid" by Devaney [2–5] and simply "cardioid" by

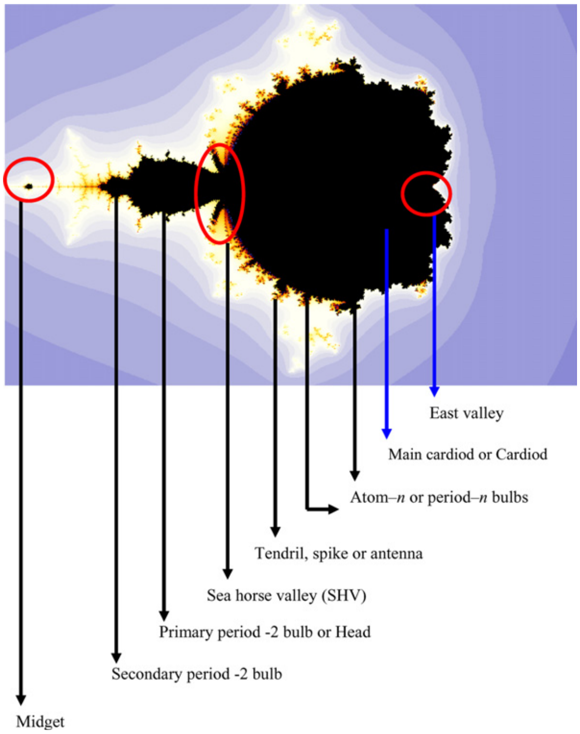

Fig. 1. Nomenclature of superior Mandelbrot set for $n = 2$, $s = 0.5$.

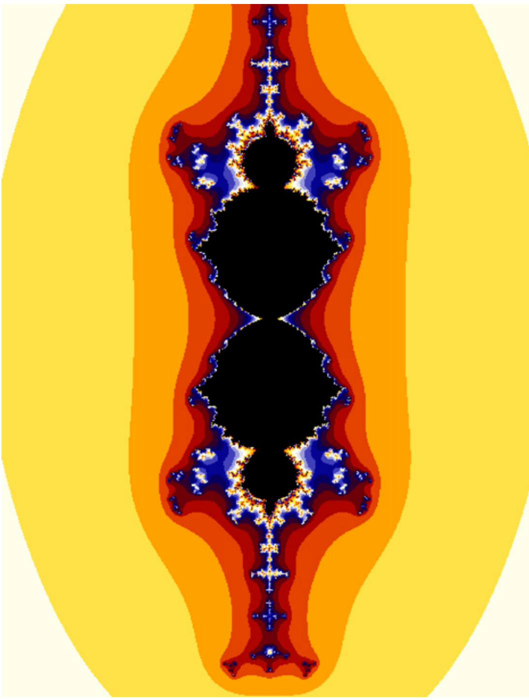

Fig. 2. Superior Mandelbrot set for $s = 0.1$ and $n = 3$.

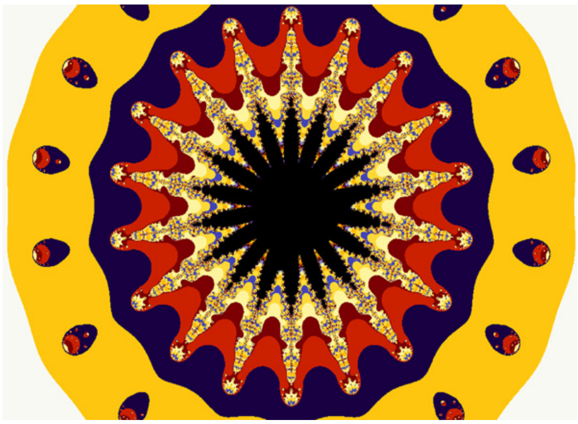

Fig. 3. Superior Mandelbrot set for $s = 0.1$ and $n = 19$.

Philip [11]. Similarly, the biggest bulb attached to the main cardiod is named as "period-2 bulb" by Devaney [2–5] and "head" by Philip [11].

Further, the small decorations attached to the main cardiod is called "atom-n" by Philip [11], and "period-n bulb" by Devaney [3], where n is the period of the bulb. Each period-n bulb has a main antenna attached to it. This antenna is named as "spikes" or "tendrils" by Philip [11]. Devaney has shown that the main antenna consists of a number of spokes attached; the number of spokes is the same as on the period of the corresponding bulb. In our study, we largely follow Devaney's nomenclature [3] and occasionally that of Philip [11]. For the relevant details concerning an SM set, see Fig. 1.

The SM set for $Q_c(z)$, with degree n contains $n - 1$ primary bulbs. We find that a period-2 bulb is connected to each of these $(n - 1)$ primary bulbs. Other period-n bulbs vanish as the value of s comes nearer to 0. Further, the main antenna looks disconnected for small values of s.

