ANTIPODE FORMULAS FOR PATTERN HOPF ALGEBRAS

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Abstract. Computing antipodes in Hopf algebras is notoriously difficult. Takeuchi's classical formula applies broadly but is computationally unwieldy, riddled with unnecessary cancellations and groupings. This work develops systematically **cancellation-free and grouping-free** antipode formulas for a rich family of Hopf algebras constructed from combinatorial structures—the pattern Hopf algebras.

The key object is the **permutation pattern Hopf algebra**, where the product counts permutation patterns (via quasi-shuffle signatures) and the coproduct respects decomposition. This algebra fits a general framework via species with restrictions, unifying permutation patterns, graphs, parking functions, and beyond.

Our main contribution is a cancellation-free and grouping-free antipode formula for the permutation pattern Hopf algebra, derived via the sign-reversing involution method. Rather than merely theoretical, this formula has concrete power: it directly yields a new polynomial invariant on permutations—the Multiple Occurrences Polynomial—and reveals reciprocity interpretations when evaluating this polynomial at negative integers.

We also develop formulas for the packed word pattern Hopf algebra and introduce an original species with restrictions structure on parking functions, recovering recent results on parking function avoidance. Throughout, the interplay between algebraic structure and combinatorial form creates an elegant and powerful framework.

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1. Introduction

A central problem in combinatorics is to extract combinatorial interpretations of algebraic invariants. Hopf algebras provide a powerful framework for this, encoding combinatorial structures with algebraic operations. However, computing antipodes in Hopf algebras—the algebraic analogue of inversion—remains notoriously difficult. Takeuchi's celebrated formula (see [Tak71], Lemma 14) offered the first general approach, applying to any filtered Hopf algebra. Yet this formula is computationally unwieldy: it produces intermediate cancellations and requires grouping terms to simplify the final result.

In this paper, we develop economical antipode formulas for **pattern Hopf algebras**—a rich family of Hopf algebras constructed from combinatorial structures via species with restrictions. These algebras generalize classical objects like the permutation pattern Hopf algebra, and connect combinatorics directly to algebra. Our main contribution is a systematic method to derive **cancellation-free and grouping-free** antipode formulas, which not only simplify computation but reveal hidden combinatorial structure.

An antipode formula is **cancellation-free** if every term in its expansion contributes fully to the final answer, with no intermediate cancellations. Intuitively, if you sum contributions from many combinatorial objects, each with a ± 1 sign, you want every term to "survive" in the final sum—no term should be negated by another, wasting computational effort. Similarly, a formula is **grouping-free** if combinatorial objects are counted independently: you never need to aggregate or combine multiple objects to simplify the computation. These properties are not just aesthetically pleasing; they reveal that the formula directly reflects the underlying combinatorial structure without algebraic noise.

Economical formulas for antipodes in Hopf algebras in combinatorics have played an important role in extracting old and new combinatorial equations, see [Sch93, HM12, BS17, AA17, XY22]. In particular, we argue Humpert and Martin stand out, as they were able to explain, in [HM12], an elusive *reciprocity relation* on graphs first presented by Stanley in [Sta75], where the number of acyclic orientations plays a role.

A fundamental insight in enumerative combinatorics is that algebraic morphisms often correspond to important combinatorial invariants. For instance, the chromatic polynomial χ_G of a graph counts proper vertex colorings; evaluating it at $x = -1$ yields the number of acyclic orientations—a nontrivial result discovered by Stanley. Such evaluations at negative integers are called **reciprocity relations**, and they often reveal surprising combinatorial structure.

More generally, a polynomial χ that is a Hopf algebra morphism commutes with the antipode: $\chi(-x) = \chi(S(h))$. This means that if we have a combinatorial interpretation of the antipode $S(h)$, we automatically obtain a combinatorial interpretation of χ evaluated at negative integers. Hence, developing cancellation-free antipode formulas directly yields new reciprocity results. This is the engine driving our application in permutation patterns: the formula we develop reveals a new polynomial invariant and its reciprocity interpretation.

The *sign-reversing involution method* (introduced by Sagan and Benedetti in [BS17]) offers a systematic approach: design a combinatorial involution that pairs most terms with opposite signs, causing them to cancel; the surviving terms yield a cancellation-free formula. This elegant technique has succeeded for several Hopf algebras: the shuffle Hopf algebra, the incidence Hopf algebra on graphs, and quasisymmetric functions (see [BS17]). However, the method is far from automatic. It requires deep understanding of the specific algebra's structure, and constructing the involution is often highly nontrivial.

Notably, computing the antipode for permutation patterns has resisted these methods. While Foissy's bialgebra of cointeraction approach yielded results [Foi22], and Xu achieved partial results for the Malvenuto–Reutenauer algebra [XY22], only restricted families of permutations were handled. The permutation pattern algebra remains largely out of reach—until now. Our main contribution is the first cancellation-free and grouping-free antipode formula for permutation patterns, achieved by carefully adapting the sign-reversing involution method to this previously intractable family.

Another method for finding cancellation-free formulas arose in [AA17]. There, a cancellation-free antipode formula for a Hopf structure on generalized permutahedra was found. Because several interesting combinatorial structures can be embedded in the Hopf algebra of the generalized permutahedra, this also yields a cancellation-free formula for these Hopf subalgebras. Specific examples are graphs, matroids, posets and set partitions. Yet another method was found in [Foi22], where the notion of *bialgebra of cointeraction* was used to obtain antipode formulas for the permutation pattern Hopf algebra.

Permutation patterns—the study of which smaller patterns appear within larger permutations—is a rich classical subject with roots in computer science. Knuth's pioneering work on stack-sortable permutations [Knu68] sparked decades of research, establishing permutation pattern avoidance as a central problem in enumerative combinatorics. However, classical permutation pattern theory focuses on counting and avoiding patterns; algebraic structure remained elusive.

This motivated the second author to introduce the **permutation pattern Hopf algebra** [Var14], elevating permutation patterns to the level of algebraic Hopf algebras. This construction revealed new structure and opened the door to using tools from Hopf algebra theory—including antipode formulas—to understand permutation patterns. Our paper extends this program: we develop a cancellation-free antipode formula specifically for the permutation pattern Hopf algebra, and show how it yields new reciprocity results. Specifically, we consider finite sums of functions $\mathbf{pat}_\pi$ of the form

$$\binom{\tau}{\pi} \coloneqq \mathbf{pat}_\pi(\tau) \coloneqq \#\{ \text{ ways to fit } \pi \text{ in } \tau \ \},$$

the central functions in the study of permutation patterns. These functions span a vector space that is closed for pointwise product, see (1), and a compatible coproduct, see (2). The corresponding Hopf algebra is the **permutation pattern Hopf algebra** $\mathcal{A}(\mathtt{Per})$, and is shown to be free in [Var14]. For a definition of patterns in permutations, or of "ways to fit π in σ", see [Pen20]

The permutation pattern Hopf algebra construction was generalized to other combinatorial objects by the first author [Pen22], for instance graphs, marked permutations or set partitions. This was done in such a way that any combinatorial object enriched with restriction functions, a structure that we call a **species with restrictions**, yields a **pattern Hopf algebra** (a construction presented in [Pen22], that we recover here in Theorem 4.9). The algebraic structure of species with restrictions is a generalisation of the classical algebraic structure of combinatorial species from Joyal, see [AM10].

We present a general antipode formula (Theorem 6.11) for any pattern Hopf algebra, derived via the sign-reversing involution method. While this general formula does not guarantee cancellation-free and grouping-free expressions, the method reveals deeper structure that enables specializations to specific algebras. We exploit this to derive our main theorems: **cancellation-free and grouping-free antipode formulas for both packed word patterns** (Theorem 7.4) **and permutation patterns** (Theorem 7.6).

These concrete formulas are not merely theoretical—they directly yield new reciprocity results.

Onwards we introduce a new polynomial invariant in permutations, by considering the evaluation character in the permutation pattern Hopf algebra, see Definition 8.1. As far as we know, this is a new family of invariants in permutations. We apply the cancellation-free formula to interpret this polynomial in negative values, leading to a reciprocity result via the antipode formula. Specifically, we show that the value of this polynomial at $x = -1$ counts so called **interlacing quasi-shuffle signatures** of specific permutations—a restricted type of decomposition where pattern occurrences maintain a specific ordering, see definition below.

A key question in the broader program is: which combinatorial objects can be faithfully represented within the species with restrictions framework? Some objects are naturally suited—graphs and set partitions have intrinsic labelings that match the framework directly. However, many unlabeled objects resist naive encoding. Permutations, for instance, required an ingenious idea: interpreting them as double orders to capture their inherent structure.

Parking functions present a similar challenge. These objects—sequences encoding how cars park along a road by occupying the first available spot—are rich combinatorial structures with deep connections to Dyck paths and the Catalan numbers. Yet their lack of intrinsic labeling makes representing them as species with restrictions nontrivial. We resolve this by exploiting a beautiful bijection between labeled Dyck paths and parking functions [Loe11, BDL$^+$23]. This yields a novel species with restrictions on parking functions, enabling us to study parking function patterns within our framework. As a bonus, our pattern definition recovers recent results on parking function avoidance [AP22].

Structure of This Paper. We organize our contributions as follows. The remainder of this introduction develops the permutation pattern Hopf algebra and states our main theorems formally (Theorem 7.4 and **??**), emphasizing the combinatorial significance of interlacing quasi-shuffle signatures.

Section 2 reviews Takeuchi's general formula and illustrates the sign-reversing involution method through examples, providing context for our specialized approach.

Section 3 and Section 4 build the foundational machinery: combinatorial species and the general pattern Hopf algebra construction. Readers familiar with this material may skip ahead.

Section 5 introduces species with restrictions, developing permutation patterns, packed words, and our novel encoding of parking functions.

Section 6 and Section 7 present our main technical contributions: the general antipode formula and its specialization to cancellation-free formulas for packed words and permutations.

Finally, Section 8 demonstrates the power of our formulas by introducing the Multiple Occurrences Polynomial and deriving new reciprocity results that interpret this polynomial at negative integers.

Roadmap of Results. Before diving into technical details, let us sketch the key ideas:

(1) We derive a general antipode formula for any pattern Hopf algebra using the sign-reversing involution method, though it lacks the cancellation-free property in general.
(2) We specialize this to permutation patterns and carefully construct a combinatorial involution that yields cancellation-free and grouping-free formulas.
(3) We introduce a new polynomial invariant on permutations—the **Multiple Occurrences Polynomial** (MOP)—and apply our formula to derive reciprocity results interpreting this polynomial at negative integers.

The power of this program is that the algebra (Hopf structure) and combinatorics (cancellation-free formulas, interlacing quasi-shuffle signatures) reinforce each other, creating a coherent and elegant narrative.

1.1. **The permutation pattern Hopf algebra and the main result.** In this section we introduce the Hopf algebra structure on permutation patterns, formalize the notion of cancellation-free and grouping-free formulas, and present the main result of this paper: a cancellation-free and grouping-free antipode formula for the permutation pattern Hopf algebra.

Let $\mathbf{pat}_\pi$ be a function on permutations, so that $\mathbf{pat}_\pi(\sigma)$ counts the number of restrictions of the permutation σ that fit the pattern π. In this way, the collection of permutation pattern functions $\{\mathbf{pat}_\pi\}$ is linearly independent, so it is a basis of a vector space $\mathcal{A}(\texttt{Per})$. Further, in [Var14], it was shown that the pointwise product of two such functions can be expressed as a sum of other permutation pattern functions:

$$\mathbf{pat}_{\pi_1}\,\mathbf{pat}_{\pi_2} = \sum_\sigma \binom{\sigma}{\pi_1, \pi_2} \mathbf{pat}_\sigma\,, \tag{1}$$

where the sum runs over all permutations σ of any size. If $|\sigma| > |\pi_1| + |\pi_2|$ we have $\binom{\sigma}{\pi_1,\pi_2} = 0$, in fact, the coefficients $\binom{\sigma}{\pi_1,\pi_2}$ that arise in this product formula count the so called **quasi-shuffle signatures**, or **QSS**, of σ from π_1, π_2. Let us make this concept precise:

Definition 1.1 (QSS on permutations)**.** A quasi-shuffle signature, or QSS, of σ from $\pi_1, \ldots, \pi_n$ is a tuple $\vec{\mathbf{I}} = (I_1, \ldots, I_n)$ of sets on the ground set of the permutation σ, that cover this ground set in such a way that each I_i is a pattern of π_i in σ — that is the restricted permutations $\sigma|_{I_i}$ are the permutation π_i for any i. Unlike standard shuffles, these sets can overlap and must jointly cover the entire ground set.

The sets $I_1, \ldots, I_n$ are not necessarily disjoint but their union must equal the ground set I of σ. We let $\binom{\sigma}{\pi_1,\ldots,\pi_n}$ denote the number of such QSS. It was shown in [Pen22] that $\binom{\sigma}{\pi_1,\ldots,\pi_n}$ is precisely the coefficient that arises in the iterated product of n elements. This fact justifies the notation choice in (1).

This algebra $\mathcal{A}(\texttt{Per})$ can be endowed with a Hopf algebra structure with the help of the diagonal sum of permutations, $\oplus$, also called shifted concatenation of permutations. If one lets $\pi = \pi_1 \oplus \cdots \oplus \pi_n$ be the decomposition of π into $\oplus$-indecomposable permutations under the $\oplus$ product, we define the shifted deconcatenation coproduct

$$\Delta\, \mathbf{pat}_\pi = \sum_{k=0}^{n} \mathbf{pat}_{\pi_1 \oplus \cdots \oplus \pi_k} \otimes \mathbf{pat}_{\pi_{k+1} \oplus \cdots \oplus \pi_n} \in \mathcal{A}(\texttt{Per}) \otimes \mathcal{A}(\texttt{Per})\,. \tag{2}$$

To present a cancellation-free antipode formula we need to introduce a restricted subclass of QSS. For sets A, B of integers, we write $A < B$ if $a < b$ for any $a \in A, b \in B$.

Definition 1.2 (Interlacing QSS on permutations)**.** A QSS $\vec{\mathbf{I}} = (I_1, \dots, I_n)$ is said to be **non-interlacing** if there exists an index $i \in \{1, \dots, n-1\}$ where both orderings align: $I_i < I_{i+1}$ and $\sigma(I_i) < \sigma(I_{i+1})$. Intuitively, this means the sets appear in the same relative order in both the positions and values.

Otherwise, we say that the QSS is **interlacing**. In this case, the sets do not maintain a consistent ordering across both positions and values—they are "mixed" or "crossed" in some way. This property is crucial because it allows cancellation-free antipode formulas to work: the constraint of interlacing reduces the complexity of the sum.

Importantly, reordering an interlacing QSS does not in general give an interlacing QSS. This is in contrast with a simple QSS, where a reordering does result in a QSS. This distinction is what makes interlacing QSS a powerful tool for deriving efficient antipode formulas.

We let $\left[{\sigma \atop \pi_1,\dots,\pi_n}\right]$ denote the number of interlacing QSS.

Theorem 1.3 (Antipode formula for permutation pattern Hopf algebra)**.** Let π be a permutation that factors $\pi = \pi_1 \oplus \cdots \oplus \pi_n$ into $\oplus$-indecomposable permutations. Then, we have the following formula for the antipode of $\mathbf{pat}_\pi$:

$$S(\mathbf{pat}_\pi) = (-1)^n \sum_{\sigma} \left[{\sigma \atop \pi_1,\dots,\pi_n}\right] \mathbf{pat}_\sigma\,,$$

where the sum runs over all permutations σ, and the coefficients count the number of **interlacing QSS** of σ from $\pi_1, \dots, \pi_n$.

Note that for $|\sigma| > \sum_i |\pi_i|$, the coefficient $\left[{\sigma \atop \pi_1,\dots,\pi_n}\right]$ vanishes, so this sum is finite.

In the following we present some examples that help explain QSS and interlacing QSS.

Example 1.4 (Interlacing QSS)**.** The permutation 2314 has several QSS from $1, 21, 1, 1$, for instance $(4, 13, 2, 1)$ and $(1, 13, 4, 2)$, but from these two, only the first is interlacing. In Fig. 1, one can see these two QSS. Further computations can show that $\binom{2314}{1,21,1,1} = 36$ and $\left[{2314 \atop 1,21,1,1}\right] = 8$.

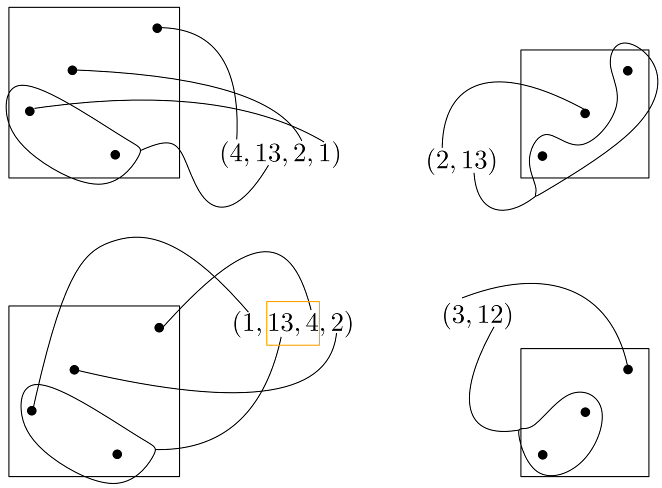


FIGURE 1. **Left:** the permutation 2314, along with a labelling of two of its QSS from $1, 21, 1, 1$. In orange the two sets that do not interlace. **Right:** the permutation 123, along with its two interlacing QSS from $1, 12$.

