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On the cusp and the tip of a midget in the Mandelbrot set antenna

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Abstract

We study the limit of the harmonics and antiharmonics of a pattern that originated from a tangent bifurcation. As a result, two simple formulae to calculate the pattern of the cusp and the tip of a midget in the Mandelbrot set antenna are deduced, and the nature of the tip (a noncharacteristic Misiurewicz point) is shown.

Keywords: One-dimensional quadratic maps; Mandelbrot set; Midget; Misiurewicz points; Tip; Cusp; Antenna

1. Introduction

One-dimensional quadratic maps, like the logistic map $x_{n+1} = \lambda x_n(1 - x_n)$ or the real Mandelbrot map $x_{n+1} = x_n^2 + c$, normally are graphically studied by means of their bifurcation diagram. We have proposed to use the real axis neighbourhood (the antenna) of the Mandelbrot-like set of the corresponding complex quadratic map, which offers graphic advantages [1]. In fact, a narrow window is then replaced for a midget whose main filament gives us valuable information about the period of the midget (the period of the window) when the Mandelbrot-like set is drawn with the escape line method [2]. As is well known, all the one-dimensional quadratic maps are topologically conjugate, and the parameter values of the real Mandelbrot map and the logistic map are related by $\lambda = 1 + \sqrt{1 - 4c}$.

In a previous paper [3] we have introduced the F-harmonics and the F-antiharmonics of a pattern (in memory of Fourier), and we have compared them with the MSS-harmonics and MSS-antiharmonics of Metropolis, Stein and Stein [4]. The F-harmonics and F-antiharmonics are very important in the ordering of one-dimensional quadratic maps [5].

Equally well known is that the *Feigenbaum point* [6] *pattern of a midget is the limit of the MSS-harmonics of the midget pattern when the period of the harmonics increases indefinitely* [4]. This pattern exhibits a self-similar symbolic dynamics, as pointed out by Schroeder [7]. In Section 2 we will see that the limit of the F-harmonics of the pattern of a midget, when the period of the harmonics increases indefinitely, is the pattern of the midget antenna tip (the closing of the window in the bifurcation diagram). In Section 3 we will see that the limit of MSS-antiharmonics and F-antiharmonics of the pattern of a midget, when the period of the antiharmonics increases indefinitely, is the pattern of the midget

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cusp (the opening of the window in the bifurcation diagram).

2. On the limit of F-harmonics

We will name a midget with the pattern of its main cardioid. In the upper part of Fig. 1 we can see a sketch of the period-3 CLR midget in the Mandelbrot set antenna. This primary midget has its own antenna with an infinity of microscopic secondary midgets of periods $3 \cdot 3$, $3 \cdot 4$, $3 \cdot 5, \ldots$ whose sizes are smaller and smaller. The end of the midget antenna, the *tip* [8, p. 59] is a point (a midget of null size, whith no attracting periodic interval). Note also that there are a lot of microscopic primary midgets on the left, near the tip of the CLR midget antenna, but their periods are not $\dot{3}$. According to Grebogi et al. [9] the closing of the CLR window (the CLR midget antenna tip) is located at $-1.790327492\ldots$.

In the lower left part of Fig. 1, the patterns of the microscopic midgets (until period-24) near the CLR midget antenna tip can be seen. We have obtained these patterns by means of an adaptation of the MSS algorithm [4, Theorem 1] from the logistic map to the real Mandelbrot map. So, starting from period-3 CLR and period-4 CLR^2 midgets we obtain the intermediate midget $\mathrm{CLR}^2\mathrm{L}$ by using $H_{\mathrm{MSS}}(\mathrm{CLR}) = \mathrm{CLR}^2\mathrm{LR}$, $A_{\mathrm{MSS}}(\mathrm{CLR}^2) = \mathrm{CLR}^2\mathrm{L}^2\mathrm{R}^2$ and $H_{\mathrm{MSS}}(\mathrm{CLR}) \cap A_{\mathrm{MSS}}(\mathrm{CLR}^2) = \mathrm{CLR}^2\mathrm{L}$. Note that on the left of the tip we have obtained period-$(\dot{3}+1)$ and period-$(\dot{3}+2)$ midgets, but on the right of the tip we have obtained period-$\dot{3}$ midgets whose patterns have the general form $\mathrm{CLR}^2\overline{\mathrm{LRL}}^{n-1}\mathrm{LR}$ ($n = 1, 2, 3, \ldots$), where we write $n-1$ times LRL as $\overline{\mathrm{LRL}}^{n-1}$.

