ANTIPODES IN PATTERN HOPF ALGEBRAS

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ABSTRACT. An abstract

CONTENTS

1. INTRODUCTION

In [Tak71], Takeuchi obtained a quite general formula for the antipode of a Hopf algebra. This is an antipode formula for any filtered Hopf algebra, and the methods used in its proof can be applied in much more generality. However, we can observe that it is not the most economical formula, as it leaves some cancellations to be made.

A method for obtaining a cancellation-free formula that seemed to work with a large family of Hopf algebras was brought forth by [BS17], called the **sign-reversing involution method**. This is a classic method in enumerative combinatorics, that was successfully brought to the realm of Hopf algebras by providing, in some cases, an easy avenue to obtain a cancellation-free antipode formula. This method revolves around a careful treatment of the antipode formula of Takeuchi, but it varies widely according to the combinatorial Hopf algebra at hand. Therefore, for each Hopf algebra a new and original application of the method is needed. Notwithstanding, this has been shown to

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work in the incidence Hopf algebra on graphs and in the Malvenuto–Reutenauer Hopf algebra (see [MR95]), among many others.

In [AA17], a cancellation-free antipode formula for the Hopf monoid on generalized permutahedra was found. This was particularly striking, because several interesting combinatorial Hopf monoids are Hopf submonoids of the generalized permutahedra. Specific examples are graphs, matroids, posets, set partitions, paths, simplicial complexes and building sets. Thus, this allows us to readily determine a cancellation-free formula for all these Hopf algebras.

We present in this section yet another antipode formula. This is an antipode formula for the pattern Hopf algebra on permutations, and uses neither of the methods above. However, it is clear to the author that this antipode formula can be also obtained by the sign-reversing involution method, despite not having explicitly done so.

In this section we start by describing the sign-reversing involution method to obtain a cancellation-free formula for the antipode of a Hopf algebra. Thus, we present some examples where we compute some antipodes via *ad hoc* methods. Then we present the main result of this section in Theorem 1.3, a cancellation-free formula for the antipode of $\mathcal{A}(\texttt{Per})$.

1.1. **The sign-reversing involution method.**

The application of the **sign-reversing involution method** to compute antipodes of Hopf algebras was first presented in [BS17]. This is a method to find cancellation-free formulas for the antipode of a Hopf algebra. It starts in the formula given by Takeuchi, and keeps track of all the terms to be summed, usually by means of compositions, that arise in this formula. Thus, the sum obtained runs over a collection of objects, say $\mathcal{O}$, that is partitioned into families indexed by compositions. These compositions play an important role, as their length determines the sign of the corresponding objects.

Recall that an involution ζ is a map such that $\zeta \circ \zeta$ is the identity. The sign-reversing involution method simply points out some ways in which we can describe an involution in $\mathcal{O}$, call it ζ, in such a way that if $\zeta(x) \neq x$, then x and $\zeta(x)$ contribute with opposite signs to the antipode, and can therefore be canceled. An example where this method is applied is given in Section 2.

1.2. **Examples of antipodes in the polynomial Hopf algebra and pattern algebras.**

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Let us first discuss an example on the Hopf algebra on polynomials. There, the formula from Takeuchi gives us that

On the other hand, we can go ahead and compute explicitly $S(x^3)$ via **??**. Indeed, we have

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We now present another example, this time on the permutation pattern Hopf algebra. Consider $\pi = 132 = 1 \oplus 21$. Then

$$\begin{aligned} S(\mathbf{p}_{132}) &= \sum_{k=0}^{2} (-1)^k \mu^{\circ k-1} \circ (\mathrm{id}_{\mathbb{K}[x]} - \iota \circ \varepsilon)^{\otimes k} \circ \Delta^{\circ k-1}(\mathbf{p}_{132}) \\ &= -(\mathrm{id}_{\mathbb{K}[x]} - \iota \circ \varepsilon)(\mathbf{p}_{132}) + \mu \circ (\mathrm{id}_{\mathbb{K}[x]} - \iota \circ \varepsilon)^{\otimes 2}(\mathbf{p}_1 \otimes \mathbf{p}_{12}) \\ &= -\mathbf{p}_{132} + \mathbf{p}_1 \, \mathbf{p}_{21} = 2\,\mathbf{p}_{321} + 2\,\mathbf{p}_{231} + \mathbf{p}_{213} + \mathbf{p}_{132} + 2\,\mathbf{p}_{312} + \mathbf{p}_{21} \, . \end{aligned} \tag{1}$$

1.3. **The antipode in the pattern algebra on permutations.** We will now introduce some notation, in order to present the cancellation-free formula for the antipode of the permutation pattern Hopf algebra.