One can observe that there are three QSS of 123 from 1, 12, but one of them is non-interlacing (the QSS $(1, 23)$), so $\left[{123 \atop 1,21}\right] = 2$. In Fig. 1, one can see these two interlacing QSS.

2. Computing the antipode

In this section we explore the antipode of a Hopf algebra using Takeuchi's formula. We explore examples on the polynomial algebra and on the permutation pattern Hopf algebra. We start by recalling Takeuchi's formula, in the form that is presented in [GR14], as well as some convenient notation. Let us define the $\star$ notation (convolution product) on linear maps $a, b : C \to A$. Whenever A is an algebra, and C is a coalgebra, we define:

$$a \star b := \mu_A \circ (a \otimes b) \circ \Delta_C \,,$$

which is an associative and unitary product on linear maps from C to A. We will be focused on the case when $C = A = H$ is a Hopf algebra, so this defines a convolution operation on $\mathrm{End}(H)$.

The following result is from [Tak71, Lemma 14].

Proposition 2.1 (Takeuchi's formula)**.** If $H = (H, \mu, \iota, \Delta, \epsilon, S)$ is a Hopf algebra such that $(\iota \circ \epsilon - \mathrm{id}_H)$ is $\star$-nilpotent (as in any filtered Hopf algebra), then

$$S = \sum_{k \geq 0} (\iota \circ \epsilon - \mathrm{id}_H)^{\star k} = \sum_{k \geq 0} (-1)^k \mu^{\circ (k-1)} \circ (\mathrm{id}_H - \iota \circ \epsilon)^{\otimes k} \circ \Delta^{\circ (k-1)} \,, \tag{3}$$

where we use the convention that $\Delta^{\circ(-1)} = \epsilon$ and $\mu^{\circ(-1)} = \iota$. Notice that the $\star$-nilpotent property ensures that this sum is finite.

Note that any pattern algebra is a filtered Hopf algebra (see [Pen20, Theorem A.2.4]), so for our objects of study we can always apply Takeuchi's formula.

In the Hopf algebra of polynomials, this gives us the following:

$$S(x^3) = \underbrace{0}_{k=0} - \underbrace{x^3}_{k=1} + \underbrace{3x^2 \cdot x + 3x \cdot x^2}_{k=2} - \underbrace{6x \cdot x \cdot x}_{k=3} = -x^3 \,.$$

We now present another example, this time on the permutation pattern Hopf algebra $\mathcal{A}(\texttt{Per})$. Consider $\pi = 132 = 1 \oplus 21$. Then Takeuchi's formula gives us:

$$\begin{aligned} S(\mathbf{pat}_{132}) &= \sum_{k=0}^{2} (-1)^k \mu^{\circ(k-1)} \circ (\mathrm{id}_{\mathcal{A}(\texttt{Per})} - \iota \circ \epsilon)^{\otimes k} \circ \Delta^{\circ(k-1)}(\mathbf{pat}_{132}) \\ &= - (\mathrm{id}_{\mathbb{K}[x]} - \iota \circ \epsilon)(\mathbf{pat}_{132}) + \mu \circ (\mathrm{id}_{\mathbb{K}[x]} - \iota \circ \epsilon)^{\otimes 2}(\mathbf{pat}_1 \otimes \mathbf{pat}_{12}) \\ &= - \underbrace{\mathbf{pat}_{132}}_{k=1} + \underbrace{\mathbf{pat}_1 \, \mathbf{pat}_{21}}_{k=2} \\ &= 3\, \mathbf{pat}_{321} + 2\, \mathbf{pat}_{231} + 2\, \mathbf{pat}_{312} + \mathbf{pat}_{213} + 2\, \mathbf{pat}_{21} \;. \end{aligned}$$

These coefficients can be seen as enumerating quasi-shuffle signatures of 132 from 1 and 12 that are **interlacing**, according to Theorem 1.3.

2.1. **The sign-reversing involution method.** The sign-reversing involution method is a combinatorial technique that uses involutions with opposite signs to simplify sums and find cancellation-free formulas for the antipode of a Hopf algebra. This method was first applied in the context of Hopf algebra antipodes in [BS17]. It starts in the formula given by Takeuchi, and results in a ± 1 sum that runs over a collection of objects, say $\mathcal{O}$. An involution ζ is an endomorphism such that $\zeta \circ \zeta$ is the identity. A sign-reversing involution ζ, is an involution on $\mathcal{O}$ where for non-fixed-point elements x (i.e., $\zeta(x) \neq x$), the elements x and $\zeta(x)$ contribute with opposite signs to the sum over $\mathcal{O}$ that results in the antipode. As a consequence, when applying Takeuchi's formula, we can cancel terms that are not fixed points of ζ.

An example of how this method is applied in the Hopf algebra $\mathbb{K}[x]$ is given in [BS17]. This example demonstrates how to systematically apply the involution to obtain the cancellation-free formula.

Theorem 2.2 (The antipode formula for the polynomial Hopf algebra)**.** The antipode S for $\mathbb{K}[x]$ is

$$S(x^n) = (-x)^n \,.$$

3. Combinatorial species

In this section we provide the preliminaries on monoids in species. This will follow closely [AM10] and [Sch93]. Specifically, we introduce both vector species and set species. We also introduce species with restrictions, and we will clarify the meaning of a monoid, comonoid, bimonoid and Hopf monoid in each of these monoidal categories.

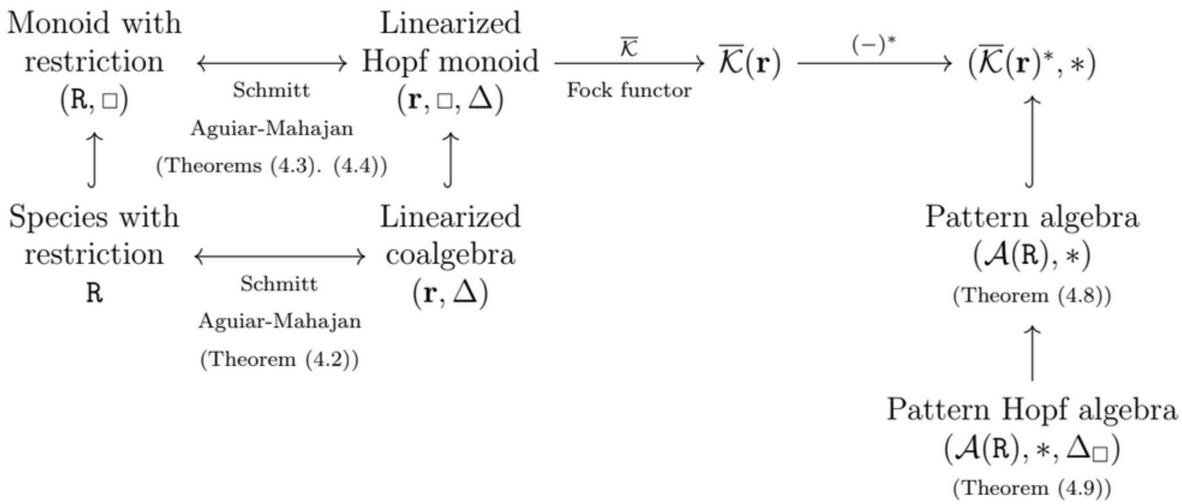


FIGURE 2. Diagram reflecting the context of the pattern Hopf algebra. Given a restriction species $\mathbf{r}$, its linearisation is denoted $\mathbf{r} = \mathbb{K}\mathbf{r}$. The product $*$ in the top-right corner corresponds to the pointwise product of functions in the dual space $\overline{\mathcal{K}}(\mathbf{r})$. The coproduct $\Delta_\square$ of the pattern Hopf algebra corresponds to the dual of the concatenation product $\square$, after applying the Fock functor $\overline{\mathcal{K}}$.

We finally present some examples of species with restrictions that will be important in the remaining paper.

3.1. **Species.** In this section we recall the basic definitions of the general theory of *combinatorial species*. Following [AM10], we will focus first on *vector species* and *set species*.

Let $\mathbb{K}$ be a field of arbitrary characteristic. Let $\mathsf{Set}^\times$ be the category of finite sets and bijections between finite sets, and $\mathsf{Vect}_\mathbb{K}$ be the category of $\mathbb{K}$-vector spaces and linear maps between vector spaces. A **vector species**, or simply a **species**, is a functor $\mathbf{p} : \mathsf{Set}^\times \to \mathsf{Vect}_\mathbb{K}$. A morphism between species $\mathbf{p}$ and $\mathbf{q}$ is a natural transformation between the functors $\mathbf{p}$ and $\mathbf{q}$; that is, a collection of linear maps $\mathbf{p}[I] \to \mathbf{q}[I]$ that are natural with respect to bijections of finite sets. For clarity, we denote the vector species with a bold lowercase Latin letter, with few exceptions.

A species $\mathbf{p}$ is said to be **positive** if $\mathbf{p}[\emptyset] = 0$. The **positive part** of a species $\mathbf{q}$ is the positive species $\mathbf{q}_+$ given by

$$\mathbf{q}_+[I] = \begin{cases} \mathbf{q}[I], & \text{if } I \neq \emptyset \\ \emptyset, & \text{otherwise} \end{cases}.$$

Given a vector space V, let $\mathbf{1}_V$ be the vector species defined by

$$\mathbf{1}_V[I] = \begin{cases} V, & \text{if } I = \emptyset \\ \emptyset, & \text{otherwise} \end{cases}.$$

We write $\mathsf{Sp}_{\mathbb{K}}$ for the category of vector species over the field $\mathbb{K}$. We consider two monoidal structures on this category: the **Cauchy** and **Hadamard** products $\cdot$ and $\times$, respectively: for any finite set I,

$$(\mathbf{p} \cdot \mathbf{q})[I] := \bigoplus_{I = S \sqcup T} \mathbf{p}[S] \otimes \mathbf{q}[T];$$

$$(\mathbf{p} \times \mathbf{q})[I] := \mathbf{p}[I] \otimes \mathbf{q}[I].$$

We denote by $(\mathsf{Sp}_{\mathbb{K}}, \cdot)$ the resulting monoidal category obtained from the Cauchy operation.

We can also consider **set species**, these are functors $\mathrm{P} : \mathsf{Set}^{\times} \to \mathsf{Set}$, where Set is the category of arbitrary sets and arbitrary maps between sets. Given a set species P, the notions of **positive part** P_+ of P, **positive set species** are defined analogously as for vector species. If C is a set, let $\mathbf{1}_C$ be the set species defined by

$$\mathbf{1}_C[I] = \begin{cases} C, & \text{if } I = \emptyset \\ \emptyset, & \text{otherwise} \end{cases},$$

for any finite set I. For clarity, we denote a set species with a capital Latin letter, with few exceptions.

The Cauchy and Hadamard products of vector species have their analogues in this context. For instance, if P and Q are two set species, let

$$(\mathrm{P} \cdot \mathrm{Q})[I] := \bigsqcup_{I = S \sqcup T} \mathrm{P}[S] \times \mathrm{Q}[T];$$

$$(\mathrm{P} \times \mathrm{Q})[I] := \mathrm{P}[I] \times \mathrm{Q}[I],$$

on any finite set I, where the $\times$ symbol in the right-hand sides refers to the Cartesian product.

It is possible to relate set species to vector species via the *linearisation functor* $\mathbb{K}(-) : \mathsf{Set} \to \mathsf{Vect}_{\mathbb{K}}$, which sends a set to the vector space generated by the given set. Composing a set species P with the linearisation functor gives a vector species, denoted by $\mathbb{K}\mathrm{P}$. A **linearized species** is a vector species $\mathbf{p}$ of the form $\mathbf{p} = \mathbb{K}\mathrm{P}$, for some set species P. We have natural isomorphisms

$$\mathbb{K}(\mathrm{P} \cdot \mathrm{Q}) \simeq \mathbb{K}\mathrm{P} \cdot \mathbb{K}\mathrm{Q} \qquad , \qquad \mathbb{K}(\mathrm{P} \times \mathrm{Q}) \simeq \mathbb{K}\mathrm{P} \times \mathbb{K}\mathrm{Q}.$$

3.2. **Algebraic structures on vector species.**

3.2.1. *Monoids.* A **monoid** in $(\mathsf{Sp}_{\mathbb{K}}, \cdot)$ consists of a species $\mathbf{p}$ equipped with morphisms of species

$$\mu : \mathbf{p} \cdot \mathbf{p} \to \mathbf{p} \qquad \text{and} \qquad \iota : 1_{\mathbb{K}} \to \mathbf{p}.$$

That is, for each finite set I and for each decomposition $I = S \sqcup T$, we have a linear map

$$\mu_{S,T} : \mathbf{p}[S] \otimes \mathbf{p}[T] \to \mathbf{p}[I] \text{ and } \iota_{\emptyset} : \mathbb{K} \to \mathbf{p}[\emptyset].$$

If $x \in \mathbf{p}[S]$, $y \in \mathbf{p}[T]$, let

$$x \cdot y \in \mathbf{p}[I]$$

denote the image of $x \otimes y$ under $\mu_{S,T}$.

The collection of linear maps $\mu = (\mu_{S,T})$, called the **product** of the monoid, must satisfy the following axioms.

(i) Naturality axiom: for finite sets I, J, a bijection $\sigma : I \to J$, a decomposition $I = S \sqcup T$ and elements $x \in \mathbf{p}[S]$ and $y \in \mathbf{p}[T]$, we have

$$\mathbf{p}[\sigma](x \cdot y) = \mathbf{p}[\sigma(S)](x) \cdot \mathbf{p}[\sigma(T)](y)\,.$$

(ii) Associativity axiom: for finite set I, a decomposition $I = R \sqcup S \sqcup T$ and for elements $x \in \mathbf{p}[R]$, $y \in \mathbf{p}[S]$ and $z \in \mathbf{p}[T]$, we have

$$(x \cdot y) \cdot z = x \cdot (y \cdot z). \tag{4}$$

(iii) Unit axiom: for each finite set I and $x \in \mathbf{p}[I]$, we have

$$x \cdot \iota_{\emptyset}(1) = x = \iota_{\emptyset}(1) \cdot x,$$

where $1 \in \mathbb{K}$ is the unit of the field $\mathbb{K}$.

A monoid $(\mathbf{p}, \mu, \iota)$ in $(\mathsf{Sp}_{\mathbb{K}}, \cdot)$ is **commutative** if

$$x \cdot y = y \cdot x,$$

for all $I = S \sqcup T$, $x \in \mathbf{p}[S]$ and $y \in \mathbf{p}[T]$.

Let $(\mathbf{p}, \mu, \iota)$ be a monoid. From the associativity axiom, for any decomposition $I = S_1 \sqcup \cdots \sqcup S_k$ with $k \geq 2$ there is a unique map

$$\mu_{S_1,\ldots,S_k} : \mathbf{p}[S_1] \otimes \cdots \otimes \mathbf{p}[S_k] \to \mathbf{p}[I], \tag{5}$$

called the **higher product map** of $\mathbf{p}$, obtained by iterating the product maps in any meaningful way. This is well defined from (4). We can extend the definition of higher product map for all $k \geq 0$: for $k = 1$, μ_I is defined as the identity map of $\mathbf{p}[I]$ (I is the only decomposition of itself in one block); if $k = 0$, then $\mu_{\emptyset} := \iota_{\emptyset}$.

Monoids are closed under the Cauchy product. Also, if $(\mathbf{p}, \mu, \iota)$ is a monoid, then $\mathbf{p}[\emptyset]$ is an algebra with product $\mu_{\emptyset,\emptyset}$ and unit $\iota_{\emptyset}(1)$ (see [AM13], section 2.3).

3.2.2. *Comonoids.* A **comonoid** in $(\mathsf{Sp}_{\mathbb{K}}, \cdot)$ corresponds to the dual of a monoid in $(\mathsf{Sp}_{\mathbb{K}}, \cdot)$. Specifically, a comonoid consists of a species $\mathbf{p}$ equipped with morphisms of species

$$\Delta : \mathbf{p} \to \mathbf{p} \cdot \mathbf{p} \qquad \text{and} \qquad \varepsilon : \mathbf{p} \to 1_{\mathbb{K}}.$$

That is, for each finite set I and for each decomposition $I = S \sqcup T$, we have a linear map

$$\Delta_{S,T} : \mathbf{p}[I] \to \mathbf{p}[S] \otimes \mathbf{p}[T] \text{ and } \varepsilon_{\emptyset} : \mathbf{p}[\emptyset] \to \mathbb{K}.$$

The collection of linear maps $\Delta = (\Delta_{S,T})$, called the **coproduct** of the comonoid, must satisfy the following axioms.