The patterns of these period-$\dot{3}$ midgets can also be found by the Derrida et al. [10] composition rule (a deduction of this rule can be seen in Ref. [11]). It is well known that the last appearance primary midgets have patterns of the form CLR^n [7,11]. For these midgets, the composition rule gives $\mathrm{CLR} * \mathrm{CLR}^n = \mathrm{CLR}^2\overline{\mathrm{LRL}}^{n-1}\mathrm{LR}$. Then the patterns of the midgets of periods $6, 9, 12, 15\ldots$ on the right of the CLR midget antenna tip correspond to the last appearance midgets generated by CLR. Moreover, the pattern of a F-harmonic of CLR has also the general form $H_{\mathrm{F}}^{(n)}(\mathrm{CLR}) = \mathrm{CLR}^2\overline{\mathrm{LRL}}^{n-1}\mathrm{LR}$.

Then, *we can obtain the pattern of the midget antenna tip as the limit of the F-harmonics of the pattern of the midget main cardioid.* Let P be the pattern of the main cardioid of a period-p midget, and let Q be the same pattern without its initial letter C. Then $P = \mathrm{C}Q$. We consider two cases:

(a) If the pattern P has odd L-parity [3], we have $H_{\mathrm{F}}^{(1)}(P) = P\mathrm{R}Q$, $H_{\mathrm{F}}^{(2)}(P) = P\mathrm{R}Q\mathrm{L}Q$, $H_{\mathrm{F}}^{(3)}(P) = P\mathrm{R}Q\mathrm{L}Q\mathrm{L}Q, \ldots$ and the pattern of the midget antenna tip can be written as

$$\mathrm{tip}(P) = (P\mathrm{R})Q\mathrm{L}. \tag{1}$$

(b) If the pattern P has even L-parity, we have $H_{\mathrm{F}}^{(1)}(P) = P\mathrm{L}Q$, $H_{\mathrm{F}}^{(2)}(P) = P\mathrm{L}Q\mathrm{R}Q$, $H_{\mathrm{F}}^{(3)}(P) = P\mathrm{L}Q\mathrm{R}Q\mathrm{R}Q, \ldots$ and the pattern of the midget antenna tip can be written as

$$\mathrm{tip}(P) = (P\mathrm{L})Q\mathrm{R}. \tag{2}$$

Eqs. (1) and (2) show that the tip of the antenna of a midget is a $(p+1)$ preperiodic and p periodic noncharacteristic Misiurewicz point [5]. For example, the Mandelbrot set antenna tip is the preperiod-2 and period-1 Misiurewicz point tip(C) = (CL)R (a special case, because the Mandelbrot set ends in this tip), and the CLR midget antenna tip is a preperiod-4 and period-3 noncharacteristic Misiurewicz point $\mathrm{tip(CLR)} = (\mathrm{CLR}^2)\mathrm{LRL}$.

We can obtain the parameter value of a midget antenna tip by two methods. The first makes use of the Feigenbaum algorithm [6, Appendix B] for the adequate scaling constant. The scaling of the last appearance secondary midgets should be the same scaling of the last appearance primary midgets on account of the self-similarity. The value of this scaling constant is 4 [12]. The second method is based on obtaining the parameter value of the Misiurewicz point $M_{p+1,p}$ near the midget, solving $\mathrm{P}_{2p+1} = \mathrm{P}_{p+1}$ [5,9]. In this way, we have obtained $M_{4,3} = -1.79032749199934\ldots$ for the CLR midget antenna tip.