Definition 1.1. Given permutations $\pi_1, \dots, \pi_k$ and σ, where σ is a permutation on the set I, we say that a k-tuple $(I_1, \dots, I_k)$ is a **quasi-shuffle signature** (or simply a QSS) of σ from $\pi_1, \dots, \pi_k$ if we have that $\sigma|_{I_i} = \pi_i$ for $i = 1, \dots, k$, and $\bigcup_i I_i = I$. We will simply refer to a quasi-shuffle signature of σ from $\pi_1, \dots, \pi_k$ as a QSS of σ, whenever $\pi_1, \dots, \pi_k$ are clear from the context.

Given two sets of integers A and B, we write $A < B$ if for any $a \in A, b \in B$ we have that $a < b$. Recall that for a permutation $\sigma = (\leq_P, \leq_V)$ in a set I it corresponds a bijection $\tilde{\sigma}$ of I, see **??**. If additionally we are given a subset $J \subseteq I$, we write $\sigma(J) \subseteq I$ for the elements of the form $\tilde{\sigma}(j)$ for some $j \in J$. We say that a QSS of σ from $\pi_1, \dots, \pi_k$ is **non-interlacing** if there exists some $i = 1, \dots k-1$ such that $I_i < I_{i+1}$ and $\sigma(I_i) < \sigma(I_{i+1})$. Otherwise, we say that a QSS is **interlacing**.

Furthermore, we define the **interlacing quasi-shuffle coefficient** as

$$\left[{\sigma \atop \pi_1, \dots, \pi_k} \right] = |\{\text{interlacing QSS of } \sigma \text{ from } \pi_1, \dots, \pi_k\}| \, .$$

Note that the quasi-shuffle coefficient on permutations can be written as

$$\binom{\sigma}{\pi_1, \dots, \pi_k} = |\{\text{QSS of } \sigma \text{ from } \pi_1, \dots, \pi_k\}| \, . \tag{2}$$

Example 1.2. If we consider $\sigma = 312$, a permutation in $\{1, 2, 3\}$, $\pi_1 = 1$ and $\pi_2 = 21$, then there are two QSS of σ, specifically $(\{2\}, \{1, 3\})$ and $(\{3\}, \{1, 2\})$. Therefore, $\binom{\sigma}{\pi_1, \pi_2} = 2$. Note that all these QSS are interlacing, thus $\left[{\sigma \atop \pi_1, \pi_2} \right] = 2$.

On the other hand, if $\sigma = 132$, and π_1, π_2 as before, we can conclude that $\binom{\sigma}{\pi_1, \pi_2} = 1$ and $\left[{\sigma \atop \pi_1, \pi_2} \right] = 0$.

We introduce now the main result of this section.

Theorem 1.3 (Cancellation-free formula for the antipode of $\mathcal{A}(\mathtt{Per})$)**.** Let π be a permutation, and suppose that $\pi_1, \dots, \pi_k$ are $\oplus$-indecomposable permutations such that $\pi = \pi_1 \oplus \cdots \oplus \pi_k$. Then

$$S(\mathbf{p}_\pi) = (-1)^k \sum_{\sigma} \mathbf{p}_\sigma \left[{\sigma \atop \pi_1, \dots, \pi_k} \right] .$$

Example 1.4. With this, we can compute the antipode of $\mathbf{p}_{132}$ directly. First, observe that $132 = 1 \oplus 21$ so we just need to compute some interlacing quasi-shuffle coefficients for $\pi_1 = 1$ and $\pi_2 = 21$. Specifically:

$$S(\mathbf{p}_{132}) = (-1)^2 \left(\left[{21 \atop 1,21} \right] \mathbf{p}_{21} + \sum_{|\sigma|=3} \left[{\sigma \atop 1,21} \right] \mathbf{p}_\sigma \right) = 2\,\mathbf{p}_{321} + 2\,\mathbf{p}_{231} + \mathbf{p}_{213} + \mathbf{p}_{132} + 2\,\mathbf{p}_{312} + \mathbf{p}_{21} \,.$$

In this section we exhibit an application of the sign-reversing involution method in the Hopf algebra of polynomials. Then, we present and prove the cancellation-free formula for the permutation pattern Hopf algebra.