(i) Naturality axiom: for finite sets I, J, bijection $\sigma : I \to J$, a decomposition $I = S \sqcup T$ and an element $x \in \mathbf{p}[I]$.

$$(\mathbf{p}[\sigma|_S] \otimes \mathbf{p}[\sigma|_T]) \circ \Delta_{S,T}(x) = \Delta_{\sigma(S),\sigma(T)} \circ \mathbf{p}[\sigma](x).$$

(ii) Coassociativity axiom: for a finite set I, a decomposition $I = R \sqcup S \sqcup T$ and for each $x \in \mathbf{p}[I]$, we have

$$(\Delta_{R,S} \otimes \mathrm{id}_{\mathbf{p}[T]}) \circ \Delta_{R \sqcup S,T}(x) = (\mathrm{id}_{\mathbf{p}[R]} \otimes \Delta_{S,T}) \circ \Delta_{R,S \sqcup T}(x). \tag{6}$$

(iii) Counit axiom: for each finite set I and $x \in \mathbf{p}[I]$, we have

$$(\varepsilon_{\emptyset} \otimes \mathrm{id}_{\mathbf{p}[I]}) \circ \Delta_{\emptyset,I}(x) = x = (\mathrm{id}_{\mathbf{p}[I]} \otimes \varepsilon_{\emptyset}) \circ \Delta_{I,\emptyset}(x).$$

A comonoid $(\mathbf{p}, \mu, \iota)$ in $(\mathsf{Sp}_{\mathbb{K}}, \cdot)$ is **cocommutative** if for any finite disjoint sets S, T, and element $x \in \mathbf{p}[S \sqcup T]$, we have

$$\Delta_{S,T}(x) = \Delta_{T,S}(x).$$

Let $(\mathbf{p}, \mu, \iota)$ be a comonoid. Dually to the case of monoids, every decomposition $I = S_1 \sqcup \cdots \sqcup S_k$ with $k \geq 0$ gives rise to a unique linear map

$$\Delta_{S_1,\ldots,S_k} : \mathbf{p}[I] \to \mathbf{p}[S_1] \otimes \cdots \otimes \mathbf{p}[S_k], \tag{7}$$

called the **higher coproduct map** of $\mathbf{p}$, obtained by iterating the coproducts map $\Delta_{S,T}$. This is well defined because of Eq. (6). We extend this definition to $k = 1$ as the identity of $\mathbf{p}[I]$ and for $k = 0$, this map is the counit map $\varepsilon_{\emptyset}$.

Comonoids are closed under the Cauchy product. If $(\mathbf{p}, \Delta, \varepsilon)$ is a comonoid, then $\mathbf{p}[\emptyset]$ is a coalgebra with coproduct $\Delta_{\emptyset,\emptyset}$ and counit $\varepsilon_{\emptyset}$ (see [AM13], section 2.4).

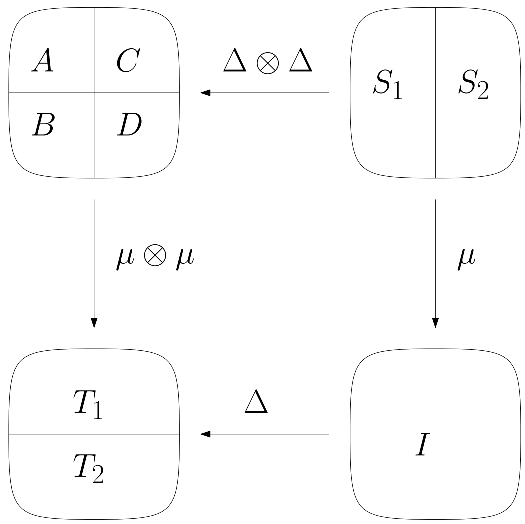


FIGURE 3. The bimonoid compatibility axiom.

3.2.3. *Bimonoids and Hopf monoids.* A **bimonoid** $(\mathbf{h}, \mu, \Delta, \iota, \varepsilon)$ in $(\mathsf{Sp}_{\mathbb{K}}, \cdot)$ is a monoid and comonoid such that the diagram

$$\begin{tikzcd}
\mathbf{h}[A] \otimes \mathbf{h}[B] \otimes \mathbf{h}[C] \otimes \mathbf{h}[D] \arrow[rr, "\cong"] & & \mathbf{h}[A] \otimes \mathbf{h}[C] \otimes \mathbf{h}[B] \otimes \mathbf{h}[D] \arrow[d, "\mu_{A,C} \otimes \mu_{B,D}"] \\
\mathbf{h}[S_1] \otimes \mathbf{h}[S_2] \arrow[u, "\Delta_{A,B} \otimes \Delta_{C,D}"] \arrow[r, "\mu_{S_1,S_2}"'] & \mathbf{h}[I] \arrow[r, "\Delta_{T_1,T_2}"'] & \mathbf{h}[T_1] \otimes \mathbf{h}[T_2]
\end{tikzcd}$$

commutes, where $I = S_1 \sqcup S_2 = T_1 \sqcup T_2$ are two decompositions of a finite set I with the following resulting pairwise intersections:

$$A := S_1 \cap T_1 \quad , \quad B := S_1 \cap T_2 \quad , \quad C := S_2 \cap T_1 \quad , \quad D := S_2 \cap T_2.$$

This is also schematically presented in Fig. 3.

We define the convolution algebra $\mathrm{End}_{\mathsf{Sp}_{\mathbb{K}}}(\mathbf{h})$ as the monoid of natural transformations $l : \mathbf{h} \to \mathbf{h}$ with the product $\star$. Note how $\iota \circ \epsilon$ is the identity, by definition of bialgebra.

A morphism of species $s : \mathbf{h} \to \mathbf{h}$, is called and **antipode** of $\mathbf{h}$, if $\mathbf{h}[\emptyset]$ is a Hopf algebra with antipode $s_\emptyset : \mathbf{h}[\emptyset] \to \mathbf{h}[\emptyset]$, and for each nonempty set I, we have

$$\sum_{S \sqcup T = I} \mu_{S,T}(\mathrm{id}_S \otimes s_T)\Delta_{S,T} = 0 = \sum_{S \sqcup T = I} \mu_{S,T}(s_S \otimes \mathrm{id}_T)\Delta_{S,T}. \tag{8}$$

In this case, (8) can be rephrased as the star inverse of the identity $s = \mathrm{id}_{\mathbf{h}}^{-1}$.

A **Hopf monoid** in $(\mathsf{Sp}_{\mathbb{K}}, \cdot)$ is a bimonoid along with an antipode $s : \mathbf{h} \to \mathbf{h}$. Recall that this is also the case in the classical Hopf algebras.

3.3. **Algebraic structures on set species.** The notions of monoid, comonoid, bimonoid and Hopf monoid for set species can be described in terms similar to those in the previous section.

3.3.1. *Monoids.* Set species are defined analogously to vector species, with $\times$ denoting Cartesian products instead of tensor products.

A **monoid in set species** is a set species P equipped with morphisms of species

$$\mu : \mathrm{P} \cdot \mathrm{P} \to \mathrm{P} \qquad \text{and} \qquad \iota : 1_{\{\emptyset\}} \to \mathrm{P}. \tag{9}$$

That is, for each decomposition $I = S \sqcup T$, we have maps

$$\mu_{S,T} : \mathrm{P}[S] \times \mathrm{P}[T] \to \mathrm{P}[I] \text{ and } \iota_{\emptyset} : \{\emptyset\} \to \mathrm{P}[\emptyset]. \tag{10}$$

If $x \in \mathrm{P}[S]$, $y \in \mathrm{P}[T]$, let

$$x \cdot y \in \mathrm{P}[I]$$

denote the image of x under $\mu_{S,T}$. Also, let $e \in \mathrm{P}[\emptyset]$ denote the image of $\emptyset$ under $\iota_{\emptyset}$.

The collection of maps $\mu = (\mu_{S,T})$, called the **product** of the comonoid, must satisfy naturality, associativity and unit axioms analogue to the ones defined for monoids in vector species.

Note that, for any monoid on a set species (P, μ, ι), then $(\mathrm{P}[\emptyset], \mu_{\emptyset,\emptyset}, \iota_{\emptyset})$ is a set theoretical monoid.

A monoid in set species (P, μ, ι) is **commutative** if

$$x \cdot y = y \cdot x,$$

for all $I = S \sqcup T$, $x \in \mathrm{P}[S]$ and $y \in \mathrm{P}[T]$.

3.3.2. *Comonoids.* A **comonoid in set species** consist of a species P equipped with morphisms of species

$$\Delta : \mathrm{P} \to \mathrm{P} \cdot \mathrm{P} \qquad \text{and} \qquad \varepsilon : \mathrm{P} \to 1_{\{\emptyset\}}.$$

That is, for each decomposition $I = S \sqcup T$ we have maps $\Delta_{S,T} : \mathrm{P}[I] \to \mathrm{P}[S] \times \mathrm{P}[T]$, and $\varepsilon_{\emptyset} : \mathrm{P}[\emptyset] \to \{\emptyset\}$. If $x \in \mathrm{P}[I]$, let

$$(x|_S, x/_S) \in \mathrm{P}[S] \times \mathrm{P}[T]$$

denote the image of (x, y) under $\Delta_{S,T}$. The map $x \mapsto x|_S$ corresponds to the restriction of the structure x from I to S in examples such as sets, graphs, and matroids, while $x \mapsto x/_S$ corresponds to the contraction of S from x in such examples, resulting in a structure on T.

The collection of maps $\Delta = (\Delta_{S,T})$, called the **coproduct** of the monoid, must satisfy naturality, coassociativity and counit axioms analogues to the ones defined for monoids in vector species. These axioms can be described explicitly:

- Naturality axiom: for each bijection $\sigma : I \to J$, we have
$$\Big(\mathrm{P}[\sigma](x)\Big)\Big|_{\sigma(S)} = \mathrm{P}[\sigma|_S](x|_S) \qquad , \qquad \Big(\mathrm{P}[\sigma](x)\Big)\Big/_{\sigma(S)} = \mathrm{P}[\sigma|_T](x/_S),$$
for all $x \in \mathrm{P}[I]$.
- Coassociativity axiom: for all decomposition $I = R \sqcup S \sqcup T$,
$$(x|_{R\sqcup S})|_R = x|_R \qquad , \qquad (x|_{R\sqcup S})/_R = (x/_R)|_S \qquad , \qquad x/_{R\sqcup S} = (x/_R)/_S,$$
for all $x \in \mathrm{P}[I]$
- Counit axiom: we have
$$x|_I = x = x/_\emptyset,$$
for each finite set I and for each $x \in \mathrm{P}[I]$. In particular, $\Delta_{\emptyset,\emptyset}(x) = (x, x)$, for each $x \in \mathrm{P}[\emptyset]$.

A comonoid in set species $(\mathrm{P}, \Delta, \varepsilon)$ is **cocommutative** if
$$x|_S = x/_T,$$
for any disjoint finite sets S, T and $x \in \mathrm{P}[S \sqcup T]$.

3.3.3. *Bimonoids.* A bimonoid in set species $(\mathrm{H}, \mu, \Delta, \iota, \varepsilon)$ is a monoid and comonoid in set species such that the diagram

$$\begin{tikzcd}
\mathrm{H}[A] \times \mathrm{H}[B] \times \mathrm{H}[C] \times \mathrm{H}[D] \arrow[rr, "\cong"] & & \mathrm{H}[A] \times \mathrm{H}[C] \times \mathrm{H}[B] \times \mathrm{H}[D] \arrow[d, "\mu_{A,C}\times\mu_{B,D}"] \\
\mathrm{H}[S_1] \times \mathrm{H}[S_2] \arrow[u, "\Delta_{A,B}\times\Delta_{C,D}"] \arrow[r, "\mu_{S_1,S_2}"'] & \mathrm{H}[I] \arrow[r, "\Delta_{T_1,T_2}"'] & \mathrm{H}[T_1] \times \mathrm{H}[T_2]
\end{tikzcd}$$

commutes, where $I = S_1 \sqcup S_2 = T_1 \sqcup T_2$ are two decompositions of a finite set I with the following resulting pairwise intersections:
$$A := S_1 \cap T_1 \quad , \quad B := S_1 \cap T_2 \quad , \quad C := S_2 \cap T_1 \quad , \quad D := S_2 \cap T_2.$$

The compatibility axiom in the definition of bimonoid can be reformulated as
$$x|_A \cdot y|_C = (x \cdot y)|_{T_1} \qquad , \qquad x/_A \cdot y/_C = (x \cdot y)/_{T_1},$$
for any disjoint sets S_1, S_2, for elements $x \in \mathrm{H}[S_1]$, $y \in \mathrm{H}[S_2]$ and for any set $T_1 \subseteq S_1 \sqcup S_2$, by letting $A = S_1 \cap T_1$ and $C = S_2 \cap T_1$.

A **Hopf monoid** in $(\mathsf{Sp}, \cdot)$ H is a bimonoid in set species such that the monoid $\mathrm{H}[\emptyset]$ is a group. Its antipode on $\emptyset$ is the map $s_\emptyset : \mathrm{H}[\emptyset] \to \mathrm{H}[\emptyset]$ given by $s_\emptyset(x) := x^{-1}$. We define an antipode map $s : H \to H$ via the Takeuchi formula, adapted to set species.

3.4. **Fock functor.** The Fock functor is a functor that constructs graded Hopf algebras from species.

In [AM10] (Part III), a construction is presented allowing to produce a (graded) Hopf algebra from a Hopf monoid. This is a categorical approach of a construction due to Stover ([Sto93], Section 14), studied later by Patras, Schocker and Reutenauer in [PR04], [PS06] and [PS08].

We recall briefly this construction. Let $\mathbb{K}$ be a field of characteristic zero. If $\mathbf{p} \in \mathsf{Sp}_{\mathbb{K}}$, then there is an action of the symmetric group $\mathfrak{S}_n$ on $\mathbf{p}[n]$ by relabeling, for each $n \geq 0$. We denote by $\mathbf{p}[n]_{\mathfrak{S}_n}$ the *space of* $\mathfrak{S}_n$*-coinvariants of* $\mathbf{p}[n]$ (the quotient by the subspace generated by differences $x - \mathbf{p}[\alpha](x)$ for $\alpha \in \mathfrak{S}_n$):

$$\mathbf{p}[n]_{\mathfrak{S}_n} := \mathbf{p}[n] / \langle\, x - \mathbf{p}[\alpha](x) \,|\, \alpha \in \mathfrak{S}_n, x \in \mathbf{p}[n] \,\rangle \,. \tag{11}$$

Consider gVec be the category of graded vector spaces over $\mathbb{K}$. The functors $\mathcal{K}, \overline{\mathcal{K}} : \mathsf{Sp}_{\mathbb{K}} \to \mathsf{gVec}$ given by

$$\mathcal{K}(\mathbf{p}) := \bigoplus_{n \geq 0} \mathbf{p}[n] \qquad , \qquad \overline{\mathcal{K}}(\mathbf{p}) := \bigoplus_{n \geq 0} \mathbf{p}[n]_{\mathfrak{S}_n} \tag{12}$$

are referred in [AM10] as *full Fock functor* and *bosonic Fock functor*, respectively, which account for labeled and unlabeled elements respectively. From any monoid (resp. comonoid, Hopf monoid) $\mathbf{p}$, it is possible to obtain algebras (resp. coalgebras, Hopf algebras) $\mathcal{K}(\mathbf{p})$ and $\overline{\mathcal{K}}(\mathbf{p})$ from those of $\mathbf{p}$, together with certain canonical transformations (see [AM10], section 15.2).

4. The pattern Hopf algebra

4.1. **Species with restrictions.** To construct Hopf algebras from combinatorial objects, we need a framework that combines the algebraic structure of species with the ability to extract components from larger objects. Species with restrictions provide this essential unifying structure, serving as the central foundation for all pattern Hopf algebras presented in this paper. They introduce restriction maps that allow us to systematically decompose objects, a capability that is particularly important for studying patterns, where we need to understand how smaller patterns appear within larger ones. This framework also naturally accommodates the monoidal structure and coproduct operations needed to give Hopf algebra structures to pattern algebras.

The general setting for our approach to patterns is given by the notion of *species with restrictions*, a terminology due to Schmitt (see [Sch93]) and used by the first author in [Pen22], where these were called combinatorial presheaves.

Let $\mathsf{Set}^{\times}_{\hookrightarrow}$ be the category of finite sets with injections as morphisms. A (set) **species with restrictions** is a contravariant functor $\mathtt{R} : \mathsf{Set}^{\times}_{\hookrightarrow} \to \mathsf{Set}$. Given a species with restrictions $\mathtt{R}$ and a couple of finite sets I, J such that $J \subseteq I$, the **restriction map** $\mathrm{res}_{I,J}$ is the image under the functor $\mathtt{R}$ of the inclusion $J \hookrightarrow I$. We will use the notation

$\mathrm{res}_{I,J}$ exclusively when J is a subset of I, and drop the more general species notation:

$$\mathrm{res}_{I,J} : \mathtt{R}[I] \to \mathtt{R}[J].$$

By functoriality, these maps satisfy the contravariant axioms

$$\mathrm{res}_{J,K} \circ \mathrm{res}_{I,J} = \mathrm{res}_{I,K} \qquad , \qquad \mathrm{res}_{I,I} = \mathrm{id}_{\mathtt{R}[I]}, \tag{13}$$

for any finite sets $I \supseteq J \supseteq K$. Since any arbitrary injection equals a bijection followed by an inclusion, any species with restrictions is equivalent to a set species together with restriction maps satisfying the axioms (13). Going forward, we use the notation $a|_J = \mathrm{res}_{I,J}(a)$ for simplicity.

Species with restrictions form a category Spr, where the morphisms are natural transformations between species. For clarity, we denote species with restrictions with a typewriter typescript. Note how a species with restrictions is also a set species.

Notice that, for any finite set C, the set species 1_C is also a species with restrictions, where $\mathrm{res}_{\emptyset,\emptyset} = \mathrm{id}_C$.

4.2. **Schmitt's comonoid.** In [Sch93] (Section 3), Schmitt gave a construction of coalgebras and bialgebras from certain species. We describe the coalgebra construction, following the notation of [AM10] (Section 8.7).

Given a species with restrictions $\mathtt{R}$, we construct a linearized comonoid in $(\mathsf{Sp}, \cdot)$ as follows. Let $\mathbf{r} = \mathbb{K}\mathtt{R}$ be the linearisation of $\mathtt{R}$. Given a decomposition $I = S \sqcup T$, consider the linear map

$$\Delta_{S,T} : \mathbf{r}[I] \to \mathbf{r}[S] \otimes \mathbf{r}[T]$$

given by

$$\Delta_{S,T}(x) := \mathrm{res}_{I,S}(x) \otimes \mathrm{res}_{I,T}(x), \tag{14}$$

for any $x \in \mathtt{R}[I]$. Let $\epsilon_\emptyset : \mathbf{r}[\emptyset] \to \mathbb{K}$ be the linear extension of the map sending every element of $\mathtt{R}[\emptyset]$ to 1. Hence, we have the following result.