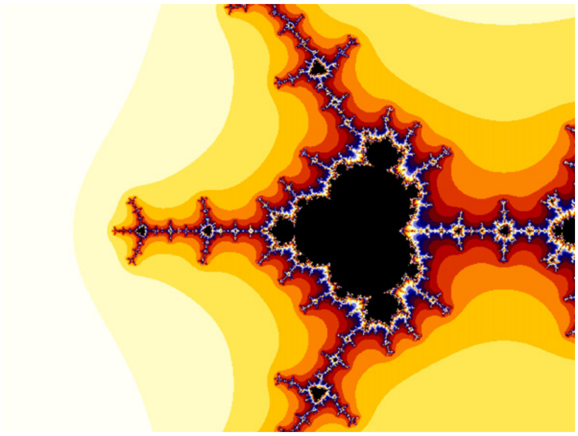

Fig. 4. Midgets of Mandelbrot set for $n = 4$.

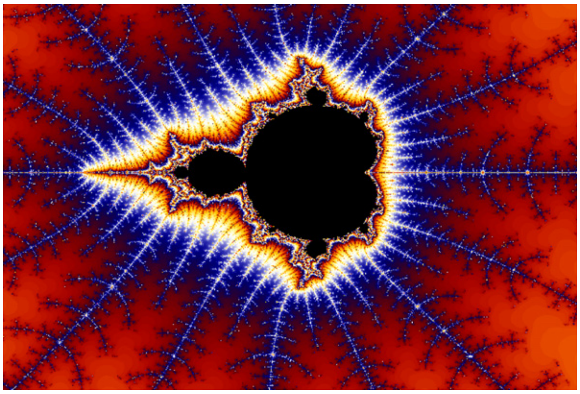

Fig. 5. Midgets of superior Mandelbrot set for $n = 4$ and $s = 0.1$.

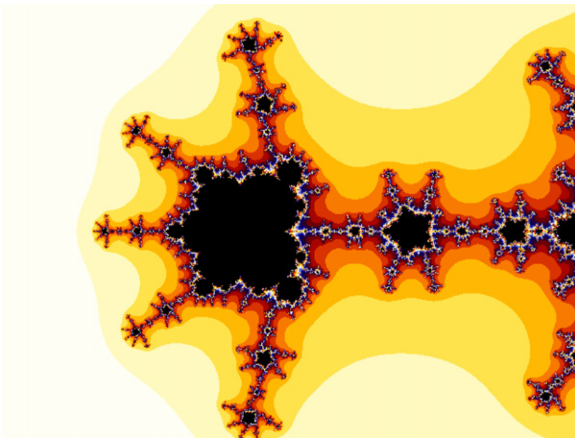

Fig. 6. Midgets of Mandelbrot set for $n = 6$.

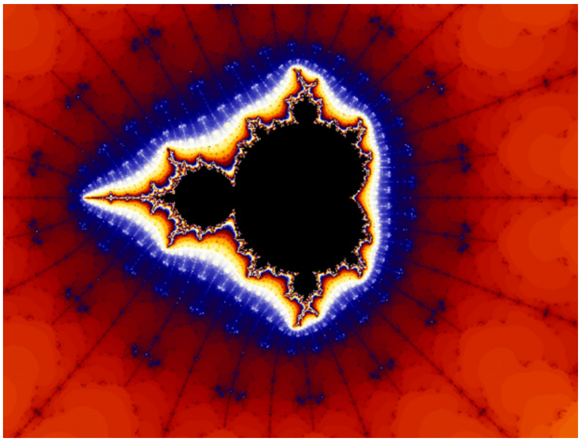

Fig. 7. Midgets of superior Mandelbrot set for $n = 6$ and $s = 0.1$.

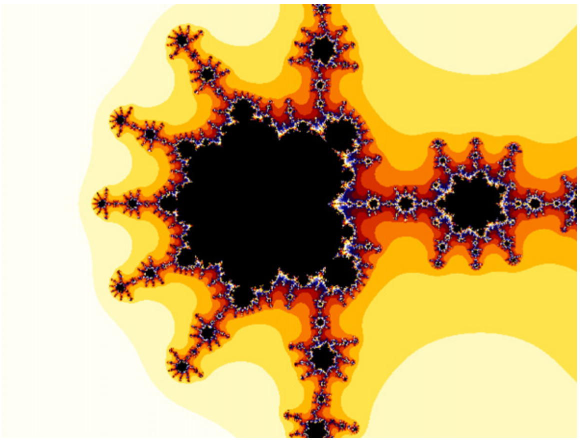

Fig. 8. Midgets of Mandelbrot set for $n = 8$.