2. The antipode formula for the polynomial Hopf algebra and the sign-reversing involution method

The following is a computation from [BS17]. We would like to point out that this is the most complicated proof of an antipode formula for the polynomial Hopf algebra, and much simpler methods exist, as the interested reader can find directly, or by consulting [GR14]. However, it shows the power of Takeuchi's formula, as the whole process can be done with little recourse to intuition.

Theorem 2.1 (The antipode formula for the polynomial Hopf algebra)**.** The antipode S for $\mathbb{K}[x]$ is

$$S(x^n) = (-x)^n \,.$$

First, we introduce weak compositions. A weak composition of n is a list of non-negative integers $\alpha = (\alpha_1, \dots, \alpha_l)$ that may be zero, such that they sum up to n. We denote its length by $l(\alpha) = l$, and write $\alpha \models^0 n$.

A weak set composition of a set I is a list of disjoint sets $\vec{\boldsymbol{\pi}} = A_1 | \dots | A_l$ such that $\biguplus_j A_j = I$. These sets may be empty. We denote its length by $l(\vec{\boldsymbol{\pi}}) = l$, and write $\vec{\boldsymbol{\pi}} \models^0 I$.

Proof. We use the Takeuchi formula given in **??**, which holds for graded Hopf algebras, as established in **??**:

$$S(x^n) = \sum_{k=0}^{n} (-1)^k \mu^{\circ k-1} \circ (\mathrm{id}_{\mathbb{K}[x]} - \iota \circ \varepsilon)^{\otimes k} \circ \Delta^{\circ k-1}(x^n) \,.$$

However, observe that the following holds by induction:

$$\Delta^{\circ k-1}(x^n) = \sum_{(A_1,\dots,A_k) \models^0 [n]} (x^{|A_1|} \otimes \cdots \otimes x^{|A_k|}) \,,$$

where the sum runs over weak set compositions of $[n]$ of size k, that may contain empty sets. Thus, the antipode of x^n is given by

$$
\begin{aligned}
S(x^n) &= \sum_{j=0}^{n}(-1)^j \sum_{(A_1,\dots,A_j)\models^0[n]} \mu^{\circ j-1}\circ(\mathrm{id}_{\mathbb{K}[x]} - \iota\circ\varepsilon)^{\otimes j}(x^{|A_1|}\otimes\cdots\otimes x^{|A_j|}) \\
&= \sum_{j=0}^{n}(-1)^j \sum_{(A_1,\dots,A_j)\models[n]} \mu^{\circ j-1}(x^{|A_1|}\otimes\cdots\otimes x^{|A_j|}) \qquad (3)\\
&= \sum_{j=0}^{n}(-1)^j \sum_{(A_1,\dots,A_j)\models[n]} x^n = x^n \sum_{\vec{\boldsymbol{\pi}}\in\mathbf{C}_n}(-1)^{l(\vec{\boldsymbol{\pi}})}\,.
\end{aligned}
$$

Consider for the moment the following involution $\zeta:\mathbf{C}_n\to\mathbf{C}_n$. For $\vec{\boldsymbol{\pi}}=(A_1,\dots,A_k)$, let $j_{\vec{\boldsymbol{\pi}}}$ be the smallest index such that $|A_j|\neq 1$ or $\max A_j<\max A_{j+1}$. Then, there are three cases

1. The set A_j is a singleton with $j\leq k-1$, then let $\zeta(\vec{\boldsymbol{\pi}})$ be the set composition resulting from merging A_j and A_{j+1}.
2. The set A_j is not a singleton, then we define $\zeta(\vec{\boldsymbol{\pi}})$ to be the set composition resulting from splitting A_j into $\{\max A_j\}$ and $A_j\setminus\{\max A_j\}$.
3. There is no such j. Then $\vec{\boldsymbol{\pi}}=(\{n\},\dots,\{1\})$ and we define $\zeta(\vec{\boldsymbol{\pi}})=\vec{\boldsymbol{\pi}}$.