Lemma 4.1 (Schmitt)**.** The vector species $\mathbf{r}$ is a linearized comonoid in $(\mathsf{Sp}, \cdot)$. In particular, the comonoid $\mathbf{r}$ is cocommutative.

Consider now the converse, a linearized comonoid $\mathbf{p} = \mathbb{K}\mathrm{P}$ in $(\mathsf{Sp}, \cdot)$. In this case, the coproduct gives a pure tensor

$$\Delta_{S,T}(x) = x|_S \otimes x/_S,$$

for each $x \in \mathrm{P}[I]$ and for each decomposition $I = S \sqcup T$. In the examples below (sets, graphs, matroids) the maps $x \mapsto x|_S$ and $x \mapsto x/_S$ will correspond to the usual notions of restrictions and contractions.

We may then define restriction maps on $\mathbf{p}$ either by

$$\begin{aligned} \mathrm{res}^{(1)}_{I,J} : \mathbf{p}[I] &\to \mathbf{p}[J] \\ x &\mapsto x|_J \end{aligned} \qquad \text{or} \qquad \begin{aligned} \mathrm{res}^{(2)}_{I,J} : \mathbf{p}[I] &\to \mathbf{p}[J], \\ x &\mapsto x/_{I\setminus J} \end{aligned}$$

for $x \in \mathbf{p}[I]$. Each restriction map $\mathrm{res}^{(1)}$ or $\mathrm{res}^{(2)}$ turns $\mathbf{p}$ into a species with restrictions. When $\mathbf{p}$ is cocommutative, then both restriction maps coincide. We have then the following characterisation of species with restrictions (see [AM10], Proposition 8.29, for other characterisations).

Theorem 4.2 (Schmitt, Aguiar-Mahajan)**.** There is an equivalence between the category of species with restrictions and the category of linearized cocommutative comonoids.

4.3. **Monoids with restrictions.** We see now that the restriction structures are stable for the Cauchy product. Let $\mathtt{P}, \mathtt{Q}$ be two species with restrictions. Given two finite sets I and J with an inclusion $J \hookrightarrow I$, consider the map $\mathrm{res}_{I,J}$ defined as the sum of the maps running over all decompositions $I = S \sqcup T$:

$$\mathtt{P}[S] \times \mathtt{Q}[T] \xrightarrow{\mathrm{res}_{S,S\cap J} \times \mathrm{res}_{T,T\cap J}} \mathtt{P}[S \cap J] \times \mathtt{Q}[T \cap J] \subseteq (\mathtt{P} \cdot \mathtt{Q})[J],$$

where the first and second restrictions on the arrow above are the restrictions maps corresponding to $\mathtt{P}$ and $\mathtt{Q}$, respectively, from the inclusions $S \cap U \hookrightarrow S$ and $T \cap U \hookrightarrow T$, respectively. This defines restriction maps on $\mathtt{P} \cdot \mathtt{Q}$.

Therefore, the category of species with restrictions is a monoidal category $(\mathsf{Spr}, \cdot, 1_{\{\emptyset\}})$.

We describe monoids in the monoidal category $(\mathsf{Spr}, \cdot, 1_{\{\emptyset\}})$. A monoid $(\mathtt{P}, \mu, 1)$ in species with restrictions is a monoid in set species such that for each $J \subseteq I = S \sqcup T$, the diagram

$$\begin{tikzcd}
\mathrm{P}[S] \times \mathrm{P}[T] \arrow[r, "\mathrm{res}_{S,S\cap J} \times \mathrm{res}_{T,T\cap J}"] \arrow[d, "\mu_{S,T}"'] & \mathrm{P}[S \cap J] \times \mathrm{P}[T \cap J] \arrow[d, "\mu_{S\cap J, S\cap J}"] \\
\mathrm{P}[I] \arrow[r, "\mathrm{res}_{I,J}"'] & \mathrm{P}[J]
\end{tikzcd}$$

commutes, ensuring that restrictions respect the monoid structure.

Given a monoid $\mathtt{P}$ in the monoidal category of species with restrictions $(\mathsf{Spr}, \cdot)$, let $\mathbf{p} := \mathbb{K}\mathtt{P}$ be the linearisation of the underlying set species of $\mathtt{P}$. By Theorem 4.2, $\mathbf{p}$ is a cocommutative comonoid. Since $\mathtt{P}$ is a monoid, then $\mathbf{p}$ is a monoid in the category of vector species. Moreover, the above diagram implies that the product and coproduct of $\mathtt{P}$ are compatible, meaning that $\mathtt{P}$ is a cocommutative bimonoid. This proves the following, which was proved by Aguiar-Mahajan (see [AM10]):

Theorem 4.3 (Aguiar-Mahajan)**.** There is an equivalence between the category of monoids in $(\mathsf{Spr}, \cdot)$ and the category of linearized cocommutative bimonoids.

Theorem 4.4 (Aguiar-Mahajan)**.** If $\mathtt{P}$ is a connected monoid in species with restrictions (i.e., $\mathtt{P}[\emptyset]$ is a singleton), then $\mathbb{K}\mathtt{P}$ is a Hopf monoid in vector species.

4.4. **Pattern functions and the pattern Hopf algebra.** Given a species with restrictions $\mathtt{R}$ and two finite sets I and J, two objects $a \in \mathtt{R}[I], b \in \mathtt{R}[J]$ are said to be **isomorphic objects**, or $a \sim b$, if there is a bijection $\sigma : I \to J$ such that $\mathtt{R}[\sigma](b) = a$.

The collection of equivalence classes of a species with restrictions $\mathtt{R}$ is denoted by

$$\mathcal{G}(\mathtt{R}) := \bigcup_{n \geq 0} \mathtt{R}[n]_{\mathfrak{S}_n}. \tag{15}$$

In this way, the set $\mathcal{G}(\mathtt{R})$ is the collection of all the $\mathtt{R}$-objects up to isomorphism. It is straightforward to show that $\mathcal{G}(\mathtt{R})$ is a basis for the vector space $\overline{\mathcal{K}}(\mathtt{R})$.

Definition 4.5 (Pattern coefficients)**.** Let $\mathtt{R}$ be a species with restrictions. Given two finite sets I, J such that $J \subseteq I$ and two objects $a \in \mathtt{R}[I], b \in \mathtt{R}[J]$, we say that the subset $J \subseteq I$ is a **pattern** of b in a if $a|_J \sim b$. More precisely, $J \subseteq I$ is a pattern of b in a if there exists a bijection $\sigma : J \to J$ such that

$$\mathtt{R}[\sigma](b) = \mathrm{res}_{I,J}(a).$$

We define the **pattern coefficient** of b in a as

$$\binom{a}{b}_{\mathtt{R}} := |\{J \subseteq I \,:\, a|_J \sim b\}| \,. \tag{16}$$

This definition only depends on the isomorphism classes of $b \in \mathtt{R}[J]$ and $a \in \mathtt{R}[I]$; it can be directly seen from the definition that the number of patterns only depends on the isomorphism class (see [Pen22]). This motivates the following notion.

Definition 4.6 (Patterns functions)**.** Let $\mathtt{R}$ be a species with restrictions. Given a finite set I, we define the **pattern function associated to** $b \in \mathtt{R}[I]$ as the function

$$\mathbf{pat}_b : \mathcal{G}(\mathtt{R}) \to \mathbb{K}$$

given by

$$a \mapsto \binom{a}{b}_{\mathtt{R}}, \tag{17}$$

for all $a \in \mathcal{G}(\mathtt{R})$.

By definition $\mathbf{pat}_b \in \mathcal{F}(\mathcal{G}(\mathtt{R}), \mathbb{K})$, where $\mathcal{F}(\mathcal{G}(\mathtt{R}), \mathbb{K})$ denotes the set of functions from $\mathcal{G}(\mathtt{R})$ to $\mathbb{K}$. Without loss of generality, we denote by $\mathbf{pat}_b$ the linear extension to $\mathcal{A}(\mathtt{R})$ of the pattern function associated to b. Hence, we can consider $\{\mathbf{pat}_b\}_{b \in \mathcal{G}(\mathtt{R})}$ as a family of linear functions from $\overline{\mathcal{K}}(\mathtt{R})$ to $\mathbb{K}$, indexed by $\mathcal{G}(\mathtt{R})$.

Definition 4.7 (Pattern spaces)**.** If $\mathtt{R}$ is a species with restrictions, then the linear span of the pattern functions is denoted by

$$\mathcal{A}(\mathtt{R}) := \mathbb{K}\{\mathbf{pat}_a \,:\, a \in \mathcal{G}(\mathtt{R})\}. \tag{18}$$

We write $\mathcal{A}(\mathtt{R})$ for the **pattern space** associated to the species $\mathtt{R}$. By definition, $\mathcal{A}(\mathtt{R})$ is a linear subspace of the space of linear functions $\overline{\mathcal{K}}(\mathtt{R})^*$ from $\overline{\mathcal{K}}(\mathtt{R})$ to $\mathbb{K}$. The following was proven in [Pen22].

Theorem 4.8. The subspace $\mathcal{A}(\mathtt{R})$ of $\overline{\mathcal{K}}(\mathtt{R})^*$ is closed under pointwise multiplication and has a unit. It forms an algebra, called the **pattern algebra associated to** $\mathtt{R}$. More precisely, if $a, b \in \mathcal{G}(\mathtt{R})$,

$$\mathbf{pat}_a\,\mathbf{pat}_b = \sum_c \binom{c}{a,b}_{\mathtt{R}} \mathbf{pat}_c\,, \tag{19}$$

where the coefficients $\binom{c}{a,b}_{\mathtt{R}}$ are the number of "quasi-shuffles" of a, b that result in c, specifically, if we take $c \in \mathtt{R}[C]$ to be a representative of the equivalence class c, then:

$$\binom{c}{a,b}_{\mathtt{R}} = |\{(I,J) \text{ such that } I \cup J = C\,,\ c|_I \sim a,\ c|_J \sim b\}|\ .$$

Consider now a positive species with restrictions $\mathtt{R}$ endowed with an associative product $(\mathtt{R}, \square)$. We require $\mathtt{R}$ to be positive (meaning $\mathtt{R}[\emptyset] = 0$) to ensure that the resulting Hopf algebra has a well-defined unit element. Specifically, this restriction ensures that the empty set corresponds to "no object," which naturally identifies with the multiplicative identity in the Hopf algebra structure. Examples of associative operations on species with restrictions are the direct sum of permutations $\oplus$, introduced above. Recall by Theorem (4.4) that $(\mathtt{R}, \square)$ is equivalent to the linearized cocommutative Hopf monoid $(\mathbf{r}, \square, \Delta)$, where $\mathbf{r} = \mathbb{K}\mathtt{R}$ and Δ is defined as in (14). If $*$ denotes the pointwise product in $\overline{\mathcal{K}}(\mathbf{r}^*)$, then the space of pattern function $\mathcal{A}(\mathtt{R})$ is a subalgebra of $\overline{\mathcal{K}}(\mathbf{r}^*, *, \Delta_\square)$, where $\Delta_\square$ denotes the dual map of $\square$.

Under the natural identification of the function algebra $\mathcal{F}(\mathcal{G}(\mathtt{R}), k)^{\otimes 2}$ as a subspace of $\mathcal{F}(\mathcal{G}(\mathtt{R}) \times \mathcal{G}(\mathtt{R}), k)$, we have

$$\Delta_\square\, \mathbf{pat}_a(b \otimes c) = \mathbf{pat}_a(b \,\square\, c)\,. \tag{20}$$

This is shown in Theorem 4.9. Therefore, we have the following coproduct in the pattern algebra $\mathcal{A}(\mathtt{R})$:

$$\Delta_\square\, \mathbf{pat}_a = \sum_{\substack{b,c \in \mathcal{G}(\mathtt{R}) \\ a = b \,\square\, c}} \mathbf{pat}_b \otimes \mathbf{pat}_c\ . \tag{21}$$

where the sum runs over isomorphism classes b, c that factor into a via the operation $\square$. The relation (20) is central in establishing that the coproduct $\Delta_\square$ is compatible with the product in $\mathcal{A}(\mathtt{R})$.

Theorem 4.9. Let $(\mathtt{R}, \square, 1)$ be an associative species with restrictions. Then the pattern algebra of $\mathtt{R}$ together with the coproduct $\Delta_\square$, and a natural choice of counit, forms a bialgebra (a structure with compatible product and coproduct, see [AM10]). If additionally $|\mathtt{R}[\emptyset]| = 1$, the pattern algebra forms a filtered Hopf algebra (with respect to the size of objects).

Well-studied Hopf algebras, including some classical ones, can be constructed as the pattern algebra of a species with restrictions. An example is *Sym*, the Hopf algebra of *symmetric functions*. This Hopf algebra has a basis indexed by partitions, and corresponds to the pattern Hopf algebra of the species on set partitions.

We end this section with a relevant theorem on functors of species with restrictions.

Theorem 4.10 (Theorem 3.8. in [Pen22])**.** If $\mathtt{f} : \mathtt{R} \Rightarrow \mathtt{S}$ is a morphism of associative species with restrictions, between connected species, then $\mathcal{A}(\mathtt{f})$ is a Hopf algebra morphism that maps $\mathcal{A}(\mathtt{S}) \to \mathcal{A}(\mathtt{R})$. Note the reversal of direction due to the contravariant nature of restrictions.

5. Examples of species with restrictions

5.1. **Species on total orders.** The first relevant species with restrictions to define is the species of total orders $\mathtt{L}$. This has $\mathtt{L}[I] = \{$ total orders on I $\}$, a set with $|I|!$ total orders. The restriction of an order $\leq$ of $\mathtt{L}[I]$ to a subset J is the induced order, which is a total order. This defines a species with restrictions.

We also define an associative product on $\mathtt{L}$. If p is a total order on I and r is a total order on J, and if I, J are disjoint, then we can define a total order $\mathsf{p} * \mathsf{r}$ a total order on $I \sqcup J$ as:

- if $a, b \in I$ such that $a \, \mathsf{p} \, b$, then $a \, (\mathsf{p} * \mathsf{r}) \, b$.
- if $a, b \in J$ such that $a \, \mathsf{r} \, b$, then $a \, (\mathsf{p} * \mathsf{r}) \, b$.
- if $a \in I$ and $b \in J$, then $a \, (\mathsf{p} * \mathsf{r}) \, b$.

This definition is independent of the choice of labeling and respects restriction to subsets. In particular, the product $*$ on orders depends only on the order structures, not on the specific labels chosen, and it respects restrictions to subsets, so this builds a connected associative species with restrictions.

If $\mathsf{p} \in \mathtt{L}[I]$, we write $\mathbb{X}(\mathsf{p}) = I$. The following fact is an immediate observation.

Proposition 5.1. If $\mathbb{X}(\mathsf{p}) = I$, then I is an order ideal in $\mathsf{p} * \mathsf{r}$. That is, if $a \in I$ and $b(\mathsf{p} * \mathsf{r})a$, then $b \in I$.

5.2. **Species on permutations.** To fit the framework of species with restrictions, we use a rather unusual definition of permutations introduced in [ABF20].

This definition is motivated by the fundamental need to represent permutations as a combinatorial species with restrictions—that is, we need a definition where we can systematically understand how smaller permutation patterns fit into larger ones via restriction maps. Rather than viewing a permutation as a bijection (which doesn't naturally support a restriction structure because a bijection's "pattern" is not obviously defined as a sub-bijection), we represent it as a pair of total orders.

Specifically, a permutation on a set I is seen as a pair of total orders $(\leq_P, \leq_V)$ on I, called the "position" and "value" orders respectively. Write $\mathbb{X}(\pi) = I$.

This relates to the usual notion of a permutation as a bijection in the following way: if we order the elements of $I = \{a_1 \leq_P \cdots \leq_P a_k\} = \{b_1 \leq_V \cdots \leq_V b_k\}$, then the natural pairing $a_i \leftrightarrow b_i$ defines a bijection via the mapping $a_i \mapsto b_i$. Conversely, for any bijection f on I, there exist multiple pairs of orders $(\leq_P, \leq_V)$ that correspond to the same bijection f; all of these pairs are isomorphic as permutations in the sense that they differ only by relabeling of the set I.

Crucially, the pair-of-orders representation allows restriction in a natural way: if $J \subseteq I$ is an injection, we can restrict both the position and value orders to J, yielding a permutation on J. This restriction structure is what enables permutations to form a species with restrictions. The resulting species with restrictions structure is denoted by $\texttt{Per}$.

It will be useful to represent permutations in I as square diagrams labeled by I. This is done in the following way: we place each element $a \in I$ at grid position (i, j) where i is the position of a in the $\leq_P$ (position) order and j is the position of a in the $\leq_V$ (value) order. This visual representation directly encodes the bijection: the row index shows where an element comes from (position), and the column index shows where it goes (value). We always start counting from the bottom left, following the Cartesian coordinates style.

For instance, the permutation $\pi = \{1 <_P 2 <_P 3, 2 <_V 1 <_V 3\}$ in $\{1, 2, 3\}$ can be represented as

$$\begin{array}{|c|c|c|} \hline & & 3 \\ \hline 1 & & \\ \hline & 2 & \\ \hline \end{array} \,. \tag{22}$$

In this way, there are $(n!)^2$ elements in $\texttt{Per}[n]$. Up to relabelling, we can represent a permutation as a diagram with one dot in each column and row. Thus, $\texttt{Per}[n]/_{\mathfrak{S}_n}$ (the set of isomorphism classes), has $n!$ isomorphism classes of permutations of size n, as expected.