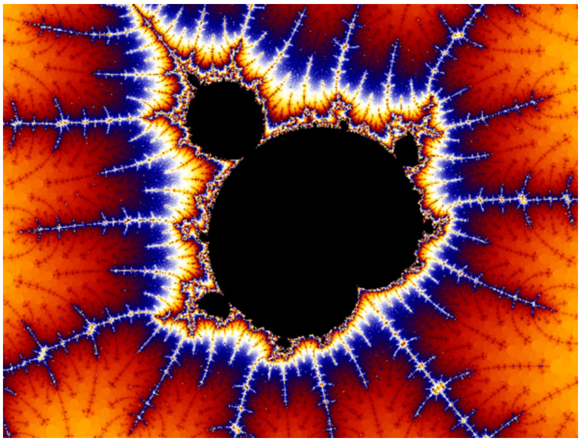

Fig. 9. Midgets of superior Mandelbrot set for $n = 8$ and $s = 0.1$.

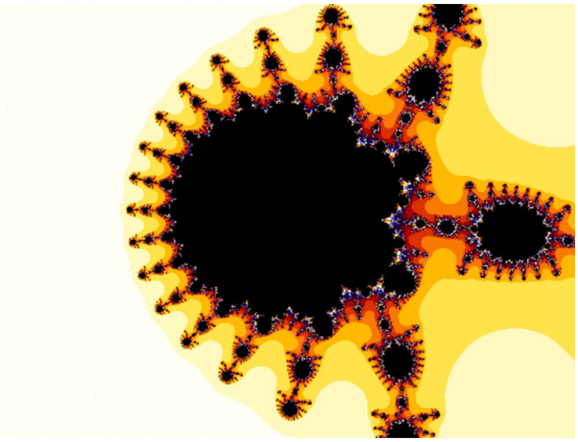

Fig. 10. Midgets of Mandelbrot set for $n = 17$.

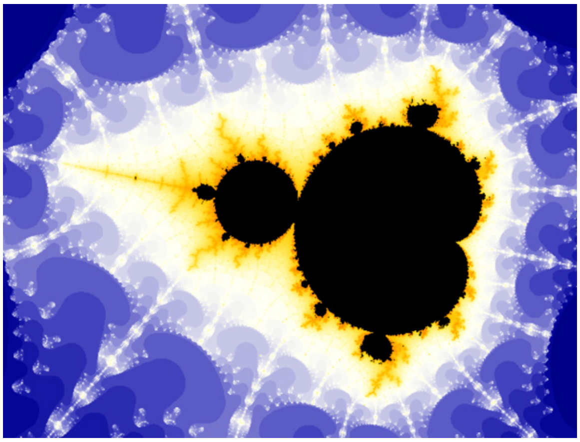

Fig. 11. Midgets of superior Mandelbrot set for $n = 17$ and $s = 0.1$.

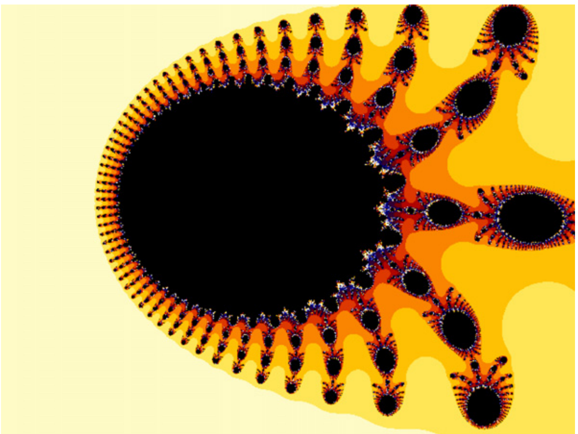

Fig. 12. Midgets of Mandelbrot set for $n = 52$.

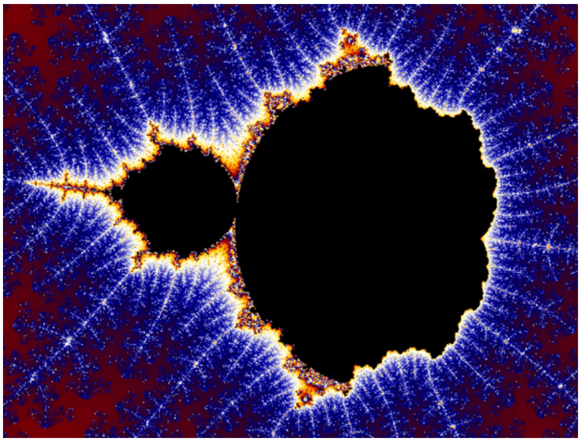

Fig. 13. Midgets of superior Mandelbrot set for $n = 52$ and $s = 0.1$.