It is a direct observation that ζ is an involution. In fact, the only fixed point is $\vec{\boldsymbol{\pi}}=(\{n\},\dots,\{1\})$, and for any other set composition $\vec{\boldsymbol{\pi}}$, whenever the smallest index $j_{\vec{\boldsymbol{\pi}}}$ in $\vec{\boldsymbol{\pi}}$ behaves as described in case 1, then the smallest index $j_{\zeta(\vec{\boldsymbol{\pi}})}$ in $\zeta(\vec{\boldsymbol{\pi}})$ behaves as described in case 2, in which case we have $l(\vec{\boldsymbol{\pi}})=1+l(\zeta(\vec{\boldsymbol{\pi}}))$ and we can easily see that $\zeta(\zeta(\vec{\boldsymbol{\pi}}))=\vec{\boldsymbol{\pi}}$. We also have the converse statement.

Thus, we have that $x^n\sum_{\vec{\boldsymbol{\pi}}\in\mathbf{C}_n}(-1)^{l(\vec{\boldsymbol{\pi}})}=x^n(-1)^n=(-x)^n$, as desired. □

3. The antipode formula for the pattern algebra on permutations

Fix a permutation π, such that $\pi=\pi_1\oplus\cdots\oplus\pi_k$ is its $\oplus$-factorization into $\oplus$-indecomposable permutations. We will assume this notation for the remaining of this section.

The intermediate step to conclude Theorem 1.3 is Lemma 3.4, which is a description of the interlacing coefficients via an inclusion-exclusion sum. In order to establish this, we define a classical poset structure in compositions, which we will see is isomorphic to a Boolean poset. This allows us to simplify sums quite drastically, which leads us to the desired formula.

Definition 3.1 (Order on compositions)**.** We can endow the set of compositions of n with an order $\leq$ as follows: We say that $\alpha\leq\beta$ if β results from α by merging two consecutive entries. By taking the transitive closure, we have a poset in $\mathcal{C}_n$. Note that (n) is the unique minimal element, whereas $(1,\dots,1)$ is the unique maximal element.

Remark 3.2. This results in the **opposite** order of the coarsening order, defined in **??** for set partitions and set compositions, and extended in **??** to an order $\leq'$ in compositions.

We now descibe the bijection between $\mathcal{C}_n$ and the Boolean poset $[k-1]$.

Definition 3.3 (Cumulative sum map)**.** Assume $n \geq 1$. Then we define an explicit bijection between subsets of $[n-1]$ and the set $\mathcal{C}_n$: if $\alpha = (\alpha_1, \dots, \alpha_j)$ is a set composition, then we define

$$\mathbf{CS}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \cdots + \alpha_{j-1}\}\,.$$

This is a subset of $[n-1]$. That this is a bijection is immediate, as an inverse can be readily constructed.

We further define the integers $\mathbf{CS}(\alpha)_1 < \mathbf{CS}(\alpha)_2 < \cdots < \mathbf{CS}(\alpha)_{j-1}$ such that

$$\mathbf{CS}(\alpha) = \{\mathbf{CS}(\alpha)_1, \mathbf{CS}(\alpha)_2, \dots, \mathbf{CS}(\alpha)_{j-1}\}\,.$$

and set $\mathbf{CS}(\alpha)_0 = 0$ and $\mathbf{CS}(\alpha)_j = n$.

Furthermore, this map is a poset isomorphism, where it maps $\leq$, the partial order in $\mathcal{C}_n$ introduced in Definition 3.1, to the inclusion of sets. Details on this map are given in [Sta00].

Recall that we fix a permutation π with $\pi = \pi_1 \oplus \cdots \oplus \pi_k$. Given a composition $\alpha \models n$, and an integer $i \in \{1, \dots, l(\alpha)\}$, we write

$$\pi_\alpha^{(i)} = \pi_{\mathbf{CS}(\alpha)_{i-1}+1} \oplus \cdots \oplus \pi_{\mathbf{CS}(\alpha)_i}\,.$$

This notation can be further extended to weak compositions α by setting $\pi_\alpha^{(i)} = \pi_\beta^{(i)}$, where β is the composition resulting from erasing the zeros from α.

Lemma 3.4.

$$\left[{\sigma \atop \pi_1, \dots, \pi_k}\right] = \sum_{\alpha \models n} (-1)^{l(\alpha)} \binom{\sigma}{\pi_\alpha^{(1)}, \dots, \pi_\alpha^{(l(\alpha))}}\,.$$

We now see that this result suffices to establish the main result of this section.