The $*$ operation on orders can be extended to a **diagonal sum** operation $\oplus$ on permutations, which places all elements of the first permutation before all elements of the second permutation in both the position and value orders. This is usually referred as the shifted concatenation of permutations. This endows $\texttt{Per}$ with an associative species with restrictions structure.

5.3. **Species on packed words.** For packed words, we will mimic the framework produced above for permutations. First, recall that a linear partial order $\leq$ on a set I is the pullback order of a surjective map $I \to [m]$, called the **rank function** of $\leq$, or $rk_{\leq}$. Equivalently, it is an order $\leq$ where for any two distinct elements $a, b \in I$ such that $a \leq b$, there is a disjoint partition $I = A \uplus B$ such that all elements x in A and all elements y in B satisfy $x \leq y$.

We call the integer m the rank of the order $\leq$.

For instance, if $f = \{a \mapsto 3, b \mapsto 1, c \mapsto 3, d \mapsto 2\}$ is a surjective map $\{a, b, c, d\} \to [3]$, the pullback order is $\{b < d < \{a, c\}\}$ and has rank three. Its Hasse diagram is presented in Fig. 4. In this way, a packed word ω on I is a pair of orders $(\leq_P, \leq_V)$ where $\leq_P$ is a total order on I, and $\leq_V$ is a linear partial order on I. This generalizes permutations by allowing ties (incomparabilities) in the value order, represented as ties in the vertical

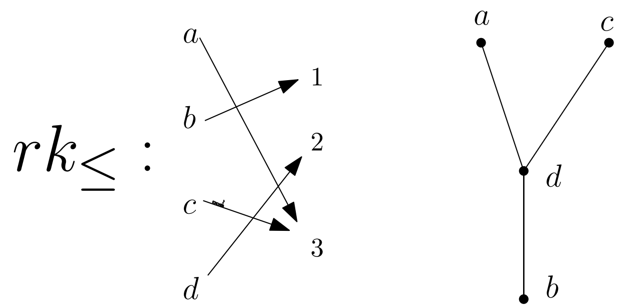


FIGURE 4. **Left:** The description of the function $rk_{\leq}$. **Right:** The Hasse diagram of the linear order inherited by f.

direction of the visual representation. In particular, note that any permutation on I can be seen as a packed word on I, as any total order is a partial linear order. This relates to the usual notion of packed words (a word in $[m]$ where $m \leq |I|$ represents the rank) in the following way: if we order the elements of $I = \{a_1 \leq_P \cdots \leq_P a_k\}$, then the corresponding packed word is

$$rk_{\leq_V}(a_1)rk_{\leq_V}(a_2)\dots rk_{\leq_V}(a_k)\,.$$

Conversely, for any packed word $\omega = p_1 \dots p_k$, there are several pairs of orders $(\leq_P, \leq_V)$ that correspond to the word ω, all of which are isomorphic. Specifically, we can fix any total order $\leq_P$, and construct $\leq_V$ from ω and $\leq_P$: if the i-th entry of ω is $\omega(i)$ and the i-th element according to $\leq_P$ is a_i, then we set $rk_{\leq_V}(a_i) = \omega(i)$.

Consider for instance the packed words $\omega_1 = 13123$ and $\omega_2 = 32413$. These correspond to packed words on $\{a, b, c, d, e\}$. For the examples given ω_1 corresponds to $(d \leq_P e \leq_P c \leq_P a \leq_P b, \{d, c\} \leq_V a \leq_V \{b, e\})$ and ω_2 corresponds to $(a \leq_P b \leq_P c \leq_P d \leq_P e, d \leq_V b \leq_V \{a, e\} \leq_V c)$.

If $I \hookrightarrow J$, by restricting the orders on I to orders on J, we obtain a restriction to a packed word on J. The resulting species with restrictions structure is denoted by `PWo`.

It will be useful to represent packed words ω in I as rectangle diagrams labeled by I. This is done in the following way: let $1 \leq m \leq |I|$ be the rank of ω, we place the elements of I in an $m \times |I|$ grid so that the elements are placed horizontally according to the $\leq_P$ order, and vertically according to the $\leq_V$ order. For instance, the packed word $\omega_1 = 13123 = (d < e < c < a < b, \{d, c\}) < \{a\} < \{b, e\}$ in $\{a, b, c, d, e\}$ can be represented as

	e			b
			a	
d		c		

. (23)

In this way, there are $(n!) \times \sum_{m=1}^{n} \mathrm{SurFunc}(I, [m])$ elements in $\texttt{PWo}[n]$, where $\mathrm{SurFunc}(I, [m])$ denotes the number of surjections (surjective functions) from I to $[m]$. Because each packed word ω has an isomorphism class of size $n!$, there are $\sum_{m=1}^{n} \mathrm{SurFunc}(I, [m])$ many packed words, which is expected.

If we consider a packed word $\omega = (\leq_P, \leq_V)$ on a set I, we write $\mathbb{X}(\omega) = I$. If $f : J \to I$ is an injective map, the preimage of each order $\leq_P, \leq_V$ is well defined. Furthermore, the preimage of $\leq_P$ is a total order on J, whereas the preimage of $\leq_V$ is a linear order on J. Thus, this defines the packed word $\texttt{PWo}[f](\omega)$. The $*$ operation on orders can be extended to a **diagonal sum** operation $\oplus$ on packed words. This is usually referred to the shifted concatenation of packed words. This endows $\texttt{PWo}$ with an associative species with restrictions structure.

5.4. **Relation between parking functions and labelled Dyck paths.** Before we make the goal of this section explicit, let us first recall the definition of a parking function and of a Dyck path.

Definition 5.2 (Parking function)**.** A parking function $\mathfrak{p} = a_1 \dots a_n$ is a sequence of integers in $[n]$ such that, after reordering $a^{(1)} \leq a^{(2)} \leq \cdots \leq a^{(n)}$, we have $a^{(i)} \leq i$ for all i.

Intuitively, parking functions model the following scenario: n cars arrive at a street with n parking spots labeled $1, \dots, n$. Each car i has a preferred spot a_i. Cars park greedily: car i parks at its preferred spot if available, otherwise it parks at the next available spot to the right. A sequence is a parking function if and only if all cars can find a parking spot using this greedy algorithm.

Examples of parking functions are 12, 131 and 3114.

Definition 5.3 (Dyck path)**.** Given an $n \times n$ square grid, a **Dyck path** is an edge path from $(0, 0)$ to (n, n) staying weakly above the line $y = x$ (the main diagonal). It is a classical result that Dyck paths are enumerated by Catalan numbers.

To describe species on parking functions we need to use the construction of parking functions as labelled Dyck paths. Specifically, if I is a set of size n, we label a Dyck path $\mathcal{D}$ on an $n \times n$ square grid by a function f that assigns unique values on the set I to each of the **up** segments of the Dyck path. An 'up' segment is an edge from (i, i) to $(i + 1, i + 1)$; a 'down' segment is an edge from $(i, i + 1)$ to $(i + 1, i)$. If we enrich I with a total order $\leq$ in such a way that in each sequence of **up** segments, the labels arise in **increasing** order, Bergeron et al. construct in [BDL+23] a parking function $\mathfrak{p} = \mathfrak{p}(I, \mathcal{D}, f, \leq)$. We recover here such construction, adapted to the language in this article, for convenience.

There are two steps in this construction. The first is to realize the labelled Dyck path as a weak set composition $\vec{\boldsymbol{\pi}}$ of I in $|I|$ parts. The second is to translate the weak set composition $\vec{\boldsymbol{\pi}}$ and the order $\leq$ in I as a parking function. We will use an example on $I = \{1, 2, 3, 4, 5\}$ to help us highlight the most important details of this construction, presented in Fig. 5. There, we have a Dyck path $\mathcal{D}$ and a corresponding assignment f to each **up** segment. We assume the usual integer order on I.

In the first step we group each of the labels that occur on the i-th vertical line in the i-th block of $\vec{\boldsymbol{\pi}}$. This gives us a weak set composition of I with exactly $|I|$ parts. Notice that some parts may be empty sets, as it happens in the example given in Fig. 5.

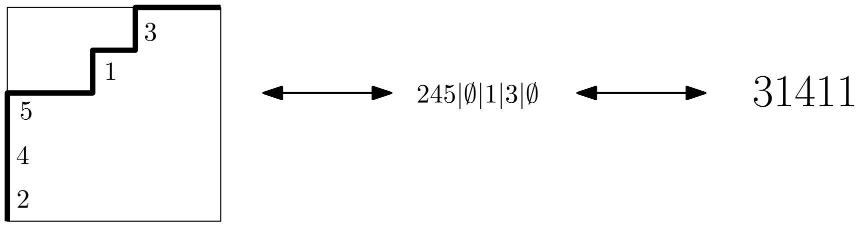
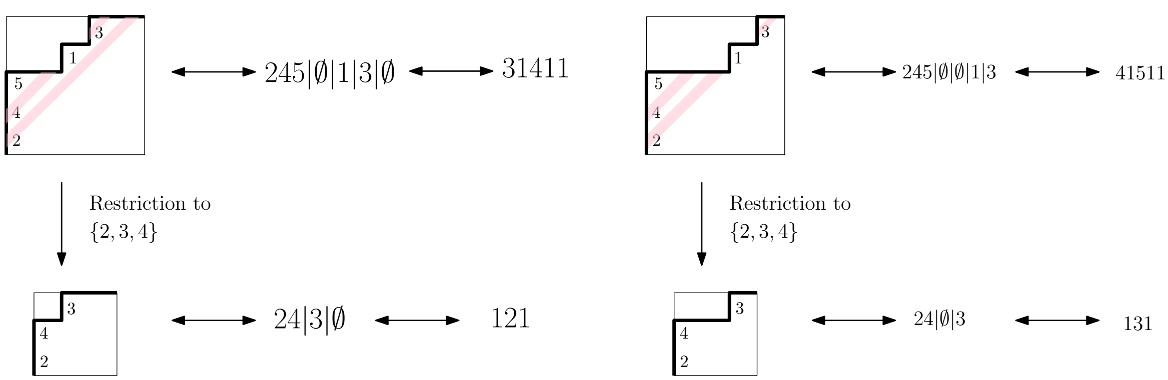


FIGURE 5. This is an example of a Dyck path labelled by the set $I = \{1, 2, 3, 4, 5\}$. We enrich this set with the usual order on the integers.

In the second step we read off the position of each of the labels of I, writing down in which part these occur. The resulting sequence is a parking function of size $|I|$. In the example in Fig. 5, for instance, we notice that there are no entries in the second block, so 2 does not occur in the corresponding parking function. Because there are three elements in the first block, namely 245, in the parking function $\mathfrak{p}$ the character 1 arises three times, on the second, fourth and fifth position.

5.5. **Species on parking functions.** The species of **parking functions**, $\mathtt{PF}$, arises by letting $\mathtt{PF}[I]$ be the collection of all parking functions $\mathfrak{p} = \mathfrak{p}(I, \mathcal{D}, f, \leq)$, that is Dyck paths $\mathcal{D}$ in an $|I| \times |I|$ grid, labelled by elements of I along with an order $\leq$ of I. We can give it a notion of species with restrictions: for each inclusion $\iota : I \hookrightarrow J$ and a labelled Dyck path $(J, \mathcal{D}, f, \leq)$ on J, the restriction $\mathtt{PF}[\iota](J, \mathcal{D}, f, \leq) = (I, \mathcal{D}|_I, f|_I, \leq|_I)$ has an intuitive meaning, except perhaps for $\mathcal{D}|_I$, which we clarify in the following. This is done via a notion of **tunnels** — pairs of an up and down segment at the same level, that have no other segment at that level in between — introduced by Deutsch and Elizalde in [ED03]. Specifically, the corresponding Dyck path is defined to be the restricted Dyck path by taking the **tunnels** labelled by elements in I, relabelling sequences of **up** segments if necessary to preserve the increasing property. One can see in Fig. 6 how this works on the example given above, as well as the corresponding parking function.



(A) The parking function 31411 and its restriction

(B) The parking function 41511 and its restriction

FIGURE 6

pat	111	112	121	211	113	131	311	122	212	221	123	132	213	312	231	321
11	3	2	2	2	1	1	1	1	1	1	0	0	0	0	0	0
12	0	1	0	0	2	1	0	2	1	0	3	2	2	1	1	0
21	0	0	1	1	0	1	2	0	1	2	0	1	1	2	2	3

TABLE 1. Pattern functions evaluated at parking functions of length three.

One defines a shifted concatenation $\oplus$ on parking functions, defined via concatenation of the underlying Dyck paths and orders (similar to the diagonal sum operation for permutations). The following claim can be established by the same methods presented in [Pen22].

Proposition 5.4 (Species with restrictions on parking functions)**.** $\mathtt{PF}$ forms a species with restrictions structure. The proof follows the same approach as for permutations: the operations on Dyck paths and orders respect the restriction to subsets. Furthermore, the shifted concatenation $\oplus$ endows $\mathtt{PF}$ with a monoid structure, and the resulting Hopf algebra $\mathcal{A}(\mathtt{PF})$ is free.

For the last part, we observe that because $\mathtt{PF}$ is NCF (see below in Definition 6.2), we have from Corollary 6.4 that this algebra is free.

Example 5.5. The five smallest parking functions are $\emptyset, 1, 11, 12$ and 21, where the empty sequence $\emptyset$ is the unique parking function of size 0. The sixteen parking functions of size three are displayed in Fig. 7, along with its corresponding labelled Dyck paths.

For any parking function $\mathfrak{p}$ of size three, we have $\mathbf{pat}_\emptyset(\mathfrak{p}) = 1$ and $\mathbf{pat}_1(\mathfrak{p}) = 3$. The values of $\mathbf{pat}_{11}, \mathbf{pat}_{12}$ and $\mathbf{pat}_{21}$ in parking functions of size three are represented below in Table 1. There, one can also check the relation

$$\mathbf{pat}_1^2 = \mathbf{pat}_1 + 2(\mathbf{pat}_{11} + \mathbf{pat}_{12} + \mathbf{pat}_{21}),$$

predicted from (19).

6. THE ANTIPODE FORMULA FOR PATTERN ALGEBRAS

In this section we give a general formula for the antipode of a pattern algebra, whenever our connected species with restrictions is of the form $\mathtt{L} \times -$, that is, species that factor as the Hadamard product (see Section 3) of total orders $\mathtt{L}$ with another species. This antipode formula is a dual analogue of the antipode formula in [BB19], where a similar formula was obtained for linearized Hopf species. We note explicitly that species on parking functions and packed words, as well as $\mathtt{Per}$, are of the form $\mathtt{L} \times -$ (see examples in Section 5).

The formula obtained is not cancellation free, but it serves as a starting platform to explore the cancellation free formulas for the cases presented above: permutations, packed words and parking functions. The requirement that our species with restrictions

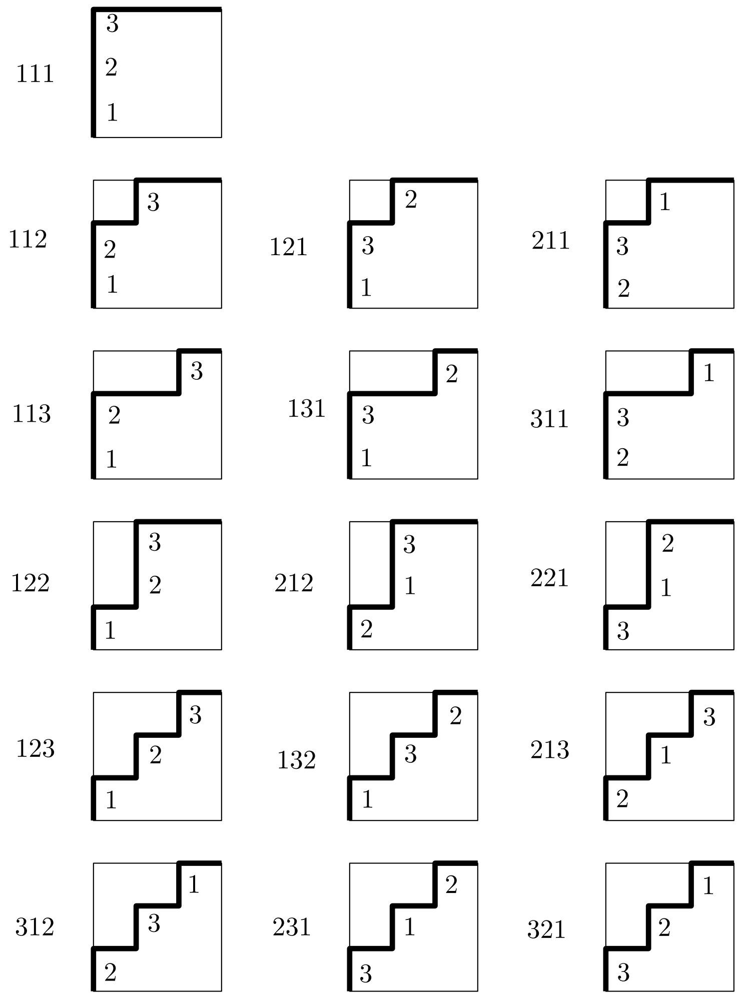


Figure 7

is of the form $\mathtt{L}\times -$ explains why no cancellation free formulas for other species with restrictions, for instance in marked permutations, introduced in [Pen22], were found.