3. On the common limit of MSS and F-antiharmonics

Let us consider the real Mandelbrot map. At $c = \frac{1}{4} = 0.25$ a tangent bifurcation occurs, where a sta-

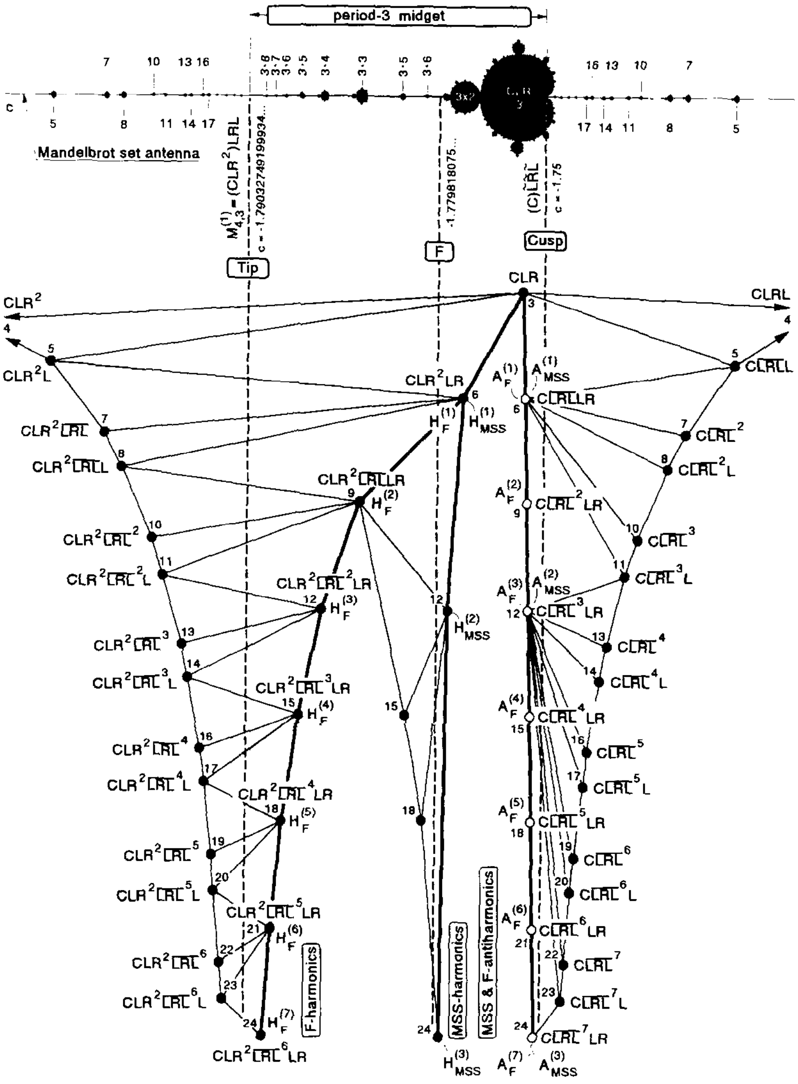

Fig. 1. The cusp and the tip of the CLR midget in the Mandelbrot set antenna.

ble and an unstable fixed point are created. As has been pointed out [9], the logistic map, when $\lambda = 1$, does not possess an analogous bifurcation because the Mandelbrot-like set of its complex form does not have a main cardioid but a main disc: let us compare Ref. [13, Fig. 2a] and Ref. [14, Fig. 1]. At $c = -\frac{7}{4} = -1.75$ another tangent bifurcation occurs where a stable period-3 orbit appears, in the same way as the logistic map when $\lambda = 1 + \sqrt{8}$ [15].

Let us now consider the Mandelbrot set. At each of $c = 0.25 + 0i$ and $c = -1.75 + 0i$ there is a *cusp* [16,17 (p. 859),18,19]. In the first case we refer to the cusp in the heart-shaped part of the Mandelbrot set, the Mandelbrot set cusp. In the second one we refer to the main cardioid cusp of the period-3 midget, the period-3 midget cusp. The cusp has been also named *butt* [17, p. 859], *root point* [16] and *apex* [20]. In the upper part of Fig. 1 we can see the cusp of the period-3 midget. On the left of the cusp is the Feigenbaum region (a large attracting interval), but on the right of the cusp there are a lot of microscopic primary midgets.