Proof of Theorem 1.3. We simply apply Takeuchi's formula and use Lemma 3.4 when needed:

$$\begin{aligned}
S(\mathbf{p}_\pi) &= \sum_{j=0}^{k} (-1)^j \mu^{\circ j-1} \circ (\mathrm{id}_{\mathcal{A}(\mathtt{Per})} - \iota \circ \varepsilon)^{\otimes j} \circ \Delta^{\circ j-1}(\mathbf{p}_\pi) \\
&= \sum_{\alpha \models^0 k} (-1)^{l(\alpha)} \mu^{\circ l(\alpha)-1} \circ (\mathrm{id}_{\mathcal{A}(\mathtt{Per})} - \iota \circ \varepsilon)^{\otimes l(\alpha)} (\mathbf{p}_{\pi_\alpha^{(1)}} \otimes \cdots \otimes \mathbf{p}_{\pi_\alpha^{(l(\alpha))}}) \\
&= \sum_{\alpha \models k} (-1)^{l(\alpha)} \mu^{\circ l(\alpha)-1} (\mathbf{p}_{\pi_\alpha^{(1)}} \otimes \cdots \otimes \mathbf{p}_{\pi_\alpha^{(l(\alpha))}}) = \sum_{\alpha \models k} (-1)^{l(\alpha)}\, \mathbf{p}_{\pi_\alpha^{(1)}} \cdots \mathbf{p}_{\pi_\alpha^{(l(\alpha))}} \\
&= \sum_{\alpha \models k} (-1)^{l(\alpha)} \sum_\sigma \mathbf{p}_\sigma \binom{\sigma}{\mathbf{p}_{\pi_\alpha^{(1)}}, \dots, \mathbf{p}_{\pi_\alpha^{(l(\alpha))}}} \\
&= \sum_\sigma \mathbf{p}_\sigma \sum_{\alpha \models k} (-1)^{l(\alpha)} \binom{\sigma}{\mathbf{p}_{\pi_\alpha^{(1)}}, \dots, \mathbf{p}_{\pi_\alpha^{(l(\alpha))}}} = \sum_\sigma \mathbf{p}_\sigma \left[{\sigma \atop \pi_1, \dots, \pi_k}\right],
\end{aligned} \tag{4}$$

where in the last equality we used Lemma 3.4. ∎

In order to establish Lemma 3.4, we proceed as follows: For a composition α, we introduce a notion of α-QSS of a permutation σ. We see that any interlacing QSS of σ cannot be "extended" to a non-trivial α-QSS of a permutation σ. Finally, we see that in Lemma 3.10, all the QSS of σ that can be extended will cancel its contribution to the interlacing quasi-shuffle coefficient.

Definition 3.5. Fix a permutation σ and a composition α. We say that $\vec{\mathbf{I}}$ is an α-QSS of σ from $\pi_1, \dots, \pi_k$ if $\vec{\mathbf{I}}$ is a QSS of σ from $\pi_\alpha^{(1)}, \dots, \pi_\alpha^{(l(\alpha))}$. When the permutations $\pi_1, \dots, \pi_k$ are clear from context, we simply write that $\vec{\mathbf{I}}$ is an α-QSS of σ.

Assume that $\vec{\mathbf{I}}$ is an α-QSS of σ, then we can construct a canonical QSS of σ, which we write $\vec{\mathbf{J}} = \sigma(\vec{\mathbf{I}})$, as follows: For each $i = 1, \dots, l(\alpha)$, let $J_1^{(i)}, \dots, J_{\alpha_i}^{(i)}$ be the unique choice of sets such that

- $\biguplus_{j=1}^{\alpha_i} J_j^{(i)} = I_i$;
- $J_j^{(i)} < J_{j+1}^{(i)}$ for any $j = 1, \dots, \alpha_i - 1$;
- $\sigma|_{J_j^{(i)}} = \pi_{\mathbf{CS}(\alpha)_{i-1}+j}$ for any $j = 1, \dots, \alpha_i$.

This is possible, because $\sigma|_{I_i} = \pi_\alpha^{(i)} = \pi_{\mathbf{CS}(\alpha)_{i-1}+1} \oplus \cdots \oplus \pi_{\mathbf{CS}(\alpha)_i}$. Then, by letting $\vec{\mathbf{J}} = (J_1^{(1)}, \dots, J_{\alpha_1}^{(1)}, J_1^{(2)}, \dots)$ we obtain a QSS of σ.