We start by recalling Takeuchi's formula, from above in Proposition 2.1. If H is a Hopf algebra such that $(\iota\circ\epsilon - \mathrm{id}_H)$ is $\star$-nilpotent, then

$$S = \sum_{k\geq 0} (\iota\circ\epsilon - \mathrm{id}_H)^{\star k}\,.$$

Recall that for any pattern Hopf algebra $\mathcal{A}(\mathtt{h})$, $(\iota\circ\epsilon - \mathrm{id}_H)$ is $\star$-nilpotent.

6.1. $\mathtt{L}\times -$ **species with restrictions.** In [Pen22, Corollary 4.4.] it was shown that in species with restrictions $\mathtt{h}$, any factorization into $*$-indecomposables is unique possibly up to order of the factors. In some species with restrictions, we can also drop the "up to order" requirement: some species have a unique factorization, not just unique up to reordering of the factors. We make this precise in the *non-commuting factorization* definition.

Remark 6.1 (Product notation clarification)**.** For clarity, we use $\star$ for the convolution operation on $\mathrm{End}(H)$ (the antipode operations) and $*$ for the associative operation on a species (the combinatorial operation). This distinction is important to avoid confusion between operations in different contexts.

Definition 6.2 (Non-Commuting factorization on pattern Hopf algebras)**.** A monoidal species with restrictions is called a **non-commuting factorization** species, or simply an NCF species, if any element x has a unique factorization into $*$-indecomposables elements $x = x_1 * \cdots * x_n$. This property is crucial for obtaining cancellation-free antipode formulas.

Lemma 6.3 (Linear species with restrictions have NCF)**.** Let $\mathtt{S}$ be a connected species with restrictions. Then $\mathtt{L} \times \mathtt{S}$ has NCF.

We call species of the form $\mathtt{L} \times \mathtt{S}$ an **ordered species with restrictions**. In what follows, we will focus on associative and connected ordered species with restrictions.

Proof. Let $x = x_1 * \cdots * x_k \in (\mathtt{L} \times \mathtt{S})[I]$ and $y = y_1 * \cdots * y_n \in (\mathtt{L} \times \mathtt{S})[J]$ such that $x \sim y$. From [Pen22, Corollary 3.4.], we know that $k = n$ and the multisets $\{x_i\}_{i=1}^{k}, \{y_i\}_{i=1}^{n}$ are the same. It remains to show that the factorizations order coincide and that we have $x_i \sim y_i$ for $i = 1, \ldots, n$. To that effect we act by induction, where $n = 1$ is trivial.

We write $x_i = (l_i, p_i) \in (\mathtt{L} \times \mathtt{S})[I_i]$ and $y_i = (m_i, q_i) \in (\mathtt{L} \times \mathtt{S})[J_i]$, where (l_i, p_i) denotes the pair of a total order and the object in the original species. Write $x = (l, p) \in (\mathtt{L} \times \mathtt{S})[I]$ and $y = (m, q) \in (\mathtt{L} \times \mathtt{S})[J]$. Assume wlog that $|I_1| \geq |J_1|$.

By hypothesis, we have that $x \sim y$, so there exists some bijection $\phi : J \to I$ such that $(\mathtt{L} \times \mathtt{S})[\phi](x) = y$. This bijection yields a correspondence between two linear orders $\mathtt{L}[\phi](l_1 * \cdots * l_n) = m_1 * \cdots * m_n$. Observe that I_1 and J_1 are ideals in $l_1 * \cdots * l_n$ and $m_1 * \cdots * m_n$, respectively, as seen in Proposition 5.1. So either $J_1 = \phi(I_1)$ or $J_1 \subsetneq \phi(I_1)$. Assume for sake of contradiction that $J_1 \subsetneq \phi(I_1)$, and consider the following factorization of x_1:

$$x_1 = x|_{I_1} \sim y|_{\phi(I_1)} = y_1|_{\phi(I_1) \cap J_1} * (y_2 * \cdots * y_n)|_{(J \setminus J_1) \cap \phi(I_1)} . \tag{24}$$

From $|I_1| > |J_1|$ and ϕ is a bijection, we have that $|(J \setminus J_1)| + |\phi(I_1)| = |J| - |J_1| + |I_1| > |J|$, so the intersection $(J \setminus J_1) \cap \phi(I_1)$ is non-empty. On the other hand, $\phi(I_1) \cap J_1 = J_1$ is also non-empty.

Going back to (24), we get a non-trivial factorization of x_1, which contradicts the fact that we started with a factorization into indecomposables. Thus $|I_1| = |J_1|$, which shows that $\phi(I_1) = J_1$. Therefore, $\phi(I \setminus I_1) = J \setminus J_1$, $\mathtt{R}[\phi](p|_{I_1}) = q|_{J_1}$ and $\mathtt{R}[\phi](p|_{I \setminus I_1}) = q|_{J \setminus J_1}$. We get that $x_1 \sim y_1$ and $x_2 * \cdots * x_n \sim y_2 * \cdots * y_n$, via ϕ. By induction hypothesis, this tells us that $x_2 \sim y_2, \cdots, x_n \sim y_n$, as desired. $\square$

Corollary 6.4. If $\mathtt{R}$ is an ordered species with restrictions, then $\mathcal{A}(\mathtt{R})$ is free.

This follows from [Var14, Theorem 5.2], as this proof only uses the fact that `Per` is NCF.

The pattern Hopf algebra on permutations, on packed words and on parking functions all satisfy the NCF property. In fact, permutations, packed words and parking functions are of the form $\mathtt{L} \times \mathtt{R}$. This property allows for an easy manipulation of the coproduct, and results in a tractable approach to the antipode formula.

Definition 6.5 (Composition poset and cumulative sum)**.** Recall that we write $\mathcal{C}_n$ for the set of compositions of size n. We define **CS**, a bijection between $\mathcal{C}_n$ and $2^{[n-1]}$, as follows. If $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathcal{C}_n$, define $f_i = \sum_{j=1}^{i} \alpha_j$ and

$$\mathbf{CS}(\alpha) := \{ f_i \,|\, i = 1, \dots, \ell - 1 \} . \tag{25}$$

Intuitively, the cumulative sum bijection maps compositions to subsets by recording where "cuts" occur in the composition — each element of $\mathbf{CS}(\alpha)$ represents a position where the composition could be split. This bijection allows us to define an order $\leq$ in $\mathcal{C}_n$, via the pullback from the boolean poset in $2^{[n-1]}$. This order can also be defined as follows: we say that $\alpha \leq \beta$ if α arises from β after merging and adding consecutive entries.

Recall that a QSS $\vec{\mathbf{I}} = (I_1, \dots, I_n)$ of $y \in \mathtt{R}[I]$ from $x_1, \dots, x_n$ satisfies $y|_{I_i} \sim x_i$ for all $i = 1, \dots, n$, and $I = \bigcup_i I_i$ (see Section 1 for the definition of QSS).

Definition 6.6 (Compositions and QSS)**.** Consider again $\mathtt{R}$ an ordered species with restrictions. Let $x \in \mathtt{R}[J], y \in \mathtt{R}[I]$, and $x = x_1 * \cdots * x_n$ be the unique factorization of x into indecomposables. Further say that $y = (\leq_y, \iota)$, where $\leq_y$ is a linear order on I. Let $\vec{\mathbf{I}} = (I_1, \dots, I_n)$ be a QSS of y from $x_1, \dots, x_n$ and consider a composition $\alpha \models n$.

Suppose that $\mathbf{CS}(\alpha) = \{f_1, \dots, f_{\ell(\alpha)-1}\}$ and use the convention that $f_0 = 0$ and $f_{\ell(\alpha)} = n$. Then we define for $i = 1, \dots, \ell(\alpha)$:

$$I_i^\alpha := I_{f_{i-1}+1} \cup \cdots \cup I_{f_i} \quad , \quad x_i^\alpha := x_{f_{i-1}+1} * \cdots * x_{f_i} .$$

For a partial order $\leq$ on a set I, and two sets $A, B \subseteq I$, we write $A \lneq_{\leq} B$ to mean that A, B are disjoint and $a \leq b$ for any $a \in A$ and $b \in B$.

Two indices $i < j$ are said to be **merged** by α if there is some k in $\{1, \dots, n\}$ such that $f_{k-1} < i < j \leq f_k$.

Intuitively, a QSS is α-stable if merging its components according to α respects the underlying structure and pattern relationships. Formally, we say that a QSS $\vec{\mathbf{I}}$ of y from $x_1, \dots, x_n$ is α-stable if $(I_i^\alpha)_i$ is a QSS of y from $(x_i^\alpha)_i$ and, whenever $x_i \sim x_j$ and $i < j$ are merged with α, then $I_i \lneq_y I_j$.

Finally, we define

$$\mathcal{I}_{\vec{\mathbf{I}}}^{x,y} := \left\{ \alpha \models n \,\middle|\, \vec{\mathbf{I}} \text{ is an } \alpha\text{-stable QSS of } y \text{ from } (x_i)_i \right\} . \tag{26}$$

Example 6.7 (α-stable QSS on $\mathtt{PWo}$)**.** Consider the packed word $\rho = 21 \oplus 111 = 21333 = (1 < 2 < 3 < 4 < 5, 2 < 1 < \{3, 4, 5\})$ on the set $[5]$. This packed word has three QSS from $21, 1, 11$, precisely $\vec{\mathbf{I}}_1 = (12, 3, 45)$, $\vec{\mathbf{I}}_2 = (12, 4, 35)$ and $\vec{\mathbf{I}}_3 = (12, 5, 34)$. All three are $(1, 1, 1)$-stable.

We now observe that $\vec{\mathbf{I}}_1, \vec{\mathbf{I}}_2$ and $\vec{\mathbf{I}}_3$ are $(2, 1)$-stable, but not $(1, 2)$-stable. For $(1, 2)$-stability: The composition $(1, 2)$ merges indices 2 and 3. We compute $\vec{\mathbf{I}}_1^{(1,2)} = \vec{\mathbf{I}}_2^{(1,2)} = \vec{\mathbf{I}}_3^{(1,2)} = (12, 345)$. However, $\rho|_{345} = 111$ is not $\rho|_{I_2} \oplus \rho|_{I_3}$ for any of the QSS, so $(1, 2)$-stability fails for all three. For $(2, 1)$-stability: The composition $(2, 1)$ merges indices 1 and 2. We have $\vec{\mathbf{I}}_1^{(2,1)} = (123, 45)$, $\vec{\mathbf{I}}_2^{(2,1)} = (124, 35)$ and $\vec{\mathbf{I}}_3^{(2,1)} = (125, 34)$. In each case, it is straightforward to verify that $\rho|_{I_1^{(2,1)}} = 21 \oplus 1$ and $\rho|_{I_2^{(2,1)}} = 11$, making all three QSS $(2, 1)$-stable.

Therefore, in each case we can see that

$$\mathcal{I}_{\vec{\mathbf{I}}}^{21\oplus 1\oplus 11, 21333} = \{(1, 1, 1), (2, 1)\} .$$

The following example portrays the consequences of the additional requirement that $I_i \lneq_y I_j$ whenever $x_i \sim x_j$ and i, j are merged by α, in the context of packed words.

Example 6.8 (α-stable on $\mathtt{PWo}$, with $x_i \sim x_j$)**.** Let us consider now the packed word $\rho = 2133 = (1 \leq_P 2 \leq_P 3 \leq_P 4, 2 \leq_V 1 \leq_V \{3, 4\})$, we will be considering QSS of ρ from $\omega_1 = 12, \omega_2 = 1, \omega_3 = 1$. Namely, $\vec{\mathbf{I}}_1 = (13, 2, 4)$ and $\vec{\mathbf{I}}_2 = (13, 4, 2)$. We have that $\rho|_{I_2 \cup I_3} = \rho|_{I_2} \oplus \rho|_{I_3} = 1 \oplus 1$ in either of the cases. However, note that $\omega_2 \sim \omega_3$, and the composition $(1, 2)$ merges 2 and 3, so $(1, 2)$-stability requires $I_2 \lneq_P I_3$. Indeed, because $2 \leq_P 4$, we have that $\vec{\mathbf{I}}_1$ is $(1, 2)$-stable, whereas $\vec{\mathbf{I}}_2$ is not $(1, 2)$-stable.

Define the composition $\mu_i \coloneqq (\underbrace{1, \dots, 1}_{i-1 \text{ times}}, 2, 1, \dots, 1)$. The composition μ_i is important because it represents merging positions i and $i + 1$. If $\vec{\mathbf{I}} = (I_1, \dots, I_n)$ is a μ_i-stable QSS, then $y|_{I_1 \cup I_{i+1}} = y|_{I_i} * y|_{I_{i+1}}$. This motivates the following lemma:

Lemma 6.9. Let $\mathtt{R}$ be an ordered species with restrictions. Consider x, y objects in $\mathtt{R}$, such that $y = (\leq_y, m)$ and $x = x_1 * \cdots * x_n$ a factorization into indecomposables. If $\vec{\mathbf{I}} = (I_1, \dots, I_n)$ is a QSS of y from $x_1, \dots x_n$ that is μ_i-stable, then $I_i \lneq_y I_{i+1}$.

We observe latter that in the case of packed words, a stronger claim can be used to compute α-stability. See Lemma 7.2.

Proof. If $x_i \sim x_{i+1}$, the μ_i stability implies that $I_i \lneq_y I_{i+1}$, because μ_i merges i and $i + 1$. We can now assume that $x_i \not\sim x_{i+1}$. Let $X_j = \mathbb{X}(x_j)$ for $j = i, i + 1$. Note that because $y|_{I_j} \sim x_j$, we have that $|X_j| = |I_j|$. The stability condition further gives us that $y|_{I_i \cup I_{i+1}} \sim x_i * x_{i+1}$. Let $\phi : X_i \sqcup X_{i+1} \to I_i \cup I_{i+1}$ be bijection such that $\mathtt{R}[\phi](y|_{I_i \cup I_{i+1}}) = x_i * x_{i+1}$.

Because ϕ is a bijection, and $|X_j| = |I_j|$, this means that I_i, I_{i+1} are disjoint. We will consider the case that $|I_i| \geq |I_{i+1}|$ here. The proof on the $|I_i| \leq |I_{i+1}|$ case can be done in a similar way.

Let inc be the injection $I_i \to I_i \cup I_{i+1}$. Observe that

$$\begin{aligned} \mathtt{h}[\phi \circ \mathrm{inc}](y|_{I_i \cup I_{i+1}}) =& (x_i * x_{i+1})|_{\phi^{-1}(I_i)} \\ \mathtt{h}[\phi](y|_{I_i}) =& (x_i)|_{X_i \cap \phi^{-1}(I_i)} * (x_{i+1})|_{X_{i+1} \cap \phi^{-1}(I_i)} \,. \end{aligned} \tag{27}$$

However, $\mathtt{R}[\phi](y|_{I_i}) \sim y|_{I_i} \sim x_i$, which is $*$-indecomposable, we conclude that either $|X_i \cap \phi^{-1}(I_i)| = 0$ or $|X_{i+1} \cap \phi^{-1}(I_i)| = 0$. Assume the first for sake of contradiction, and because $\phi^{-1}(I_i) \subseteq X_i \cap X_{i+1}$, we have that $\phi^{-1}(I_i) \subseteq X_{i+1}$. But $|\phi^{-1}(I_i)| = |I_i| \geq |I_{i+1}| = |X_{i+1}|$, therefore we have equality, that is $\phi^{-1}(I_i) = X_{i+1}$

This together with (27) and connectedness of $\mathtt{R}$ yields $\mathtt{h}[\phi](y|_{I_i}) = x_{i+1}$, which implies $x_i \sim x_{i+1}$, a contradiction. We conclude that $|X_{i+1} \cap \phi^{-1}(I_i)| = 0$, and so $\phi^{-1}(I_i) = X_i$. Because I_i and I_{i+1} are disjoint, it follows that $\phi^{-1}(I_{i+1}) = X_{i+1}$.

If we write $y = (\leq_y, m)$ and $x_i * x_{i+1} = (\leq_x, n)$, then $X_i \lneq_x X_{i+1}$ from Proposition 5.1. Because $\mathtt{R}[\phi^{-1}](x_i * x_{i+1}) = y|_{I_i \cup I_{i+1}}$, this gives us $\phi(X_i) \lneq_y \phi(X_{i+1})$, as desired. □

Observation 6.10. Let $\mathtt{R}$ be a species with restrictions, $y, x_1, \dots, x_n \in \mathcal{G}(\mathtt{R})$ and $\vec{\mathbf{I}}$ a QSS of y from $x_1 \dots, x_n$. Call $x \coloneqq x_1 * \cdots * x_n$. Then, $\mathcal{I}_{\vec{\mathbf{I}}}^{x,y}$ has a unique maximal element, $\mathbb{1} = (1, \dots, 1)$.

The following is the main theorem in this section. The antipode is given by summing over all possible objects y, all quasi-shuffle signatures of y, and all α-stable compositions, with an alternating sign determined by the composition length.