In the lower part of Fig. 1, the pattern of the microscopic midgets (until period-24) near the CLR midget cusp can be seen. We have obtained these patterns by means of the MSS algorithm (as in Section 2). On the left of the cusp (but on the right of the CLR center) we obtain period-$(3 \cdot 2^n)$ ($n = 1, 2, 3 \ldots$) patterns but, as is well known, they are MSS-antiharmonics and they do not correspond to any midget. The general form of one of these patterns is $A_{\rm MSS}^{(n)}(\rm CLR) = C\overline{LRL}^{2^n-1}LR$ ($n = 1, 2, 3, \ldots$). We have drawn the MSS-antiharmonics as empty dots (they are ghosts) between the center of the midget main cardioid and the midget cusp. On the right of the cusp we have obtained period-$(\dot{3}+1)$ and period-$(\dot{3}+2)$ midgets, whose patterns have the general forms $\rm C\overline{LRL}^{\it m}$ and $\rm C\overline{LRL}^{\it m}L$ ($m = 1, 2, 3, \ldots$) respectively. In the figure we have also drawn the F-antiharmonics $A_{\rm F}^{(m)}(\rm CLR) = C\overline{LRL}^{\it m}LR$ ($m = 1, 2, 3, \ldots$) as empty dots. The MSS-antiharmonics and the F-antiharmonics have the same pattern when they have the same period [3], as can be seen in the figure. Then, *we can obtain the pattern of the cusp of a midget as the common limit of the MSS and F-antiharmonics of the pattern of the midget main cardioid.*

Let P be the pattern of the main cardioid of a period-p midget, and let Q be the same pattern without its initial letter C. Then $P = CQ$. We consider two cases:

(a) If the pattern P has odd L-parity, we have $A_{\rm F}^{(1)}(P) = CQLQ$, $A_{\rm F}^{(2)}(P) = CQLQLQ$, $A_{\rm F}^{(3)}(P) = CQLQLQLQ$, ... and finally $A_{\rm MSS}^{(\infty)}(P) = C\overline{QL}^{\infty}$. Note that $A_{\rm MSS}^{(\infty)}(P)$ is an antiharmonic, and therefore $C\overline{QL}^{\infty}$ is a periodic orbit that cannot exist. This is known as well, and we can verify it with a simple pocket calculator iterating the real Mandelbrot map for the initial value $x_0 = 0$, that in a cusp there is an *asymptotically fixed point* (i.e. $c = 0.25$) or an *asymptotically periodic* [21] *orbit* (i.e. $c = -1.75$). It is also well known that a tangent bifurcation occurs in a neutral (indifferent) point [16,17], and that in a tangent bifurcation intermitency can occur [22,23]. Then, we write the midget cusp pattern expression as

$$\text{cusp}(P) = (\text{C})\tilde{Q}\tilde{\text{L}}. \tag{3}$$

where $\tilde{Q}\tilde{\rm L}$ is an asymptotically periodic orbit that begins after the first iteration C.

(b) If the pattern P has even L-parity, we have

$$\text{cusp}(P) = (\text{C})\tilde{Q}\tilde{\text{R}}. \tag{4}$$

For example, the pattern of the Mandelbrot set cusp is $\text{cusp(C)} = (\text{C})\tilde{\text{R}}$, and the pattern of the CLR midget cusp is $\text{cusp(CLR)} = (\text{C})\tilde{\text{L}}\tilde{\text{R}}\tilde{\text{L}}$.

Obtaining the parameter value of the cusp of a midget is not easy. We can try to draw the zone and magnify it until reaching a measurement of the desirable precision: the number of iterates must be very large, and the computation time very high. We can use the algebraic method due to Stephenson [20]: the size of polynomials is enormous. We can also use a two-dimensional modified Newton method, as did Farmer [24]: the method is difficult. Finally, we can apply the Feigenbaum algorithm to microscopic midgets on the right of the cusp: the scaling constant is very low (it is possible that it tends to one) and the method perhaps cannot be used.

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