Finally, fix a permutation σ, and let $\vec{\mathbf{J}}$ be a QSS of σ. Then, define

$$\mathcal{I}_{\vec{\mathbf{J}}}^{\sigma,\pi} = \{\alpha \models k \mid \exists \vec{\mathbf{I}} \text{ QSS of } \sigma \text{ s.t. } \sigma(\vec{\mathbf{I}}) = \vec{\mathbf{J}}\}.$$

Lemma 3.6. Fix a permutation π and σ, and let $\pi_1, \dots, \pi_k$ be as above. Given $\vec{\mathbf{J}}$ a QSS of σ, and α a composition of k, then there is at most one $\vec{\mathbf{I}}$ that is an α-QSS of σ such that $\sigma(\vec{\mathbf{I}}) = \vec{\mathbf{J}}$.

Furthermore, $\vec{\mathbf{J}}$ is a $(1, \dots, 1)$-QSS of σ and $\sigma(\vec{\mathbf{J}}) = \vec{\mathbf{J}}$.

Proof. That $\vec{\mathbf{J}}$ is a $(1, \dots, 1)$-QSS of σ and $\sigma(\vec{\mathbf{J}}) = \vec{\mathbf{J}}$ is immediate by definition. The fact that there is at most one such $\vec{\mathbf{I}}$, follows from the fact that the procedure $\sigma(\vec{\mathbf{I}}) = \vec{\mathbf{J}}$ is invertible if we know α: if $\sigma(\vec{\mathbf{I}}) = \vec{\mathbf{J}}$, then $\vec{\mathbf{I}}$ results from $\vec{\mathbf{J}} = (J_1, \dots, J_k)$ by taking the union of consecutive sets J_i according to α. Thus it follows that such QSS $\vec{\mathbf{I}}$ is unique. □

Remark 3.7. Having established the uniqueness, one can wonder if we can also guarantee the existence. That is in fact not the case, because the procedure of "taking the union of consecutive sets J_i according to α" does not guarantee that this union will satisy the properties of QSS, namely that $\sigma|_{I_i} = \pi_\alpha^{(i)}$.

In fact, we will see now that if $\vec{\mathbf{J}}$ is an interlacing QSS of σ, then there is no such α and such $\vec{\mathbf{I}}$, except the trivial choice $\vec{\mathbf{I}} = \vec{\mathbf{J}}$ and $\alpha = (1, \dots, 1)$.

Lemma 3.8. Fix a permutation π and σ, and let $\pi_1, \dots, \pi_k$ be as above. Fix $\vec{\mathbf{J}}$ a QSS of σ. Then $|\mathcal{I}_{\vec{\mathbf{J}}}^{\sigma,\pi}| = 1$ if and only if $\vec{\mathbf{J}}$ is interlacing QSS of σ.

Proof. Assume that $\vec{\mathbf{J}} = (J_1, \dots, J_k)$ is non-interlacing QSS of σ. Then, there is some $i = 1, \dots, k-1$ such that $J_i < J_{i+1}$ and $\sigma(J_i) < \sigma(J_{i+1})$. We claim that $\alpha = (\underbrace{1, \dots, 1}_{i-1 \text{ times}}, 2, \underbrace{1, \dots, 1}_{k-i-1 \text{ times}}) \in \mathcal{I}^{\sigma,\pi}_{\vec{\mathbf{J}}}$.

In fact, we have that $\vec{\mathbf{I}} = (J_1, \dots J_{i-1}, J_i \cup J_{i+1}, J_{i+2}, \dots, J_k)$ is precisely the desired α-QSS, since we have by construction that $\sigma(\vec{\mathbf{I}}) = \vec{\mathbf{J}}$. Further, the non-interlacing condition gives us that $\sigma|_{J_i \cup J_{i+1}} = \pi_i \oplus \pi_{i+1}$.

Thus $|\mathcal{I}^{\sigma,\pi}_{\vec{\mathbf{J}}}| \neq 1$, as desired.

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On the other hand, if $\vec{\mathbf{J}}$ is interlacing, suppose that there is some non-trivial $\alpha \in \mathcal{I}^{\sigma,\pi}_{\vec{\mathbf{J}}}$, and consider $\vec{\mathbf{I}}$ its corresponding α-QSS of σ. Then $\vec{\mathbf{J}} = \sigma(\vec{\mathbf{I}})$, and this contradicts the assumption that $\vec{\mathbf{J}}$ is interlacing by construction of $\sigma(\vec{\mathbf{I}})$. □

Lemma 3.9. The set $\mathcal{I}^{\sigma,\pi}_{\vec{\mathbf{J}}}$ is an ideal with a unique minimum.