Theorem 6.11. For a species with restrictions $\mathtt{R}$ that is a multiple of $\mathtt{L}$, and an element x, along with its factorization $x = x_1 * \cdots * x_n$, we have the following antipode formula in $\mathcal{A}(\mathtt{R})$:

$$S(\mathbf{pat}_x) = \sum_y \mathbf{pat}_y \sum_{\substack{\vec{\mathbf{I}} \text{ QSS of } y \\ \text{from } x_1, \dots, x_n}} \sum_{\alpha \in \mathcal{I}_{\vec{\mathbf{I}}}^{x,y}} (-1)^{\ell(\alpha)} \,.$$

Proof.

$$\Delta^{\circ(k-1)}(\mathbf{pat}_x) = \sum_{x = \chi_1 * \cdots * \chi_k} \mathbf{pat}_{\chi_1} \otimes \cdots \otimes \mathbf{pat}_{\chi_k}$$

Because the species with restrictions $\mathtt{R}$ has NCF, the only ways to factorize x into k factors is to start from the original factorization $x = x_1 * \cdots * x_n$ and bracket these factors into k blocks, possibly empty, of n. Therefore, we can enumerate these factorizations using weak compositions, as follows:

$$\begin{aligned}(\iota \circ \epsilon - \mathrm{id}_{\mathcal{A}(\mathtt{R})})^{\star k}(\mathbf{pat}_x) =& (-1)^k \mu^{\circ(k-1)} (\mathrm{id}_{\mathcal{A}(\mathtt{R})} - \iota \circ \epsilon)^{\otimes k} \Delta^{\circ(k-1)}(\mathbf{pat}_x) \\ =& (-1)^k \mu^{\circ(k-1)} (\mathrm{id}_{\mathcal{A}(\mathtt{R})} - \iota \circ \epsilon)^{\otimes k} \left(\sum_{\substack{\alpha \models^0 n \\ \ell(\alpha)=k}} \mathbf{pat}_{x_1^\alpha} \otimes \cdots \otimes \mathbf{pat}_{x_k^\alpha} \right) \\ =& (-1)^k \mu^{\circ(k-1)} \sum_{\substack{\alpha \models n \\ \ell(\alpha)=k}} \mathbf{pat}_{x_1^\alpha} \otimes \cdots \otimes \mathbf{pat}_{x_k^\alpha} \\ =& (-1)^k \sum_{\substack{\alpha \models n \\ \ell(\alpha)=k}} \mathbf{pat}_{x_1^\alpha} \cdots \mathbf{pat}_{x_k^\alpha}\end{aligned}$$

Note how we used that $(\mathrm{id}_{\mathcal{A}(\mathtt{R})} - \iota \circ \epsilon)(\mathbf{pat}_x) = \mathbb{1}[x \neq 1]\, \mathbf{pat}_x$. Takeuchi's formula gives:

$$\begin{aligned} S(\mathbf{pat}_x) &= \sum_{k \geq 0} (\iota \circ \epsilon - \mathrm{id}_{\mathcal{A}(\mathtt{R})})^{\star k}(\mathbf{pat}_x) \\ &= \sum_{\alpha \models n} (-1)^{\ell(\alpha)} \mathbf{pat}_{x_1^\alpha} \cdots \mathbf{pat}_{x_{\ell(\alpha)}^\alpha} \\ &= \sum_{\alpha \models n} (-1)^{\ell(\alpha)} \sum_y \mathbf{pat}_y \binom{y}{x_1^\alpha, \dots, x_{\ell(\alpha)}^\alpha} \\ &= \sum_y \mathbf{pat}_y \sum_{\alpha \models n} \sum_{\substack{\vec{\mathbf{I}} \text{ QSS of } y \\ \text{from } x_1^\alpha, \dots, x_{\ell(\alpha)}^\alpha}} (-1)^{\ell(\alpha)} \\ &= \sum_y \mathbf{pat}_y \sum_{\substack{\vec{\mathbf{I}} \text{ QSS of } y \\ \text{from } x_1, \dots, x_n}} \sum_{\alpha \in \mathcal{I}_{\vec{\mathbf{I}}}^{x,y}} (-1)^{\ell(\alpha)}. \end{aligned}$$

All equalities but the last one are simple rearrangements. To motivate the last equality, we will construct a series of bijections, for each composition $\alpha \models n$:

$$\Phi_\alpha^{x,y} = \Phi_\alpha : \left\{ \vec{\mathbf{J}} \text{ QSS of } y \text{ from } x_1^\alpha, \dots, x_{\ell(\alpha)}^\alpha \right\} \to \left\{ \vec{\mathbf{I}}\ \alpha\text{-stable QSS of } y \text{ from } x_1, \dots, x_n \right\}, \\ (J_1, \dots, J_{\ell(\alpha)}) \mapsto (I_1, \dots, I_n).$$

Recall that $y = (\leq_y, \iota_y)$ for certain objects $\leq_y$ and ι_y. Note that $\leq_y$ induces a linear order m in I. The sets $(I_i)_i$ are defined so that $J_s = I_{f_{s-1}+1} \uplus \cdots \uplus I_{f_s}$, that $|I_i| = |x_i|$ and that $I_i \lneq_y I_j$ whenever $f_{s-1} < i < j \leq f_s$. There is clearly a unique way to choose such $(I_i)_i$. This is easily seen to be an α-stable QSS of y from $x_1, \dots, x_n$.

Inversely, we define:

$$\Psi_\alpha^{x,y} = \Psi_\alpha : \left\{ \vec{\mathbf{I}} \ \alpha\text{-stable QSS of } y \text{ from } x_1, \dots, x_n \right\} \to \left\{ \vec{\mathbf{J}} \text{ QSS of } y \text{ from } x_1^\alpha, \dots, x_{\ell(\alpha)}^\alpha \right\} ,$$
$$(I_1, \dots, I_n) \mapsto (I_1^\alpha, \dots, I_{\ell(\alpha)}^\alpha),$$

where we recall that $I_i^\alpha = I_{f_{s-1}+1} \cup \cdots \cup I_{f_s}$.

The maps Ψ_α and Φ_α are easily seen to be inverses of each other, establishing the bijection and the desired result. □

Proposition 6.12 (Filtered structure of $\mathcal{I}$)**.** For an ordered species with restrictions $\mathtt{R}$, and elements $x \in \mathtt{R}[J], y \in \mathtt{R}[I]$, along with a factorization into $*$-indecomposables $x = x_1 * \cdots * x_n$ and a QSS $\vec{\mathbf{I}}$ of y from $x_1, \dots, x_n$, we have that $\mathcal{I}_{\vec{\mathbf{I}}}^{x,y}$ is a filter. That is, if $\alpha \in \mathcal{I}_{\vec{\mathbf{I}}}^{x,y}$ and $\beta \geq \alpha$ then $\beta \in \mathcal{I}_{\vec{\mathbf{I}}}^{x,y}$.

Furthermore, if $\mathcal{I}_{\vec{\mathbf{I}}}^{x,y}$ has a unique minimal element distinct from $\mathbb{1}$, then

$$\sum_{\alpha \in \mathcal{I}_{\vec{\mathbf{I}}}^{x,y}} (-1)^{\ell(\alpha)} = 0 \,.$$

Proof. Say that $y = (\leq_y, \iota)$. To establish the first fact, assume that $\alpha \in \mathcal{I}_{\vec{\mathbf{I}}}^{x,y}$ and let $\beta \geq \alpha$. Because $\beta \geq \alpha$, if β merges $i < j$ then α merges $i < j$, thus we have that $I_i \not\leq_y I_j$. It remains to show that $(I_i^\beta)_i$ is a QSS of y from $(x_i^\beta)_i$, or equivalent that $y|_{I_i^\beta} \sim x_i^\beta$.

However, because $\beta \geq \alpha$, for each i we have $I_i^\beta \subseteq I_j^\alpha$ so

$$y|_{I_i^\beta} = \left(y|_{I_j^\alpha} \right)|_{I_i^\beta} = x_j^\alpha|_{I_i^\beta} = x_i^\beta \,.$$

For the concluding part, we simply observe that if $\mathcal{I}_{\vec{\mathbf{I}}}^{x,y}$ has a unique minimal element $\mathbb{0}$, then it is the interval of a boolean poset $\mathcal{I}_{\vec{\mathbf{I}}}^{x,y} = [\mathbb{0}, \mathbb{1}]$. Let K be the collection of sets $A \subseteq [n-1]$ that contain $X \coloneqq \mathbf{CS}(\mathbb{0})$.

Consider $\zeta : K \to K$. Choose $\chi \in [n-1] \setminus X$, if $X \neq [n-1]$, and consider the following symmetric difference:

$$\zeta(A) = A \,\Delta\, \{\chi\}$$

If no such χ exists, define $\zeta(A) = A$. This is an involution on K, which corresponds to a sign-reversing involution in compositions in $[\mathbb{0}, \mathbb{1}]$:

$$\sum_{\alpha \in \mathcal{I}_{\vec{\mathbf{I}}}^{x,y}} (-1)^{\ell(\alpha)} = \mathbb{1}[[n-1] \setminus X \text{ is empty }] = \mathbb{1}[\mathbb{1} = \mathbb{0}] \,.$$

This concludes the proof. □

The consequence is, if $\mathcal{I}^{x,y}_{\vec{\mathbf{I}}}$ has a unique minimal element for any two objects x, y and QSS $\vec{\mathbf{I}}$, the antipode formula reduces to counting how many of these minimal elements are the composition $\mathbb{1}$. Indeed, these will contribute with $(-1)^n$ to the total coefficient of $\mathbf{pat}_y$, giving us a cancellation-free formula.

7. Antipode formulas for packed words and permutations

7.1. **The antipode formula for the pattern algebra on packed words.** For a partial order $\leq$ on a set I, and two sets $A, B \subseteq I$, recall that we say that $A \lneq B$ if $A \cap B = \emptyset$ and $a \leq b$ for any $a \in A$ and $b \in B$. We recall the definition of non-interlacing QSS on packed words (see Section 6 for context).

Definition 7.1 (Interlacing QSS on packed words)**.** Generalizing the notion of interlacing QSS from Section 1, we say that a QSS $\vec{\mathbf{I}} = (I_1, \dots, I_n)$ of a packed word $\rho = (\leq_P, \leq_V)$ from packed words $\omega_1, \dots, \omega_n$ is **non-interlacing** if there exists some $i = 1, \dots, n-1$ such that $I_i \lneq_P I_{i+1}$ and $I_i \lneq_V I_{i+1}$. If no such i exists, we say that the QSS is **interlacing**.

Additionally, let $\left[{\rho \atop \omega_1, \dots, \omega_n}\right]$ be the number of interlacing QSS of ρ from $\omega_1, \dots, \omega_n$.

Our goal in this section is to analyse $\mathcal{I}^{\omega,\rho}_{\vec{\mathbf{I}}}$, show that it always has a unique minimal element, and that it is precisely $\mathbb{1}$ whenever $\vec{\mathbf{I}}$ is interlacing. Recall that $\mu_i = (\underbrace{1, \dots, 1}_{i-1 \text{ times}}, 2, 1, \dots, 1)$.

Lemma 7.2. Let $\omega, \rho = (\leq_P, \leq_V)$ be packed words, such that $\omega = \omega_1 \oplus \cdots \oplus \omega_n$ is its factorization into $\oplus$-indecomposables. Let $\vec{\mathbf{I}}$ be a QSS of ρ from $\omega_1, \dots, \omega_n$. Then $\vec{\mathbf{I}}$ is μ_i-stable if and only if $I_i \lneq_P I_{i+1}$ and $I_i \lneq_V I_{i+1}$.

Proof. Let us take care of the **forward direction** first. From Lemma 6.9, we have that $I_i \lneq_P I_{i+1}$, therefore I_i and I_{i+1} are disjoint and I_i is the unique ideal of $(\leq_P)|_{I_i \cup I_{i+1}}$ of size $|I_i|$. We have that $\rho|_{I_i \cup I_{i+1}} \sim \omega_i \oplus \omega_{i+1}$, so the unique ideal of $(\leq_V)|_{I_i \cup I_{i+1}}$ of size $|I_i|$ is also an ideal of $(\leq_P)|_{I_i \cup I_{i+1}}$, so it has to be I_i. We conclude that I_i is an ideal of $(\leq_V)|_{I_i \cup I_{i+1}}$, so $I_i \lneq I_{i+1}$, concluding this part of the proof.

For the **backwards direction**, if $I_i \lneq_P I_{i+1}$ then I_i and I_{i+1} are disjoint. Therefore, by the definition of $*$ in partial orders,

$$((\leq_P)|_{I_i \cup I_{i+1}}) = ((\leq_P)|_{I_i} * (\leq_P)|_{I_{i+1}}) .$$

Similarly for $\leq_V$. So we conclude that

$$\rho|_{I_i \cup I_{i+1}} = ((\leq_P)|_{I_i} * (\leq_P)|_{I_{i+1}}, (\leq_V)|_{I_i} * (\leq_V)|_{I_{i+1}}) = \rho|_{I_i} \oplus \rho|_{I_{i+1}} .$$

If $\omega_i \sim \omega_{i+1}$, we also have that $I_i \lneq_P I_{i+1}$, so $\vec{\mathbf{I}}$ is μ_i-stable. $\square$

Lemma 7.3. Let ω be packed word, along with $\omega = \omega_1 \oplus \cdots \oplus \omega_n$, its unique factorization into $\oplus$-indecomposable packed words. Let ρ be another packed word, and $\vec{\mathbf{I}}$ a QSS of ρ from $\omega_1, \ldots, \omega_n$. Then $\mathcal{I}^{\omega,\rho}_{\vec{\mathbf{I}}}$ has a unique minimal element and, therefore, it is an interval in $\mathcal{C}_n$.

Proof. We give a concrete description of $\mathcal{I}^{\omega,\rho}_{\vec{\mathbf{I}}}$. Consider the following set:

$$J = \{i \in [n-1] | I_i \lneq_P I_{i+1} \text{ and } I_i \lneq_V I_{i+1}\} = \{i \in [n-1] | \, \vec{\mathbf{I}} \text{ is } \mu_i\text{-stable } \} ,$$

where the second equality is due to Lemma 7.2. Let $\beta = \mathbf{CS}^{-1}(J)$. We claim that $\beta \in \mathcal{I}^{\omega,\rho}_{\vec{\mathbf{I}}}$ and that this is its smallest element.

- **That β is in $\mathcal{I}^{\omega,\rho}_{\vec{\mathbf{I}}}$** we prove now.
  Indeed, we just need to establish that for each I_i^β we have that $\rho|_{I_i^\beta} = x_i^\beta$. Because $I_i^\beta = I_{f_{i-1}+1} \cup \cdots \cup I_{f_i}$, and $I_{f_{i-1}+1} \lneq_P \cdots \lneq_P I_{f_i}$, $I_{f_{i-1}+1} \lneq_V \cdots \lneq_V I_{f_i}$, we get that
  $$\rho|_{I_i^\beta} = \rho|_{I_{f_{i-1}+1}} \oplus \cdots \oplus \rho|_{I_{f_i}} = x_{f_{i-1}+1} \oplus \cdots \oplus x_{f_i} = x_i^\beta \, ,$$
- **That β is the smallest element in $\mathcal{I}^{\omega,\rho}_{\vec{\mathbf{I}}}$** we prove now.
  Let α be a composition of n such that $\alpha \not\geq \beta$. Assume that $\alpha \neq \beta$. Then $J^c \cap \mathbf{CS}(\alpha) \neq \emptyset$, so pick some $i \in J^c \cap \mathbf{CS}(\alpha)$. Because $i \notin J$, is such that $I_i \not\lneq_P I_{i+1}$ or $I_i \not\lneq_V I_{i+1}$.

  If $\omega_i \not\sim \omega_{i+1}$, one can see that $\rho|_{I_i \cup I_{i+1}} \neq \omega_i \oplus \omega_{i+1}$. If $\omega_i \sim \omega_{i+1}$, stability would require $I_i \lneq_P I_{i+1}$, so $I_i \not\lneq_V I_{i+1}$, and we conclude again that $\rho|_{I_i \cup I_{i+1}} \neq \omega_i \oplus \omega_{i+1}$.

  So we can never have $\rho|_{I_j^\alpha} = \omega_j^\alpha$ for j such that ω_j^α includes both $\omega_i \oplus \omega_{i+1}$ in its factorization. If $\omega_i \sim \omega_{i+1}$, stability would require $I_i \lneq_P I_{i+1}$, so $I_i \not\lneq_V I_{i+1}$, and we conclude again that $\rho|_{I_i \cup I_{i+1}} \neq \omega_i \oplus \omega_{i+1}$. This is the contradiction that we are aiming for.

With the construction of the minimal element, we have that $\mathcal{I}^{\omega,\rho}_{\vec{\mathbf{I}}}$ is an interval in $\mathcal{C}_n$. $\square$

Theorem 7.4. Let ω be a packed word, and $\omega = \omega_1 \oplus \cdots \oplus \omega_n$ be its decomposition into $\oplus$-indecomposable packed words. Then, on the pattern Hopf algebra of packed words, we have the following cancellation free and grouping free formula:

$$S(\mathbf{pat}_\omega) = (-1)^n \sum_\rho \left[{}^{\;\;\rho}_{\omega_1, \ldots, \omega_n}\right] \mathbf{pat}_\rho \, .$$

Proof. From Theorem 6.11, we only need to establish that

$$\sum_{\alpha \in \mathcal{I}^{\rho,\omega}_{\vec{\mathbf{I}}}} (-1)^{\ell(\alpha)} = (-1)^n \mathbb{1}[\vec{\mathbf{I}} \text{ is interlacing QSS of } \rho \text{ from } \omega_1, \ldots, \omega_n] \, . \tag{28}$$

Further, from Proposition 6.12 we know that the sum $\sum_{\alpha \in \mathcal{I}_{\vec{\mathbf{I}}}^{\rho,\omega}} (-1)^{\ell(\alpha)}$ vanishes whenever $\mathcal{I}_{\vec{\mathbf{I}}}^{\rho,\omega}$ is an interval with more than one element. From Lemma 7.3, we know that $\mathcal{I}_{\vec{\mathbf{I}}}^{\rho,\omega}$ is indeed an interval. The minimal interval is $\mathbb{1}$ if and only if $\vec{\mathbf{I}}$ is an interlacing QSS from Lemma 7.3. This concludes the proof. $\square$

Notice that this proof hides a sign-reversing involution in it. Specifically, it was used in establishing Proposition 6.12.