Now we consider the following two cases.

Case I ($s = 1$, special case)
We observe that, when $s = 1$, the SM set for $n \geqslant 2$ is the Mandelbrot set of nth order having $n - 1$ main cardiods. On zooming, we find only one kind of midgets, i.e., mini Mandelbrot sets of nth order, which are near the apex or end of the branch. Further, we hasten to conclude from Figs. 4, 6, 8, 10 and 12 that the nature of midgets of Mandelbrot set of nth order is the mini Mandelbrot set of nth order.

Case II ($0 < s < 1$, general case)
For $n \geqslant 2$, we observe that, the SM set of nth order contains $n - 1$ distinct main cardioids (see Figs. 2 and 3). On zooming the SM set for $n \geqslant 2$, we get the midgets of mini Mandelbrot set on the main antennas of period-n bulbs. A remarkable feature is observed that, all the midgets are of order 2, i.e., the order of each of the midgets does not depend on the degree of the polynomial $Q_c(z)$, (see Figs. 5, 7, 9, 11 and 13). Notice that the order of midgets is not independent of the degree of the polynomial $Q_c(z)$ when $s = 1$ (*cf.* Case I). These observations show that, at least, some of the SM sets are effectively different from the usual Mandelbrot sets.

4. Conclusions

In the foregoing numerical analysis, we have considered the trend of appearance of midgets in SM sets for the polynomial $Q_c(z)$ when $n = 2, 4, 6, 8, 17$ and 52. We find that the number of primary bulbs is $(n - 1)$. Following Devaney's [3] system of counting the periodicity, we assert that these are indeed period-2 bulbs. Another remarkable phenomenon is that the number of iterations required to achieve convergence for a particular choice of the parameter c, is much less in the superior orbit as is evident from Table 1.

Table 1
Number of iterations required to achieve convergence for $Q_c(z)$, taking $c = (-0.4, 0)$ for different values of s

s	Number of iterations needed		Fixed point of convergence
	Function iterates	Superior iterates	
1.0	24	24	$(-0.30623, 0.0)$
0.9	–	14	$(-0.30623, 0.0)$
0.8	–	8	$(-0.30623, 0.0)$
0.7	–	5	$(-0.30623, 0.0)$
0.6	–	4	$(-0.30623, 0.0)$

Once again following Devaney [3], we know that the antennas of Mandelbrot sets are well connected to their corresponding bulbs. However, in our figures (see Figs. 15–17), antennas are somewhat disconnected with an exception of period-2 bulb (see Fig. 14).

We believe that there is a wide scope of further study of the geometry of SM sets. The work presented in this paper is just a mild beginning.

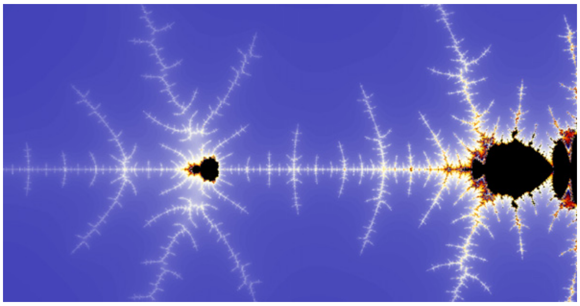

Fig. 14. Connected antenna of period-2 bulb.

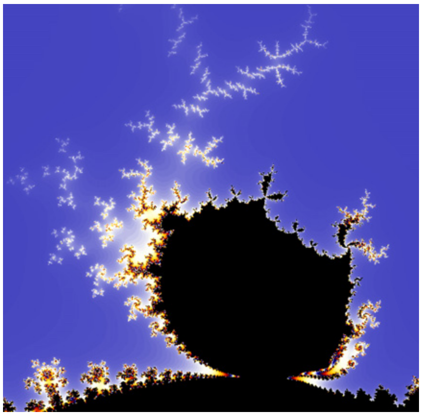

Fig. 15. Disconnected antenna of period-3 bulb.

Fig. 16. Disconnected antenna of two different period-5 bulbs.

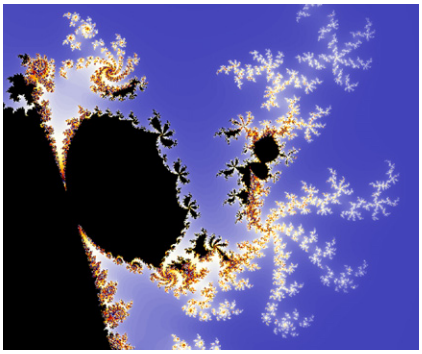

Fig. 17. Disconnected antenna of two different period-5 bulbs.

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