Proof. Let $P = \{i \in [k-1] | J_i < J_{i+1} \text{ and } \sigma(J_i) < \sigma(J_{i+1})\}$. We claim that $\mathcal{I}^{\sigma,\pi}_{\vec{\mathbf{J}}} = \{\alpha \models [k-1] | \mathbf{CS}(\alpha) \cup P = [k-1]\}$. That is, $\mathbf{CS}^{-1}([k-1] \setminus P)$ is the unique minimal element in $\mathcal{I}^{\sigma,\pi}_{\vec{\mathbf{J}}}$

First, let $\alpha \geq \mathbf{CS}^{-1}([k-1] \setminus P)$. We observe that it is straightforward to construct an α-QSS of σ, say $\vec{\mathbf{I}}$, such that $\sigma(\vec{\mathbf{I}}) = \vec{\mathbf{J}}$. The definition of P gives us that we indeed obtain an α-QSS of σ.

On the other hand, suppose that $\vec{\mathbf{J}} = \sigma(\vec{\mathbf{I}})$, where $\vec{\mathbf{I}}$ is an α-QSS of σ such that $\mathbf{CS}(\alpha) \cup P \neq [k-1]$. Say $j \in [k-1]$ is such that $j \notin \mathbf{CS}(\alpha) \cup P$. Then either $J_j \not< J_{j+1}$ or $\sigma(J_j) \not< \sigma(J_{j+1})$.

However, from $j \notin \mathbf{CS}(\alpha)$, we know that there is some $i \in \{1, \dots, l(\alpha)\}$ such that $J_j, J_{j+1} \subseteq I_i$. From $\vec{\mathbf{J}} = \sigma(\vec{\mathbf{I}})$ we know that J_j, J_{j+1} are obtained from the $\oplus$-decomposition of $\sigma|_{I_i}$. This contradicts the assumption that either $J_j \not< J_{j+1}$ or $\sigma(J_j) \not< \sigma(J_{j+1})$. □

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Lemma 3.10. Fix a permutation π and σ, and let $\pi_1, \dots, \pi_k$ be as above. Fix $\vec{\mathbf{J}}$ a QSS of σ. Then

$$\sum_{\alpha \in \mathcal{I}^{\sigma,\pi}_{\vec{\mathbf{J}}}} (-1)^{l(\alpha)} = \begin{cases} (-1)^k, & \text{if } |\mathcal{I}^{\sigma,\pi}_{\vec{\mathbf{J}}}| = 1, \\ 0, & \text{otherwise} \end{cases} .$$

Proof. The case where $|\mathcal{I}^{\sigma,\pi}_{\vec{\mathbf{J}}}| = 1$ is trivial, because in this case we have $\mathcal{I}^{\sigma,\pi}_{\vec{\mathbf{J}}} = \{\underbrace{(1, \dots, 1)}_{k \text{ times}}\}$. Otherwise, from Lemma 3.9 we have that $\mathcal{I}^{\sigma,\pi}_{\vec{\mathbf{J}}}$ is an ideal with a unique minimum that contains $\{\underbrace{(1, \dots, 1)}_{k \text{ times}}\}$. Thus, $\mathbf{CS}(\mathcal{I}^{\sigma,\pi}_{\vec{\mathbf{J}}})$ is an ideal with a unique minimum, so there is a set $M \subsetneq [k-1]$ such that

$$\mathbf{CS}(\mathcal{I}^{\sigma,\pi}_{\vec{\mathbf{J}}}) = \{K \subseteq [k-1] | M \subseteq K\} .$$

Thus, we have that

$$\sum_{\alpha\in\mathcal{I}^{\sigma,\pi}_{\vec{\mathbf{J}}}}(-1)^{l(\alpha)} = \sum_{\substack{M\subseteq \mathbf{CS}(\alpha)\\ \mathbf{CS}(\alpha)\subseteq[k-1]}}(-1)^{l(\alpha)} = \sum_{\substack{M\subseteq K\\ K\subseteq[k-1]}}(-1)^{|K|+1} = 0\,, \tag{5}$$

where the last equality is a simple binomial identity. □

With this we can finally prove Lemma 3.4, which concludes the proof of the cancellation-free formula of the antipode of $\mathcal{A}(\mathtt{Per})$.