7.2. **The antipode formula for the pattern algebra on permutations.** We start by recalling the definition of interlaced QSS on permutations.

Definition 7.5 (Interlacing QSS on permutations)**.** Let $\sigma, \pi_1, \dots, \pi_n$ be permutations, where $\sigma = (\leq_P, \leq_V)$ is a permutation on I. Let $\vec{\mathbf{I}} = (I_1, \dots, I_n)$ be a QSS of σ from $\pi_1, \dots, \pi_n$. We say that $\vec{\mathbf{I}}$ is **non-interlacing** if there exists $i = 1, \dots, n-1$ such that $I_i \lneq_P I_{i+1}$ and $I_i \lneq_V I_{i+1}$. If no such i exists, we say that the QSS is **interlacing**.

Additionally, let $\left[{}_{\pi_1, \dots, \pi_n}^{\quad \sigma} \right]$ be the number of interlacing QSS of σ from $\pi_1, \dots, \pi_n$

Theorem 7.6. Let π be a permutation, and $\pi = \pi_1 \oplus \cdots \oplus \pi_n$ be its decomposition into irreducible permutations. Then, on the pattern Hopf algebra of permutations, we have the following cancellation free and grouping free formula:

$$S(\mathbf{pat}_\pi) = (-1)^n \sum_{\sigma} \left[{}_{\pi_1, \dots, \pi_n}^{\quad \sigma} \right] \mathbf{pat}_\sigma \ .$$

Although we can obtain the antipode formula by showing that a relevant poset of compositions is an interval (see Lemma 7.3), we present here a different proof. Specifically, we will use a surjective Hopf algebra morphism $\mathcal{A}[\mathrm{inc}] : \mathcal{A}(\mathtt{PW}) \to \mathcal{A}(\mathtt{Per})$ and the previous result on packed words.

First, observe that any permutation (a pair of total orders) is a packed word, since a packed word is a pair of partial orders such that $\leq_P$ is a total order and $\leq_V$ is a partial linear order. This gives us an inclusion map $\mathrm{inc} : \mathtt{Per} \to \mathtt{PW}$ that preserves restrictions and the monoidal structure. Therefore, this gives us a surjective Hopf algebra morphism $\mathcal{A}[\mathrm{inc}] : \mathcal{A}(\mathtt{PW}) \to \mathcal{A}(\mathtt{Per})$.

Proof. From Theorem 7.4, we know that for any permutation π seen as a packed word we have that

$$S(\mathbf{pat}_\pi) = (-1)^n \sum_{\omega} \left[{}_{\pi_1, \dots, \pi_n}^{\quad \omega} \right] \mathbf{pat}_\omega \ .$$

If ω is a packed word, we compute $\mathcal{A}[\,\mathrm{inc}](\mathbf{pat}_\omega) = \mathbf{pat}_\omega \mathbb{1}[\omega \text{ is a permutation }]$. Thus, applying $\mathcal{A}[\mathrm{inc}]$ to both sides of the equation above, we get the desired result.

Note that $\mathcal{A}[\text{inc}]$ is a Hopf algebra morphism from Theorem 4.10, so it commutes with the antipode. □

8. Reciprocity results

In this section, let $\mathtt{R}$ be a connected associative species with restrictions. On this setting, we define a new polynomial invariant associated to any object in $\mathtt{R}$, the **multiple occurrences polynomial**, or MOP for short. The MOP construction in this section is rather general and corresponds to an infinite family of polynomial maps $\mathcal{A}(\mathtt{R}) \to \mathbb{K}[x]$, one for each choice of object $\mathrm{z} \in \mathtt{R}$.

After the definition of the MOP, we present a motivation for this construction. Our main theorem is that this construction is in fact a Hopf algebra morphism, and we leave the proof of this result to the end of the section. We also display an application of the cancellation free formula above, Theorem 7.6 to produce a reciprocity result, following the steps in [HM12], on the permutation pattern Hopf algebra.

Definition 8.1 (Multiple occurrence polynomial on a species with restrictions $\mathtt{R}$)**.** Fix an object z of the connected associative species with restrictions $\mathtt{R}$. We construct a map $\chi^{\mathrm{Z}} : \mathcal{A}(\mathtt{R}) \to \mathbb{K}[x]$. For that, fix a positive integer x and an object y in $\mathtt{R}$, χ^{Z} has

$$\chi^{\mathrm{z}}(\mathbf{pat}_{\mathrm{y}})(x) \coloneqq \mathbf{pat}_{\mathrm{y}}(\mathrm{z}^{\star x}) \,,$$

where $\mathrm{z}^{\star x}$ denotes the $\oplus$-product of z with itself x times.

That this indeed defines a polynomial, and that this is a Hopf algebra morphism, is the content of the main theorem below Theorem 8.6. Let us now see what values this invariant takes. For notation simplicity we write $\chi^{\mathrm{x}}_{\mathrm{y}} \coloneqq \chi^{\mathrm{x}}(\mathbf{pat}_{\mathrm{y}})$.

Example 8.2. We compute this invariant on particular objects for $\mathtt{R} = \mathtt{Per}$, defined above in Section 5.2. Take $\mathtt{z} = 1$, the unique permutation of size one. It is easy to see that $\chi^{\mathrm{z}}_{\mathrm{y}}(x) = 0$ whenever y is not an increasing permutation, and $\chi^{\mathrm{z}}(\mathbf{pat}_{\mathrm{z}^{\star k}})(x) = \binom{x}{k}$, a polynomial of degree k. In fact, we can observe something much more general: if π, τ have disjoint sets of $\oplus$-indecomposable factors, then $\chi^{\tau}_{\pi} = 0$

Let's now take $\mathtt{z} = 21$, the unique indecomposable permutation of size two. In this case, $\ker \chi^{\mathrm{z}} = \operatorname{span}\{\mathbf{pat}_{\pi} \,| \oplus\text{-factorization of } \pi \text{ contains a permutation } \tau \text{ with } |\tau| > 2\}$ from the observation above. If π decomposes into k_1 copies of $\tau_1 = 1$ and k_{21} copies of $\tau_2 = 21$, then $\chi^{\mathrm{z}}_{\pi}(x) = 2^{k_1}\binom{x}{k_1+k_{21}}$.

The MOP encapsulates the complexity of the species with restrictions quite well. One can easily observe that, for the species with restrictions $\mathtt{R} = \mathtt{Gr}$ on graphs, a general formula for $\chi^{\mathrm{z}}_{\mathrm{y}}(x)$ can be found for indecomposable y, depending only on $\mathbf{pat}_{\mathrm{y}}(\mathrm{z})$. This is not so simple for the permutation case. On $\mathtt{R} = \mathtt{MPer}$, the species with restrictions of marked permutations, the story is even more complicated as one does not have a unique factorization into indecomposables, making the description of all possible factorizations more complex. In this way, we ask the following question.

Conjecture 8.3. For any marked permutations $\mathtt{y}, \mathtt{z}$, the polynomial $\chi^{\mathrm{z}}_{\mathrm{y}}$ is a binomial multiplied by some constant, that is it maps $x \mapsto a\binom{x}{b}$ for a, b independent of x.

Observation 8.4. Fixing $x = 1$ we get a multiplicative map $\mathcal{A}(\mathtt{R}) \to \mathbb{K}$. This map is in fact a **character**, that is, a unital algebra homomorphism to the field $\mathbb{K}$.

In [ABS06], a Hopf algebra morphism was constructed from any connected graded Hopf algebra to $QSym$ using only a character. The astute reader may note that there is a classical *specialisation* map $QSym \to \mathbb{K}[x]$, so one should wonder if this χ^{z} arises from lifting a character $\mathcal{A}(\mathtt{R}) \to \mathbb{K}$ to a map $QSym \to \mathbb{K}[x]$, followed by composing with said specialisation.

Indeed, the evaluation map $\mathcal{A}(\mathtt{R}) \to \mathbb{K}$ defined by $\mathbf{pat}_{\mathrm{y}} \mapsto \mathbf{pat}_{\mathrm{y}}(\mathrm{z})$ is this character. We leave the proof of this fact to the interested reader, because it is mostly tedious unwravelling of definitions that closely relates to the proof of Theorem 8.7 below.

We now apply Theorem 7.6 to extract a reciprocity result for permutations.

Example 8.5. Let us get back to the case $\mathrm{z} = 21$. Because the antipode on polynomials simply flips the sign of the variable, we have

$$\chi^{21}_{\pi}(-x) = S(\chi^{21}(\mathbf{pat}_{\pi})(x)) = \chi^{21}(S(\mathbf{pat}_{\pi}))(x)$$

Setting $x = 1$ and using Theorem 7.6, we get

$$\chi^{21}_{\pi}(-1) = S(\mathbf{pat}_{\pi})(21) = (-1)^n \sum_{\sigma} \begin{bmatrix} \sigma \\ \pi_1, \dots, \pi_n \end{bmatrix} \mathbf{pat}_{\sigma}(21)$$

This allows us to quickly compute simple cases for $\pi = 12$, for instance $\chi^{21}_{\pi}(-1) = 1 \times \mathbf{pat}_{21}(21) = 1$.

8.1. **Proof of polynomiality and Hopf morphism.**

We present and prove the main theorem of this section.

Theorem 8.6 (MOPs in pattern algebras)**.** Let $\mathtt{R}$ be a connected associative species with restrictions and x an object. The map χ^{x} takes patterns to polynomials and is a Hopf algebra morphism $\mathcal{A}(\mathtt{R}) \to \mathbb{K}[x]$.

We recall that for each object y in a connected associative species with restrictions, any factorization into $*$-indecomposable elements has the same multiset of factors and, therefore, the same size, which we call $\ell(\mathrm{y})$.

Theorem 8.7. The invariant $\chi^{\mathrm{x}}_{\mathrm{y}}(n)$ is a polynomial in n. The degree of this polynomial is at most $\ell(\mathrm{y})$, the number of $*$-indecomposable factors of y (with repetition).

Proof. We act by induction on $|\mathrm{y}|$. If $|\mathrm{y}| = 0$, recall that $\mathtt{R}$ is connected, so $\chi^{\mathrm{x}}_{\mathrm{y}}(n) = 1$ for all n, which is a polynomial of degree zero.

For the induction hypothesis, observe that

$$\Delta\, \mathbf{pat}_{\mathrm{y}} = 1 \otimes \mathbf{pat}_{\mathrm{y}} + \mathbf{pat}_{\mathrm{y}} \otimes 1 + \sum_{\substack{a*b=y \\ \ell(a),\ell(b)<\ell(\mathrm{y})}} \mathbf{pat}_a \otimes \mathbf{pat}_b\,,$$

thus we have for $n > 1$. The key insight is that differences of consecutive values are polynomials of lower degree:

$$\mathbf{pat}_{\mathrm{y}}(\mathrm{x}^{*n}) = \mathbf{pat}_{\mathrm{y}}(\mathrm{x}^{*n-1}) + \mathbf{pat}_{\mathrm{y}}(\mathrm{x}) + \sum_{\substack{a*b=y\\|a|,|b|<|\mathrm{y}|}} \mathbf{pat}_a(\mathrm{x}) \otimes \mathbf{pat}_b(\mathrm{x}^{*n-1})$$

$$\mathbf{pat}_{\mathrm{y}}(\mathrm{x}^{*n}) - \mathbf{pat}_{\mathrm{y}}(\mathrm{x}^{*n-1}) = \mathbf{pat}_{\mathrm{y}}(\mathrm{x}) + \sum_{\substack{a*b=y\\|a|,|b|<|\mathrm{y}|}} \mathbf{pat}_a(\mathrm{x}) \chi^{\mathrm{x}}_b(n-1)$$

The right hand side is, by induction hypothesis, a polynomial of degree at most $\ell(\mathrm{y})-1$. The left hand side is $\chi^{\mathrm{x}}_{\mathrm{y}}(n) - \chi^{\mathrm{x}}_{\mathrm{y}}(n-1)$, which shows that $\chi^{\mathrm{x}}_{\mathrm{y}}(n)$ has degree at most $\ell(\mathrm{y})$. □

Lemma 8.8. The map $\mathbf{pat}_{\mathrm{y}} \mapsto \chi^{\mathrm{x}}_{\mathrm{y}}$ is a Hopf algebra morphism.

Proof. Observe the product is preserved precisely because the product on $\mathcal{A}(\mathtt{R})$ is the pointwise product: that is, if $\mathbf{pat}_{\mathrm{y}_1} \mathbf{pat}_{\mathrm{y}_2} = \sum_{\mathrm{z}} \binom{\mathrm{z}}{\mathrm{y}_1,\mathrm{y}_2}_{\mathtt{R}} \mathbf{pat}_{\mathrm{z}}$, then passing in the argument x^{*n} of both sides yields a true equality $\chi^{\mathrm{x}}_{\mathrm{y}_1}(n)\chi^{\mathrm{x}}_{\mathrm{y}_2}(n) = \sum_{\mathrm{z}} \binom{\mathrm{z}}{\mathrm{y}_1,\mathrm{y}_2}_{\mathtt{R}} \chi^{\mathrm{x}}_{\mathrm{z}}(n)$.

For the coproduct, we want to show that $\Delta \chi^{\mathrm{x}}_{\mathrm{y}} = \chi^{\mathrm{x}} \otimes \chi^{\mathrm{x}}(\Delta\, \mathbf{pat}_{\mathrm{y}})$. It is enough to show that $\Delta \chi^{\mathrm{x}}_{\mathrm{y}}(m \otimes n) = \chi^{\mathrm{x}} \otimes \chi^{\mathrm{x}}(\Delta\, \mathbf{pat}_{\mathrm{y}})(m \otimes n)$ for any non-negative integers m, n.

Write $\chi^{\mathrm{x}}_{\mathrm{y}} = \sum_{k\geq 0} a_k x^k$. On the left hand side, we get $\Delta\chi^{\mathrm{x}}_{\mathrm{y}}(m \otimes n) = \sum_{k\geq 0} a_k \sum_{j=0}^{k} \binom{k}{j} x^j \otimes x^{k-j}(m\otimes n) = \sum_{k\geq 0} a_k (m+n)^k = \chi^{\mathrm{x}}_{\mathrm{y}}(m+n)$.

On the right hand side we have

$$\begin{aligned}
(\chi^{\mathrm{x}} \otimes \chi^{\mathrm{x}})\Delta\, \mathbf{pat}_{\mathrm{y}}(m\otimes n) &= \sum_{a*b=\mathrm{y}} \chi^{\mathrm{x}}_a(m) \otimes \chi^{\mathrm{x}}_a(n) \\
&= \sum_{a*b=\mathrm{y}} \mathbf{pat}_a(x^{*m})\, \mathbf{pat}_b(x^{*n}) \\
&= \sum_{a*b=\mathrm{y}} \mathbf{pat}_a \otimes \mathbf{pat}_b(x^{*m} \otimes x^{*n}) \\
&= \Delta\, \mathbf{pat}_{\mathrm{y}}(x^{*n} * x^{*b}) = \Delta\, \mathbf{pat}_{\mathrm{y}}(x^{*m+n}) = \chi^{\mathrm{x}}_{\mathrm{y}}(n+m)\,.
\end{aligned}$$

This is exactly what we obtained earlier, so this map preserves coproducts. □

Proof of Theorem 8.6. This is a consequence of Theorem 8.7 and Lemma 8.8. □

Glossary of Key Terms and Abbreviations

This glossary provides brief definitions of frequently used terms and abbreviations throughout the paper.

- **QSS** (Quasi-Shuffle Signature): A tuple $\vec{\mathbf{I}} = (I_1, \dots, I_n)$ of subsets whose union covers the ground set of a permutation (or more generally, an object in a species with restrictions), where each subset represents a pattern occurrence. Unlike standard shuffles, the sets may overlap. See Section 1.
- **MOP** (Multiple Occurrences Polynomial): A polynomial invariant associated to objects in a species with restrictions, defined by evaluating pattern elements

at repeated copies of a fixed object. This invariant is a Hopf algebra morphism. See Definition 8.1.

- **NCF** (Non-Commuting Factorization): A property of species with restrictions where every element has a unique factorization into indecomposable elements. This property is crucial for obtaining cancellation-free antipode formulas. See Section 6.
- **Cancellation-free formula**: An antipode formula expressed as a sum where each term contributes its full value without intermediate cancellations. Formally, a formula where the combinatorial objects can be partitioned such that non-fixed-point pairs cancel completely. See introduction of this paper.
- **Grouping-free formula**: An antipode formula where each combinatorial object contributes independently, without requiring aggregation or grouping of multiple objects for simplification. See introduction of this paper.
- **Interlacing QSS**: A restricted subclass of quasi-shuffle signatures where the sets do not maintain a consistent ordering across both position and value orders. This constraint is essential for cancellation-free antipode formulas. See Section 1.
- **Species with restrictions**: A contravariant functor from the category of finite sets with injections to the category of sets, equipped with restriction maps that allow systematic decomposition of objects. This structure unifies the construction of all pattern Hopf algebras discussed in this paper.
- **Pattern Hopf algebra**: A Hopf algebra constructed from combinatorial objects via species with restrictions, where the product counts pattern occurrences (via quasi-shuffles) and the coproduct respects the decomposition of objects.
- **Sign-reversing involution method**: A combinatorial technique for obtaining cancellation-free antipode formulas using involutions that reverse the signs of non-fixed-point elements in a sum, allowing terms to cancel completely. See Section 2.

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