Proof of Lemma 3.4. We use (2) to develop the sum at hand, yielding:

$$\begin{aligned}\sum_{\alpha\models n}(-1)^{l(\alpha)}\binom{\sigma}{\pi_\alpha^{(1)},\dots,\pi_\alpha^{(l(\alpha))}} &= \sum_{\alpha\models n}(-1)^{l(\alpha)}\sum_{\vec{\mathbf{I}}\text{ is }\alpha-\text{QSS of }\sigma} 1\\ &=\sum_{\alpha\models n}(-1)^{l(\alpha)}\sum_{\vec{\mathbf{J}}\text{ is QSS of }\sigma}\mathbb{1}[\exists\ \vec{\mathbf{I}}\,\alpha-\text{QSS s.t. }\sigma(\vec{\mathbf{I}})=\vec{\mathbf{J}}]\\ &=\sum_{\vec{\mathbf{J}}\text{ is QSS of }\sigma}\ \sum_{\substack{\alpha\models n\\ \alpha\in\mathcal{I}^{\sigma,\pi}_{\vec{\mathbf{J}}}}}(-1)^{l(\alpha)}\\ &=\sum_{\substack{\vec{\mathbf{J}}\text{ is QSS of }\sigma\\ |\mathcal{I}^{\sigma,\pi}_{\vec{\mathbf{J}}}|=1}} 1\,,\end{aligned} \tag{6}$$

where in the last sum we used Lemma 3.10.

Thus, we are to compute

$$|\{\vec{\mathbf{J}}\text{ is QSS of }\sigma\text{ s.t.}|\mathcal{I}^{\sigma,\pi}_{\vec{\mathbf{J}}}|=1\}|\,,$$

which is precisely $\left[{\sigma \atop \pi_1,\dots,\pi_k}\right]$, according to Lemma 3.8. This concludes the proof. □

Problem 3.11. Can we obtain a cancellation-free formula for the antipode of $\mathcal{A}(\mathtt{Per})$ via the sign-reversing involution method?

4. Antipode problem on marked permutation

We start by recovering the unique factorization theorem on marked permutations

Theorem 4.1. Let $\pi^* = \rho_1^*\star\cdots\star\rho_j^*$ be a factorization of a marked permutation into irreducible marked permutations. Then this factorization is unique, up to $\oplus$-relations and $\ominus$-relations.

We start by defining the $\times$ operation and α map marked permutations. For that we consider the notion of QSS ina general presheaf-

Definition 4.2 (QSS on associative presheaves)**.** Consider an associative presheaf $(h, *, 1)$. Given objects $p_1, \ldots, p_k$ and s, where $\mathbb{X}(s) = I$, we say that a k-tuple $(I_1, \ldots, I_k)$ is a **quasi-shuffle signature** (or simply a QSS) of s from $p_1, \ldots, p_k$ if we have that $s|_{I_i} = p_i$ for $i = 1, \ldots, k$, and $\bigcup_i I_i = I$. We will simply refer to a quasi-shuffle signature of s from $p_1, \ldots, p_k$ as a QSS of s, whenever $p_1, \ldots, p_k$ are clear from the context.

We say that a QSS of s from $p_1, \ldots, p_k$ is **non-interlacing** if there exists some $i = 1, \ldots k-1$ such that $s|_{I_i \cup I_{i+1}} \neq s|_{I_i} * s|_{I_{i+1}}$ Otherwise, we say that a QSS is **interlacing**.

Furthermore, we define the **interlacing quasi-shuffle coefficient** as

$$\left[{s \atop p_1, \ldots, p_k}\right] = |\{\text{interlacing QSS of } s \text{ from } p_1, \ldots, p_k\}| \, .$$

Definition 4.3 (Dual operation)**.** Given objects $r_1, \ldots, r_k$, we define

$$r_1 \times \cdots \times r_k = \sum_{s \in \mathcal{A}(h)} s \left[{s \atop r_1, \ldots, r_k}\right] .$$

This can be extended as a multilinear function, so it defines an operation $\mathcal{A}(h)^k \to \mathcal{A}(h)$.

Theorem 4.4. This defines an associative operation $\times$ on $\mathcal{A}(h)$.

Theorem 4.5. Let α^* be a marked permutation, and assume that $\alpha^* = \gamma_1^* \star \cdots \star \gamma_k^*$ is the *short-Q* factorization of α^*.

$$S(\alpha^*) = \alpha(\gamma_1^*) \times \cdots \times \alpha(\gamma_k^*) \, .$$

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